PRINCIPLES OF GEOMETRY Boxed Set

H. F. BAKER

Introductory Note by
John Clark

Introduction to the Principles of Grammar

During the evolution of a mind, it is not, nor can be, rational, i.e. cannot function by, or produce, functional grammar systems…

One of the earliest sources which point out the fact humanity will be functionally illiterate is today known, cumulatively, as the Judeo-Christian Scripture. This is possibly the only book in our history that repeatedly tells the reader humanity will not even be able to read that book until a certain time in human history, i.e., a stage in human evolution. Language does separate mankind from the other animals, however, mankind is not yet functionally literate and cannot take advantage of that distinction to the point he will survive.

Proving this is rather simple, unless one is hopelessly illiterate. Ask your computer: *How many parts of speech are possible*? Every moment that a computer is functioning, it produces all the things it can do by simple binary recursion. The Bible put this fact into a metaphor about life:—

Adam and Eve are a Conjugate Binary Pair that, *by Complete Induction and Deduction*, *produce the human race*.

As written, if one cannot read the book, then literacy is the key to opening it, just like any other book. The Key to the House of David, a simple puzzle, is simply literacy, something most of mankind believe that they are capable of, but by example prove to be the opposite.

If the metaphor is too esoteric then one has the *Dialogs* of Plato where Plato called binary recursion *Dialectic*, speech as a binary function. Or one can study Euclid’s *Elements*. Or one can refer to the definition of any thing as a relative constrained by correlatives, which again, Plato called nouns and verbs. One can even examine their own body, for any particular sense can abstract either the form, limit, of a thing, or it can abstract the material difference which is again a simple binary distinction.

Binary recursion, when applied to the construction of grammar systems, produce methods of naming, or one can say counting, in order to manage memory. Binary recursion, used for the construction of grammar systems, produce four, and only four, categories of grammar, or four distinct methods of binary recursion: Common Grammar, Arithmetic, Algebra and Geometry. The functionally illiterate cannot comprehend or take advantage of this, of being literate.

These works give a study some material by which to study illiteracy and literacy alike.

Mankind actually believes that one grammar can contradict another, a stance championed by the Einsteins of the world, however, binary recursion is not different from itself, and binary recursion can only produce a binary result. In the words of a famous prophet, “Let your yea’s be yea’s and your nay’s be nay’s.” Which, only the functionally literate can manage.

Your computer is a machine that contradicts you every day, but long before that, your thoughts and actions, have contradicted simple reason.

Language is Universal and Intelligible, i.e. it requires intelligence to grasp it. It cannot be taught, spoken, written, or seen. While Grammar is Particular and Perceptible, which, by the simple binary recursion exampled by Language, produces exactly four methods of binary recursion for human survival: Common Grammar, Arithmetic, Algebra and Geometry. These four are also called Standards of Mental Behavior that produce Standards of Social Behavior. Psychology is commensurate with the principles of Language functionally resident in the mind of man as simple grammar systems.

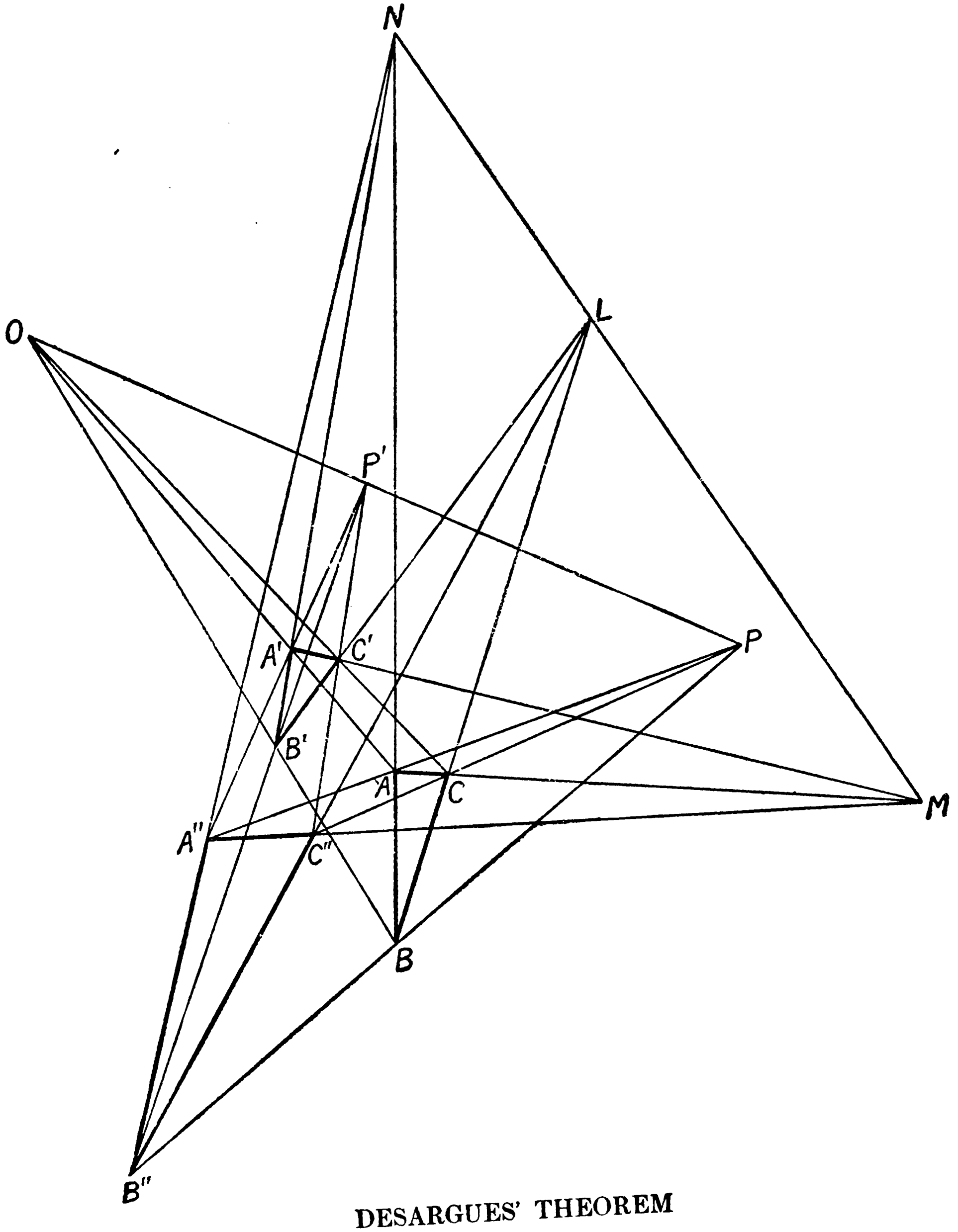
PRINCIPLES OF GEOMETRY

Cambridge University Press
Fetter Lane, London

New York
Bombay, Calcutta, Madras
Toronto
Macmillan

Tokyo
Maruzen-Kabushiki-Kaisha

All rights reserved


N
L
O
P'
P
A'
C'
B'
A
C
M
A''
C''
B
B''

DESARGUES' THEOREM
(see p. 6)

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., F.R.S.,
LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY, AND FELLOW OF
ST JOHN'S COLLEGE, IN THE UNIVERSITY OF CAMBRIDGE

VOLUME I
FOUNDATIONS

Amplissima est et pulcherrima
scientia figurarum

CAMBRIDGE
AT THE UNIVERSITY PRESS
1929

First Edition 1922
Second Edition 1929

PRINTED IN GREAT BRITAIN

PREFACE

THE volumes of which this is the first have the purpose of introducing the reader to those parts of geometry which appear most important in developments taking place to-day. The present volume is devoted to the indispensable logical preliminaries. It assumes only those relations of position, for points, lines and planes, which, furnished with a pencil, a ruler, some rods and some string, a student may learn by drawing diagrams and making models. It seeks to set these relations in an ordered framework of deduction, gradually rendered comprehensive and precise enough to include all the subsequent theory; to this end it puts aside, at first, most of those intricate details which make up the burden of what is generally called elementary geometry. That such a plan can be carried through, thanks to the work of many generations of thinkers, is well enough known; and experience has shewn that many students, especially of the class who look forward to becoming Engineers or Physicists, to whom the geometry of the usual textbooks is tiresome, find such a course stimulating and easy, when the matter is properly presented to them. The mathematician who has followed such a course will find that he has no cause to think he has learnt the wrong things. The preliminary theorems in this method of approaching the subject are indeed of Greek origin; only, these are here made to lead to general principles, giving a command of detail unknown to the Greeks. Subsequent volumes will deal, on the basis of the results obtained in this volume, with conics (and circles), with quadric surfaces and cubic curves in space, and with cubic surfaces and certain quartic surfaces. These volumes are ready to print; it is hoped that they may appear in no long time.

Speaking in more detail of the present volume, it rejects the consideration of distance, and of congruence, as fundamental ideas; these are, in effect, replaced by a theory of related ranges; the geometry usually described with the help of the notion of distance appears later, in a more general, but not more difficult, form. By what means it is possible, so to dispense with this notion, should be of interest to others than the student of geometry. An account is given, however, of the consequences of accepting as fundamental the continuity of the real points of the line. As it is necessary to provide for the consideration of the so-called imaginary elements, room has been given to the justification of these; the ideas to which they lead are indeed an essential part of the power which belongs to the point of view adopted. The references to the theory of space of more than three dimensions are also vital to the plan of the work; it will be seen later with what simplicity this theory enables us to deal with curves and surfaces whose properties can otherwise be developed only at great length, with complicated analysis. The geometrical theory is accompanied by an algebraic symbolism, which serves to help to fix ideas, and for purposes of verification; it is necessary also to include the proof that this symbolism is appropriate to the purpose. It is held, however, that the geometrical argument should be complete in itself, independently of the symbols; a geometry should have such a comprehensive grasp of geometrical relations that all its results are clear by consideration of the geometrical entities alone.

The writer is under great obligations to Mr J. B. Peace, M.A., for his interest in the work, and to the Staff of the University Press for their ready cooperation, especially at this time of difficulty in the production of books.

H. F. B.

26 *September* 1921

The present Reprint gives the opportunity of taking account of many kind suggestions made for the improvement of the text. And three brief Notes have been added at the end of the Volume, the first of these mainly with the intention of making more explicit the point of view adopted for the so-called imaginary elements; this seems the natural view, and logically satisfactory.

H. F. B.

22 *December* 1928

Erratum. In Ex. 9, p. 175, *instead of* "by taking three positions of P upon p," *read* "in general, by taking three positions of P upon the real line p through the intersection of the two given lines, so chosen that the corresponding points U, V, W are in line"

TABLE OF CONTENTS

PAGES

INTRODUCTORY . . . . 1—3

CHAPTER I. ABSTRACT GEOMETRY

Section I. Propositions of Incidence

Propositions of Incidence in three dimensions 4, 5

The transversal to two lines from a point 5

Desargues' theorem . . . . 6—8

Remarks in regard to Desargues' theorem . . . . . . 8—10

The fourth harmonic point . . . . 10—12

First properties of the harmonic relation . . . . . . 12—14

An important construction determining a sixth point from five given points of a line . . . . 14—16

Remarks 16, 17

A theorem in regard to a chain of perspectivities; reduction to two links 17, 18

Proof for the case of ranges on skew lines . . . . . . 18—21

General proof of the theorem . . . . 21—25

Examples in regard to related ranges . . . . 25—33

Propositions of Incidence in space not limited to three dimensions 33—36

Examples of the Propositions of Incidence in space of four and five dimensions . . . . 36—42

On the Principle of Duality in general 42, 43

On projection from one space to another 43

Section II. The correspondence of the points of two lines. Pappus' theorem

Preliminary remarks . . . . 43—45

Conditions that the correspondence of two related ranges should be of threefold freedom 45, 46

Brianchon hexagons and another form of the conditions . . 46, 47

Pappus' theorem. Is true if the conditions are satisfied . . 47, 48

Pappus' theorem. The conditions follow if Pappus' theorem be assumed 48, 49

Alternative proof of the equivalence of Pappus' theorem with the conditions . . . . 49—51

Equivalence of Pappus' theorem with the conditions for the case of ranges in one plane . . . . 51—53

PAGES
Related ranges on the same line 53, 54
Pappus' theorem and the Principle of Duality 54, 55
Related ranges in space of more than three dimensions . . 55, 56
Examples in connexion with Pappus' theorem . . . . 56—62

Section III. Introduction of algebraic symbols

The laws of operation of the symbols employed . . . . 62—66
Examples of symbols obeying these laws . . . . . . 66—69
Preliminary remarks in regard to the use of the symbols in geometry 69, 70
The symbolical representation of the Propositions of Incidence . 70—72
Examples of the application of the symbolism . . . . 72—74
Interpretation of the fundamental laws of the algebraic symbols . 74—81
Representation of the algebraic effect of Pappus' theorem in three dimensions . . . . 81—83
Examples of the symbols; the commutative law of multiplication. 83—93

CHAPTER II. REAL GEOMETRY

Section I. The Propositions of Incidence. Introduction of a plane, and of a space

Preliminary remarks 94, 95
Betweenness for points of a line. Segments. 95, 96
Points determined by three points. Intersections of lines determined therewith . . . . 96—100
Points of a plane system . . . . 100—103
Separation of the points of a plane effected by a line therein . 103, 104
Space systems, defined by four points not lying in a plane . . 104—106
Extension to fourfold and higher space 106

Section II. The extension of the Real Geometry by means of Postulated Elements

Purpose of the Section. 107
Extended form of Desargues' theorem, for lines and planes with a common point . . . . 107—109
Fundamental theorem for the correspondence of the lines and planes of two stars . . . . 109—114
Postulated Points . . . . 114—116
Remark on the theory of postulated elements 117
Application of the theory of postulated elements. Separation . 117, 118
Separation for four points in harmonic relation . . . . . 118—120
The impossibility of Desargues' theorem from plane Propositions of Incidence 120, 121

Section III. A deduction of Pappus' theorem from the extended Real Geometry

PAGES

The abstract notion of order 121, 122
Remarks in regard to the preceding formulation . . . . 122—125
A theorem in regard to corresponding positions of an abstract order 125—127
Application of the theory of an abstract order. Pappus' theorem . 128—130
Alternative deduction of related ranges, based on a harmonic net 130—136
Consequences of the equable distribution of points of the harmonic net 136, 137
Introduction of symbols, in particular of the iterative symbols . 137—140

CHAPTER III. ABSTRACT GEOMETRY, RESUMED

Related spaces; justification of the symbols. Geometrical assumption for imaginary elements; replacement of imaginary elements by series of real elements

General discussion of the problems of this chapter . . . 141—148
Theory of two related plane systems . . . . 148—150
Two related threefold systems, or spaces . . . . . 150—152
Case of two spaces both of n dimensions 152
Comparison of the correspondence between two spaces with the correspondence between either space and the symbols used to represent it . . . . 152—155
A geometrical existence theorem for the solution of an algebraic equation . . . . 155—164
Statement of the further Axiom henceforth assumed in the Abstract Geometry 164, 165
Aggregates of real elements with the properties of imaginary elements 165—172
The preceding aggregates in the general Abstract Geometry . 172—175

BIBLIOGRAPHICAL . . . . 176—183

NOTE I . . . . 184—187

NOTE II . . . . 188—191

NOTE III . . . . 192—194

INDEX TO THE TEXT 195

INTRODUCTORY

THE present is the first volume of a brief attempt to place the reader in touch with the main ideas dominant in contemporary geometry. For this purpose an account of many of the preliminary facts of geometry is indispensable, but in later stages a good deal of variety was possible in the selection, and no complete recital of geometrical theorems is aimed at. In several respects the views taken are not those usual in current textbooks; but it is believed that the system here suggested is logically complete, and does not require that long preliminary study of elementary geometry to which at present so much time is devoted. In particular it is desired to enter a protest against the custom of regarding so-called projective geometry as based upon metrical geometry; in the present account *distance*, as a primary conception, does not enter at all. And an attempt is made to include as soon as possible the indispensable ideas of geometry of more than three dimensions, and of geometry of so-called imaginary points. While the view is taken that all geometrical deduction should finally be synthetic, it is also held that to exclude algebraic symbolism would be analogous to preventing a physicist from testing his theories by experiment; and it becomes part of the task to justify the use of this symbolism.

It was impossible on such a plan to avoid discussing the logical foundations—and to this the present volume is devoted. Here difficulties of exposition arise from the desire to be brief, and not to be pedantic. It is not so much aimed at to give a faithful transcript of views elsewhere stated to be final, as to suggest a position which may appeal to the reader's logical sense as likely to be satisfying with further analysis. Logically, Chapter II, headed *Real Geometry*, should have come first, and should have been developed in greater detail. To the writer, after much experience as a teacher, that seemed unpractical. He has thought it better to state at once, in Chapter I, rather as working hypotheses, the greater part of the foundations on which the whole is to be based, reserving however, for Chapter III, an account of the grounds on which imaginary elements are introduced, as well as a more formal *résumé* of the logical position adopted. Thus Chapter II is a less abstract, but so far as the writer can judge the reader's view, a more toilsome analysis of fundamental conceptions than Chapter I. Incidentally, the particular geometrical propositions of Chapter I furnish a good deal of practice in the habit of geometrical construction in three dimensions, which, to the writer, appears logically prior to a detailed consideration of plane figures.

A Science grows up from the desire to bring the results of observation, of the relations of a class of facts which appear to be connected, under as few general propositions as possible. Into these propositions it is generally found necessary, or convenient, when the science has reached a sufficient development, to introduce abstract entities, transcending actual observation, whose existence is only asserted by the postulation of their mutual relations. If the science is to be arranged as a body of thought developed deductively, it is necessary to begin by formulating fundamental relations connecting *all* the entities which are to be discussed, from which other properties are to follow as a logical consequence. If this is done we may in the first instance regard all the entities involved in these fundamental propositions as being abstract, even those to which we attach names agreeing with the names borne by entities which we regard as subject to actual observation. The usefulness of the science, for the purpose for which it was undertaken, will depend on the agreement of the relations obtained for these latter entities with those which we can observe. It would seem that this process of substituting conceived entities, limited by supposed interrelations, for those which are regarded as objects of experience, belongs to every science. But it is clear that the degree of abstractness which may usefully and safely be applied is a matter for judgment and choice, conditioned by knowledge of the matter in hand, even in dealing with the same experiences. In geometry, as applied to the external world, we cannot but be conscious, for instance, in dealing with the points of a line, of the difference between those points which we regard as accessible and those which we regard as the others; and then, given two points of the line, of the difference between those points which we think of as between these, and those not between these; and then, finally, of the order in which several points of the line, which we can think of simultaneously, are arranged. None of these considerations however is taken account of in Chapter I; nor indeed do they enter finally into the abstract system of geometry which we set up. But it is to be expected that, if we take account of them, we may be able to analyse further some of the conceptions which we have adopted without analysis in Chapter I. It is to shew how this can be carried through that Chapter II is written. In particular this Chapter gives grounds on which the fundamental theorem for the correspondence of points on two straight lines may be based. The theory is built up in this Chapter with the use only of points supposed accessible and lying in a limited (unclosed) region, and there are some striking differences between the propositions and those of Chapter I; for instance it is not assumed that two lines in a plane necessarily have a point of intersection; and a line in a plane divides the points of the plane into two distinct sets. It is therefore of importance to shew that there is no inconsistency in afterwards disregarding these special results, as we do; this is effected by shewing that we may consider the limited space as a portion of a closed space in which the more general, if less definite, propositions of Chapter I are true. This step is of interest too as shewing how the detailed considerations which make up the usual elementary expositions of the so-called non-Euclidian geometry are superseded by the more general point of view. To some extent the comparison between the two points of view is analogous to the comparison between the theory of quadratic equations in which there may or may not be two roots, and the theory in which there are always two roots.

The writer is not of those who hold that the process of analysing the fundamental conceptions and setting out the sufficient axioms can ever be final; any such formulation, it would seem, must be subject to the possibility that other logical alternatives of consecutive thought may be revealed. Indeed the progress of science appears to consist in the very gradual unfolding of such alternative, or more embracing, conceptions. The history of Mathematics furnishes many examples, none more instructive and interesting than in the case of non-Euclidian geometry. But the instinct to such analysis is not less imperative than the instinct to all scientific thinking, and in geometry as in other subjects the attempt has been fruitful in a better understanding of the subject-matter. From those who would base upon the confession of the incompleteness of the analysis a refusal to enter upon it the writer would appeal to this instinct, which is more fundamental than formal logic.

Such a volume as this would be impossible save for the work and insight of a long line of other writers. In the Bibliography at the end an attempt is made to give the most necessary acknowledgments. In the text the view is sometimes taken that, in Mathematics, an indication may be more acceptable than an exposition; one remembers the words of Desargues (1636, *Oeuvres* I, 420) "la démonstration est...en notes sans discours pour le soulagement de la veuë et de l'esprit."

CHAPTER I

ABSTRACT GEOMETRY

SECTION I. PROPOSITIONS OF INCIDENCE

Propositions of incidence in three dimensions. The entities with which we deal in the first instance we call by the names, *point*, *line*, and *plane*. These are any objects which are subject to the following laws of combination, which we call the Propositions of Incidence, together with another law, explained below (Sect. II). It is provisionally assumed that these laws are self-consistent and, when properly explained, are sufficient to enable the reader to form a clear impression whether any statement made in regard to these entities is a consequence, or not, of the fundamental laws. It is also very frequently assumed provisionally, when, in the course of a geometrical construction, two points are obtained by certain rules and it is desired to continue the construction with the help of the line joining these points, that these points do not coincide. In many cases it may be easy to shew that the coincidence of the points would involve an undesired limitation in the given points of the figure. But there are cases in which it may seem to be consistent with the assumed fundamental propositions to assume either that the two points always coincide, or that they do not always coincide. The former is excluded if shewn invalid when, to the original hypotheses, others, consistent with these, are added (cf. pp. 11, 119). We adopt words which are in common use, but it is not to be assumed that in effect the meanings which we attach to these are co-extensive with the usual meanings. As is customary we frequently employ diagrams to fix ideas and clarify statements, representing a line by a mark made on the paper and a point by a dot thereon. Notwithstanding this formal and abstract method of statement, the whole object of the theory is of course, as has been stated, to deal with the ideas suggested by our ordinary experience; *it is this experience itself which has gradually suggested the abstract statement.* The Propositions may be stated as follows:

Through an arbitrary point there pass an infinite number of lines, of which one passes through any other arbitrary point. Thus a line is determined uniquely by any two points. The line contains an infinite number of points beside the two determining ones, and is determined by any two of these. Through any given line there pass an infinite number of planes, of which one passes through any point not lying on the given line. Thus a plane is determined uniquely by any three points which do not lie in line. The plane contains an infinite number of points beside the three determining ones, and is determined by any three of these which do not lie in line. The plane entirely contains the line which is determined by any two of its points. Thus if two lines have a common point, and we take two other points, one on each of the lines, the plane determined by the three points contains both the lines; that is, two lines with a common point determine a plane, containing both the lines. It is also true that two lines which are in the same plane have a common point. We have said that there is an infinite number of planes all containing the points of any given line; it is also true that any two planes have common points, lying on a line, or that any two planes intersect in a line. It will be seen below that this statement ceases to be true when we are not limiting ourselves to space of three dimensions. Further it is true (still in space of three dimensions), that any plane contains a point of any line (and contains all the points of the line if it contains two of these), or that a line and a plane intersect in one point. Thus any three planes, which have not in common all the points of a line, have a point in common, that namely where the line of intersection of two of the planes meets the third plane. If now we compare this with the earlier statement that three points not in line determine a plane, and compare the other statements with one another in pairs, as for instance the statement that two points determine a line with the statement that two planes determine a line, the statement that a line and a point determine a plane with the statement that a line and a plane have a point in common, the statement that two lines in a plane have a point in common with the statement that two lines intersecting in a point lie in a plane, we see that there is a complete descriptive correspondence between the relations of points, lines and planes on the one hand, and the relations of planes, lines and points on the other hand. Two propositions obtained one from the other by this interchange will be said to be *dual*, or *reciprocal*, or *correlative*. If we limit our consideration to the points and lines of a single plane it is at once seen that there is a similar reciprocity between propositions relating to points and lines and other propositions relating to lines and points. And a similar duality will be found to subsist in any space of higher dimensions.

We now at once deduce several immediate consequences of these Propositions of Incidence.

The transversal to two lines from a point. Given any two lines a, b, which do not intersect, or, as we shall often say, are *skew* to one another, and a point, P, which does not lie on either of these lines, a single line can be drawn from P to meet both a and b. For

P and a determine a plane, as do P and b; and P is among the points common to these planes, that is, it is on their line of intersection. But this line, being on the plane Pa, intersects the line a, and, being on the plane Pb, intersects the line b; it is thus such a line as is desired. Conversely, any line through P which meets the line a and b must be on both the planes Pa and Pb, and must therefore be the line we have found. The correlative proposition is that there lies in any plane not containing either of two given skew lines a line which meets both; the join namely of the points where the plane intersects the respective lines.

Desargues' theorem. If two triads of points, A, B, C and A', B', C', be such that the three joining lines AA', BB', CC' meet in one point, say O, then the two lines BC, $B'C'$ meet, say in the point P, and similarly the lines CA, $C'A'$ meet, say in Q, and the lines AB, $A'B'$ meet, say in R, and the three points P, Q, R are in line. Conversely, given two triads of points, A, B, C and A', B', C', such that the lines BC, $B'C'$ intersect, say in P, that the lines CA, $C'A'$ intersect, say in Q, and that the lines AB, $A'B'$ intersect, say in R, while P, Q, R are in line, then the lines AA', BB', CC' cointersect. (See the Frontispiece of the Volume.)

It is supposed that the points A, B, C are not in line, so that they determine a plane, and similarly that A', B', C' determine a plane. Take first the case when these planes are different. Suppose further that no one of the joins BC, CA, AB is in the plane $A'B'C'$, and no one of the joins $B'C'$, $C'A'$, $A'B'$ is in the plane ABC. Then the fact that the lines BB', CC' intersect involves that the points B, B', C, C' are in one plane, and hence that the lines BC, $B'C'$ meet. Their point of intersection, say P, lies then on BC which is in the plane ABC, and on $B'C'$ which is in the plane $A'B'C'$, so that P lies on the line of intersection of these two planes. The point, Q, of intersection of the lines CA, $C'A'$, similarly found, as well as the point, R, of intersection of the lines AB, $A'B'$, are equally on the line of intersection of the planes ABC, $A'B'C'$. So that the former statement made is clearly true. Conversely if the lines BC, $B'C'$ meet in the point P, while CA, $C'A'$ meet in Q and AB, $A'B'$ meet in R, then, still assuming the planes ABC, $A'B'C'$ to be different, and assuming that no one of the lines BC, $B'C'$, CA, $C'A'$, AB, $A'B'$ is common to both planes, the points P, Q, R are evidently in the line of intersection of these planes. The fact that the lines BC, $B'C'$ intersect, involves that the lines BB' and CC' intersect, while similarly the line AA' intersects both of these. Now the lines BB', CC' determine a plane, and the line AA' does not lie in this plane, since we have assumed that the planes ABC, $A'B'C'$ are different; wherefore the line AA' meets this plane in a point and can only intersect both BB' and CC' by passing through their point of intersection. Thus the lines AA', BB', CC' meet in a point, as we desired to shew.

We have explicitly excluded, for the sake of simplicity, the possibility of one of the six lines $BC, \ldots$ lying in both planes, which is easily seen to be unimportant. Incidentally we see that if three lines be such that every two of them have a common point, then the three lines lie all in one plane, or pass through one point.

Next suppose the two triads A, B, C and A', B', C' to be in one plane, it being assumed as before that A, B, C are not in line and A', B', C' are not in line. Then, first, let the lines AA', BB', CC' meet in one point, O. Draw through O any line not lying in the plane of the two triads, and let P, P' be any two points on this line. Then the intersecting lines $AA'O$ and $PP'O$ determine a plane, and the lines AP, $A'P'$, lying therein, intersect in a point, say A''. Similarly the lines BP, $B'P'$ meet in a point, say B'', and the lines CP, $C'P'$ meet in a point, say C''. As P is not in the plane of ABC, $A'B'C'$, it is clear that the points A'', B'', C'' are not in this plane; further A'', B'', C'' are not in line, since otherwise, as the plane containing this line and P would contain the points A, B, C, these would be in line; finally the plane which is thus determined by the points A'', B'', C'' does not contain, for instance, the line AB, for otherwise the plane of AB and $A''B''$, which contains P, would contain C'' and C, and A, B, C would be in line. Hence it follows from what is proved above, as the lines AA'', BB'', CC'' meet in P, that there is, in the plane of ABC and $A'B'C'$, a line, where this plane is met by the plane $A''B''C''$, and upon this line three points, say L, M and N, such that the lines BC, $B''C''$ meet in L, the lines CA, $C''A''$ meet in M and the lines AB, $A''B''$ meet in N. The point L is then the point where the plane of ABC and $A'B'C'$ is met by the line $B''C''$, and M, N are similarly the points where this plane is met by $C''A''$ and $A''B''$. By considering that the three lines $A'A''$, $B'B''$, $C'C''$ meet in the point P', we prove however, in the same way, that $B'C'$ passes through L, and that $C'A'$, $A'B'$ respectively pass through M and N. We have thus proved that if two triads of points ABC, $A'B'C'$, in the same plane, be such that the joining lines AA', BB', CC' meet in a point, then the three points of intersection $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ are in line. Conversely, for two triads of points ABC, $A'B'C'$ in the same plane, assume that the three points $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ are in line. Denote these points respectively by L, M, N. Draw through the line LMN a plane, other than the original plane of ABC, $A'B'C'$; draw through the points L, M, N, in this new plane, respectively the lines $LB''C''$, $MC''A''$, $NA''B''$, giving by their intersections a further triad A'', B'', C'' in this new plane. This new plane does not contain any one of the lines BC, CA, AB, $B'C'$,

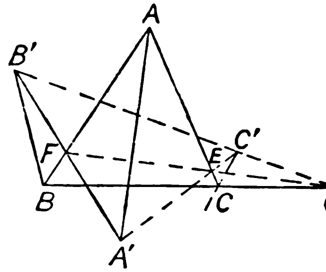
$C'A'$, $A'B'$, and it is supposed that the lines $LB''C''$, $MC''A''$, $NA''B''$ are all drawn so as not to lie in the original plane of ABC, $A'B'C'$. Then, as the lines BC, $B''C''$ meet in the point L, the lines CA, $C''A''$ meet in the point M, and the lines AB, $A''B''$ meet in the point N, it follows from what has preceded that the three lines AA'', BB'', CC'' meet in a point, say P, which does not lie in either the original plane or the new plane. It follows similarly, as the lines $B'C'$, $B''C''$ meet in the point L, and $C'A'$, $C''A''$ in M, and $A'B'$, $A''B''$ in N, that the lines $A'A''$, $B'B''$, $C'C''$ meet in a point, say P', not lying in either of the two planes described. Now let O be the point where the line PP' meets the original plane of ABC, $A'B'C'$. Then, since the lines AP, $A'P'$ intersect, in the point A'', and the points A, A', P, P' are not in line, these lines determine a plane, and therefore the line AA' intersects the line PP'. This intersection can only be at O, where the line PP' meets the original plane containing A and A'. Thus the line AA' passes through O, as, similarly, do also the lines BB' and CC'. We have thus proved that if two triads ABC, $A'B'C'$, of points in one plane, neither triad being in line, be such as to have three intersections (BC, $B'C'$), (CA, $C'A'$), (AB, $A'B'$) lying in line, then the three joining lines AA', BB', CC' meet in a point.

The aggregate of the four theorems now established will be referred to as Desargues' theorem. (See *Oeuvres de Desargues, par M. Poudra*, Paris, 1864, t. I, pp. 413, 430.)

Remarks in regard to Desargues' theorem. (*a*) The reader will notice that the theorem for two triads of points in one plane is proved by considerations involving the existence of points not lying in this plane. From the previously given Propositions of Incidence it is possible to select those which relate only to the geometry of points and lines lying in one plane. It is an interesting fact that the theorem of Desargues for two triads of points lying in this plane cannot be proved as a consequence only of the propositions of incidence relating to geometry in this plane—as we shall prove below (Chap. II). The propositions of incidence for one plane appear to furnish no criterion for deciding whether three constructed points lie in line; such criterion arises only when the line enters as the intersection of two planes. And they appear to furnish no criterion for deciding whether three constructed lines meet in one point; such criterion arises only when the point enters as the intersection of the plane with a line not lying therein.

(*b*) We have said that two lines intersect one another when they lie in one plane; thus they intersect one another when there are two points A, A' on one of the lines, a, and two points B, B' on the other line, b, such that the cross joins AB, $A'B'$ meet; for then the two lines a, b lie in the plane of these lines AB, $A'B'$. Let

F be the intersection of AB and $A'B'$; and let C be any other point not lying on either of the lines a, b. Desargues' theorem furnishes a construction for the line joining C to the point of intersection of the lines, a, b; and this construction depends only on the points A, A' and B, B' by which these lines may be supposed to be determined. For, on the given line BC, in the plane ABC, wherein A is also given, take a point O and join it to the given point F of the line AB. Let the joining line meet AC in the point E. The plane determined by the points O, B', F contains the points A' on $B'F$, and E on FO; thus the line $A'E$ intersects OB', say in C'. We now have two triads of points, ABC and $A'B'C'$, such that the lines BC, $B'C'$ meet in O, the lines CA, $C'A'$ meet in E, and the lines AB, $A'B'$ meet in F, while the points O, E, F are in line. It therefore follows that the line CC' passes through the intersection of the given lines AA' and BB'.



That this construction should be useful when the point of intersection of the lines AA', BB' is not accessible depends on the possibility of taking O in such a position upon BC that the two points E, C' should be accessible (see Chapter II). The construction is valid when C is in the plane of the two lines AA', BB'; when C is not in this plane the line from C to the intersection of these lines may also be defined as the line of intersection of the two planes CAA' and CBB'.

(c) If we have two tetrads of points, A, B, C, D and A', B', C', D', with the property that the four lines AA', BB', CC', DD' meet in one point O, it being supposed that neither tetrad consists of four points lying on a plane, then it can be shewn that every two corresponding joining lines in the two figures, such as BC and $B'C'$, intersect one another, and that the six points of intersection so arising are in one plane, and are the intersections in pairs of four lines of that plane.

In fact, the lines DA, $D'A'$, in the plane of the lines ODD', OAA', must intersect, say in P; so the lines DB, $D'B'$ must intersect, say in Q, and the lines DC, $D'C'$ must intersect, say in R. The points P, Q, R are then not in line, since the points D, A, B, C are not in a plane; and thus the points P, Q, R define a plane. Again, as the joins of corresponding points of the two triads DBC and $D'B'C'$, namely DD', BB' and CC', meet, in O, it follows from Desargues' theorem that the lines BC, $B'C'$ intersect in a point of the line QR, where Q is the point $(DB, D'B')$ and R is the point $(DC, D'C')$, say in X. Similarly the lines CA, $C'A'$ meet in a point, say Y, of the line *RP*, and *AB*, *A'B'* meet in a point, say *Z*, of the line *PQ*. The points *X*, *Y*, *Z* are on the plane *ABC*, and are thus in line, the line of intersection of the planes *ABC* and *PQR*. This proves the result stated.

(*d*) Two figures, such as the triads *ABC*, *A'B'C'* of Desargues' theorem, or the two tetrads just considered, are said *to be in perspective*, when to any point *P* of one figure there corresponds a point *P'* of the other figure, such that all lines, like *PP'*, which join two corresponding points, pass through the same point.

We may notice that the figure which arises in the proof of Desargues' theorem for the case of two triads in one plane, consists of three triads, of which every two are in perspective, the three centres of perspective being in line; while each set of three corresponding joining lines, such as *BC*, *B'C'*, *B''C''*, is formed of three lines which meet in a point. In all there are fifteen points and twenty lines; each line contains three of the points, and through each point there pass four of the lines. If we denote the points *P*, *P'*, *A''*, *B''*, *C''* respectively by the binary symbols, each formed with two numbers, 04, 05, 01, 02, 03, then it is appropriate to denote the points *A*, *B*, *C*, *A'*, *B'*, *C'* respectively by the symbols 14, 24, 34, 15, 25, 35, the point of meeting of the three lines *BC*, *B'C'*, *B''C''* by 23, the point of meeting of the three lines *CA*, *C'A'*, *C''A''* by 31, and that of *AB*, *A'B'*, *A''B''* by 12, using 45 for the point of meeting of *AA'*, *BB'*, *CC'*. The points are then denoted by all the combinations of six symbols two at a time. Each line is met by nine others, lying in threes in three planes; the three triads so formed, one in each of these planes, are two and two in perspective, the centres of perspective being in line.

The figure arising in the proof of Desargues' theorem for two triads not lying in the same plane is simpler. Here there are five planes, ten lines, consisting of the lines of intersection of the planes in pairs, and ten points, consisting of the intersections of the planes in threes.

Example. If two triads of points, *A*, *B*, *C* and *A'*, *B'*, *C'*, be such that the lines *AB'*, *A'B* intersect, say in *R*, also *AC'*, *A'C* intersect, say in *Q*, while *AB*, *A'B'* intersect in *H*, and *BC*, *B'C'* intersect, in *F*, then prove that *BC'*, *B'C* intersect, say in *P*, and that *AC*, *A'C'* intersect, say in *G*, and that each of the four sets *FGH*, *FQR*, *GRP*, *HPQ* consists of three points in line.

The fourth harmonic point. We come now to an important construction by which we pass from three given points of a line to another unique point of this line. For the construction itself it is assumed that a plane can be drawn through the line, and for the proof that the fourth point of the line which is constructed is unique it is assumed that two planes can be drawn through the line. Thus the construction supposes that the line lies in space of three dimensions, though the result of the construction relates only to points lying on a line.

Let A, B, C be three arbitrary points lying on a line. Through this line draw an arbitrary plane. In this plane draw three lines, one through each of the points A, B, C, but otherwise arbitrary, forming by their intersections a triad P, Q, R, so that the line QR passes through the point A, while RP passes through B and PQ passes through C. *Now regard A, B, C as consisting of a pair of points, and a third point*; let A, B be the pair. Join to A the point of intersection of the line (RP) drawn through B with the line (PQ) drawn through C; that is, join AP. Similarly join to B the point of intersection of the line (QR) drawn through A with the line (PQ) drawn through C; that is, join BQ. Let the lines AP and BQ meet in the point U. Join U to the point of intersection of the line (QR) drawn through A with the line (RP) drawn through B; that is, join UR. Let UR meet AB in D. Then D is the point to be constructed corresponding to the separation of the points A, B, C into the pair A, B and the third point C. Temporarily, assuming D, C do not coincide, it may be denoted by

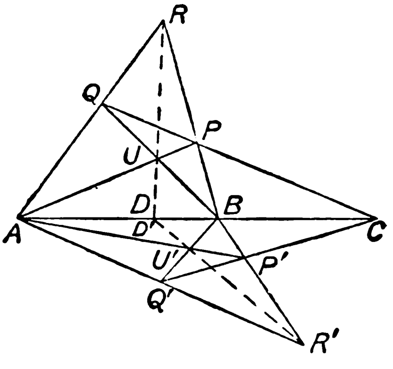
$$D=(A, B)/C.$$

We may similarly construct two other points, appropriately represented respectively by

$$(B, C)/A \text{ and } (C, A)/B,$$

with which we are not concerned for the moment.

To prove that the point D is the same whatever be the lines QR, RP, PQ we proceed as follows. Draw through the given line another plane, and therein make a similar construction, with lines $Q'R'$, $R'P'$, $P'Q'$ passing respectively through A, B and C. We shew first that the points D, D' in which RU, $R'U'$ meet AB are the same. In fact the planes RAR', RBR', are intersected by the plane $QPCP'Q'$ respectively in the lines QQ' and PP'; these lines, lying in a plane, must then meet, and their point of meeting must be on the line RR' containing all the points common to the planes RAR', RBR'. Let this common point of the lines QQ', PP' be called O. Again the planes PAP', QBQ' are met by the plane $QQ'OP'P$ respectively in the lines PP' and QQ';



these lines, lying in a plane, must then meet in a point on the line of intersection UU' of the planes PAP', QBQ'.

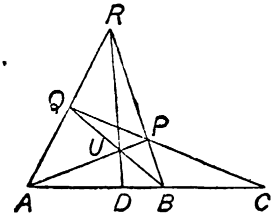
Thus the line UU' passes through the intersection, O, of the lines PP' and QQ', through which the line RR' also passes. But, from the fact that UU' and RR' meet, it follows that the lines RU and $R'U'$ lie in a plane, so that these intersect; and their intersection, being common to the planes RAB, $R'A'B'$ in which they respectively lie, must be on the line AB.

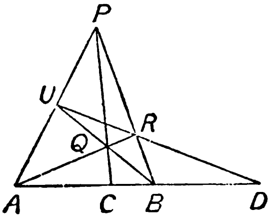
Next suppose that in the plane PQR other lines are drawn through A, B, C, namely AQ_1R_1 through A, and BR_1P_1 through B, and CP_1Q_1 through C; and let AP_1, BQ_1 meet in U_1, while R_1U_1 meets AB in D_1. By the argument just given it follows that D_1 coincides with the point, D', in which $R'U'$ meets AB. And this has been shewn to coincide with the point, D, in which RU meets AB.

The position of D is thus independent of the lines AQR, BRP, CPQ.

First properties of the harmonic relation. (*a*) It is clear from the construction that no change is made in D by the interchange of the points A and B. Thus we may equally write

$$D=(B, A)/C.$$



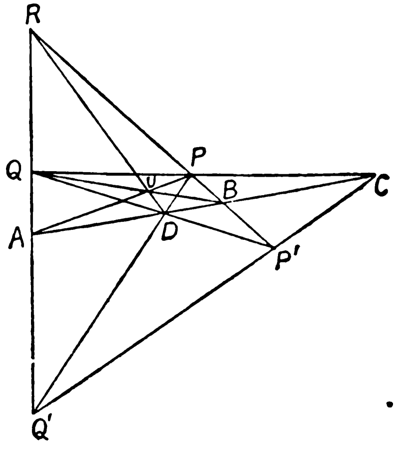


(*b*) Again, as we may see using the same figure but calling the points originally named P, Q, R, U, C, D respectively by the names R, U, P, Q, D, C, we can pass from the point D to the point C by a construction which is descriptively identical with that by which we passed from C to D. Thus we may also write

$$C=(A, B)/D,$$

and, like the couple A, B, the points C, D form a couple of interchangeable points.

(*c*) And in fact one of these couples may be interchanged with the other. For, it is clear from the original construction that if U_1 be any point on the line RD, and the lines AU_1, BU_1 meet RB and RA respectively in P_1 and Q_1, then P_1Q_1 passes through C; we have only to replace the line CPQ by the line CQ_1; if the intersection of this with RB be joined to A, the joining line must meet BQ_1 in a point lying on RD, that is, in U_1. Hence, interchanging the parts played by the lines CBA and CPQ, if the line PD meet RA in Q', and the line QD meet RB in P', it follows from the fact that the lines PA, QB meet on RD that the line $Q'P'$ passes through C.



Now consider the point which would be expressed by

$$(D, C)/B.$$

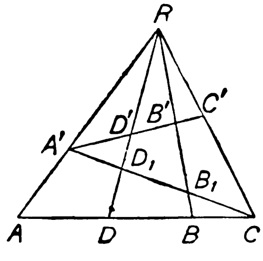
To obtain this point by the rule, we may draw respectively through the points D and C the lines DP' and CP, and then through the point B the line $P'BP$, meeting the two former respectively in the points P' and P; then, if the lines $P'C$ and PD meet in Q', we are to join Q' to the point of intersection of the lines DP' and CP, that is to the point Q. The joining line QQ' will then meet DC in the point required; and this is the point A. Hence we have

$$A = (D, C)/B,$$

which shews that the couples A, B and C, D may be interchanged.

We may then speak of *the points of either couple as being harmonic conjugates of one another in regard to the points of the other couple.*

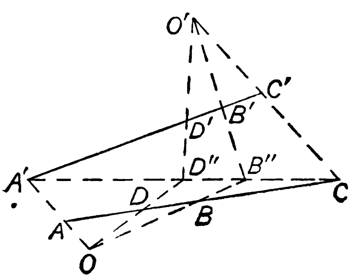
(d) We have stated that two figures are said to be in perspective when to any point, P, of one figure there corresponds a point, P', of the other, such that the joining lines, PP', all pass through the same point. It is easy to see that if four points A', D', B', C' of one line be in perspective with four points A, D, B, C respectively, of another line, which are such that C, D are harmonic conjugates in regard to A, B, then also C', D' are harmonic conjugates in regard to A', B'.



If, in the original figure, the line RU meet the line CPQ in the point X, then X is the harmonic conjugate of C in regard to P and Q. For, from P, Q are drawn, respectively, the lines PA, QB, meeting in U, and the line CAB, drawn from C, meets these respectively in A and B; then the lines AQ, BP meet in the point R, and the line RU meets the line PQ in the point X.

Hence taking the points A', D', B', C' respectively in perspective with the points A, D, B, C, from the centre R, where C, D are harmonic conjugates in regard to A and B, let the line CA' meet the lines RD and RB respectively in D_1 and B_1. From the remark just made, it follows first that C, D_1 are harmonic conjugates in regard to A' and B_1; and then it similarly follows that C', D' are harmonic conjugates in regard to A' and B'.

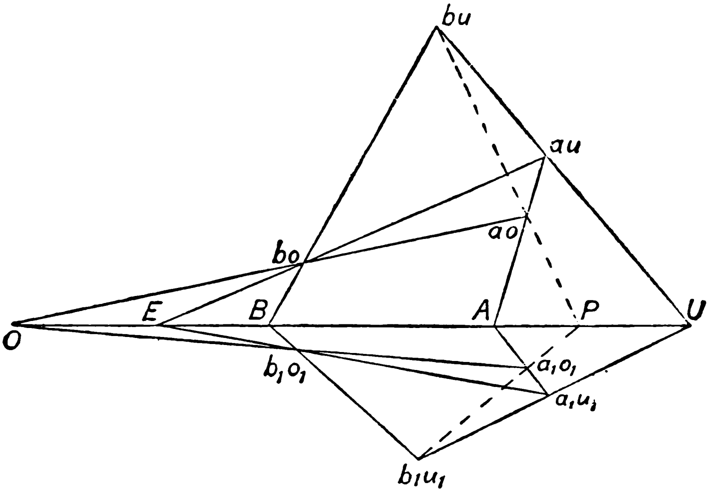
(e) Conversely if on each of two lines we have a set of four points in harmonic relation, they can be regarded as both arising by perspective from the same set of four points in harmonic relation lying on another line.



For let C, D be harmonic conjugates in regard to A, B, on one line, and C', D' be harmonic conjugates in regard to A', B', on another line; join $A'C$ and take O any point on $A'A$; let OD, OB meet $A'C$ in D'', B'' respectively; let CC' and $B''B'$ meet in O' and let $O'D'$ meet $A'C$ in D_1. Then, by what is remarked under (d), C and D'' are harmonic conjugates in regard to A' and B'', while C and D_1 are also harmonic conjugates in regard to A' and B''. Thus D_1 is the same as D'', and the two given sets are both in perspective with the harmonic set A', D'', B'', C.

(f) When the points that can exist on a line are further conditioned as in Chapter II below, it will be found that if the points C, D are harmonic conjugates in regard to the points A, B, then C and D are necessarily separated from one another, in the sense there explained, by A and B. Without these conditions, to speak of a separation would be unmeaning. But we shall, in accordance with a general assumption referred to at the beginning (above, p. 4), assume provisionally that the point D does not coincide with C unless C coincides with A or with B. A geometry, of fifteen points in all, has been suggested in which D always does coincide with C (Fano, "Sui postulati fondamentali...," *Giorn. di Mat.*, XXX, 1892, p. 106). See the Bibliography, and p. 119, below.

An important construction determining a sixth point from five given points of a line. The five given points are to be regarded as consisting of two couples and another point; for clearness sake we shall at present speak of this last point as the determining point. Denote the points of one couple by A, B, those of the other couple by O, U; let E be the determining point, and P the point to be determined. Draw any plane through the line, and therein draw lines, denoted respectively by a, b, o, u, one through each of the points A, B, O, U of the two given couples. We associate one of the two lines a, b with one of the two lines o, u, and the other of a, b with the remaining one of o, u; this is possible in two ways; take the association of a with u and of b with o, denoting the intersection of a and u by (a, u), and the intersection of b and o by (b, o). The lines a, b, o, u are to be any lines of the plane taken subject to the condition that the two points (a, u), (b, o) shall lie on a line, say e, which passes through the determining point E. Then by the other possibility of association we have two other points which may be respectively denoted by (a, o) and (b, u). The line, say p, which joins these, gives, by its intersection with the original line, the point, P, which is determined, in the one of two possible ways chosen, as the result of the construction. The other method of association would require the lines a, b, o, u to be such that the points (a, o) and (b, u) shall lie on a line, e', passing through E, and then the line, p', containing the points (a, u) and (b, o), would meet the original line in a point, P', which would be the point determined by the construction. For our purpose it will be found to be very important that the Propositions of Incidence assumed do not enable us to prove that P and P' coincide.



That, subject to the condition that the points (a, u), (b, o) shall lie on a line through E, the point P is the same whatever be the lines a, b, o, u, may be proved as in a previous case.

Draw through the original line any other plane, and therein, through the points A, B, O, U, respectively, any lines a_1, b_1, o_1, u_1, subject to the condition that the points (a_1, u_1) and (b_1, o_1) shall lie on a line through E, completing the figure as before. We shew that the line joining the points (a_1, o_1) and (b_1, u_1) also passes through P. For, first, the three lines joining the respective pairs of points

(a, u) and (a_1, u_1), (b, o) and (b_1, o_1), (a, o) and (a_1, o_1)

meet in a point, namely in the point of intersection of the three planes

$$[e, e_1], \quad [a, a_1], \quad [o, o_1],$$

where e, e_1 are the lines through E in the two planes respectively; this follows because each of these planes contains two of the three joining lines in question, or each of the lines is the line of intersection of two of these planes. Similarly, the three lines joining the respective pairs of points

$$(b, o) \text{ and } (b_1, o_1), \quad (a, u) \text{ and } (a_1, u_1), \quad (b, u) \text{ and } (b_1, u_1)$$

meet in a point, namely in the point of intersection of the three planes

$$[e, e_1], \quad [b, b_1], \quad [u, u_1],$$

each of these lines being common to two of these planes.

From these two facts it follows that the two lines joining the respective pairs of points

$$(a, o) \text{ and } (a_1, o_1), \quad (b, u) \text{ and } (b_1, u_1)$$

must intersect one another. From this it follows that the two lines joining the respective pairs of points

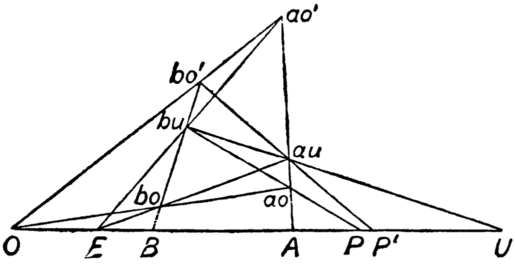
$$(a, o) \text{ and } (b, u), \quad (a_1, o_1) \text{ and } (b_1, u_1)$$

must meet; and they can only meet in a point P of the original line which is the line of all the points common to the two planes in which the two figures are constructed.

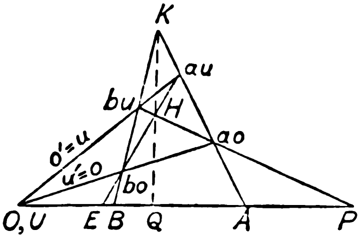
Having proved that the same point P is obtained when we construct figures in two different planes passing through the original line, it follows as before that the same point P is obtained when we construct two figures in the same plane, by varying the lines a, b, o, u, preserving the same method of association of the couples a, b and o, u. It is clear that the two couples A, B and O, U play similar parts in the construction, and that the two points of a couple play similar parts; we may therefore represent the points P, P' obtained, respectively, by

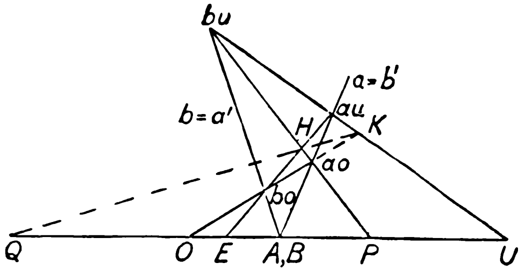
$$P = \begin{pmatrix} A, & B \\ O, & U \end{pmatrix} \Big/ E = \begin{pmatrix} O, & U \\ A, & B \end{pmatrix} \Big/ E, \quad P' = \begin{pmatrix} A, & B \\ U, & O \end{pmatrix} \Big/ E = \begin{pmatrix} U, & O \\ A, & B \end{pmatrix} \Big/ E.$$

Remarks. (*a*) We may make a figure containing both methods of association, using the same lines a, b, u, but two lines o, o' through O.



(*b*) As the two methods of association are distinguished by either the interchange of the points A, B, or the interchange of the points O, U, the points P', P will coincide when either A, B coincide, or O, U coincide. In either case the figures appropriate to the two methods of association may be taken identical. When A, B coincide, taking lines a, b, o, u, e for one association, we may take lines $a'=b$, $b'=a$, o, u, e for the other. When O, U coincide, taking lines a, b, o, u, e





for one association, we may take lines a, b, $o'=u$, $u'=o$, e for the other. By a construction indicated in the diagrams we may prove that when A, B coincide, if $Q=(O, U)/A$, then $P=(A, Q)/E$; and that when O, U coincide, if $Q=(A, B)\ O$, then $P=(O, Q)/E$. Hence also, when A, B coincide, if $U=(E, A)/O$, then P coincides with E, as a diagram at once makes clear.

Further, when E coincides with O, both P and P' coincide with U.

A theorem in regard to a chain of perspectivities; reduction to two links. There is another result of great importance to us in the sequel, which may be deduced from the Propositions of Incidence. Though the proof is somewhat intricate we give it at once; the reader may prefer, when he has understood what is the effect of the theorem, to pass over the details of the proof for the present.

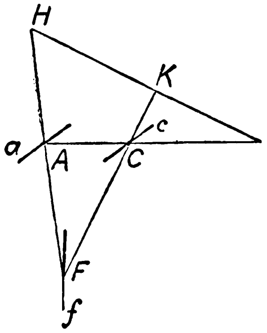
It has been seen that two ranges of points $AA'\ldots$, $BB'\ldots$, lying respectively on two lines a, b, are said to be in perspective when the joins of corresponding points of these ranges, that is the lines AB, $A'B'$,..., all pass through another point. It is clearly necessary that the lines a, b should be in one plane. Suppose now that we have a set of lines a, b, c,..., k, finite in number, arranged in order such that every consecutive two intersect; and that there are ranges of points upon these lines, such that those upon the first and second are in perspective with one another from some point, that the ranges upon the second and third lines are in perspective with one another from some point, and so on, for every consecutive two, to the last. *It can then be proved that the ranges upon the first and last lines, a and k, can be obtained from one another by means of only two perspectivities*, namely that there is a line m, and upon this a range which is in perspective with the given range upon the line a from some point, and is also in perspective with the given range upon the line k from some point. It will appear that the line m, which must intersect both a and k, is by no means unique.

Let us agree to say that two ranges of points upon two different lines, p, q, which may or may not intersect one another, are *related*, when these ranges are both in perspective with the same range upon another line, n, from appropriate centres. Two ranges which are in perspective with one another are then also related, in a very particular way, but the converse is not generally true. The proposition to be proved is then that the ranges on the lines a, k are related. And the proposition will follow if we prove the theorem that *when two ranges are related, any range which is in perspective with one of these is related to the other.* We address ourselves then to proving this last theorem.

It may prevent confusion of thought if we add at once that we shall also be led to speak of two ranges of points on the *same* line as being related; but, for a reason which will appear, the definition will be that two such ranges are related when one of them is in perspective with a range on another line which is related, in the above sense, to the other range of the given line, not being directly in perspective with it.

Proof for the case of ranges on skew lines. Before entering upon a general proof of the theorem, we give a proof, which is very direct, for the case when the two given related ranges are on lines which do not intersect one another, and the range which is given as being in perspective with one of the two ranges lies on a line not intersecting the line containing the other of these. Logically it is unnecessary to give this, the general proof which follows being applicable in all cases.

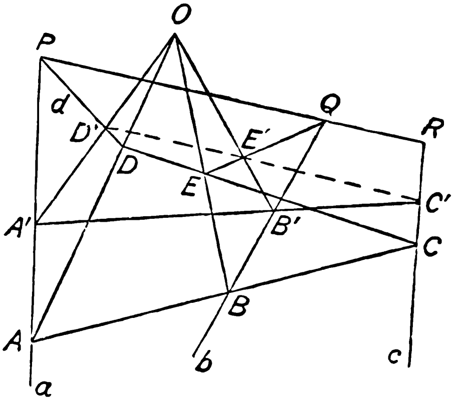
But in this case there is a simple way of describing when the ranges on the two (non-intersecting) lines are related. For if ranges $(A, A', \ldots)$, $(C, C', \ldots)$, on two lines a, c, which do not intersect, be both in perspective with a range $(F, F', \ldots)$, on a line f, so that there is a point H through which the lines AF, $A'F'$, ... all pass, and a point K through which the lines CF, $C'F'$, ... all pass, then the lines AC, $A'C'$, ..., joining corresponding points of the two related ranges on a, c, all intersect the line HK. This line HK will not intersect the lines a, or c; for if it intersected a, the plane Ha, which contains f, would contain the line HK, and the plane Kc, containing f and KH, would coincide with the former plane Ha; so that the lines a and c would intersect. Thus a simple construction by which the range on the line a gives rise to



the related range on the line c is, to draw from the points A, A',..., of the range on a, the transversals which meet both c and the line HK; their intersections with the line c are the points of the related range thereon. Conversely, if we draw from points A, A',... of a line a, the transversals to two other lines, c and m, which do not meet one another, or meet a, and if these transversals meet c in C, C',... respectively, then the ranges (A, A',...), (C, C',...) are related in the sense first defined; for a range with which they are both in perspective is obtained by taking any two points H, K on the line m, remarking that the lines HA, KC meet in a point, F, of the line, f, in which the planes Ha, Kc intersect one another. We may therefore define two ranges on two non-intersecting lines as being related when the joins of corresponding points of these ranges have a common transversal.

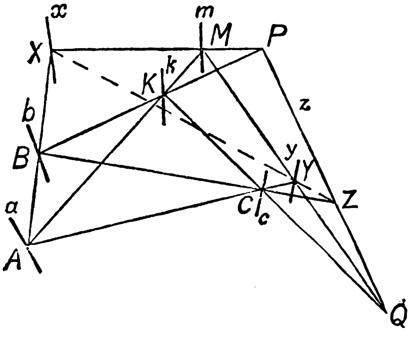
Now consider two related ranges (A, A', ...), (C, C', ...), on two lines a, c which do not intersect one another, the joins AC, $A'C'$,... of corresponding points of these ranges having the common transversal b, which they meet in (B, B',...). Suppose also that the range (D, D',...), on the line d, is in perspective with the range (A, A',...), from the point O, the line d not intersecting the line c. We wish to shew that the range (D, D', ...) is related to the range (C, C', ...).

Let the lines a, d meet in P; from P draw the transversal PQR to the lines b, c, meeting these respectively in Q, R. Let D, C be two points, respectively of the ranges on the lines d, c, which correspond to a point A of the range on a; thus DC is in the plane joining O to AC. Wherefore, if AC meet b in B, then OB meets DC, say in E. Next, let B', D' be the points of the ranges on the lines b, d which similarly arise from *any* point A' of the range on a, so that O, A', D' are in line, and the line $A'B'$ meets c, in C'. The lines OB', QE both lie in the plane Ob; let them meet in E'; we can then shew that D', E', C' are in line. For consider the two tetrads of points A', A, B, B' and D', D, E, E'; as they are in perspective from O any two joins of corresponding pairs of points in the two tetrads meet in a point lying on a plane which contains all such points of meeting, as was remarked above in connexion with Desargues' theorem. Such corresponding joins are AA', DD', meeting in P, and BB', EE', meeting



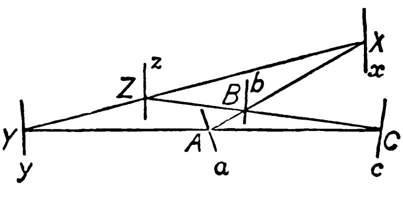
in Q, and AB, DE, meeting in C; whence it follows that $D'E'$ meets $A'B'$ in a point of the plane PQC, and the only point of $A'B'$ lying on this plane is the point C'. Thus $D'E'$ passes through C'. Therefore, as D' is an arbitrary point of the range on d, we see that this range is related to the range on c, the joins of corresponding points of these ranges all meeting a line QE; this proves what is required. The range thus arising on the line QE is in perspective with the range on the line b, from the point O.

Example. It follows from what has just been given that, if a, b, c be three lines, of which no two intersect, and a range of points on the first be related to a particular range on the second, and this in turn be related to a range on the third, then the range on the first is related to the range on the third; regarding ranges on two non-intersecting lines as being related to one another by the fact that the joins of corresponding points of these ranges have a common transversal, we may speak of this transversal as the *supplementary* line, and the range thereon as the supplementary range. A further exercise in the ideas is then obtained by proving the following particular result: Let a, b, c be three skew (non-intersecting) lines, there being ranges on a and b which are related to one another by a supplementary range on a line x, the range on b being related to a range on c by a supplementary range on a line z. Then if, in accordance with the preceding theorem, the ranges on a and c be related to one another by a supplementary range (y), on a line y, it can be shewn that the ranges (x), (z), which arise on the supplementary lines x and z, are also related to one another by the same supplementary range (y). It will be seen that, if the figure formed by four lines in a plane and their six intersections in pairs, or *vertices*, be called a complete plane quadrilateral, the result is the same as that, *if of the six vertices of a complete plane quadrilateral, five move on arbitrary skew lines, then the locus of the remaining vertex is also a line.*



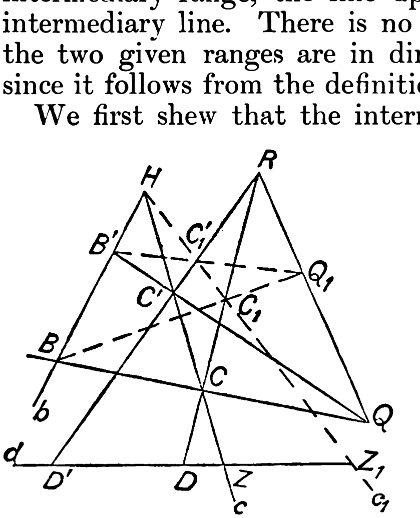
The diagram by which we illustrate the proof of this result will be of a symbolical character, a range being represented by a single general point of the range, except in the case of the range (z). Take any points, P, Q, on the supplementary line, z, of the related ranges (b), (c). Then, as has been remarked above, the joins of P, Q, respectively to corresponding points, B, C, of the ranges (b), (c), meet in the points of a range (k), lying on the intersection of the planes Pb, Qc. As the ranges (a), (b) are related, with (x) as supplementary range, and the range (k) is in perspective with the range (b), from P as centre, it follows, from the result just proved, that the joins of corresponding points, A, K, of the ranges (a), (k), meet the plane Px in the points, M, of a range (m), in perspective with (x) from P, which is the supplementary range of the two related ranges (a) and (k). Then, as the ranges (a), (k) are related, with (m) as supplementary range, and the range (c) is in perspective with (k), from Q, it follows, by the same result, that the joins of corresponding points, A, C, of the ranges (a), (c), meet the plane Qm in the points, Y, of a range (y), in perspective with (m) from Q, which is the supplementary range of the ranges (a), (c). These are therefore related. Considering the triads A, B, C and M, P, Q, we see that the joins of corresponding points, AM, BP and CQ, meet in a point K; wherefore, by Desargues' theorem, the three points of intersection of corresponding sides, namely X, from AB and MP, Y from AC and MQ, and Z, from BC and PQ, are in line. Thus any two of the ranges (x), (y), (z) are related, with the third as supplementary range; as we desired to prove. The diagram does not put in evidence the facts that the line k intersects both b and c, and the line m intersects both x and y.

The result is evidently equivalent to saying that if a, b, c, z, x be any five lines of which no two intersect, and from a variable point, B, of the line b, the transversals, ABX, CBZ, be drawn to meet the lines a, x and c, z, respectively in A, X, C, Z, then the lines AC, ZX meet in a point Y lying on a sixth line. This is another form of the statement above made in regard to a complete plane quadrilateral. More generally, given four fixed lines a, c, x, z, of which no two intersect, if through an arbitrary point B we draw the transversal to one pair of these, say a and x, meeting these in A and X respectively, and also draw the transversal to the other pair, meeting these in C and Z, it is clear that the lines AC, XZ will intersect, say in Y (as will also the lines AZ, XC). Taking all the possibilities of grouping there are thereby six points Y obtainable from B, and from any one of these, when the association of the lines is specified, we can proceed back to B by a similar construction.



General proof of the theorem. We now pass to the general proof of the theorem referred to, that, if two ranges be related, any range in perspective with either is related to the other.

By definition two ranges, on different lines, are related when they are both in perspective with a third range, from appropriate centres. For the sake of clearness let this other range be called the intermediary range, the line upon which it lies being called the intermediary line. There is no need to prove the theorem when the two given ranges are in direct perspective with one another, since it follows from the definition of related ranges.



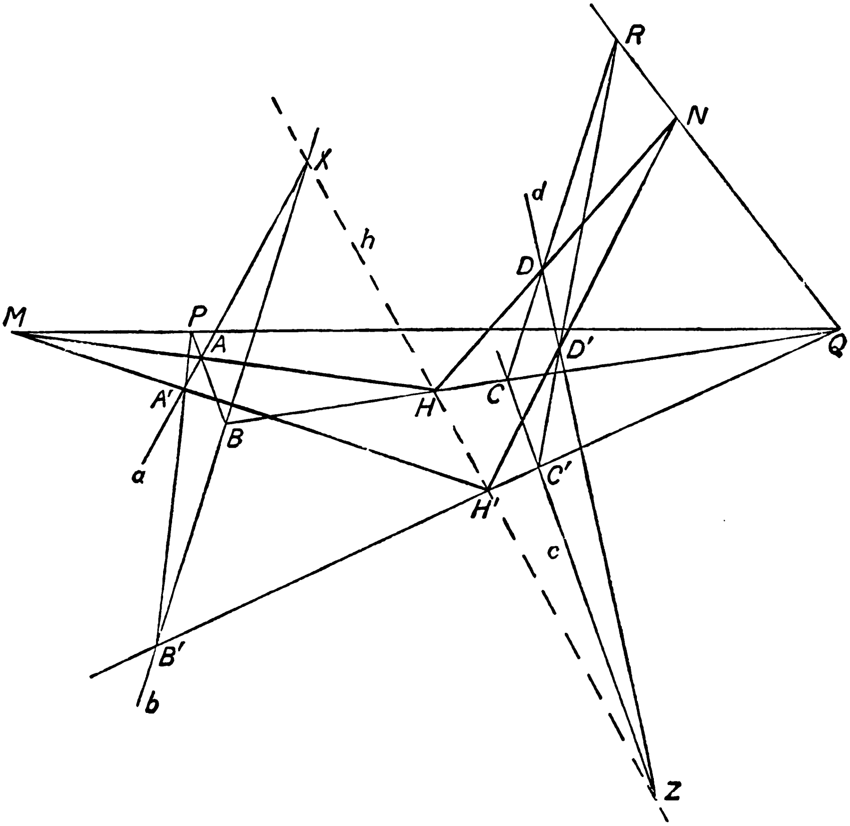
We first shew that the intermediary line, which necessarily intersects both the given lines, can be taken to be any line whatever intersecting these which does not either (*a*) intersect the lines of the related ranges in corresponding points, or (*b*) pass through the point of intersection of these lines in case they intersect. Let b, d be the lines of two related ranges, not in perspective with one another, with an intermediary line c, intersecting both b and d. Through the point H where c intersects b draw a line c_1, meeting d, in a point, Z_1, not corresponding to the point H, or (b, c), of the range (b). The possibility of this requires the assumption that if b and d intersect, the intermediary line c does not pass through their intersection; and this assumption may be made because it is easy to see, from Desargues' theorem, that if the intermediary line c passed through the intersection of b and d, then the ranges on b and d would be in perspective with one another, which we suppose not to be the case. It is also supposed that c_1 does not coincide with b (and therefore does not pass through the intersection of b and d, if they intersect). We shew that c_1 may be used, instead of c, as an intermediary line. From this the general arbitrariness of the intermediary line, asserted above, can be deduced at once. Let the ranges (b) and (c) be in perspective from the centre Q, the points C, C' of c corresponding respectively to the points B, B' of b; also let the ranges (c), (d) be in perspective from the centre R, the points D, D' of d corresponding respectively to the points C, C' of c. The line QR may be called the supplementary line. In the plane $[c, d]$, which contains R, and also contains the line c_1 drawn to meet c and d, let the lines RC, RC' meet c_1 respectively in C_1 and C_1'. The triads B, C, C_1 and B', C', C_1' are in perspective from the point H; thus, by Desargues' theorem, the lines BC, $B'C'$ intersect in a point, Q_1, of the line QR. Whence, taking for B' any point of the range (b), with corresponding positions of C' and C_1', the range (b) is in perspective from Q_1, which is determined by the definite line BC_1, with the range $(C_1, C_1', \ldots)$ lying on the line c_1, while this is in perspective from R with the range (d). Thus c_1 is a new intermediary line for the two given ranges (b), (d). The supplementary line QR is unchanged by the replacement of the line c by c_1, being determined however, now, not by R and Q, but by R and Q_1. When we further, in order to get a general replacement of the intermediary line, replace c_1 by a line c', meeting (b), passing through the point (c_1, d), the point R will be replaced by another point, say R', also on QR; the final supplementary line will be the same as the original supplementary line.

For what follows it is necessary to add the following further remark in regard to the choice of the line c_1. Let c and c_1 meet d in Z and Z_1; as the point Q lies in the plane $[b, c]$, the line QZ meets b, say in X, and X, Z are corresponding points of the ranges (b), (d) respectively. Thus X, Z_1 are not corresponding points of the ranges (b), (d). Hence Q_1, Z_1, X are not in line.

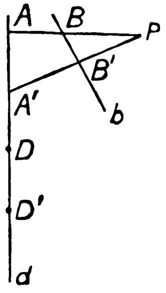
Now suppose the ranges (b), (d), on different lines b, d, to be related, not being in perspective with one another, and (a) to be a range in perspective with (b), the line a not being the same as d. We are to prove that the ranges (a), (d) are related. When (b), (d) are in perspective with one another this is true by definition. As before, let (c) be the intermediary range of (b) and (d), the centre of perspective for (b) and (c) being Q. Let X be the point of intersection of the lines a, b, and Z the intersection of the lines c, d. If Q, Z, X are in line, we can, as we have just proved, suppose c and Q respectively replaced by a line c_1 and a point Q_1 such that, if Z_1 be the point (c_1, d), the points Q_1, Z_1, X are not in line. We may, therefore, without loss of generality, suppose Q, Z, X not to be in line. Let h be the line XZ, joining the points (a, b) and (c, d). [See the diagram on page 24.]

Let A, B, C, D be corresponding points of the ranges (a), (b), (c), and (d), respectively, and A', B', C', D' other corresponding points of these ranges. The line h, joining a point of the line b to a point of the line c, is in the plane $[b, c]$, and therefore intersects the lines BC and $B'C'$, say in H and H' respectively. The triads A, B, H and A', B', H' are in perspective from X; thus, if P be the centre of perspective of the ranges (a), (b), lying at the intersection of BA and $B'A'$, it follows, by Desargues' theorem, that $H'A'$ and HA meet on PQ, say in M; this would fail if Q, Z, X were in line, the points H, H' then coinciding at Q. Again the triads H, C, D and H', C', D' are in perspective from Z; if R be the centre of perspective for the ranges (c), (d), it follows, by Desargues' theorem, that $H'D'$ and HD meet on QR, say in N.

This in effect establishes what is required; for the range on (h) is shewn to be in perspective with the range (a), from M as centre, and in perspective with the range (d), from N as centre, so that the ranges (a), (d) are related; the line h is the intermediary line and the line MN the supplementary line. The points M, N are determined when A, B, C, D are assigned.

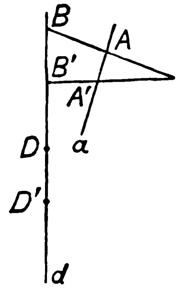


This proof is valid whether the lines a, c intersect or not, and also whether the lines b, d intersect or not. But it is necessary to consider whether the proposition is true when the lines b, d, a are not all different. The proposition has no meaning when a coincides with b; when d coincides either with b or with a, the line h, joining the intersection of a, b with the intersection of c and d, becomes the line d, and the previous construction fails. We must therefore consider the proposition independently. For this purpose it is necessary to define when two ranges on the same line are to be said to be related. If we take the natural course, and define two ranges on the same line as being related when they are both in perspective with the same range from different centres, we cannot at this stage give a proof to enable us to maintain the proposition above when the lines a, d are the same. We cannot in fact prove that if two ranges (b), (d) on different lines be related, then a range (a) on the line d which is in perspective with the range (b) is necessarily related to (d), in the sense that these two ranges (a), (d) are both in perspective with the same range. (The line upon which this last range lies would in fact meet the line d in a point belonging to both the ranges (a), (d); and until we have introduced the so-called complex points it is not necessarily true that the ranges (a), (d) on the line d have any common corresponding points.) Thus we shall define two ranges on the same line as being related when one of them, (a), is in perspective with a range, (b), which is related to the other, (d); it is then at once seen that a range which is in perspective with the other is related to the one. Two ranges on the same line which are both in perspective with another range will then be related, though for not the most general possible reason.



This being understood, the above proposition is true when the lines a, d coincide, namely a range (a) on the line d is related to (d), when (a) is in perspective with a range (b) which is related to (d). And the proposition is also true when the lines b, d coincide, namely if two ranges (b), (d) on the same line are related, a range (a) in perspective with (b) is related to (d).

Thus finally the theorem that if we have a succession of ranges of which any consecutive two are related, then any range in perspective with the first is related to the last, remains true independently of the coincidence of the lines of any two related ranges.



Examples in regard to related ranges.

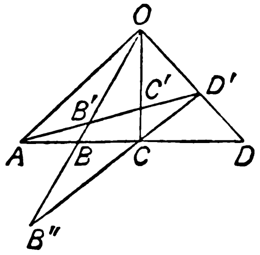
Ex. 1. If A, B, C, D be four points of a line, any two of the four ranges

$$A, B, C, D; \quad D, C, B, A; \quad C, D, A, B; \quad B, A, D, C,$$

which are obtained from the range A, B, C, D by an interchange of two of the points accompanied by an interchange of the other two points, are related.

Take any point O; join OA, OB, OC, OD; let any line through A meet the joins in A, B', C', D' respectively; let the line $D'C$ meet the join $OB'B$ in B''.

The two ranges A, B', C', D' and B, A, D, C are related, being both in perspective with the range B, B', O, B'', respectively from

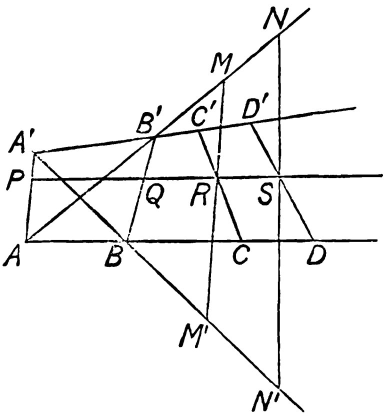


the centres C and D'; and the range A, B', C', D' is in perspective with A, B, C, D from the centre O. Thus (A, B, C, D) and (B, A, D, C) are related. From this the result follows.

Ex. 2. If on two non-intersecting lines there be two ranges A, B, C, D and A', B', C', D' which are related, so also are the ranges A, B, C, D and B', A', D', C', since (by Ex. 1) the latter is related to A', B', C', D'.

Thus if two ranges A, B, C, D and A', B', C', D', on two non-intersecting lines, be such that the four joins AA', BB', CC', DD' have a common transversal, then the four joins AB', BA', CD', DC' have a common transversal. It seems worth while to make a direct proof of this geometrical result, in verification of the foregoing theory.

The fact that the ranges A, B, C, D and A', B', C', D' are related, in accordance with what has been said above, is expressed by the fact that the lines AA', BB', CC', DD' have a common transversal, meeting them respectively, say, in P, Q, R, S. We wish to shew the existence of one or more common transversals of the cross joins AB', BA', CD', DC'. And in fact, if the transversal from R to AB' and $A'B$ meet these respectively in M and M', and the transversal from S, also to AB', $A'B$, meet these respectively in N and N', it can be shewn that the line MN', which is already a transversal of AB' and BA', is equally a transversal of CD' and DC'. The line $M'N$ is also a transversal of these four lines; and other transversals can be found by drawing from P, or from Q, the transversals to CD' and $C'D$. It can also be shewn that the point where the line MN' meets CD' lies on RA' and SA; and similarly the point $(MN', C'D)$ is the same as the point (RB, SB'), the point $(M'N, CD')$ the same as (RB', SB), and the point $(M'N, C'D)$ the same as (RA, SA').



For, first, the lines RA', SA are both in the plane $AA'Q$, and therefore meet, say in the point Y; while the line CD' lies in both the planes RCA' and SDA, and so meets both RA' and SA, but is not in the plane $AA'Q$. Thus CD' contains Y. Again, MN' lies in both the planes $RM'A'$ and SNA, and so meets both RA' and SA, but is not in the plane $AA'Q$. Thus MN' contains Y. Wherefore MN' meets CD' in Y.

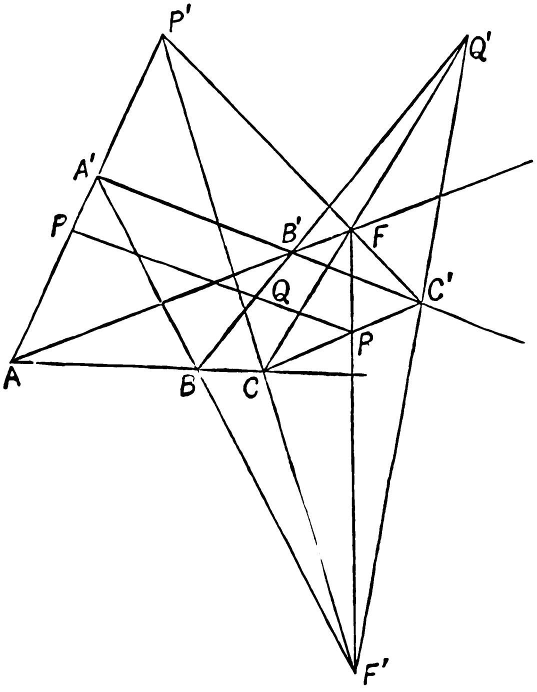
Similarly RB and SB' meet, say in X, and $C'D$ lies in both the planes BRC and $SB'D'$, while MN' lies in both the planes BRM' and $SB'N$. Thus MN' meets $C'D$ in X.

Also, SB and RB' meet, say in T, and CD' lies in both the planes BSD and $B'RC'$, while $M'N$ lies in both the planes BSN' and $B'RM$. Thus $M'N$ meets CD' in T.

Finally RA and SA' meet, say in Z, and $C'D$ lies in both the planes ARC and $A'SD'$, while $M'N$ lies in both the planes ARM and $A'SN$. Thus $M'N$ meets $C'D$ in Z.

The tetrad A, B, B', A' is in perspective with the tetrad Z, X, T, Y, from R, and in perspective with the tetrad Y, T, X, Z, from S. The four lines MC, $M'C'$, ND, $N'D'$ can be shewn to meet BB' in the same point, which is the harmonic conjugate of Q in regard to B and B'. Similarly the lines MC', $M'C$, ND', $N'D$ all meet the line AA' in the harmonic conjugate of P in regard to A and A'.

Ex. 3. It is interesting to shew that, if four skew lines AA', BB', CC', DD' have three common transversals $ABCD$, $A'B'C'D'$, $PQRS$, as in the preceding figure, we can, by the Propositions of Incidence alone, construct other transversals of these four skew lines, in infinite number, equally meeting every line having the three given transversals.



In the first place, if, on the line AA', the point P' be the harmonic conjugate of P in regard to A and A', and, on BB', the point

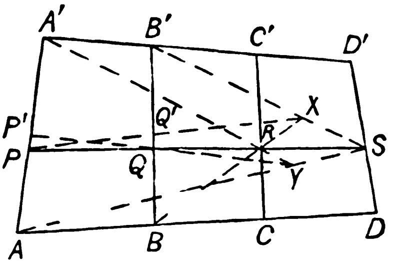
Q' be the harmonic conjugate of Q in regard to B and B', while, similarly, on CC' and DD', the points R', S' are, respectively, the harmonic conjugates of R and S in regard to C, C' and D, D', then the points P', Q', R', S' are in line. And then the line $P'Q'R'S'$ is such a transversal of AA', BB', CC', DD' as was required.

For from R, through which passes the transversal CRC' of AB and $A'B'$, draw the transversal, FRF', of AB' and $A'B$, meeting these respectively in F and F'. Then the lines CF and $F'C'$ intersect one another, and, being respectively in the planes ABB' and $A'BB'$, they meet one another on the line of intersection, BB', of these planes, say at the point Q'. Similarly the lines $F'C$ and $C'F$ meet one another in a point, P', of the line AA'. It can be shewn that Q' and P' are the harmonic conjugates of Q and P respectively in regard to the couples B, B' and A, A'. For, from the construction made, the line $P'R$ meets $F'C'$ in the point harmonically conjugate to Q' in regard to F' and C'; and this point lies on the plane $P'RA'$, which is the plane $A'QA$. Wherefore, by perspective from A', the point Q is the harmonic conjugate of Q' in regard to B and B'. A similar argument is applicable to P and P'. Next, the line $P'Q'$, in the plane of FF' and CC', meets CC', and, by perspective from P', we see that it meets CC' in the point R' which is harmonically conjugate to R in regard to C and C'. If, now, from another point S, of the line PQR, there be drawn the transversal, DSD', to AB and $A'B'$, meeting these in D and D' respectively, and the transversal, GSG', to AB' and $A'B$, meeting these in G and G' respectively, we similarly shew that the lines DG, $D'G'$ meet in Q' and the lines DG', $D'G$ meet in P', and therefore that $P'A'$ meets DD' in the point S' which is the harmonic conjugate of S in regard to D and D'.

The line $P'Q'R'S'$ may be called appropriately the harmonic conjugate of the line $PQRS$ in regard to the lines $ABCD$ and $A'B'C'D'$. We may then similarly draw the harmonic conjugate of the line $A'B'C'D'$ in regard to the lines $ABCD$ and $PQRS$, and continue this process, obtaining from every three transversals of the four lines AA', BB', CC', DD' three others, and so on.

Another method of obtaining a fourth transversal of the four lines AA', BB', CC', DD', when we are given the three transversals $ABCD$, $A'B'C'D'$ and $PQRS$, may be explained, which also leads to an infinite number of transversals. The lines so constructed to meet the four given lines are not, however, shewn to meet any line with the same three transversals.

Let the lines $B'S$ and BR, which are both in the plane SBB', meet in the point X; also the lines $A'R$ and AS, which are both in the plane SAA', meet in the point Y. The line XP, in the former plane, SBB', will meet BB', say in Q'; the line YQ, in the plane SAA', will meet AA', say in P'. It can be shewn that the line $P'Q'$, which already meets AA' and BB', also meets CC' and DD'. In fact the four lines $P'Q'$, CC', XA, YB' meet in one point, as do the lines $P'Q'$, DD', YB, XA'. A simple method of verifying this, and similar results, arises later. For the present we may argue as follows.



The lines XA, YB', in the plane ASB', intersect; the lines XA, $P'Q'$, in the plane XAP, intersect; the lines YB', $P'Q'$, in the plane YQB', also intersect. Hence the three lines $P'Q'$, XA, YB' lie in one plane or meet in one point; the former would involve that $B'Q'$ and AP', that is BB' and AA', were in one plane, and is excluded.

Again, the lines YB' and CC', in the plane $RC'A'$, intersect; and the lines AX, CC', in the plane BCR, intersect; while we have shewn that AX and YB' intersect. Hence the three lines CC', XA, YB' all lie in one plane, or meet in one point; the former would involve that $B'C'$ and AC, and therefore also BB' and CC', were in one plane, and is excluded.

Wherefore, as stated, the four lines $P'Q'$, CC', XA, YB' meet in one point.

A similar proof can be given for $P'Q'$, DD', XA', YB.

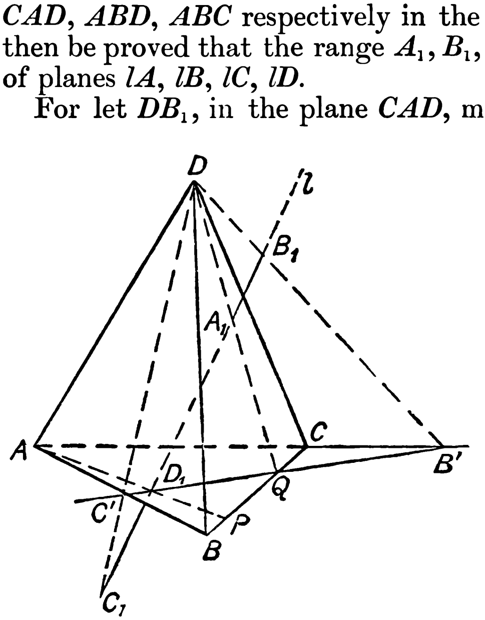
It can also be shewn that X lies on $C'D$, and that Y lies on CD'.

It cannot be proved at present that $P'Q'$ meets any other line meeting the three lines $ABCD$, $PQRS$, $A'B'C'D'$, beside the four AA', BB', CC', DD'; and it is in this respect different from the harmonic conjugate obtained above.

From $P'Q'R'S'$ we can find another line $P''Q''R''S''$ meeting AA', BB', CC', DD', just as we found $P'Q'R'S'$ from $PQRS$; and so on continually.

Ex. 4. Four planes, say α, β, γ, δ, which meet in a line, x, are said to form an axial pencil of planes. If these planes be met by any line, l, which does not meet the line x, respectively in the points A, B, C, D, and by any other line, l', not meeting x, respectively in the points A', B', C', D', then the ranges A, B, C, D and A', B', C', D' are related, since the joins AA', BB', CC', DD' have the common transversal x. We may thus speak of an axial pencil of planes, (ϖ), and a range of points, (m), as being related when the planes of (ϖ) are met by an arbitrary line, not meeting the axis of (ϖ), in a range which is related to (m).

With this definition, consider a tetrad of points A, B, C, D, not lying in a plane, and let any line, l, meet the four planes BCD, *CAD*, *ABD*, *ABC* respectively in the points A_1, B_1, C_1, D_1. It can then be proved that the range A_1, B_1, C_1, D_1 is related to the pencil of planes lA, lB, lC, lD.

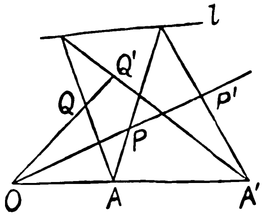


For let DB_1, in the plane *CAD*, meet *AC* in B'; let DC_1, in the plane *ABD*, meet *AB* in C'; and let DA_1, in the plane *BCD*, meet *BC* in *Q*. Then the points C', D_1, Q, B' are in the plane *Dl*, and are thus in line. Let AD_1, in the plane *ABC*, meet *BC* in *P*.

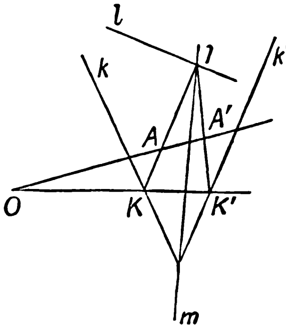
Then the ranges C_1, D_1, A_1, B_1 and B, P, Q, C, lying on lines which do not intersect, are related, because the joining lines of corresponding points, BC_1, PD_1, QA_1, CB_1, all meet the line *AD*. Wherefore, by Ex. 2 above, the ranges C_1, D_1, A_1, B_1 and C, Q, P, B are related. The latter four points lie however, respectively, upon the planes lC, lD, lA, lB. This proves the result stated.

Ex. 5. We have shewn that Desargues' theorem follows by assuming the Propositions of Incidence for space of three dimensions. Conversely it can be shewn that, if Desargues' theorem for triads of points in one plane be assumed, there are aggregates of points and lines in this plane which combine together in accordance with laws corresponding to the Propositions of Incidence of space of three dimensions. Thus, also, beings of only two-dimensional experience might, in effect, be familiar with a three-dimensional geometry.

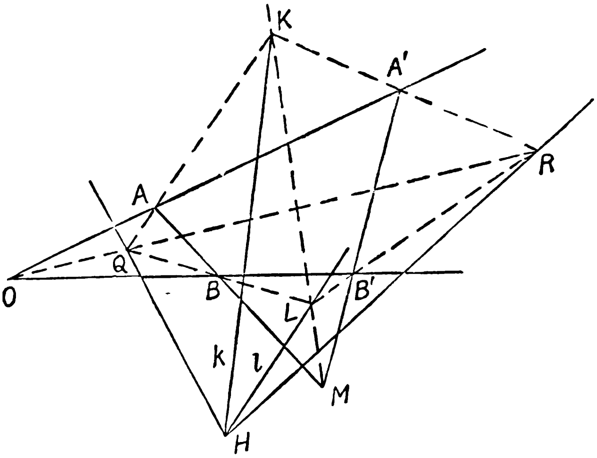
Take an arbitrary fixed point *O* of the plane. Any two points P, P' of the plane which are such that the line PP' passes through O, determine a point couple (P, P'), which, in the scheme we are explaining, plays the part of a point in a three-dimensional space of which the given plane is one plane. The couple (P', P) is to be distinguished from (P, P'). When P' coincides with P, the couple (P, P) represents the point P of the plane, regarded as belonging to the threefold space. When P' is at O, the couple (O, P), regarded as equivalent with (P, O), represents the point O itself, whatever P may be. Any two lines of the plane, taken in a definite order, give a line couple, which represents a line of the threefold space. Thus a line couple $(PQ, P'Q')$ is determined by two point couples (P, P') and (Q, Q'), and contains other point couples in infinite number, of which any two suffice for its determination. The condition that a line couple, (k, k'), should contain a point couple, (P, P'), is that the line k should contain the point P and also the line k' contain the point P'. Lastly, let the aggregate of a point couple (A, A'), and a line, l, of the given plane, represent a plane, any point couple (P, P') being regarded as lying on this plane when the lines PA and $P'A'$ meet on the line l. Thus if (Q, Q') be any other point couple of this plane, it follows, by Desargues' theorem, considering the two triads APQ, $A'P'Q'$, that the lines PQ, $P'Q'$ meet on the line l, so that the plane is equally represented by the point couple (P, P') and the line l; also any line couple $(PQ, P'Q')$ of the plane, determined by the two point couples (P, P') and (Q, Q'), is represented by two lines of the given plane which intersect on the line l. Any three point couples, (A, A'), (P, P'), (Q, Q'), determine the plane, which is given by the point couple (A, A') and a line l of the original plane containing the intersection of AP, $A'P'$, the intersection of AQ, $A'Q'$, and the intersection of PQ, $P'Q'$.



Any line couple, (k, k'), and plane (AA', l), have then a point couple in common. For, in the given plane, with the lines k, k' and the points A, A', by drawing lines through O to meet k, k', we can, in virtue of Desargues' theorem, construct a line, m, passing through the intersection of k and k'. If this line meet the line l in T, the lines TA, TA', by their intersections with k, k' respectively, define a point couple (K, K'). This lies on the line couple (k, k') and also on the plane (AA', l), and is the point couple common to these.

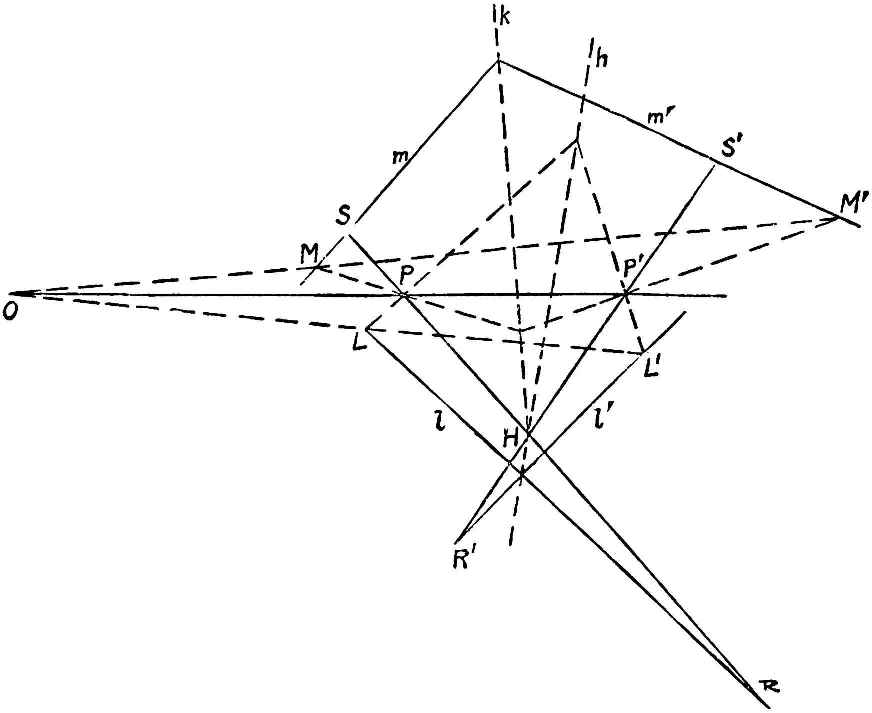


Again, any two planes, (AA', k) and (BB', l), have a line couple in common. For take the line couple containing the two point couples (A, A') and (B, B'), which is represented by the two lines AB, $A'B'$ of the given plane; let these meet in M. Draw from M an arbitrary line in this plane, meeting the two given lines k, l, of this plane respectively in K and L; let KA, LB meet in Q, and KA', LB' meet in R. Then from Desargues' theorem, applied to the two triads A, Q, B and A', R, B', it follows that QR passes through O, so that (Q, R) is a point couple. As QA, RA' meet on k, and QB, RB' meet on l, this point couple belongs to both the planes. If another line ML_1K_1 be drawn through M, and Q_1 be the position thence obtained for Q, the application of Desargues' theorem to the triads L_1, K_1, Q_1 and L, K, Q shews that the line QQ_1 passes through the point *H* where *k* and *l* intersect. Similarly the point ***R*** describes a line passing through *H*. Thus the point couple (*Q*, *R*)


K
A'
R
A
Q
O
B
B'
L
k
l
M
H

lies on a line lying in both planes, which is the line we desired to construct.

The theory may be continued. It may suffice to give one further example of its development, by drawing the line from a given point


k
h
m'
S'
m
M'
S
M
P
P'
O
L
L'
l
l'
H
R'
R

couple (P, P') to intersect each of two given lines represented by the line couples (l, l') and (m, m'). By drawing variable lines through O to meet the lines l, l' in L and L' respectively, and joining LP, $L'P'$, we define, by the locus of the intersection of these, a line h. Similarly lines through O meeting m and m' in M and M' respectively define a line k. Let these lines h, k intersect in H. Then let the lines HP, HP' meet l and l' respectively in R and R'; these will lie on a line through O and define a point couple (R, R'). So, let HP, HP' define by their intersections respectively with m and m' a point couple (S, S'). Then (HP, HP') are a line couple, representing a line, containing the given point couple (P, P'), which meets the line couple (l, l') in the point couple (R, R'), and meets the line couple (m, m') in the point couple (S, S'); as was desired.

Propositions of Incidence in space not limited to three dimensions. In general we assume that by means of two *points* a system of points is determined which can equally be determined by any two different points of the system. This system of points is said to lie on a *line*. A point is said *to be dependent upon*, or *to be in syzygy with*, two specified points, when it belongs to the system of points determined by these two. Later on we shall use the word *space* instead of the word *system*.

By means of three points, of which no one is dependent upon the others, a system of points is determined which can equally be determined by any three independent points of the system. The points of the line determined by any two points of the system all belong to the system. The system of points determined by three independent points is said to lie on a plane, or to be a twofold system. A point is said to be dependent upon, or to be in syzygy with, three specified independent points when it belongs to the system of points determined by these three.

By means of four independent points a system of points is determined which can equally be determined by any four independent points of the system. The points of a plane determined by any three independent points of the system, and therefore also the points of a line determined by two points of the system, all belong to the system. A point is said to be dependent upon, or in syzygy with, four independent points when it lies in the system determined by these four. Such a system is called a threefold system, or the points are said to lie on a threefold.

And so on in general, without limit.

Consider now an n-fold system which is determined by $(n+1)$ points, these last being such as to contain three independent points, a fourth point independent of these, a fifth point independent of these four, and so on, and, finally, an $(n+1)$th point independent of the preceding; in such an n-fold system there exist r-fold systems

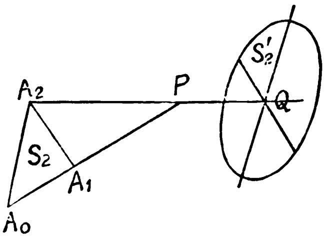
$(r < n)$, each determined by $(r+1)$ independent points, of which, by hypothesis, every point belongs to the n-fold system. Two such contained systems, for the same or different values of r, may have points in common. Denote by S_r a system determined by $(r+1)$ independent points, and in particular by S_0 a single point. The general statement is that, *if r and h be two numbers both less than n but such that their sum is equal to or greater than n, then two systems, S_r and S_h, both contained in S_n, have always common a system S_{r+h-n}, but may have common a system S_k for which k is greater than $r+h-n$. When, as before, $r<n$ and $h<n$, but $r+h<n$, two systems S_r, S_h, both contained in S_n, have not always a system common. In general, with $r<n$, $h<n$, if two systems S_r, S_h, both contained in S_n, have common a system $S_{r+h-n+t}$, then there exists a system S_{n-t}, contained in S_n, which contains both S_r and S_h.*

We shew that these results are a logical consequence of one fundamental proposition, which, like the conception of a system of points determined by given points, we assume. This proposition is that, *if in the system S_n there be contained a system S_{n-1}, determined by n independent points of S_n, and also a system S_1, determined by two points of S_n which do not belong to S_{n-1}, then S_1 and S_{n-1} have one point in common*, so that every S_{n-1} contained in S_n has with every S_1 contained therein at least one common point. For instance we assume that, in a S_2 (or plane), any two S_1, (any two lines), have a point in common; and, in a S_3, any plane has common, with a line determined by two points not in this plane, a single point.

(1) From this proposition it follows that every S_r contained in S_n, where $r<n$, has a point in common with every S_{n-r} contained in S_n.

To prove this we remark that when $r=2$ every S_{r-1} contained in S_n has a point in common with any S_{n-r+1} contained therein. Then we shew that, from the assumption that every S_{r-1} contained in S_n has a point in common with any S_{n-r+1} contained therein, it follows that every S_r contained therein has a point in common with any S_{n-r} so contained. Consider an S_r determined by the $r+1$ independent points $A_0, A_1, \ldots, A_r$, contained in S_n, and an S_{n-r} contained in S_n. It is supposed that S_r is not wholly contained in S_{n-r}; there is therefore at least one of the points $A_0, A_1, \ldots, A_r$ which is not contained in S_{n-r}; we may therefore suppose that the point A_r is not contained in S_{n-r}, and that the points $A_0, A_1, \ldots, A_{r-1}$ are independent points. The S_{n-r} is determined by $n-r+1$ independent points, and we may suppose that A_r is independent of these, since $(n-r+1)+1$, or $n+1-(r-1)$, is less than the greatest possible number, $n+1$, of independent points in S_n (for $r=2, 3, \ldots$). These $n-r+1$ points together with A_r thus determine a Σ_{n-r+1} contained in S_n, which we may speak of as determined by S_{n-r} and A_r. On the other hand the points $A_0, A_1, \ldots, A_{r-1}$ determine a Σ_{r-1} contained in S_r. By what we have supposed, this Σ_{r-1} and the Σ_{n-r+1} have a point in common (one at least), which we may denote by P. This point P, being in Σ_{r-1}, is in S_r, as also is A_r, and we can suppose P and A_r not to coincide; thus the points P and A_r determine a Σ_1 lying in S_r. On the other hand both P and A_r lie in the Σ_{n-r+1} determined by S_{n-r} and A_r, as therefore also does the Σ_1. By the fundamental proposition, therefore, the Σ_1 and S_{n-r} have a point in common, say Q, which, as Σ_1 is contained in S_r, is a point common to S_r and S_{n-r}. This is the point whose existence was to be shewn.

To explain the argument by a particular case suppose $n=4$ and $r=2$. We are to shew that any two planes in a space of four dimensions have a point in common (one at least). Let these be S_2, determined by the independent points A_0, A_1, A_2, and S_2', of which S_2' does not contain A_2. Then A_2 and S_2' determine a Σ_3, while $A_0 A_1$ determine a Σ_1' contained in S_2; this Σ_1' has with Σ_3 a point P common. Both A_2 and P are in S_2, and determine a Σ_1, the line A_2P. But A_2 and P are both in Σ_3, which contains S_2'. Thus the line A_2P meets S_2', in Q, which is then common to S_2 and S_2'. Thus it appears as a consequence of the facts that in an S_3 a line intersects a plane, and in an S_4 a line intersects an S_3, that, in an S_4, an S_2 intersects an S_2'.

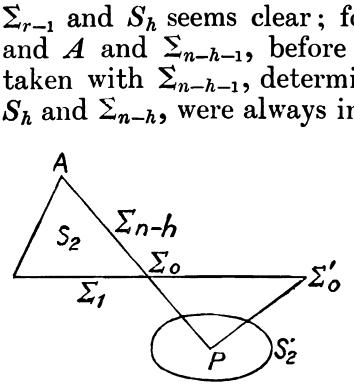


(2) We next deduce, if $r<n$ and $h<n$, but $r+h \geqq n$, that an S_r and an S_h have an S_{r+h-n} in common (at least). It is sufficient, after (1), to prove this when $r+h>n$. For this we suppose that an S_{r-1} and an S_h have an $S_{r+h-n-1}$ in common, which is true, by (1), when $r=n-h+1$.

We can, as in (1), suppose the S_r to be determined by a Σ_{r-1}, lying in S_r, and a point A, also lying therein. Then, because $r \geqq n-h$, we can, in Σ_{r-1}, find a Σ_{n-h-1}; with A this will determine a Σ_{n-h}, lying in S_r; by (1), this Σ_{n-h} will contain a point, P, of S_h; this point P lies then in S_r and S_h. By hypothesis the Σ_{r-1} and S_h have a $\Sigma_{r+h-n-1}$ in common; this, lying in Σ_{r-1}, also lies in S_r. Assuming, for a moment, that P does not lie in the $\Sigma_{r+h-n-1}$, this point P and the $\Sigma_{r+h-n-1}$, together, determine an S_{r+h-n}, and this lies both in S_r and S_h because both P and $\Sigma_{r+h-n-1}$ do so. This establishes the statement made.

That the common point, P, of S_h and the Σ_{n-h} determined by Σ_{n-h-1} and A, does not necessarily lie in the $\Sigma_{r+h-n-1}$ common to

Σ_{r-1} and S_h seems clear; for we can suppose P determined by S_h and A and Σ_{n-h-1}, before we specify the $r+h-n$ points which, taken with Σ_{n-h-1}, determine the Σ_{r-1}; if P, which is common to S_h and Σ_{n-h}, were always in Σ_{r-1}, it would be in Σ_{n-h-1}; this however has, in general, no point common with S_h.



For an illustrative example, let $n=3$, $r=2$, $h=2$; the theorem will be that in an S_3, a space of three dimensions, an S_2 has with an S_2' a line common. We suppose the S_2 determined by a line Σ_1 and a point A. In this line Σ_1 take a point Σ_0; this point, with A, determines a line Σ_{n-h}, lying in S_2. This line, Σ_{n-h}, contains a point P of S_2', lying in S_2. The line Σ_1 has, with S_2', a common point Σ_0', also in S_2. As P and Σ_0' are different, they determine a line, $P\Sigma_0'$, lying both in S_2 and S_2'. Thus the hypothesis that a line and an S_2', lying in an S_3, have a point in common, involves that two planes in S_3 have a line in common.

(3) Lastly, if in a system S_n, an S_r and S_h have common a system $S_{r+h-n+t}$, then there exists a system S_{n-t}, contained in S_n, which contains both S_r and S_h. It is not assumed that $r+h$ is as great as n.

By the hypothesis we must have

$$r+h-n+t \leqq r, \quad \text{and} \quad r+h-n+t \leqq h,$$

so that $h \leqq n-t$ and $r \leqq n-t$. We can take, in $S_{r+h-n+t}$, independent points in number $r+h-n+t+1$; these, with $n-t-h$ points, independent of the former, taken in S_r, will furnish $r+1$ independent points by which S_r may be determined; the same $r+h-n+t+1$ points in $S_{r+h-n+t}$, with $n-t-r$ points independent of those already taken in S_h, will furnish $h+1$ points by which S_h may be determined. The total number of independent points thus employed is, however, only

$$(r+h-n+t+1)+(n-t-h)+(n-t-r)$$

or $n-t+1$. The system determined by these, which is a S_{n-t}, contains then both S_r and S_h.

For example, for $n=4$, $r=h=2$, two planes in a fourfold space have not a line in common unless they both lie in the same space of three dimensions.

Examples of the Propositions of Incidence in space of four and five dimensions.

Ex. 1. In space of four dimensions, two lines which have no point in common, being each determined by two points, determine together a space of three dimensions, defined by four points two on each line. This threefold space has in common with any arbitrary third line of the fourfold space a single point; from this point can be drawn, in this threefold space, a single transversal line of the two former lines. Thus we can shew that three general lines in space of four dimensions are all met by one other definite line. If the three lines lie in a threefold space they have, of course, an infinite number of common transversals.

Ex. 2. In general a line and plane in a fourfold space have no point in common. If they have a point O in common, this with another point P of the line, and two other points A, B of the plane not lying in line with O, make four points, which determine a threefold space containing both the given line and plane.

But through one of three lines in fourfold space, having one common transversal, can be drawn a single plane to meet each of the other two lines in a point, that namely containing the chosen line and the common transversal line of the three given lines.

It follows that, in space of four dimensions, ∞^1 planes can be drawn through an arbitrary point O to meet each of three given lines of general position. For a plane can be drawn containing the line joining O to an arbitrary point of one of the three given lines which shall meet each of the other two. We can further specify three other lines each of which is met by every one of these planes. For consider one of these planes, meeting the three given lines a, b, c, respectively in A, B, C. In this plane the lines AB and OC intersect one another, say in the point P. And this point P is clearly on the line which is the intersection of the threefold defined by the lines a, b and the plane Oc. There are two other lines which arise by replacing c, in this description, by a or by b.

Ex. 3. It has already been remarked above that the necessary condition for two planes in fourfold space to have a line in common is that these planes should lie in a threefold space. This space will be determined by two points of the line and two further points, one on each plane.

Ex. 4. Three lines in most general position, being determined by six points, two on each line, lie in at most five dimensions; if the lines have a single common transversal line, they lie in the fourfold space determined by two points of this and a further point on each of the lines; if they have two common non-intersecting transversal lines they lie in the threefold space determined by these. Similarly a line and a plane, determined by two points of the line and three points of the plane, lie in a fourfold space, unless the line and plane have a common point. Further, two planes are determined by three points on each, and lie in at most five dimensions; if the planes have one common point they lie in four dimensions; if they have two common points they lie in three dimensions.

three points $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$, which, as the planes ABC, $A'B'C'$ are distinct, must be in line. The six points of intersection of corresponding lines are then on four lines of the plane ϖ. (Cf. the tetrads A, B, C, P; A', B', C', P' in the Frontispiece.)

Ex. 7. The results of Examples 2 and 5 may be generalised—and the conclusions will be of interest for an application to the theory of related ranges of points.

We prove that in space of an odd number of dimensions, S_{2n-1}, a definite S_{n-1} can be drawn through a point, O, to meet n lines of general position, each in one point. And hence, that there are ∞^1 spaces S_{n-1} meeting $(n+1)$ lines of general position; and further, when $n > 2$, that each of these S_{n-1} meets the line which is the intersection of the S_3, containing any two of the $(n+1)$ given lines, with the S_{2n-3} containing the remaining $(n-1)$ of the given lines. There are $\frac{1}{2}n(n+1)$ lines of this description.

We prove, also, that in space of an even number of dimensions, S_{2n}, a definite S_{n-1} can be drawn to meet $(n+1)$ lines of general position. And hence, that there are ∞^1 spaces S_n passing through a point, O, which meet $(n+1)$ lines of general position; and further, when $n > 1$, that each of these S_n meets the line which is the intersection, of the S_3 containing two of the $(n+1)$ given lines, with the S_{2n-2} containing O and the remaining $(n-1)$ of the given lines. There are $\frac{1}{2}n(n+1)$ lines of this description.

Consider a space S_{2n-1}, and therein a point, O, and n lines of general position. Let a be one of the lines; the remaining $(n-1)$ lines, each determined by two of its points, together with O, depend upon $(2n-1)$ points, which, as we provisionally assume, may be regarded as independent. These determine then a S_{2n-2} containing O and these $(n-1)$ lines, which we speak of as determined by O and these lines. In the S_{2n-1}, this S_{2n-2} meets the line a in a point, say A. There are other $(n-1)$ spaces, also of dimension $(2n-2)$, each of which contains the point O, the line a, and all of the remaining given lines except one. The n spaces, S_{2n-2}, thus defined, meet in a S_{n-1}, which contains the point A, this being on every one of the spaces. By symmetry this S_{n-1} contains a point of each of the n given lines; and it passes through O. Conversely a S_{n-1} through O, which meets each of the n given lines, has n points in common with a S_{2n-2} containing O and any $(n-1)$ of these lines; such S_{n-1} thus coincides with that found, which is therefore unique.

It follows that there are ∞^1 spaces S_{n-1} meeting $(n+1)$ lines of general position in S_{2n-1}, namely one such through every point of a selected one of the $(n+1)$ given lines. Let one such S_{n-1} meet two of these given lines, say a_1 and a_2, respectively in A_1 and A_2. The $(n-1)$ intersections of this S_{n-1} with the other $(n-1)$ given lines determine a S_{n-2} lying in the chosen S_{n-1}; and the line A_1A_2, lying in the S_{n-1}, meets this S_{n-2} in a point, say P. This is a point of the S_{n-1}. The line A_1A_2, however, lies in the S_3 determined by the lines a_1 and a_2; and the S_{n-2} contains $(n-1)$ points of the S_{2n-3} containing the other $(n-1)$ given lines, and so lies in this S_{2n-3}. Thus the point P is in the space common to the S_3 and S_{2n-3}; this common space is of dimension $(2n-1)-(2n-4+2)$ or 1, namely it is a line. This proves the result stated for a space S_{2n-1}. The method of proof is provisional, since we have, for simplicity, assumed without examination various sets to consist of independent points. The same remark applies to the case of a space S_{2n}, which we now consider.

In a space S_{2n} suppose $(n+1)$ lines of general position to be given. Let a be one of these lines. The remaining n lines determine a S_{2n-1}, which will meet the line a in a point, say A. The space common to this S_{2n-1} and to the other n spaces of dimension $2n-1$, each determined by the line a and $(n-1)$ of the remaining lines, is of dimension $2n-(n+1), = n-1$. This S_{n-1} contains the point A, which is on each of the $(n+1)$ spaces S_{2n-1}; and thus, by symmetry, this S_{n-1} meets every one of the $(n+1)$ given lines. Conversely, any existing S_{n-1} meeting these lines, as it contains n points of each S_{2n-1} containing n of these lines, lies in all the $(n+1)$ spaces S_{2n-1}; the S_{n-1} found is therefore unique. Hence we deduce that a single space S_n exists containing an arbitrary line, b, which meets n other lines of general position. For this line, b, and these n others, are all met by a certain S_{n-1}, as we have shewn; this S_{n-1} is determined by n points which, with a point of the line b not lying on the S_{n-1}, give $(n+1)$ points determining a S_n containing b and the S_{n-1}; this S_n therefore meets these n other lines each in a point.

It follows that ∞^1 spaces S_n can be drawn through an arbitrary point O to meet $(n+1)$ lines of general position. Take one of these S_n; let a_1, a_2 be two of the $(n+1)$ lines, the aggregate of the remaining $(n-1)$ lines being denoted by (c). On these lines (c) there are the $(n-1)$ points in which they are met by this S_n, and this also contains the point O. These n points determine a S_{n-1} lying in this S_n, and this is intersected in a point by the line, say t, also lying in this S_n, joining the two points where this S_n meets a_1 and a_2. Thus we have a point, say P, of S_n not on any of the $(n+1)$ given lines. The S_{n-1} contains n points of the S_{2n-2} determined by the $(n-1)$ lines (c) and the point O, and the line t lies in the S_3 determined by the two lines a_1, a_2. The point P therefore lies on the intersection of this S_{2n-2} and S_3, and these, in S_{2n}, meet in a space of $2n-(2+2n-3)$ dimensions, that is in a line. The ∞^1 spaces S_n thus meet the line which is the intersection of the S_3, containing two of the $(n+1)$ given lines, and the S_{2n-2} determined by O and the $(n-1)$ remaining lines. There are $\frac{1}{2}n(n+1)$ lines of this description.

Ex. 8. We have said that two ranges of points on two lines in threefold space are related when they are both in perspective with the same range on another, intermediary, line, from two appropriate centres. The threefold space may be determined by the two centres of perspective and two points of the intermediary line. As a particular case the two ranges may lie on lines which are in a plane. When this is not so the ranges are related if the joining lines of corresponding points of the ranges all meet another line. The definition gives a geometrical construction by which we may pass from a point of either range to the corresponding point of the other range *depending only on the Propositions of Incidence.* Suppose that, in a space which is of higher than three dimensions, we have a set of lines arranged in order, say a, b, c,..., k, each containing a range of points, and that the ranges (a), (b) are related by the preceding construction, which will be confined to the threefold (or twofold) space determined by the lines a, b, also the ranges (b), (c) related by this construction, in the threefold space (b, c), and so on for every consecutive two. Then we may enquire whether the ranges (a), (k) are related by the preceding construction. Next suppose that in a space of more than three dimensions we have two ranges (a), (k) for which we have a geometrical construction, only depending on the Propositions of Incidence, defining uniquely the point of either range from a corresponding point of the other. We may then enquire whether two such ranges are related according to the preceding definition. If the answer to this query be affirmative, that to the former will also be.

We do not enter into this enquiry now. But the preceding example gives a case which is illustrative. In four dimensions a geometrical correspondence is effected between the points, P, Q, of two lines, a, b, by means of planes containing these points drawn through a point O to meet a third line, c. We have shewn that then the joining line PQ meets the plane Oc in the points of a line lying in the threefold determined by a and b. In five dimensions a geometrical correspondence is effected between the points, P, Q, of two lines, a, b, by means of planes containing these points which meet two other lines, c, d. We have shewn that then the joining line PQ meets the threefold space determined by c, d in the points of a line lying in the threefold space determined by a and b. And we have shewn that this can be generalised to space of higher dimension. (See Note I, at the end of the Volume.)

Ex. 9. Another example interesting to consider in connexion with the enquiry suggested in the last example arises for the planes which meet four lines of general position in space of four dimensions. One such plane can be drawn passing through two arbitrary points one on each of two of the four given lines; so that we may say there are ∞^2 such planes. It will be proved subsequently, with the help of an assumption in addition to the Propositions of Incidence, that all these planes meet another line. (See p. 92, below.)

Let the given lines be denoted by a, b, c, d, and the line which meets a, b, c be denoted by d', the lines meeting the other sets of three of the four given lines being denoted respectively by a', b', c'; then it can be shewn from the Propositions of Incidence only that the four threefolds determined respectively by the pairs of lines (a, a'), (b, b'), (c, c') and (d, d') do not meet in a point merely, but in a line. It is this line which is shewn to be met by any plane meeting the lines a, b, c, d.

On the Principle of Duality in general. We have remarked that in three dimensions the Propositions of Incidence establish a duality whereby, to every geometrical result involving points, lines and planes, there corresponds a result involving respectively planes, lines and points. A similar duality arises from the Propositions of Incidence in space of any number of dimensions. If the space be of n dimensions the correspondence between spaces contained therein is between a space of dimension r and a space of dimension $n-r-1$. Thus to a point corresponds a S_{n-1}; to the line determined by two points corresponds the S_{n-2} which is the intersection of two spaces S_{n-1}, and so on. To our fundamental proposition that a line meets a S_{n-1} in a point corresponds then the statement that a S_{n-2} and a point together determine a S_{n-1}, which was a more fundamental part of our description of what is a space, or system.

Thus, in four dimensions, to a point corresponds a threefold and to a line corresponds a plane; and conversely in each case. For example, as two planes in four dimensions have in general a point in common, so two lines determine a threefold; the fact that when the planes have a line in common they lie in a threefold, becomes the fact that when two lines lie in a plane they have a point in common. The fact that a line and a plane which meet in a point lie in a threefold is evidently dual to its converse. Again as there is a line meeting each of three general lines in a point, so there is a plane meeting each of three given planes in a line; the fact that three lines lying in a threefold are met, each in a point, by an infinite number of lines lying in this threefold, becomes the fact that if three planes have a point in common an infinite number of planes can be drawn through this point each meeting each of the three given planes in a line. The result we have discussed above, that there is an infinite number of planes through a given point meeting each of three given lines, of general position, in a point, becomes the statement that of the lines meeting three given planes of general position there is an infinite number lying in an assigned threefold; these are evidently the common transversals of the lines in which the three given planes meet the threefold.

In five dimensions a line is dual to a threefold and a plane is dual to a plane. For example, the fact that two lines in a plane have a point in common is dual to the fact that if two threefolds, instead of meeting in a line, meet in a plane, then they lie in a fourfold space.

On projection from one space to another. It is frequently possible to make a correspondence by geometrical construction between the points of one space, α, and the points of another space, β, a single arbitrary point of either space being thus related uniquely to a single point of the other space. When through certain fixed points, or spaces, γ, variable spaces, ϖ, can be drawn, each of which intersects the space α in one point, and is made unique by the assignment of this point, and then intersects the space β in one point, and if the same is true when instead of α, β we say respectively β, α, then such a correspondence may be called a projection. The process will have so many applications that it seems unnecessary to do more at this stage than to refer to it as one of the fundamental things.

SECTION II. THE CORRESPONDENCE OF THE POINTS OF TWO LINES. PAPPUS' THEOREM

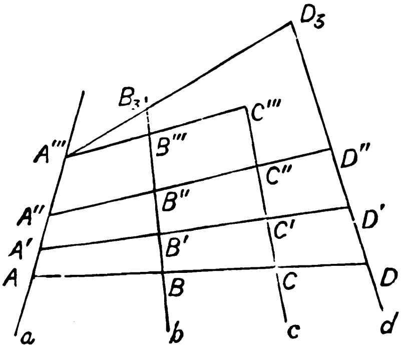
Preliminary remarks. In using the words point, line, plane, for the entities of our theory, it is clearly intended that the properties obtained shall not be widely different from those ordinarily supposed to belong to the entities, given or suggested by experience, to which these familiar names are attached. In particular, the fullness, or multiplicity, of the points that are possible on a line must not be widely different in the two cases. As will appear quite adequately in the sequel, this condition is not secured by the Propositions of Incidence alone; these leave us free to regard as distinct, because differently obtained, points which, from the suggestions of our experience, we should regard as coincident. This experience takes account in fact of other ideas than those we have embodied in the Propositions of Incidence, in particular of the idea of one point of the line being *between* two others. This idea is considered in Chapter II. As has been stated in the Introductory Remarks to this volume, we do not finally adopt this idea as fundamental; the present Section is intended to lead up to the adoption of another assumption which will enable us to avoid this.

The question of the identification of points upon a line we merge in the question of the identity of the points of two ranges of points on the line; and this we regard as a particular case of the question when two ranges of points on two different lines can be regarded as being in unambiguous correspondence, so that the specification of a point of either line leads to a definite point of the other. One way in which such a correspondence arises is suggested by our preceding work; the points of two ranges which are in perspective from a point are evidently in such unambiguous correspondence. It is natural then to make the condition that a method of correspondence adopted shall be such that ranges in perspective are a particular case; and hence that ranges which are in correspondence by means of a succession of any number of perspectivities shall also be in correspondence in virtue of the definition which we adopt. In a phraseology we have used, and by a result obtained in Section I, this comes to saying that two ranges which are *related*, in particular in perspective, shall correspond with one another. A particular consequence of this condition is that if four points of one range be in the harmonic relation to one another, this shall be true also of the four corresponding points of the other range. But, of four points in this harmonic relation, three can be taken arbitrarily. Thus the method of correspondence which we adopt must be of such generality that, having taken any three points upon one of two lines, a correspondence is possible in which any three arbitrarily chosen points of the other line correspond respectively to these; and the same must be true of two ranges upon the same line which correspond to one another. Now, we can find upon any line, b, a range which is related, in the sense used in Section I, to a given range upon another line, a, so that three arbitrary points of b shall correspond to three specified points of the given range (a); if the Propositions of Incidence were sufficient to secure an unique point of b to correspond then to an arbitrary point of the given range (a), these Propositions might be a sufficient basis for such a theory of correspondence. Unfortunately this is not so. On the other hand a range on one line, related to a given range on another line, cannot be constructed so that four arbitrary points of the constructed range shall correspond to four given arbitrary points of the original range.

Hence either we must give up the suggestion that the method of correspondence adopted, ultimately with a view to the identification of the points of a line, shall lead us to regard related ranges as being in correspondence; or else a statement that two ranges are in correspondence because they are related must always be accompanied by a statement of a particular construction rendering the method of relation unambiguous; or else finally we must add to the Propositions of Incidence some further general Proposition which, probably by acting as a restriction of what points we regard as possible on a line, shall secure that the correspondence of the points of two related ranges is unique when three points of the one are given as corresponding to three points of the other. It is the last alternative which we adopt. As will be seen, a geometrical theory can thus be framed which, avoiding many particular assumptions, obtains results exactly corresponding to those usually based on these assumptions. The merit of pointing out, in part, the very remarkable fact that such a course should be possible belongs originally to H. Wiener, *Jahresbericht d. Deutschen Math.-Verein*, Erster Bd., 1890–91, p. 47. His statement relates to geometry in a plane. For subsequent development cf. F. Schur, *Math. Annal.* LI, 1899, p. 402 and Hilbert, *Gauss-Weber Festschrift*, *Grundlagen der Geometrie*, 1899.

Conditions that the correspondence of two related ranges should be of freedom three. Consider, in space of three dimensions until the contrary is said, two lines, a, b, which do not intersect. Upon the former take three arbitrary points, A, A', A'', and upon the latter three arbitrary points, B, B', B''. Join AB; upon this take any point C. From C draw the transversal, $CC'C''$, to meet the join $A'B'$ in the point C', and the join $A''B''$ in the point C''; denote this transversal by c. Then, if A''' be any fourth point on the line a, the transversal, $A'''B'''C'''$, drawn from A''', to meet the line b in B''' and the line c in C''', makes correspond to the arbitrary point A''' of a the definite point B''' of b. Namely the assignment of three points, B, B', B'', of b, to correspond respectively to three given points, A, A', A'', of a, *with the additional assignment of a definite point* C *of the line* AB, makes correspond to any range $(AA'A''A'''\ldots)$ of the line a, a definite range $(BB'B''B'''\ldots)$ of the line b.

But it does not follow that this correspondence is independent of the position of C upon the line AB, when A, A', A'' and B, B', B'' remain the same. Let D be any other point of the line AB, and draw the transversal d to meet the line $A'B'$ in the point D', and $A''B''$ in D''; then denote by B_3 the point in which the transversal $A'''B_3D_3$, drawn from A''' to meet the lines b and d, meets the line b. It does not follow from the Propositions of Incidence alone that the point B_3 coincides with B'''.

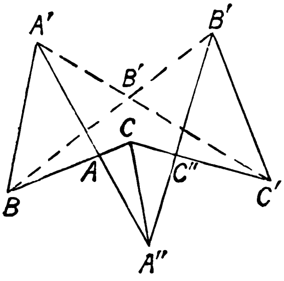


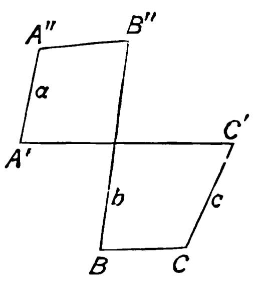
If B_3 do coincide with B''', then the line $A'''B'''C'''$ coincides with the line $A'''B_3D_3$ and meets all of a, b, c, d. If such coincidence holds for all positions of A''' upon a, the four lines a, b, c, d, simply in virtue of having three transversals, $ABCD$, $A'B'C'D'$, $A''B''C''D''$, will have an infinite number of transversals, namely, what is important, one passing through every point of the line a. And, if it be true generally that any four lines, of which no two intersect, (lying in a threefold space), which have three transversals meeting all of them, have also a common transversal passing through any point of one of them, then, in the figure just considered, the four lines ABC, $A'B'C'$, $A''B''C''$, $A'''B'''C'''$ will be intersected by the transversal drawn to the first three of them from *any* point, D, of the first.

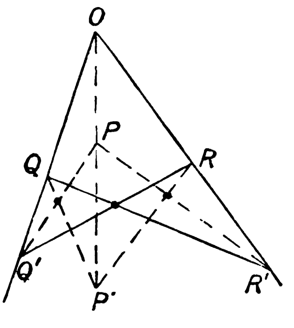
We thus have two equivalent conditions which are necessary, and are also sufficient, in order to secure that the correspondence of two related ranges, on two non-intersecting lines a, b, may be uniquely determined by the assignment of three points of one of these lines to correspond respectively to three assigned points of the other, namely:

1. That if four skew lines have three common transversals, they should possess a common transversal through any point of each of the four lines.
2. That if three lines, a, b, c, of which no two intersect, be each met by three other lines, ABC, $A'B'C'$, $A''B''C''$, then every transversal of the first three lines, a, b, c, should meet every transversal of the second set of three lines.

Brianchon hexagons and another form of the conditions. This last can be replaced by another equivalent condition. Consider six points, not lying in a plane, joined in a definite order, P, Q', R, P', Q, R', to form what we describe as a skew hexagon. It is easy to see that, if the three lines PP', QQ', RR', joining opposite angular points of the hexagon, meet in a point, say O, then any two opposite sides of the hexagon, such as PQ', $P'Q$, or $Q'R$, QR', or RP', $R'P$, intersect one another. And conversely that, if every







pair of opposite sides is in a plane, the three diagonals meet in a point. A hexagon with this particularity we may distinguish by calling it a *Brianchon* hexagon. Such a hexagon is constituted by six points of the figure we were discussing, namely, in order, A', A'', B'', B, C, C'. The three pairs of opposite sides of this hexagon, namely ($A'A''$, BC), ($A''B''$, CC'), ($B''B$, $C'A'$), meet, respectively, in the points A, C'', B'. The condition (2) above can then be stated in the form

3. That any transversal of one set of alternate sides (here the lines a, b, c) of a Brianchon hexagon should intersect every transversal of the other set of alternate sides (here the lines BC, $C'A'$, $A''B''$).

Pappus' Theorem. Is true if the conditions are satisfied. These three conditions are, in virtue of the Propositions of Incidence, equivalent to one another. We proceed to shew that they are also equivalent to a certain proposition of geometry in a plane; that is, that this proposition follows if any one of the three conditions is supposed always to be satisfied, while, conversely, the truth of this proposition would secure that every one of these conditions is satisfied. The proposition in question is

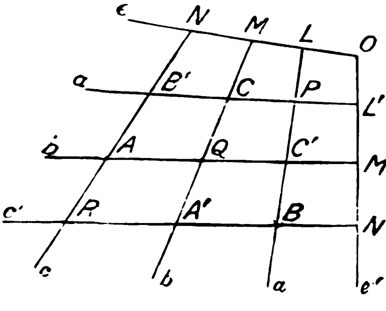
4. If L, M, N be any three points of a line, and L', M', N' be any three points of another line, intersecting the former, then the three points of intersection of cross joins

$$(MN', M'N), \quad (NL', N'L), \quad (LM', L'M)$$

are in line.

This proposition is stated in the συναγωγή of Pappus (? 340 A.D. See Ch. Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. liii, and the works of T. L. Heath on the geometry of the Greeks referred to in the Bibliography at the end of this volume). We shall refer to it as Pappus' Theorem.

We shew, first, that Pappus' theorem follows if the condition (2) above be satisfied.



Let O be the point of intersection of the lines LMN and $L'M'N'$, which we may denote respectively by e and e'. Through the points L and L' draw two lines, respectively a and a', intersecting in a point P, not lying in the plane of the lines e and e', but otherwise arbitrary. Through the point M draw any line, b, intersecting the line a', say in the point C. This line b will not intersect the line a, or the line $L'M'N'$, since, otherwise, a' would intersect the line LMN and be in the plane of e and e'. Then through the point M' draw the transversal to the lines a and b, say the line b', meeting the lines a, b respectively in C' and Q. This line b' will not meet the line a', or the line LMN, since, otherwise, the line a would intersect the line $L'M'N'$ and be in the plane of e and e'. Lastly, draw from the point N the transversal to the lines a' and b', say the line c, meeting a' and b', respectively, in the points B' and A. This line c will not meet any one of the lines b, a, $L'M'N'$, since, otherwise, the line a' would meet the line LMN. And draw from the point N' the transversal to the lines a, b, say the line c', meeting these, respectively, in B and A'. This line c' will not meet any one of the lines b', a', LMN, since, otherwise, the line a would meet $L'M'N'$. The fact that no two of the four lines a', b', c', LMN intersect, and no two of the four lines a, b, c, $L'M'N'$ intersect, is thus a consequence of taking P to be any point not lying in the plane of the two lines e and e'.

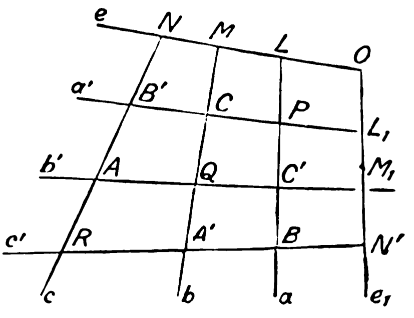
If now we assume the condition (2) above to be satisfied, it follows that the two lines c, c' intersect one another, say in the point R. Let this be so.

The line QR does not meet either the line e or the line e'; since, in the former case, b and c would intersect, and, in the latter case, b' and c' would intersect; thus QR does not lie in the plane $[e, e']$, which contains the two lines MN' and $M'N$. But QR intersects the line MN', these two lines, QR, MN', being both in the plane $[b, c']$; and QR intersects $M'N$, these two being both in the plane $[b', c]$. Therefore the line QR must pass through the point of intersection of the lines MN' and $M'N$; this intersection may thus be defined as the point of intersection of the line QR with the plane $[e, e']$. The points $(NL', N'L)$ and $(LM', L'M)$ are, similarly, the intersections, respectively, of the lines RP and PQ with the plane $[e, e']$. The points P, Q, R are not in line, since the points L, M, N, L', M', N' are supposed to be of such general positions that the three points $(MN', M'N)$, $(NL', N'L)$, $(LM', L'M)$ are different. These three points are therefore in line, namely in the line in which the plane PQR meets the plane $[e, e']$.

Thus Pappus' theorem is shewn to follow if the condition (2) is satisfied, and therefore follows equally if either of the equivalent conditions, (1) or (3), is satisfied.

Pappus' Theorem. The conditions follow if Pappus' Theorem be assumed. Conversely we now shew that if Pappus' theorem be assumed, then condition (2) is satisfied.

Let a, b, c be any three lines, of which no two intersect, and a', b', c' any other three lines of which, equally, no two intersect; let every one of the lines a', b', c' intersect every one of a, b, c, the respective intersections being P, C, B' on a', and C', Q, A on b',



and B, A', R on c'. Draw any further transversal of the lines a, b, c, say the line e, meeting these respectively in L, M, N, this line not meeting the lines a', b', or c'. Then, from any point, N', of the line c', draw the transversal, say e_1, of the lines e and a', meeting these in O and L_1, respectively. As c' does not meet e, this line e_1 will not meet a, or b, or c. As before, the line RP does not lie in the plane $[e, e_1]$, but meets both the lines $N'L$ and NL_1, lying with these, respectively, in the planes $[a, c']$ and $[a', c]$; so that the lines $N'L$ and NL_1 meet in the point where RP meets the plane $[e, e_1]$. The transversal e_1 does not lie in the plane $[a, b']$, since it does not meet the line a; let it meet this plane in the point M_1. The line PQ does not lie in the plane $[e, e_1]$, but meets the lines LM_1 and L_1M, lying with these, respectively, in the planes $[a, b']$ and $[a', b]$. Thus LM_1 and L_1M meet in the point in which the line PQ meets the plane $[e, e_1]$. The points P, Q, R do not lie in line, since the lines $N'L$, NL_1, L_1M do not meet in a point, unless N' be taken at B. There is thus a line in which the plane PQR meets the plane $[e, e_1]$. The application of Pappus' theorem to the two triads L, M, N and L_1, M_1, N' would lead to a line in the plane $[e, e_1]$ containing the three points

$$(LM_1, L_1M), \quad (MN', M_1N), \quad (NL_1, N'L);$$

of these the first and third have been shewn to lie, respectively, on PQ and PR; thus the second, the point (MN', M_1N), lies in the plane PQR. The line MN' does not lie in this plane; for, if so, it would intersect PQ, in a point other than Q, and, then, PQ, having two points in the plane $[b, c']$, would lie in this plane, so that PB, that is the line a, would intersect the line b. But MN', in the plane $[b, c']$, meets QR; therefore, as the point (MN', M_1N) lies in the plane PQR, it follows that M_1N intersects QR. Hence M_1 lies in the plane NQR, or $[b', c]$. And M_1 was constructed to be in the plane $[a, b']$. Thus, M_1 is on the line b', and the transversal e_1, drawn from N' to meet e and a', is the same as the transversal which can be drawn from N' to meet a' and b'; this last transversal therefore meets e. And this is the same as that the condition (2) is satisfied.

Alternative proof of the equivalence of Pappus' theorem with the conditions. The six points $AB'CA'BC'$ constitute a Brianchon hexagon, the pairs of opposite sides

$$(AB', A'B), \quad (B'C, BC'), \quad (CA', C'A)$$

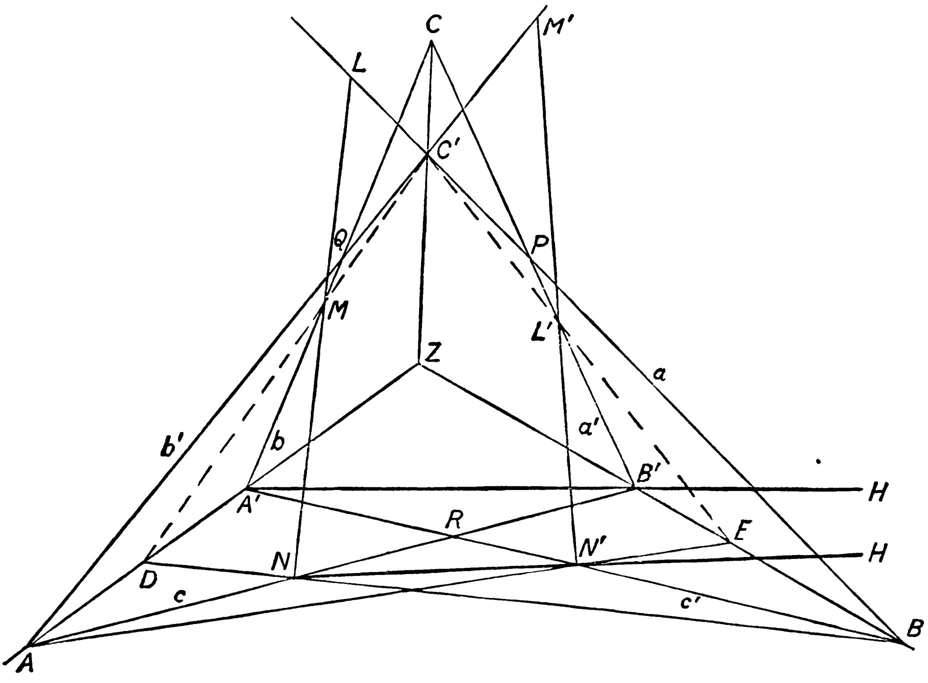
being intersecting lines, meeting, respectively, in R, P, Q. The joins of opposite angular points, namely the lines AA', BB', CC', therefore meet in a point. Taking account of this, we may substitute for the proof which has just been given, that condition (2) is satisfied if Pappus' theorem be assumed, and conversely, the following proof.

Let a, b, c be three skew lines, that is, lines of which no two intersect, and a', b', c' be three other skew lines, all intersecting every one of a, b, c. Denote the nine points

$$(a, a'),\ (a, b'),\ (a, c'),\ (b, a'),\ (b, b'),\ (b, c'),\ (c, a'),\ (c, b'),\ (c, c'),$$

respectively, by

$$P,\ C',\ B,\ C,\ Q,\ A',\ B',\ A,\ R.$$



The three planes $[a, a']$, $[b, b']$, $[c, c']$ then meet in pairs in the three lines AA', BB', CC', which thus meet in one point, say Z. Let LMN be a line, meeting the lines a, b, c, respectively, in L, M and N, but not intersecting any of the lines a', b', c'. In the plane $[c, c']$ draw the line BN, meeting the line AA' in the point D. Take N', an arbitrary point on the line c'; in the plane $[c, c']$ draw the line AN', meeting the line BB' in the point E. Let H be the intersection of the lines $A'B'$ and NN', both lying in the plane $[c, c']$. Draw from N' the transversal to meet the lines a' and b', respectively, in L' and M'.

The points D, M, C' are in the plane $[b, b']$, and are also in the plane LBN, and are therefore in line. The points C', L', E are in the plane $[a, a']$, and are also in the plane $M'AN'$, and are therefore in line. Thus the line $L'M$ lies in the two planes $CA'B'$ and $C'DE$, and, therefore, passes through the intersection of the two lines $A'B'$ and DE of the plane $[c, c']$.

Wherefore, the necessary and sufficient condition that the two transversals LMN and $L'M'N'$ should intersect one another, which is the condition that the lines $L'M$ and NN' should intersect, is that DE should pass through the intersection, H, of $A'B'$ and NN'.

If we assume Pappus' theorem, and apply it to the two triads A, N, B' and B, A', N', lying, respectively, on two lines of the plane $[c, c']$, the consequence is precisely that DE passes through H; for AA', BN meet in D, while NN', $A'B'$ meet in H, and AN', BB' meet in E.

Thus condition (2) is satisfied if Pappus' theorem be assumed.

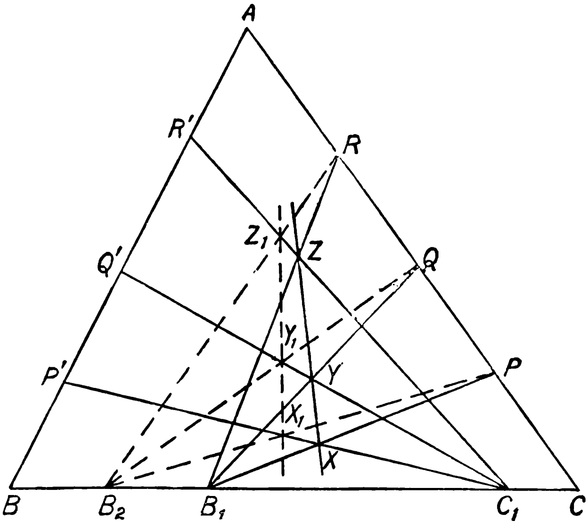
Conversely, from the assumption that the transversals LMN and $L'M'N'$ intersect one another, the truth of Pappus' theorem follows for the two triads A, N, B' and B, A', N'. Thus it will be proved that Pappus' theorem follows when condition (2) is satisfied, if we shew that the figure can be constructed when A, N, B' and B, A', N' are any two sets of three points, each in line, in a plane. For this, take Z, the intersection of the lines AA' and BB'; from Z draw the arbitrary line $ZC'C$, not in the plane of the two given triads; join the points A and B to the arbitrary point, C', of this line, and join the points A' and B' to the other arbitrary point, C, of this line. Then, from the point N, draw the transversal NML to meet $A'C$ in M and BC' in L, and, from the point N', draw the transversal $N'L'M'$ to meet $B'C$ in L' and AC' in M'. The construction of the points D, E, H then follows as before.

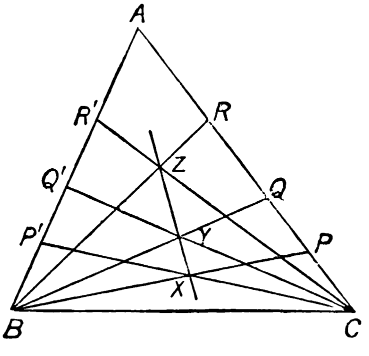
The identification of the two results is thus completed.

It is thus shewn that the adjunction of Pappus' theorem to the Propositions of Incidence enables us to regard the assignment of three points of one line, to correspond to three points of another line, not intersecting the former, as leading to a unique correspondence for all points of two related ranges upon these lines.

Equivalence of Pappus' theorem with the conditions; case of ranges in one plane. We proceed now to shew that the same is true for two related ranges upon lines which intersect one another. That this must be so is evident from what has preceded; for, of two ranges upon intersecting lines, one may be regarded, in an infinite number of ways, as being in perspective with a range lying on a line which does not intersect the other. But an independent proof is interesting, and, incidentally, will furnish information in regard to related ranges in one plane.

Suppose that the ranges $(PQR\ldots)$, $(P'Q'R'\ldots)$, respectively on the intersecting lines AC, AB, are related, being both in perspective with the range $(XYZ\ldots)$, respectively from B_1 and C_1. The line $XYZ\ldots$ must intersect both AC and AB, so that B_1C_1 lies in the plane of the lines AB, AC, meeting them, respectively, say, in B and C. It can then be shewn that these ranges are related by perspectivities whose centres are any two points on the line BC, as has indeed already appeared in Section i. For take any point, B_2, of BC; let B_2P, B_2Q, B_2R, ... respectively meet C_1P', C_1Q', C_1R', ... in X_1, Y_1, Z_1, ...; then, by Desargues' theorem applied to the two triads PXX_1, QYY_1, since corresponding joins of points of these triads meet in the points B_1, C_1, B_2, which are in line, it follows that X_1Y_1 passes through the point common to XY and AC. The same is true for X_1Z_1; and so on; so that X_1, Y_1, Z_1 are in line. In the same way, if C_2 be another point of BC, and C_2P', C_2Q', C_2R' meet B_2P, B_2Q, B_2R respectively in X_2, Y_2, Z_2, we see, from the triads $P'X_1X_2$, $Q'Y_1Y_2$, $R'Z_1Z_2$, that $X_2Y_2Z_2$ are in line.





We may thus suppose the ranges $(PQR\ldots)$ and $(P'Q'R'\ldots)$ to be in perspective with a range $XYZ\ldots$ respectively from the centres B and C.

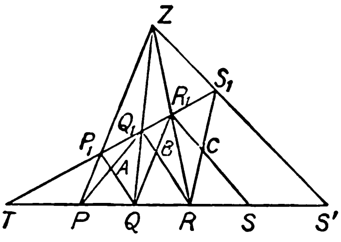
We first shew that when, of the two related ranges $(PQ\ldots)$, $(P'Q'\ldots)$, the three points P', Q', R' of one are given as corresponding to the three given points P, Q, R of the other, then the point, S', of one range which corresponds to any assigned point, S, of the other, is without ambiguity if we assume Pappus' theorem to be true.

By Pappus' theorem the two triads, P, Q, R and P', Q', R', each of points on a line, give rise to another line containing the three points of intersection $(QR', Q'R)$, $(RP', R'P)$, $(PQ', P'Q)$; and this line is without ambiguity when the two triads are given. But, also by Pappus' theorem, this line in fact is the same as the intermediary line XYZ, arising when the two ranges are related by perspectivities having B, C for centres; for, by applying Pappus' theorem to the two triads P, C, Q and P', B, Q', we infer that the point $(PQ', P'Q)$ lies on the line XY. And we can similarly infer that the points $(QR', Q'R)$ and $(RP', R'P)$ lie on the line XY. But then, if S' be the point of the range $(P'Q'R' \ldots)$ which corresponds to a point S of the range $(PQR \ldots)$, so that BS and CS' meet on XY, a similar proof shews that RS' and $R'S$ meet on XY.

Thus, the rule for the construction of S', when P, Q, R, P', Q', R', and S, are given, is, to find the Pappus line of the two triads P, Q, R and P', Q', R', take the intersection of $R'S$ with this, and draw RS' through this intersection. This construction necessarily gives a unique point, independent of the method whereby the ranges $(PQR \ldots)$, $(P'Q'R' \ldots)$ are related.

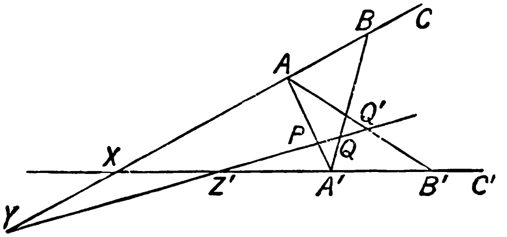
The proof of the converse theorem, that the uniqueness of S', when P, Q, R, P', Q', R', and S, are given, requires Pappus' theorem, may be omitted, in view of what has been said above.

Related ranges on the same line. Now consider two ranges on the same line which are related; by definition, this means that one of these ranges is in perspective with a range, on another line, which is related to the remaining range of the given line. Suppose that the two ranges, $(PQRS \ldots)$, $(PQRS' \ldots)$, of the given line have three corresponding points, P, Q, R, in common; and let $(PQRS' \ldots)$ be in perspective, from a centre Z, with a range $(P_1Q_1R_1S_1 \ldots)$, related to $(PQRS \ldots)$, lying on another line, which meets the former in T. Then, by the above, the ranges $(PQRS \ldots)$, $(P_1Q_1R_1S_1 \ldots)$ being related, the joins, RS_1 and R_1S, intersect on the Pappus line of the two triads P, Q, R and P_1, Q_1, R_1, if we assume Pappus' theorem. Denote the points (PQ_1, P_1Q), (QR_1, Q_1R), (RS_1, R_1S), respectively, by A, B, C. By Desargues' theorem, applied to the triads P, A, P_1 and R, B, R_1, the point T lies on AB, which is also the Pappus line of the two triads P, Q, R and P_1, Q_1, R_1. The fact that C lies on this line, involves then, by Desargues' theorem applied



to the triads Q, B, Q_1 and S, C, S_1, that the line SS_1 passes through Z. Thus S coincides with S'.

Thus it is a consequence of Pappus' theorem that two related ranges on the same line coincide entirely if they have three corresponding points in common.



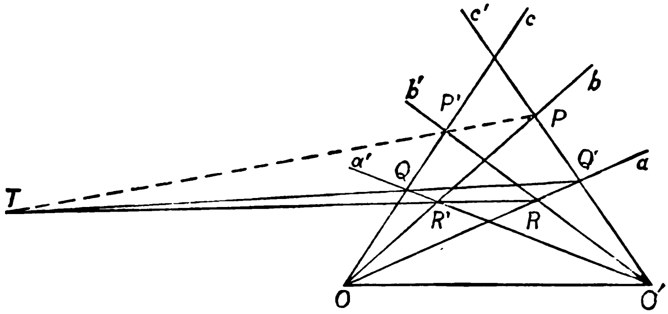
Conversely, if two related ranges on the same line coincide entirely when they have three points in common, then Pappus' theorem follows. For, let A, B, C and A', B', C' be any two triads of points respectively on two lines which intersect in X. Consider two related ranges on these lines, so constructed that the points A, B, C, of one of these, correspond respectively to the points A', B', C' of the other; it follows from preceding work that such a construction is possible. Let the point X, of intersection of the lines of the ranges, regarded as a point of the range $(ABC\ldots)$, correspond to the point Z' of the second line $A'B'C'\ldots$, in virtue of the perspectivities by which the ranges are related; similarly, let the point X, regarded as belonging to the range $(A'B'C'\ldots)$, give rise to the point Y of the line $ABC\ldots$. Join YZ', and let this joining line be met by AA', $A'B$, AB', respectively, in the points P, Q, Q'. The points Y, X, A, B, of the first range, correspond, respectively, to the points X, Z', A', B' of the second range; and, from the centre A', the former range is in perspective with Y, Z', P, Q. Therefore, by what we have proved, the last range is related to the range X, Z', A', B'; this, however, from the centre A, is in perspective with the range Y, Z', P, Q', which is then related to the range Y, Z', P, Q. If, then, we assume that two related ranges on the same line which have three points in common entirely coincide, we can infer that the points Q and Q' coincide, so that the joins AB' and $A'B$ intersect on the line YZ'. By a similar proof the joins AC' and $A'C$ intersect on this line as do the joins BC' and $B'C$. This shews that Pappus' theorem holds for the two triads A, B, C and A', B', C'.

Pappus' theorem and the Principle of Duality. It was seen that the Propositions of Incidence lead to a duality, in virtue of which, in space of three dimensions, points correspond to planes and conversely, and lines to lines. And the conditions (1), (2), (3) above are of self-reciprocal character, since two lines with a common point lie in a plane. As so far stated, Pappus' theorem has not this character, and we must enquire whether its adjunction to the Propositions of Incidence would destroy the validity of the duality of space. We shew that this is not so by deducing the reciprocal of Pappus' theorem from itself and the Propositions of Incidence. In three dimensions this reciprocal would be that, if three planes, α, β, γ, have a line, l, in common, and three planes, α', β', γ', have a line, l', in common, the lines l, l' being in one plane, and we take the lines (α, β'), and (α', β), and the plane containing these, the three planes so obtained have a line in common. By taking a section of the figure by an arbitrary plane, we see that this is equivalent to the statement that, if three lines, a, b, c, lying in a plane, meet in a point, O, and three other lines, a', b', c', of this plane, meet in a point, O', and we denote the points of intersection

$$(b, c'),\ (b', c),\ (c, a'),\ (c', a),\ (a, b'),\ (a', b)$$

respectively by $\quad P,\ P',\ Q,\ Q',\ R,\ R'$,

then the three lines PP', QQ', RR' meet in a point.



In fact, if T be the point of intersection of the lines QQ' and RR', the application of Pappus' theorem to the two triads, O, Q', R and O', R', Q, shews that the line PP' passes through T.

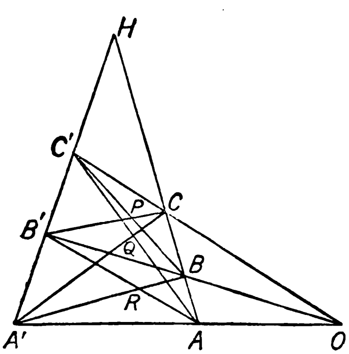
Related ranges in space of more than three dimensions. If in space of n dimensions we have a succession of ranges of which every consecutive two (which will both lie in a threefold space) are related by the definition applied before, the proof given in Section I of this Chapter avails to shew that the first and last are related, and assuming Pappus' theorem to hold generally, it follows as here that three points are sufficient to identify a range related to a given one. This leaves open the question, suggested in the examples in Section I, whether a more compendious method of relating two ranges in space of higher dimensions may not be possible than by a succession of perspectivities by lines drawn from a centre. The next Section (III) throws some light on the general question here touched. For the present we shall only add the further remark that, as in space of three dimensions, the adjunction of Pappus' theorem leads to no abandonment of the Principle of Duality, the reciprocal of Pappus' theorem in space of n dimensions being, as in that case, reducible to the theorem itself. The reciprocal in question would be as follows: In space of n dimensions we are given two spaces, each of $n-2$ dimensions, which we shall denote by λ and λ', both containing the same space S_{n-3}, say ϖ (this being the reciprocal of the plane in which the lines l, l' of the original theorem both lie). Through the S_{n-2} called λ there pass then three spaces S_{n-1}, say U, V, W, and, similarly, three spaces S_{n-1}, say U', V', W', through λ'. Then the spaces U, V' intersect in a space S_{n-2}, as do also the spaces U', V; and the two S_{n-2} so obtained both lie in a S_{n-1}, which we call w. The pairs U, W' and U', W similarly determine a S_{n-1}, which we call v; and the pairs V, W' and V', W determine a S_{n-1}, which we call u. The spaces U, V, W, U', V', W', u, v, w all contain the S_{n-3} which we call ϖ. The theorem is that the spaces u, v, w have a space S_{n-2} in common.

To see that this is true, take a general plane, σ, in the space of n dimensions, not intersecting the space S_{n-3} which we have called ϖ. This plane, σ, will be met by λ, λ' each in a point, say in the points O, O', respectively; it will be met by U, V, W in lines through O, say a, b, c respectively, and by U', V', W' in lines, a', b', c', through O'. The space w then meets the plane σ in the line joining the points (a, b') and (a', b), and similarly with the spaces u and v. We therefore have the same figure in the plane σ as was obtained in three dimensions by reciprocating the figure of Pappus' theorem. Thus the lines in the plane σ arising from the spaces u, v, w meet in a point. This point with the space ϖ, of $n-3$ dimensions, which is contained in each of the spaces u, v, w, determine a space S_{n-2} contained in all of these. This constitutes the space which proves the theorem.

Examples in connexion with Pappus' theorem.

Ex. 1. Pappus' theorem can be *proved* for the case of two triads which are in perspective.

For if AA', BB', CC' meet in O, the points of intersection $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$ lie on the line joining the point H, where the lines of the two triads intersect, to the point which is the harmonic conjugate of O in regard to A and A'.

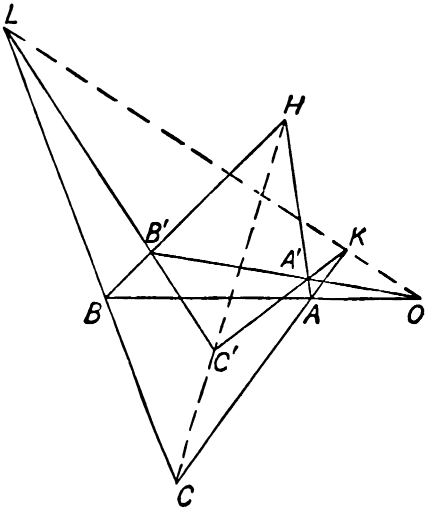


Ex. 2. We have so far regarded two ranges in perspective as being a particular case of two related ranges, of which the two centres of perspective coincide, and the intermediary line is any line through the point of intersection of the lines of the two ranges.

There are however other more definite ways in which two ranges in perspective may be related, with two distinct centres of perspective.

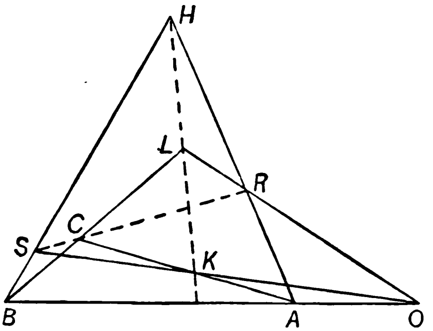
(*a*) Two ranges in perspective may be related with the intermediary line passing through the point of intersection of the lines of the ranges and the supplementary line, that is the line joining the two centres of perspective, passing through the original centre of perspective.

For if the ranges $(AA' \ldots)$, $(BB' \ldots)$ be in perspective from O, and K, L be any two points on a line through O, the lines KA, LB meeting in C, and the lines KA', LB' meeting in C', it follows, by Desargues' theorem, that C' lies on the line joining C to the point H, the point of intersection of the lines of the ranges. The original ranges are thus related with CH as intermediary line and KL as supplementary line.



(*b*) If we assume Pappus' theorem we can relate the ranges in perspective in such a way that, not the intermediary line, CH, but, the supplementary line, KL, passes through the intersection H of the lines of the ranges.

For let the ranges $(A \ldots)$, $(B \ldots)$, on lines intersecting in H, be in perspective from O, and HLK be an arbitrary line through H, while RS is an arbitrary line meeting HA, HB, respectively, in R and S, not passing through O or H; let OR, OS meet the line HLK respectively in L and K. Then, using Pappus' theorem for the two triads A, O, B and L, H, K, we see that AK and BL meet on RS. Thus the two ranges are in perspective with a range on RS, respectively from the centres K and L; the line RS is now the intermediary line, and the line KL the supplementary line.

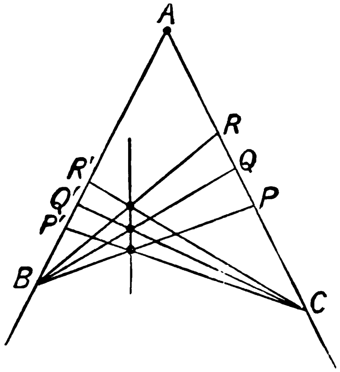


(*c*) Conversely, two related ranges are in perspective with one another when the intermediary line passes through the point of intersection of the lines of the ranges; and the new centre of perspective is on the supplementary line of the ranges. This follows by Desargues' theorem. The diagram in (*a*) may be used.

(*d*) And two related ranges are also in perspective with one another when the supplementary line passes through the point of intersection of the lines of the ranges. This follows from Pappus' theorem. The diagram in (*b*) may be used.

(*e*) It is clear that if, of two related ranges, the point of intersection of the lines of the ranges corresponds to itself regarded as belonging to both lines, then either the intermediary line or the supplementary line passes through this point. Hence, with the assumption of Pappus' theorem, it follows, from what has been said, that the two ranges are in perspective with one another.

Ex. 3. We have seen, assuming Pappus' theorem, that, in the case of two related ranges on intersecting lines for which the supplementary line does not pass through the point of intersection of the lines of the ranges, the two centres of perspective, by means of which the ranges are related, can be taken to be any two points on the supplementary line. In particular if these centres, B and C, be respectively on the lines, (P', Q', ...) and (P, Q, ...), of the two ranges, then the two triads B, P', Q' and C, P, Q have the same Pappus line as have the two triads P, Q, R and P', Q', R', this being the intermediary line for the two related ranges.



Thus the centres of perspective, B and C, are corresponding points of the two ranges.

More generally a line joining any point, S', of the range (P', Q', ...), to the corresponding point, S, of the range (P, Q, ...), is a possible supplementary line. And it will appear later (in Volume II) that, through an arbitrary point of the plane of the two ranges, two lines can be drawn either of which may serve as supplementary line; and further that any possible supplementary line for the two ranges is intersected, by the joins PP', QQ', ..., of corresponding points of the two ranges, in a range of points which is related to either of the two given ranges. This last is the same result as was found for related ranges on two lines which do not intersect, before the introduction of Pappus' theorem.

Ex. 4. Consider two related ranges (A, B, C, D, ...) and (A', B', C', D', ...), on two lines which do not intersect one another. For each pair of corresponding points of these ranges, such as A, B and

A', B', consider the cross joins (not joining corresponding points), such as AB' and $A'B$. Let O be any point of the threefold space of the two ranges; from this point a transversal can be drawn to the cross joins AB' and $A'B$. It can be shewn that all such transversals lie in a plane through O; such a plane is then determined, by the two given ranges, passing through every point O. Further, if the transversal from O to the lines of the two given ranges meet them, respectively, in P and Q', and P', Q be the points of the ranges $(A'B' \ldots)$, $(AB \ldots)$ respectively corresponding to P, Q', this plane is that joining O to the line $P'Q$.

This result, which is of subsequent interest, is a consequence of Pappus' theorem. Taking an arbitrary plane, not passing through O, the given ranges are projected from O upon this plane into two ranges, $(A_1, B_1, C_1, \ldots)$ and $(A_1', B_1', C_1', \ldots)$, related to one another, for which, as we have shewn to follow from Pappus' theorem, the points of intersection $(A_1B_1', A_1'B_1)$, $(A_1C_1', A_1'C_1)$, $(B_1C_1', B_1'C_1)$, ..., are in line.

In particular, when P, Q' and P', Q are constructed, as in the enunciation, instead of one there is an infinite number of transversals from O to the cross joins PQ', $P'Q$, namely any line in the plane $OP'Q$ is such a transversal.

Ex. 5. The reciprocal theorem is that, if $(A, B, C, \ldots)$ and $(A', B', C', \ldots)$ be related ranges on two lines, l and l', which do not intersect one another, and if an arbitrary plane, ϖ, meet these lines, l and l', respectively in P and Q', to which there correspond, on the other lines, l' and l, respectively, the points P' and Q, then the point O, in which the plane ϖ is met by the line $P'Q$, lies on the line joining the intersections of the plane ϖ with any pair of cross joins, such as AB' and $A'B$. By the two given ranges there is thus determined a point, O, upon any plane, ϖ.

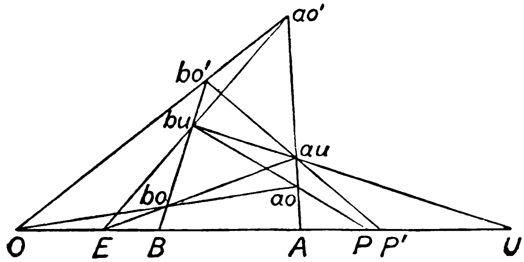
Ex. 6. If A, B, C, A', B', C' be six general points in threefold space, and O be any point on the line of intersection of the planes ABC and $A'B'C'$, and we draw from O the transversal to the pair of joins BC' and $B'C$, as also the transversal to CA' and $C'A$, and the transversal to AB' and $A'B$, these three transversals are in one plane.

Ex. 7. If, in fourfold space, the four points A, B, C, D be in one plane, and the four points A', B', C', D' be in another plane, the three transversals, respectively, of the triplets of lines $(BC', B'C, DD')$, $(CA', C'A, DD')$, $(AB', A'B, DD')$, are in one threefold space. This space then contains the line DD'; if we project the figure on to an arbitrary plane by means of planes passing through the line DD', we obtain Pappus' theorem.

Ex. 8. In Pappus' theorem for two triads, A, B, C and A', B', C', there are six ways of arranging the triad A', B', C' in association with A, B, C. Consider the figure of the six Pappus lines thus found. And reciprocate the figure.

Ex. 9. The following result is of great importance for the development of the theory.

We have seen that if five arbitrary points be taken on a line, consisting of two couples, O, U and A, B, and a determining point E, and there be drawn, in an arbitrary plane passing through this line, lines o, u, a, b respectively through the points O, U, A, B, subject to the condition that the intersections, (a, u) and (b, o), lie on a line through the point E, then the join of the points, (a, o) and (b, u), meets the original line in a point, P, independent of the lines o, u, a, b. Similarly, if the lines o, u, a, b be subject to the condition that the intersections, (a, o) and (b, u), lie on a line through E, then the join of the points, (a, u) and (b, o), meets the original line in a point, P', also independent of the lines o, u, a, b.



It was said (p. 15) that the Propositions of Incidence are insufficient to determine whether P and P' coincide. It is important to notice that, if Pappus' theorem be assumed, these points must be regarded as identical.

For we can have a figure containing both constructions, the lines a, b, u being the same in both, the line drawn through O for the second construction being o'. If then, on the lines a, b, we consider the two triads of points, respectively

$$(a, u),\quad (a, o'),\quad (a, o),$$

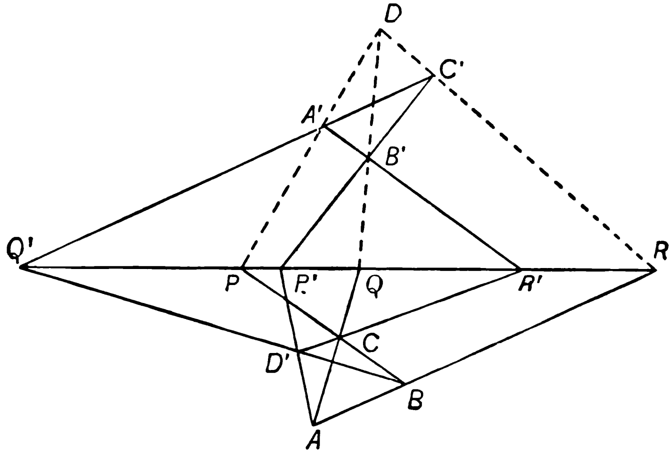
and

$$(b, u),\quad (b, o),\quad (b, o'),$$

then the first and second of these three pairs of corresponding points, by the joins of the points, (a, u) to (b, o) and (a, o') to (b, u), give the point E; the second and third, by the joins of (a, o) to (b, o) and of (a, o') to (b, o'), give the point O; but the first and third, of (a, o) to (b, u) and of (a, u) to (b, o'), are the lines respectively containing P and P'. These lines thus meet the original line, EO, in the same point.

In the original figure, before the consideration of Pappus' theorem, the points O, U, of a couple, were symmetrical, as were also the points A, B of the other couple. The introduction of Pappus' theorem renders the points E, P symmetrical in regard to one another. This is brought out well by a diagram illustrating the two cases in which lines o, u, a, b are drawn in one plane, and other lines o', u', a', b', through the same points, are drawn in another plane through the original line. The diagram so obtained is precisely that which, with different lettering, is used to illustrate the following example.

Ex. 10. Let A, B, C, D be any four points not in one plane; let A', B', C', D' be further points lying, respectively, in the planes BCD, CAD, ABD, ABC; suppose that the line $B'C'$ meets AD', the line $C'A'$ meets BD' and the line $A'B'$ meets CD'. Thus the planes $B'C'D'$, $C'A'D'$, $A'B'D'$ contain, respectively, the points A, B, C. It is supposed that the points A', B', C' are not in the plane



ABC (respectively on the lines BC, CA, AB), so that the points A', B', C', D' are not in one plane.

We can then shew, assuming Pappus' theorem, that the plane $A'B'C'$ contains the point D.

Consider the line of intersection of the two planes ABC and $A'B'C'$. As $B'C'$, AD' meet, their intersection, P', must be on this line; as must similarly be the point of intersection, Q', of $C'A'$, BD', and the intersection, R', of $A'B'$, CD'. Let BC, CA, AB meet this line, respectively, in P, Q, R.

Then we have a figure such as that spoken of in the last example. Considering the plane of the points A, B, C, D', and denoting by p, p' and r, r' the lines drawn in that plane respectively through the points P, P', R, R', the join of the points (p, r') and (p', r) passes through the point Q, and the join of the points (p, r) and (p', r') passes through the point Q'. In the other plane, consider the lines PA' and RC', denoting these, respectively, by p_1 and r_1, the lines drawn in this plane through P' and R' being respectively denoted by p_1' and r_1'. It follows then, from the fact that the join of the points (p_1, r_1') and (p_1', r_1) passes through Q', that the join of the points (p_1, r_1) and (p_1', r_1') passes through Q. Thus the lines PA', QB', RC' meet in one point. This point is evidently on each of the planes $A'BC$, $B'CA$, $C'AB$, and is therefore the point D. It is thus proved that the plane $A'B'C'$ contains the point D.

It is thus shewn that it is possible to have two tetrads of points A, B, C, D and A', B', C' D' with the relation that the plane containing any three points of either tetrad contains a point of the other tetrad. This remark was first made by Moebius, and we shall therefore sometimes refer to the tetrads as Moebius tetrads.

It is also shewn incidentally that the three pairs of opposite joins of four points in a plane meet an arbitrary line of that plane in six points of which one is to be regarded as determined uniquely by the other five. It is usually said that the three couples, of which the six points are composed, are three couples in *involution*. The relation was recognised by Pappus (cf. Ch. Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. lii). It is important for our purpose to notice that it does not follow from the Propositions of Incidence, without what we have called Pappus' theorem.

The theorem of Moebius tetrads may be stated thus: Let P_1, P_2, P_3, P_4 be four arbitrary points in a plane; through each of the six joins of two of these let an arbitrary plane be drawn, the plane through P_iP_j being denoted by α_{ij}. The set of three of these planes, passing through the pairs from three of the original points, meet in a point; for instance the planes α_{23}, α_{31}, α_{12} meet in a point which we may call P_{123}. The theorem is then that the four points P_{234}, P_{314}, P_{124}, P_{123} lie in one plane.

The reciprocal theorem, of which the generalisation to space of higher dimensions will be of interest later, is as follows: Let four planes α_1, α_2, α_3, α_4 be drawn through a point O; on the line of intersection of any two of these planes take an arbitrary point, say, the point P_{ij} on the line (α_i, α_j); through the three such points arising from three of the planes there passes a plane, for instance the plane α_{123} through the points P_{23}, P_{31}, P_{12}. The four planes α_{234}, α_{314}, α_{124}, α_{123} meet in a point, say O'. In all we have eight points and eight planes; through each point pass four of the planes, and in each plane lie four of the points.

SECTION III. INTRODUCTION OF ALGEBRAIC SYMBOLS

The laws of operation of the symbols employed. We have considered in the two preceding Sections, respectively, the Propositions of Incidence, and the consequences of supposing Pappus' theorem to hold. It will add greatly to clearness, and, finally, will amply justify itself as a means of investigation, to accompany the preceding work by an algebraic symbolism. And as an example of one kind of usefulness which this may have, it may be as well to cite at once the following, which signalises an important step in the logical development. It may perhaps be supposed that as Desargues' theorem in a plane was deduced in consequence of the Propositions of Incidence for space of three dimensions, so it may be possible to obtain Pappus' theorem as a consequence of Propositions of Incidence assumed for higher space. It will appear however that there is an algebraic symbolism corresponding in every detail with the geometry deducible from the Propositions of Incidence alone, in space of however high dimensions, which is sharply and definitely distinguished from the symbolism which is appropriate if Pappus' theorem is true. This theorem requires a precise limitation, in the character of the symbols adequate when only the Propositions of Incidence are assumed; thus Pappus' theorem cannot be a consequence of Propositions of Incidence in however high dimensions.

The symbols which we first introduce are, speaking in general terms, subject to all the laws of ordinary algebra, except the commutative law of multiplication. They are not necessarily, and will not be finally, arranged in order of magnitude, so that, in their entirety, they are wider than the real numbers of arithmetic. In order to emphasize this, we give here a formal statement of the properties assumed for them, and some examples.

We use the symbol $=$ to mean "may be replaced by." Thus if a, b be two of the symbols, the statement $a=b$ is the same as $b=a$.

From any two of the symbols, say a and b, another symbol can be formed, represented by $a+b$, or by $b+a$, independent of the order in which a, b are taken. Further, if $a=b$ and $x=y$, then also $a+x=b+y$. Also, if a, b, c be any three of the symbols, the symbol thus formed from a and $b+c$ is equivalent with that so formed from $a+b$ and c; that is

$$a+(b+c)=(a+b)+c.$$

We speak of the process of forming $a+b$ as *addition*, and of $a+b$ as the *sum* of a and b; and we may say that the addition of two symbols is *commutative*, and the addition of three symbols is *associative*. It follows that the symbol

$$a+b+c+\ldots+k,$$

formed from any number of the symbols, has a definite meaning independent of the order, and of the partial associations, of the symbols.

Among the symbols however there is one, and only one, which we denote by 0, whose addition to any other symbol leaves that unaltered, so that

$$a+0=0+a=a.$$

To every symbol, say a, there corresponds another, which we denote by $-a$, whose addition to the original gives 0 for sum, so that

$$a+(-a)=0=(-a)+a.$$

The symbols so corresponding to any two equivalent symbols are themselves equivalent, so that if $a=b$, then also $-a=-b$. Thus the symbol so corresponding to 0 is itself 0; for -0 is, by the definition of 0, equal to $0+(-0)$, and this, by the definition of -0, is itself 0. Conversely, if a be such that

$$a=-a,$$

it follows that $a+a=0$; it will be a consequence of subsequent prescriptions that thence $a=0$. We denote the sum of two symbols a and $-b$, namely $-b+a$, or $a+(-b)$, also by $a-b$. Thus if $a=b$ and $x=y$ we also have $a-x=b-y$. And, by the associative law, whatever symbols a and b may be, we have

$$(a-b)+b=a+(-b)+b=a+(-b+b)=a+0=a,$$
$$a-(a-b)=[(a-b)+b]-(a-b)=b+(a-b)-(a-b)=b.$$

Thus also, if $b=-a$, then $-b=a$, for

$$-b=0-b=a+(-a)-b=a+b-b=a,$$

and either of these follows from $a+b=0$. We may speak of $-a$ as the negative of a.

From any two symbols, a, b, can be formed, beside the sum $a+b, =b+a$, two further combinations, different from this, dependent on the order in which a, b are considered, denoted, respectively, by

$$a\,.\,b \text{ or } ab, \quad \text{and} \quad b\,.\,a \text{ or } ba.$$

It may happen that these symbols are equivalent, but this is not to be assumed in general. In particular, for the symbol 0, whatever a may be, we have $0\,.\,a=a\,.\,0=0$.

If $a=b$ and $x=y$, then $ax=by$, and $xa=yb$. Also, if three symbols a, b, c be taken, the consequence of combining, in this way, the symbol ab, taken first, with c, is the same as of combining a, taken first, with bc; namely

$$(ab)c=a(bc).$$

We may speak of ab as obtained by *multiplication* of a and b, and of ab as their *product*, and say that multiplication is *associative*. Thus if a, b, ..., k be any symbols, the symbol

$$ab\ldots k$$

is in general dependent upon the order of the component symbols, but not dependent upon their association to form partial products, being the same, for instance, as

$$(ab)(c\ldots k), \text{ or } a(bc\ldots k).$$

We may also have multiplications $(a+b)c$, or $c(a+b)$, defined respectively by

$$(a+b)c = ac + bc, \quad c(a+b) = ca + cb.$$

This leads, if a, b be any symbols, to the result

$$(-a)b = -(ab) = a(-b);$$

for

$$(-a)b + ab = (-a+a)b = 0 \,.\, b = 0,$$
$$a(-b) + ab = a(-b+b) = a \,.\, 0 = 0.$$

Beside the special symbol 0, with the two properties

$$a + 0 = 0 + a = a, \quad a \,.\, 0 = 0 \,.\, a = 0,$$

there is another special symbol, denoted by 1, with the property that, whatever symbol a may be,

$$a \,.\, 1 = 1 \,.\, a = a.$$

For reasons which will appear, the symbol 0 is said to be *singular*. There may be other singular symbols. The product of two non-singular symbols is not singular.

To every symbol, a, which is not singular, there corresponds another symbol, denoted by a^{-1}, having the property that

$$a^{-1} \,.\, a = a \,.\, a^{-1} = 1,$$

and such that, if $a = b$, then $a^{-1} = b^{-1}$. Thus

$$ab^{-1} \,.\, b = a \,.\, b^{-1}b = a \,.\, 1 = a, \quad a \,.\, a^{-1}b = aa^{-1} \,.\, b = 1 \,.\, b = b,$$

while, if a, b be symbols such that $ab = 1$, and a be not singular,

$$a^{-1} = a^{-1} \,.\, 1 = a^{-1}(ab) = a^{-1}a \,.\, b = b,$$

from which, if also b be not singular,

$$b^{-1} = 1 \,.\, b^{-1} = aa^{-1} \,.\, b^{-1} = ab \,.\, b^{-1} = a \,.\, bb^{-1} = a.$$

Further, if $x = ab$, then $x^{-1} = b^{-1}a^{-1}$; for, if $y = b^{-1}a^{-1}$,

$$xb^{-1} = a, \quad xb^{-1}a^{-1} = 1, \quad xy = 1, \quad y = x^{-1}.$$

Also, if a is not singular, there is one and only one symbol, x, such that $ax = b$, and one and only one symbol y such that $ya = b$, where b is an arbitrary symbol. For $ax = b$, $ya = b$ lead, respectively, to

$$x = a^{-1}a \,.\, x = a^{-1} \,.\, ax = a^{-1}b, \quad y = y \,.\, aa^{-1} = ya \,.\, a^{-1} = ba^{-1}.$$

This involves that the statement $ax = az$ requires $x = z$, the symbol a not being singular.

It is to be remarked that the symbol 1 is by no means necessarily identical with that so denoted in ordinary arithmetic, as subsequent examples will illustrate. But no confusion need arise, since the laws of combination agree. Similarly we may use the symbol 2 for the sum $1+1$, and 3 for the sum $1+1+1$, and so on. It may then be shewn, as in ordinary arithmetic, that these combine together just as do the symbols of ordinary arithmetic—and in particular that they are commutative in multiplication. For distinctness we shall speak of them as the *iterative* symbols. None of these symbols is singular, and to any one of them, for instance for 2, or for 3, there is a symbol 2^{-1}, or 3^{-1}, of which 1^{-1} is equivalent to 1; if m, n be any two of these symbols, we have $m^{-1}n = nm^{-1}$, this being equivalent to $nm = mn$. As regards combinations with other of our symbols, we have

$$\begin{aligned} 2\,.\,a &= (1+1)\,a = a + a = a\,(1+1) = a\,.\,2, \\ 2a\,.\,b &= (a+a)\,b = ab + ab = 2\,.\,ab = a\,(b+b) \\ &= a\,.\,2b = a2\,.\,b = a\,.\,b\,.\,2 = ab\,.\,2, \end{aligned}$$

and so on; so that if, in a product $abc\ldots k$, any of the symbols a, b, ..., k be those denoted by 1, 2, 3, ..., the places where they occur in the product are indifferent, so long as the order of the other symbols is preserved.

Examples of symbols obeying these laws. Simple examples of such symbols may be formed with aggregates of the numbers of ordinary arithmetic, the addition and multiplication of the symbols being defined by arithmetical operations carried out, according to a prescribed rule, with the numbers of the aggregate. If α, α', β, β', ... be numbers of ordinary arithmetic, including negative and fractional numbers, the numbers zero and unity being, for distinctness, denoted by $\bar{0}$, $\bar{1}$, we may, for example, have symbols formed with two such numbers, taken in a definite order,

$$a = (\alpha, \alpha'), \quad b = (\beta, \beta'), \ldots,$$

defining the fundamental symbols and operations by

$$0 = (\bar{0}, \bar{0}), \quad 1 = (\bar{1}, \bar{0}), \quad a + b = (\alpha+\beta, \alpha'+\beta'), \quad -a = (-\alpha, -\alpha'),$$
$$ab = (\alpha\beta - \alpha'\beta', \ \alpha\beta' + \alpha'\beta),$$
$$a^{-1} = \left(\frac{\alpha}{\alpha^2+\alpha'^2}, \ \frac{-\alpha'}{\alpha^2+\alpha'^2}\right).$$

Here 0 is the only singular symbol, and every two symbols are commutative in multiplication.

Another example is furnished by symbols which are aggregates of three ordinary numbers, taken in a definite order, say

$$a = (\alpha, \alpha', \alpha''), \quad b = (\beta, \beta', \beta''), \ldots,$$

with
$$0 = (\bar{0}, \bar{0}, \bar{0}), \quad 1 = (\bar{1}, \bar{0}, \bar{1}), \quad a + b = (\alpha+\beta, \alpha'+\beta', \alpha''+\beta''),$$
$$-a = (-\alpha, -\alpha', -\alpha''), \quad ab = (\alpha\beta, \ \alpha\beta' + \alpha'\beta'', \ \alpha''\beta''),$$
$$a^{-1} = \left(\frac{1}{\alpha}, \ -\frac{\alpha'}{\alpha\alpha''}, \ \frac{1}{\alpha''}\right).$$

Here every symbol (a, a', a'') for which either a or a'' vanishes is a singular symbol; and the equation $ab=ba$ is not generally true.

A very general example of such symbols as we have described is furnished by those aggregates of n^2 ordinary numbers which are called matrices; here n may be two, or three, or any positive integer number. The n^2 component numbers are thought of as arranged in n rows and n columns, so that it is convenient to denote the number in the rth row and sth column by $a_{r,s}$. Two such symbols are considered equivalent only when they have the same number of component numbers, and these are separately the same. The sum of two such symbols, a, b, of the same number of components, generally denoted by $a_{r,s}$ and $\beta_{r,s}$, is the symbol whose general component is $a_{r,s}+\beta_{r,s}$; the symbol 0 is that of which every component vanishes; the symbol $-a$ is that of which the general component is $-a_{r,s}$. The symbol ab is that of which the general component, $\gamma_{r,s}$, is formed, from the rth row of a, and the sth column of b, according to the rule

$$\gamma_{r,s}=a_{r,1}\beta_{1,s}+a_{r,2}\beta_{2,s}+\dots+a_{r,n}\beta_{n,s},$$

where, in the suffixes, the numbers 1, 2, ... are those of ordinary arithmetic. It is easy to prove from this definition that, for three such symbols a, b, c, with each n^2 components, we have

$$ab\,.\,c=a\,.\,bc$$

$$(a+b)c=ac+bc,\quad c(a+b)=ca+cb,$$

but not in general $ab=ba$. The symbol 1, when the number n^2 is specified, denotes the matrix in which every component $a_{r,s}$ is zero except the diagonal components $a_{r,r}$, of which each is unity. The symbol 2 may then denote that formed by an equal number of components, of which $a_{r,s}=0$, except $a_{r,r}$, which is the number 2 of ordinary arithmetic, and so on. It will easily be seen that this is in accordance with the general specifications above, this description being still applicable when 2 is replaced by any number of ordinary arithmetic. In particular, if h be any number of ordinary arithmetic, and a be a matrix symbol, say of n^2 components $a_{r,s}$, it will be legitimate to regard the matrix whose general component is $ha_{r,s}$ as the product of h and the matrix denoted by a. To define the symbol a^{-1}, denote by $|a|$ the determinant formed with the n^2 components, and by $A_{r,s}$ the cofactor of $a_{r,s}$ therein, with its proper sign. Then a^{-1} is the matrix of n^2 components of which the component in the rth row and sth column is given by

$$a'_{r,s}=A_{s,r}/|a|.$$

When the components are such that the determinant $|a|$ is zero, there is no symbol a^{-1}. In this case a is a singular symbol; and there may evidently, for specified n, be an infinite number of such symbols.

Particular examples of such matrix symbols may be referred to:

(1) For $n=2$, if a, a' be numbers of ordinary arithmetic, of which the numbers zero and one are denoted by $\bar{0}$, $\bar{1}$, and we put

$$1=\begin{pmatrix}1, & \bar{0}\\ \bar{0}, & \bar{1}\end{pmatrix},\quad i=\begin{pmatrix}\bar{0}, & -\bar{1}\\ \bar{1}, & \bar{0}\end{pmatrix},$$

then the symbol $$a=\begin{pmatrix}a, & -a'\\ a', & a\end{pmatrix},$$

may be replaced by

$$a\begin{pmatrix}\bar{1}, & \bar{0}\\ \bar{0}, & \bar{1}\end{pmatrix}+a'\begin{pmatrix}\bar{0}, & -\bar{1}\\ \bar{1}, & \bar{0}\end{pmatrix},\quad =a\,.\,1+a'\,.\,i;$$

we find $$1\,.\,1=1,\quad 1\,.\,i=i\,.\,1,\quad i\,.\,i=-1,$$

and hence

$$ab=(\alpha\,.\,1+\alpha'\,.\,i)(\beta\,.\,1+\beta'\,.\,i)=(\alpha\beta-\alpha'\beta')1+(\alpha\beta'+\alpha'\beta)i$$
$$=\begin{pmatrix}\alpha\beta-\alpha'\beta', & -(\alpha\beta'+\alpha'\beta)\\ \alpha\beta'+\alpha'\beta, & \alpha\beta-\alpha'\beta'\end{pmatrix},$$

so that

$$ab=ba.$$

If, for brevity, m stand for $\alpha^2+\alpha'^2$, the symbol a^{-1} is given by

$$a^{-1}=\begin{pmatrix}\dfrac{\alpha}{m}, & \dfrac{\alpha'}{m}\\ -\dfrac{\alpha'}{m}, & \dfrac{\alpha}{m}\end{pmatrix},$$

which is impossible if $\alpha=0$, $\alpha'=0$.

The symbols here described have precisely the same behaviour as the independent variables of ordinary analysis. They are therefore of the highest importance.

(2) If we similarly consider symbols given by

$$a=\begin{pmatrix}\alpha, & \alpha'\\ \bar{0}, & \alpha''\end{pmatrix},\quad b=\begin{pmatrix}\beta, & \beta'\\ \bar{0}, & \beta''\end{pmatrix},\ \ldots$$

where $\bar{0}$ is the zero and $\alpha, \alpha', \ldots, \beta, \ldots$ are numbers of ordinary arithmetic, we find

$$ab=\begin{pmatrix}\alpha\beta, & \alpha\beta'+\alpha'\beta''\\ \bar{0}, & \alpha''\beta''\end{pmatrix},\quad a^{-1}=\begin{pmatrix}\dfrac{1}{\alpha}, & -\dfrac{\alpha'}{\alpha\alpha''}\\ \bar{0}, & \dfrac{1}{\alpha''}\end{pmatrix},$$

so that ab is different from ba in general, and a^{-1} is impossible if either α or α'' be zero.

(3) As the symbols considered in Ex. (1) obey all the laws of combination of the numbers of ordinary arithmetic, they may themselves be used as components to define other symbols. Using 0, 1, i for the particular symbols of this kind defined in Ex. (1), let us consider then the four symbols

$$U=\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix},\quad I=\begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix},\quad J=\begin{pmatrix}0, & i\\ i, & 0\end{pmatrix},\quad K=\begin{pmatrix}-i, & 0\\ 0, & i\end{pmatrix};$$

it is then easy to compute that

$$JK=-KJ=I,\quad KI=-IK=J,\quad IJ=-JI=K,$$
$$I^2=J^2=K^2=-U.$$

Hence, if $\alpha, \beta, \gamma, \delta$ be iterative symbols, that is, here, matrices of two rows and columns with components zero except in the diagonal where there are equal numbers of ordinary arithmetic—so that, in brief, $\alpha, \beta, \gamma, \delta$ behave like numbers of ordinary arithmetic—, the symbol

$$a=\delta+\alpha I+\beta J+\gamma K$$

is the same as

$$a=\begin{pmatrix}\delta-\gamma i, & -\alpha+\beta i\\ \alpha+\beta i, & \delta+\gamma i\end{pmatrix}.$$

Replacing $\alpha, \beta, \gamma, \delta$ respectively by $\alpha', \beta', \gamma', \delta'$, to form a symbol a', we easily find that aa' is generally different from $a'a$. In particular, the symbol a^{-1} is obtained by replacing $\alpha, \beta, \gamma, \delta$ respectively by

$$\alpha'=-m^{-1}\alpha,\quad \beta'=-m^{-1}\beta,\quad \gamma'=-m^{-1}\gamma,\quad \delta'=m^{-1}\delta,$$

where m denotes $\alpha^2+\beta^2+\gamma^2+\delta^2$ Thus a^{-1} is impossible when $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, in which case $a=0$.

Another symbol with the same laws of combination as a may be represented by

$$\begin{pmatrix} \delta, & \gamma, & -\alpha, & -\beta \\ -\gamma, & \delta, & \beta, & -\alpha \\ \alpha, & -\beta, & \delta, & -\gamma \\ \beta, & \alpha, & \gamma, & \delta \end{pmatrix}.$$

Beside the symbols formed with aggregates of ordinary numbers, there are other symbols obeying the laws of combination which we have enunciated. Of such those introduced by Grassman are probably the most important. For a definite positive integer number n, consider symbols $e_1, e_2, \ldots, e_n$ such that

$$e_i^2=\bar{0}, \quad e_ie_j=-e_je_i, \qquad (i, j=1, 2, \ldots, n).$$

With these we can form a symbol

$$a=a+a_1e_1+\ldots+a_ne_n,$$

where $a, a_1, \ldots, a_n$ are (in the first instance) numbers of ordinary arithmetic. The symbols 0, 1 are obtained then by putting, respectively,

$$a=a_1=\ldots=a_n=\bar{0}, \text{ and } a-\bar{1}=a_1=\ldots=a_n=\bar{0};$$

while a^{-1} is given by $$a^{-1}=\frac{1}{a}-\frac{a_1}{a^2}e_1-\ldots-\frac{a_n}{a^2}e_n,$$

and is impossible, whatever $a_1, \ldots, a_n$ may be, when $a=0$.

Preliminary remarks in regard to the use of the symbols in geometry. Some brief remarks seem necessary to make clear the point of view taken at this stage in introducing the symbols into geometrical discussions. It appears to be more interesting, as well as more satisfactory, to regard this introduction at present as provisional, and illustrative, and to require, for every geometrical result, a geometrical proof independent of the symbols. In the first place we shew that the Propositions of Incidence, and the consequences of these, can be well represented by the symbols. It is more difficult to be sure that every result obtainable from the symbolism, when interpreted as a geometrical theorem, is demonstrable by geometrical reasoning; but we give in detail a geometrical representation of every one of the fundamental laws of combination of the symbols, in accordance with the Propositions of Incidence.

We then shew that the introduction of Pappus' theorem corresponds to a definite additional limiting law of combination for the symbols, namely that their multiplication is commutative. If the correspondence between the geometry and the symbols be exact, it will then follow that a geometrical result is a consequence of the Propositions of Incidence only, when its representation and deduction by means of the symbols does not require this commutative property. In attempting to construct a geometrical proof of a suspected theorem, this remark is of value, and we give several examples of its application.

Two kinds of symbols are employed in what follows: Symbols for geometrical elements, generally points, which we may call element-symbols; and symbols subject to laws of computation, which we may call algebraic symbols. It is of these last we have so far spoken in what precedes. In dealing with any geometrical figure we have as many fundamental element-symbols as is necessary to fix the symbols belonging to all the points, or elements, of the figure, together with a certain number of algebraic symbols. From the manner in which the symbols are introduced it will appear to be without doubt that two points which are identical in virtue of their geometrical construction must then have, as a consequence of computation, the same element-symbol. If we are to employ the symbolism with confidence the converse must also be established, the identity of two element-symbols, arrived at by computation, must imply the possibility of proving the identity of the elements by geometrical reasoning; we seek to induce the conviction that this is so in the present Section, reserving a final consideration for Chapter III.

The use of algebraic symbols for geometrical reasoning dates from very early times (? Apollonius B.C. 230; Ptolemy A.D. 130), and has prompted much in the development of Abstract Geometry. To some writers, therefore, it has seemed to need no justification. But first, it seems evident that a geometry prompted by an algebra must necessarily run parallel to the laws of combination of the symbols of that algebra; and, as we have sought to indicate, there is a good deal of arbitrariness in such laws. And, second, it would seem that a geometry, properly so called, must necessarily be based upon an analysis of our intuition, or experience, of geometrical relations; and that it is necessary to shew that the ideas we adopt from this experience, are properly represented by the symbols. It does not seem likely that a purely analytical geometry can convey any geometrical ideas; there must be constant appeal to geometrical conceptions to give it a meaning; it is surely proper that these conceptions should receive consideration before the symbols are introduced. And, while it would be foolish not to employ the symbols for purposes of discovery, the view taken in the present volumes is, that the object of a geometrical theory is to reach such a comprehensive scheme of conceptions, that every geometrical result may be obvious on geometrical grounds alone.

The symbolical representation of the Propositions of Incidence. In the symbolical scheme which we now proceed to construct to accompany the geometry, we represent a point by a symbol, generally indicated by the same capital letter as that by which we name the point. But we suppose that, if P is the symbol which represents a point, precisely the same point is equally represented by a symbol mP, where m is any one, other than 0, of the system of algebraical symbols which we are employing. It is supposed that this system has no singular symbol other than 0, so that m^{-1} is also a symbol of the system. If P, Q represent any two points, we regard the symbol $mP+nQ$, where m and n are any two algebraical symbols of the system employed, as equally representing a point; and regard all points so represented, for different values of m and n, as being points of the line determined by the points P and Q, making the further assumption that all points of this line can be so represented[1]. Thus a symbol $mP+nQ$ can be replaced by a symbol $-kR$, where R represents a point of the line PQ and k is an algebraic symbol of our system; and, conversely, if P, Q, R be points of a line, there is a relation between them which we may express by writing

$$mP+nQ+kR=0,$$

speaking of this expression as a *syzygy*. Here m, n, k are algebraic symbols of the system employed; in general, when such a syzygy arises, it is in the course of work in which a precise symbol, not further susceptible of being multiplied by an algebraic symbol, has been introduced for each of the points P, Q, R; when this is so, the symbols m, n, k in the syzygy are definite when the points P, Q, R are given, save for a common multiplier prefixed to each. Such syzygies we suppose capable of being combined together, by addition and subtraction, in the manner of linear algebraic equations, the order of the terms in any syzygy being indifferent. To this end it is necessary to understand that a symbol $mP+nP$ may be replaced by $(m+n)P$, and, more generally, that a symbol $m(nP)$ may be replaced by $mn\,.\,P$. This last assumption requires the associative law in multiplication for the algebraic symbols; for, in virtue of this,

$$m[n(kP)]=m[nk\,.\,P]=m(nk)\,.\,P,$$

and

$$m[n(kP)]=mn(kP)=(mn)k\,.\,P.$$

Similarly the associative law in addition is involved in the former assumption. For $[(m+n)+k]P$ is thence $(m+n)P+kP$, and this is $mP+nP+kP$, which again is $[m+(n+k)]P$.

These various suppositions receive their justification by the exact correspondence they set up between the Propositions of Incidence and their symbolical representation.

In general when a point is, in the phraseology already employed in formulating the Propositions of Incidence, *dependent* upon other points, there is a syzygy connecting its symbol with those of these other points; and, conversely, points whose symbols are connected by a syzygy are not independent. For instance, three points which are not in line determine a plane, but for any four points of a plane, represented by symbols A, B, C, D, there is a syzygy

---

[1] See Note I, at the end of the Volume.

$$aA + bB + cC + dD = 0,$$

in which a, b, c, d are algebraic symbols of the system employed; and any point, D, of the plane determined by the points A, B, C, has a symbol capable of the form

$$D = -d^{-1}aA - d^{-1}bB - d^{-1}cC,$$

or, say, $$D = mA + nB + kC.$$

Such a statement may be expressed also by saying that the point D is a *derivative* of the points A, B, C.

The formulation is applicable to space of any number of dimensions. Any $r+1$ independent points, having symbols unconnected by a syzygy, determine a space of r dimensions, in which any $r+2$ points have symbols which are so connected, and conversely. If such $(r+2)$ points be $A_1, A_2, \ldots, A_{r+2}$, this syzygy, written in the form

$$m_1A_1 + m_2A_2 = m_3A_3 + \ldots + m_{r+2}A_{r+2},$$

expresses that the line determined by the two points A_1, A_2, has a point in common with the $(r-1)$ fold determined by the r points $A_3, A_4, \ldots, A_{r+2}$.

Examples of the application of the symbolism. Particular examples may be more illustrative than any further general statement:

(*a*) That two planes in a threefold space have a line in common may be formulated thus: let the planes be respectively determined by the points A, B, C, and the points A', B', C'. As, in three dimensions, any five points are connected by a syzygy, there is such a syzygy connecting the points A', B' and A, B, C; by this is determined a point of the line $A'B'$ which also lies on the plane ABC; this point is then on both planes. Another such point is determined by the syzygy connecting the five points B', C' and A, B, C. The line joining these two points is then that common to the two planes.

(*b*) That, in threefold space, a definite transversal can be drawn through an arbitrary point to meet two lines which have no point in common, is clear because, if the point be P, and the lines be, respectively, determined by the pairs of points A, B and C, D, the syzygy connecting these five points, of the form

$$pP + aA + bB + cC + dD = 0,$$

expresses that the three points P, $aA + bB$, $cC + dD$, being connected by a syzygy, are in line. The points $aA + bB$, $cC + dD$ are upon the two lines, respectively.

(c) Desargues' theorem, that, if two triads of points, A, B, C and A', B', C', be in perspective, the joins of corresponding points, of B, C and of B', C', of C, A and of C', A', of A, B and of A', B', meet in three points which are in line, also follows easily. For, if the lines AA', BB', CC' meet in O, the symbols for A', B', C' must be, respectively, of the forms

$$A' = \alpha A + \lambda O, \quad B' = \beta B + \mu O, \quad C' = \gamma C + \nu O,$$

where α, λ, β, μ, γ, ν are algebraic symbols. There will then, in accordance with what has been said, be no loss of generality in using A', B', C' instead of $\lambda^{-1}A'$, $\mu^{-1}B'$, $\nu^{-1}C'$, respectively, so that, if a, b, c be used for $\lambda^{-1}\alpha$, $\mu^{-1}\beta$, $\nu^{-1}\gamma$, respectively, these syzygies become

$$A' = aA + O, \quad B' = bB + O, \quad C' = cC + O.$$

These give however

$$B' - C' = bB - cC, \quad C' - A' = cC - aA, \quad A' - B' = aA - bB,$$

which, respectively, determine the three points of intersection of corresponding joins, and shew, in virtue of the syzygy

$$(B' - C') + (C' - A') + (A' - B') = 0,$$

that these three intersections are in line.

This example is instructive because the proof is applicable whether the two triads A, B, C and A', B', C' are in one plane or not; in the former case there is a syzygy connecting O with A, B, C, but the proof does not need to utilise this. It has been remarked however, and a formal proof will be given in Chapter II, that Desargues' theorem is incapable of proof from the Propositions of Incidence applicable to a plane only. The symbolism we are using thus implies more than these, even when applied only to a plane. Evidently it implies that any plane through two points contains every point of the line joining these points.

(d) In space of four dimensions two planes have a point in common; for if these planes be respectively determined by the points A, B, C and A', B', C', the two members of the syzygy which necessarily connects these six points in four dimensions,

$$aA + bB + cC = a'A' + b'B' + c'C',$$

represent points lying respectively, in these planes, which, by virtue of the syzygy, are identical.

Or again, if three lines in fourfold space be respectively given by the pairs of points P and P', Q and Q', R and R', the syzygy connecting these six points,

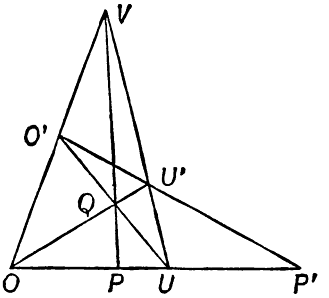
$$pP + p'P' + qQ + q'Q' + rR + r'R' = 0,$$

itself expresses that the three points $pP + p'P'$, $qQ + q'Q'$, $rR + r'R'$, lying respectively on these lines, are in syzygy. So that three lines of general position in fourfold space have a common transversal.

Interpretation of the fundamental laws of the algebraic symbols. After these illustrative examples, which could be indefinitely multiplied, we pass now to give a geometric interpretation of the laws of operation of the algebraic symbols which have been formulated.

(1) The symbol $-a$.

Let O, U be any two points, and P, represented by $O+aU$, be any point on the line joining these. As has been remarked, incidentally, in the above consideration of Desargues' theorem, there would be no gain in representing P in the form $mO+aU$. In any plane through the line OU take the points U', V, on a line passing through U; let OU', VP meet in Q, then QU and OV meet in O', and lastly $O'U'$ meets OU in P'. Thus P' is the harmonic conjugate of P in regard to O and U. We may regard the points of the plane as dependent upon O, U and V. The symbol for the point U' is of the form $U+mV$, and by putting V for mV, in accordance with our original convention, we may suppose this symbol to be $U'=U+V$. Then the point Q is in syzygy with O and U', and also in syzygy with P and V, and the symbol for Q is determined by the fact that it expresses both these circumstances; absorbing an immaterial multiplier, we thence have $Q=O+a(U+V)$, which is both $O+aU'$ and $P+aV$. Similarly, the symbol for O', which is in syzygy both with O and V, and with U and Q, is given by $O'=O+aV$, which is the same as $Q-aU$. Thence the point P', which is in syzygy as well with O' and U', as with O and U, is given by $P'=O-aU$, this being the same as $O+aV-a(U+V)$, or $O'-aU'$. Thus the point $O-aU$ is P'; and the symbols imply that P, P' are distinct.



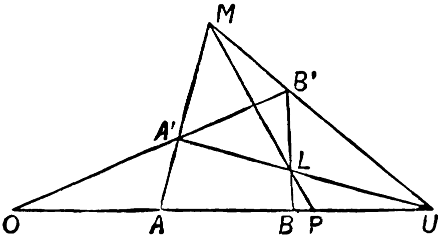
(2) The symbol $a+b$.

Let the points A, B of the line joining the points O and U, when referred to O and U, be, respectively, represented by symbols

$$A=O+aU,\quad B=O+bU.$$

We desire to construct the point $O+(a+b)U$.

In an arbitrary plane drawn through the line AB, draw two lines respectively through

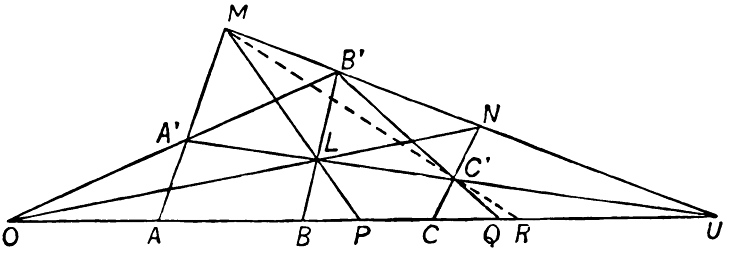


A and B, and let any line through O, in this plane, meet these respectively in A' and B'. Let UA' meet BB' in L, and UB' meet AA' in M, and let LM meet the original line OU in P.

The figure is evidently determinate when, beside O, U, A, B, the points M and A' are given. For the symbol of M we may take $M = A + mA'$, or $M = O + aU + mA'$, which gives $M - aU = O + mA'$; in this syzygy the two members are the symbols, respectively, of a point on the line MU and of a point on the line OA'; this point must then be B', and we may write, for the symbol of this, $B' = O + mA'$. As, however, we have $B = O + bU$, we thence obtain $B' - B = mA' - bU$, which states the equivalence of a point of the line $B'B$ with a point of the line $A'U$. We can, therefore, write, for the symbol of the point L the form $L = mA' - bU$. Thus, as $M = A + mA'$, and, therefore, $M - L = A + bU$, we can, similarly, infer that the point P is given by $P = A + bU$, or $P = O + (a+b)U$.

The construction of P is clearly symmetrical in regard to the points A, B, in accordance with the equation $a + b = b + a$, and, as in earlier cases considered in Section I, is independent of the lines drawn through A, B, O, U when these points are given. Further, when b is $-a$, the figure at once gives $P = O$, in accordance with $a + (-a) = 0$.

(3) The equation $(a+b)+c = a+(b+c)$.



Let p, q denote $a+b$ and $b+c$, respectively. Let A, B, C, referred to the points O, U be, respectively,

$$A = O + aU, \quad B = O + bU, \quad C = O + cU.$$

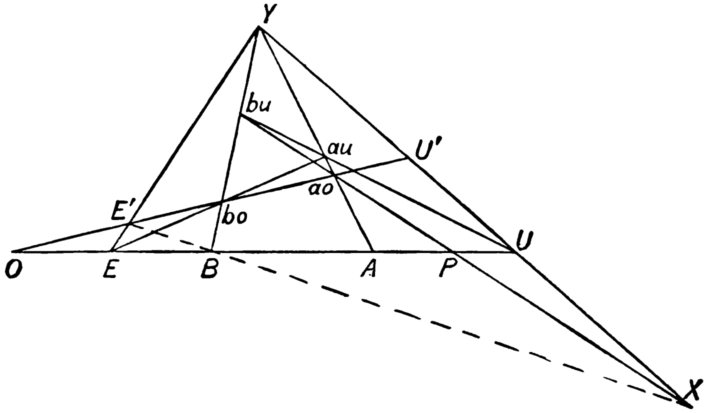
Make the same construction as in the preceding case for A and B, so obtaining the point P given by $P = O + pU$. Now let OL meet the line $MB'U$ in N; join CN meeting the line $A'LU$ in C'; join $B'C'$ meeting OU in Q; also join MC', meeting OU in R. Then as the points L, N, on BB', CC', respectively, are in line with O, it follows, as in the preceding case, that Q is $O + qU$.

Further, as the points A', B', on the lines AM, QC', are in line with O, it follows, also as in the preceding case, that $R = O + (a+q)U$. And, as the points L, N, on the lines PM, CC', are in line with O, it follows, as in the preceding case, that $R = O + (p + c)U$. Thus $p + c = a + q$, as was desired.

(4) The interpretation of the product ab.

For the interpretation of the product ab we go back to a figure which we have already been twice concerned with, first as an immediate application of the Propositions of Incidence (p. 16 above), and then as an example of the consequences of Pappus' theorem (p. 60 above).

Upon the line OU let the points A, B be respectively $O + aU$ and $O + bU$, and let E be the point $O + U$. In an arbitrary plane through this line, draw lines o, u, a, b, respectively through the points O, U, A, B, subject only to the condition that the intersections (a, u), (b, o) are in line with E. Let the joining line of the points (a, o), (b, u) meet the line OU in P. We proceed to shew that the symbol of the point P is given by $P = O + abU$. It has already been shewn that the same point P is obtained for all lines o, u, a, b.



Let the lines a, b meet in the point Y, and the line o meet YU in the point U', and meet the line YE in the point E'. The two triads of points E', E, B and (a, o), (a, u), (b, u) are in perspective from the point (b, o); and the corresponding joins, of E' to E and (a, o) to (a, u), meet in Y, while the corresponding joins, of B to E and (b, u) to (a, u), meet in U. Wherefore, by Desargues' theorem, the line $E'B$ meets the join of (a, o) to (b, u) in a point of the line UY, say X.

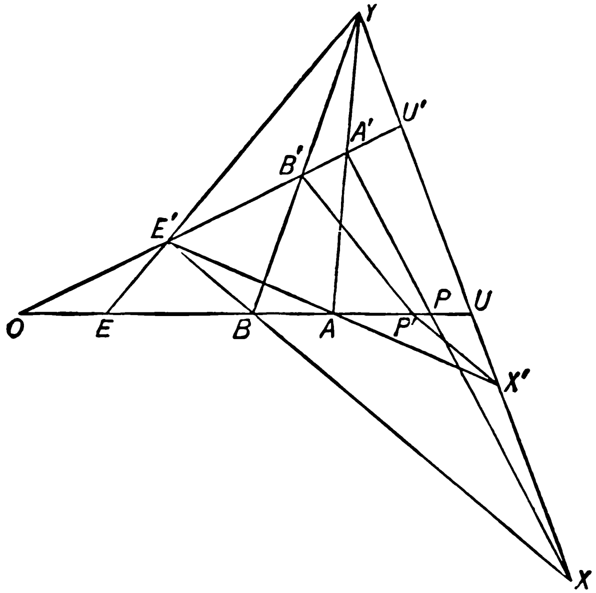
The symbol of U', by absorption of a multiplier in the symbol of Y, may be taken to be $U' = U + Y$. Then E', being both on the join of O to $U + Y$, and on the join of E, or $O + U$, to Y, is given by $E' = O + U + Y$. In a similar way, the point (a, o), being on the join of O to U', or $U + Y$, and also on the join of A, or $O + aU$, to Y, has a symbol $O + aU'$, which is $O + aU + aY$. We find, likewise, for the symbol of the point (b, o), the form $O+bU'$. Thence, the point X, being on the line $U'U$, and also on the join of B to E', that is of $O+bU$ to $O+U'$, has the symbol $U'-bU$. Thence, finally, the point P, being on the line OU, and also on the join of X to (a, o), that is of $U'-bU$ to $O+aU'$, has a symbol

$$O+aU'-a(U'-bU), \text{ or } O+abU.$$

If a line, o', be drawn through O to meet the line a in a point (a, o') which is in line with E and (b, u), and the point (b, o') in which this meets the line b, be joined to the point (a, u) by a line meeting OU in the point P', it can be similarly shewn that the line $E'A$ meets the join of (a, u) and (b, o') in a point of the line YU, and that the point P' has a symbol $O+baU$.

(5) The possibility of the equation $ab=ba$.

We have seen above that the points P, P' coincide if we assume Pappus' theorem (pp. 60, 61). In the present description of the derivation of the points P and P', we have slightly added to the original figure of p. 16, using Desargues' theorem, in order to simplify the proof of the symbols for the points P, P'. In the figure thus reached, arbitrary lines are drawn from E, B, A, U to meet in Y; an arbitrary line is drawn from O to meet these, respectively, in E', B', A', U'; then P is found by drawing $E'B$ to meet YU in X, and taking the intersection of $A'X$ with OU, while P' is found by drawing $E'A$ to meet YU in X', and taking the intersection of $B'X'$ with OU. The identity of P and P' as a consequence of Pappus' theorem can then be seen, differently from



before, by considering the two triads Y, X, X' and E', B', A'; as $YA', E'X'$ meet in A, while $YB', E'X$ meet in B, it follows then that XA' and $X'B'$, or $A'P$ and $B'P'$, meet on AB.

A direct deduction from the figure of p. 16 of the symbols for P and P' may be given, which has a value as an exhibition of the self-consistency of the non-commutative symbolism.

A further construction of P is obtainable by remarking that, in virtue of Desargues' theorem, the lines UE', UA' meet YB, YP, respectively, in two points lying in line with O.

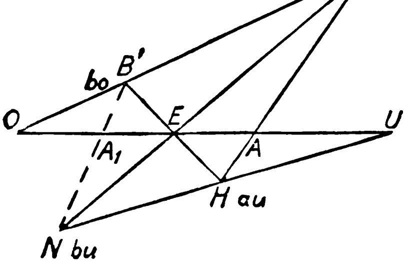
We have seen above (p. 46) that Pappus' theorem, though a theorem of geometry in a plane, is equivalent with a theorem in regard to lines in space. We shall see below (p. 82) that the corresponding expression in symbols, of this theorem in regard to lines in space, also requires that the symbols should be commutative in multiplication.

We have remarked above (p. 66) that, in the case of the iterative symbols, the commutative equation $mn = nm$ is a consequence of their definition. The question whether the geometrical representation of this identity involves Pappus' theorem, and the question whether von Staudt's introduction of numbers in Geometry involves the assumption of Pappus' theorem, are considered below (p. 87 and p. 93).

Provisionally, we may infer that *the assumption of Pappus' theorem, and the restriction of the algebraic symbols to such as are commutative in multiplication, are, subject to the other assumptions made, equivalent.*

(6) The symbol a^{-1}.

This may be represented by the same figure as that used for the representation of ab, if this be specialised so that P coincides with E.



On the line OU let A be $O+aU$; let A', H be any two points in line with A; let B', on OA', be in line with E and H, and N, on UH, be in line with E and A'. Then NB' meets OU in $O+a^{-1}U$.

For, if H have the symbol $A+\lambda A'$, or $H = O + aU + \lambda A'$, we first have

$$(1-a)\,O + \lambda A' = O + aU + \lambda A' - a\,(O+U),$$

as the symbol of a point lying both on OA' and on HE, the point B'; and then we have

$$O + U + \lambda A' = O + aU + \lambda A' + (1-a)\,U$$

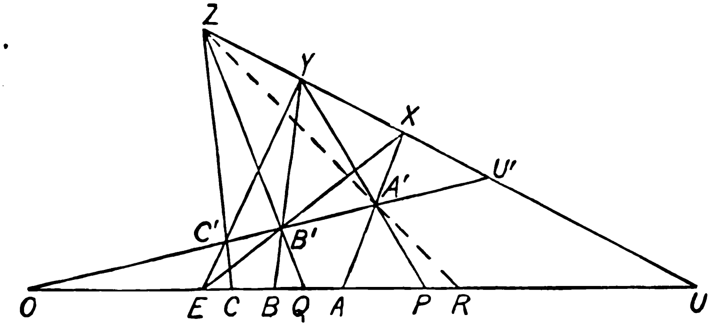
as the symbol of a point lying on EA' and also on HU, the point N. Thence we have $aO + U$ as the symbol of a point lying on $B'N$ which lies on OU; and this gives the same point, A_1, as $O + a^{-1}U$.

Evidently, by the construction, if $b = a^{-1}$, then $a = b^{-1}$, and $ab = 1$. Also when $a = 1$, then also $a^{-1} = 1$.

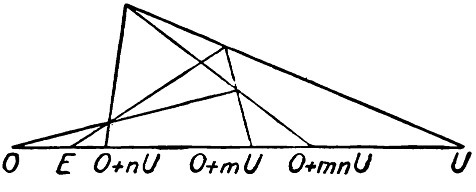
(7) The equation $(ab)c = a(bc)$.

Let p denote ab, and q denote bc.

Let the points A, B, C and E, on the line OU be respectively $O + aU$, $O + bU$, $O + cU$, and $O + U$. Draw, in an arbitrary plane through OU, two lines through O and U, intersecting in U', and let a line through B, in this plane, intersect these, respectively, in B' and Y. Let the lines joining the point E, respectively to Y and B', meet these lines, OU', UU', respectively in C' and X. Let AX meet OU' in A', and CC' meet UU' in Z. Lastly, draw YA', ZB', ZA', meeting OU respectively in P, Q, R.



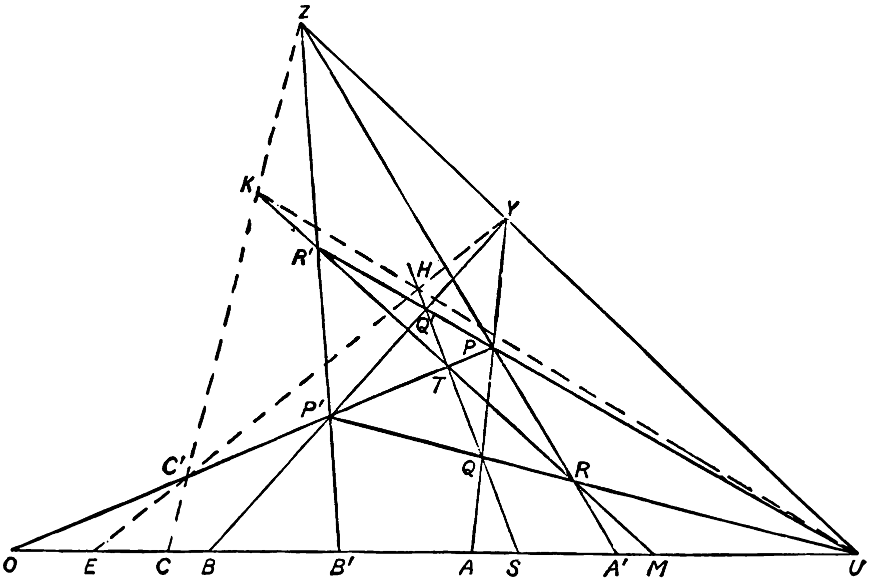
Then, recalling once more the construction for $O + mnU$ from the points $O + mU$, $O + nU$, we see (*a*) that P is $O + abU$, or $O + pU$; (*b*) that Q is $O + bcU$, or $O + qU$; (*c*) that R is $O + aqU$; (*d*) that R is also $O + pcU$. Wherefore $aq = pc$, as we desired to see.



(8) The equation $(a + b)c = ac + bc$.

First remark the following figure: let ZPR, $ZR'P'$, UPR', URP' be any four lines in a plane, with intersections indicated by the lettering, the joins PP', RR' meeting in T; take an arbitrary point, Y, upon ZU, and an arbitrary point, C', upon PP'; let the joins YP, YP' meet $P'R$, $R'P$ respectively in the points Q and Q', so that, by Desargues' theorem, as corresponding joins of the two triads P, Q, R and P', Q', R' meet, respectively, in the three points U, Z, Y, which are in line, it follows that QQ' passes through the point of intersection, T, of PP' and RR'; next, let the joins $C'Y$, $C'Z$ meet QQ'

and RR', respectively, in H and K. Then, as the corresponding joins of the two triads Y, H, Q' and Z, K, R' meet, respectively,



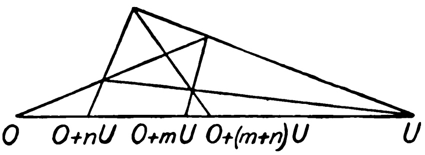
in the points T, P', C', which are in line, it follows that the line KH passes through the intersection, U, of ZY and $R'Q'$. We are concerned now with the intersections of all the lines of this figure with a line drawn through U, in the plane of the figure.

Let O, U be any two points, and E, A, B, C be points upon the line OU, whose symbols, referred to O and U, are respectively

$$E = O + U, \quad A = O + aU, \quad B = O + bU, \quad C = O + cU.$$

In an arbitrary plane through the line OU take a point, Z, and then a point, Y, upon the line ZU. Join YB and YA, and also ZC. Let YE meet ZC in C', and, then, let OC' meet YB and YA, respectively, in P' and P. Let ZP' and ZP, respectively, meet OU in B' and A'; and let UP and UP' meet ZP' and ZP, respectively, in R' and R, these same lines meeting YP' and YP, respectively, in Q' and Q. Then, by what has been remarked, the line QQ' passes through the intersection, T, of PP' and RR', and if this line QQ' meet $YC'E$ in H, while RR' meets $ZC'C$ in K, then KH passes through U. Denote the intersections of QQ' and RR', respectively, with OU by S and M.

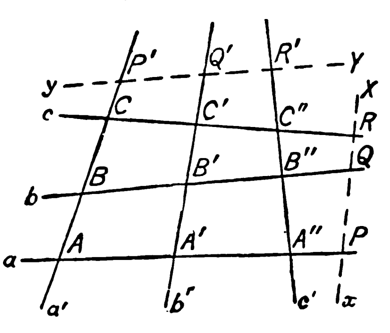
We have just recalled (in No. 7) the construction for finding the point $O + mnU$ from the points $O + mU$ and $O + nU$; if we recall also the construction for finding $O + (m+n)U$ from $O + mU$ and $O + nU$, we see that, in the figure we have constructed, the symbol of the point S is $O + (a + b) U$, and, thence, that the symbol of the point M is $O + (a + b) cU$. On the other hand, we see also that the symbol of the point A' is $O + acU$, and, similarly, of B' is $O + bcU$, and, thence, that the symbol of the point M is $O + (ac + bc) U$. Comparing these two forms for the symbol of the point M, we have the identity $(a + b) c = ac + bc$, which we desired to interpret.



A similar representation of the identity $c (a + b) = ca + cb$ may be constructed.

Representation of the algebraic effect of Pappus' Theorem in three dimensions. We have now shewn that all the fundamental laws of operation for the algebraic symbols are in accord with geometrical facts in one plane deducible from the Propositions of Incidence with the help of Desargues' theorem, which is itself deducible, as we have seen, from these Propositions, assumed not limited to two dimensions. But we have shewn that the commutative law of multiplication for the algebraic symbols involves Pappus' theorem, and conversely. We shewed however, in Section II, that Pappus' theorem, is equivalent, in virtue of the Propositions of Incidence, to a theorem of intersection of two lines in three dimensions. It is interesting to verify directly that this theorem again involves the commutative property of the symbols. The geometrical theorem is that, if we have two triads, each of non-intersecting lines, in three dimensions, every line of either triad meeting all those of the other, then any transversal of the lines of one triad intersects any transversal of the lines of the other.

Let the lines of the two triads be respectively a, b, c and a', b', c', their intersections being, A, A', A'' on a, B, B', B'' on b, and C, C', C'' on c. Let x be any transversal of a, b, c, meeting them, respectively, in P, Q, R, any point of this transversal being X; so, let y be any transversal of a', b', c', meeting them, respectively, in P', Q', R', any point of this transversal being Y.



The points A, A', B, B' are not supposed to be in a plane, and may be taken as fundamental for the threefold space in which the figure lies. Using A, A', B, B' as the symbols for these points, we may, by absorption of algebraic multiples in these symbols, suppose that the symbol of the point C'' is

$$C'' = A + A' + B + B'.$$

This expresses, however, that C'' is a derivative of the two points represented, respectively, by $A + A'$ and $B + B'$, of which the former is on the line a and the latter on the line b. For the symbols of the points A'' and B'' we thus infer

$$A'' = A + A', \quad B'' = B + B'.$$

For the symbols of the points C and C' we similarly infer

$$C = A + B, \quad C' = A' + B'.$$

The points P and Q are, respectively, derivatives of A, A' and of B, B'; thus proper symbols for these are, respectively,

$$P = A + \rho A', \quad Q = B + \lambda B',$$

with suitable values for the algebraic symbols ρ and λ. Then the point R, a derivative of P and Q, must have a symbol of the form

$$R = mP + nQ, \quad = mA + nB + m\rho A' + n\lambda B';$$

on the other hand, as R is a derivative of C and C', its symbol must be of the form

$$R = h(A + B) + h'(A' + B');$$

comparing these two forms, we infer that $m = n$ and $m\rho = n\lambda$, and hence $\rho = \lambda$. Thus the symbols of P, Q involve the same algebraic symbol, being $P = A + \rho A'$, $Q = B + \rho B'$.

Thence the point X, a derivative of P and Q, is given by a symbol

$$X = P + \sigma Q, \quad = A + \rho A' + \sigma B + \sigma\rho B'.$$

Again, the points P' and Q', respectively derivatives of A, B and of A', B', have symbols of the forms

$$P' = A + \kappa B \quad \text{and} \quad Q' = A' + \mu B';$$

and the point R', a derivative of P' and Q', is equally a derivative of A'' and B''. There exists, therefore, a syzygy of the form

$$p(A + \kappa B) + q(A' + \mu B') = r(A + A') + s(B + B');$$

hence we infer that $p = q$ and $p\kappa = q\mu$, and therefore that $\kappa = \mu$.

Thus the point Y, any point of the line $P'Q'$, has a symbol

$$Y = P' + \tau Q', \quad = A + \tau A' + \kappa B + \tau\kappa B'.$$

Comparing the symbols for the points X and Y, we see that the conditions for the lines x and y to have a common point are that, when ρ and κ have been taken, corresponding to the positions of P and P' respectively on AA' and AB, it must be possible to take σ and τ, corresponding to the positions of X and Y respectively on x and y, so that

$$\tau = \rho, \quad \kappa = \sigma, \quad \tau\kappa = \sigma\rho.$$

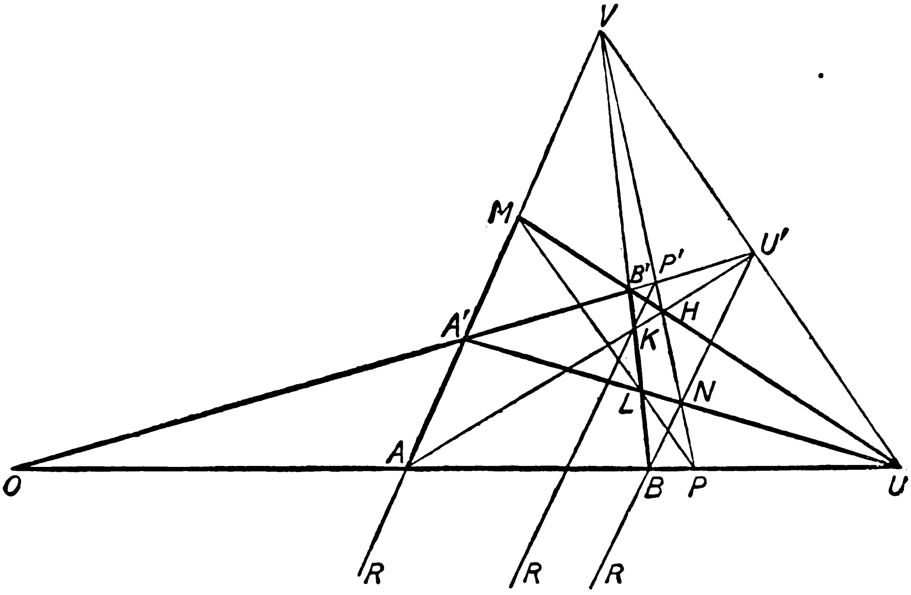
These can be satisfied if, and only if, we have

$$\rho\kappa = \kappa\rho,$$

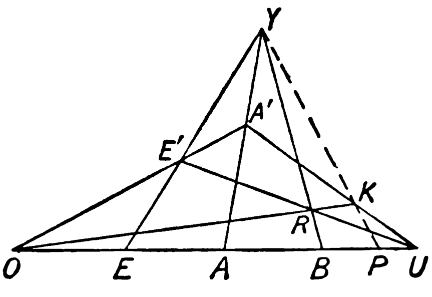
and therefore, $\sigma\tau = \tau\sigma$, $\rho\sigma = \sigma\rho$, $\kappa\tau = \tau\kappa$. The point X is determined by ρ and σ alone; only when these are such that $\rho\sigma = \sigma\rho$ is there a line y, transversal to a', b', c', which meets the line x. The condition of commutative multiplication, for every two of the algebraic symbols employed, is the necessary and sufficient condition that every line x should intersect every line y.

Examples in regard to the use of the symbols, in particular in regard to the commutative law of multiplication. A vast number of applications can be given of the symbolism explained in this Section. We limit ourselves mainly to such as seem to carry the explanation of general principles somewhat further.

Ex. 1. In the construction above given for the point $O + (a+b)U$ from the points O, U, $O + aU$ and $O + bU$, the constructed point P may be differently defined. If AA', BB' meet in V, also $A'B'$ meets VU in U', and meets VP in P', while $U'A$, $U'B$ meet VP, respectively, in H and N, and $U'A$ meets VB in K, then H and N lie, respectively, on UB' and UA', and the lines $P'K$, NB, VA meet in one point, R.

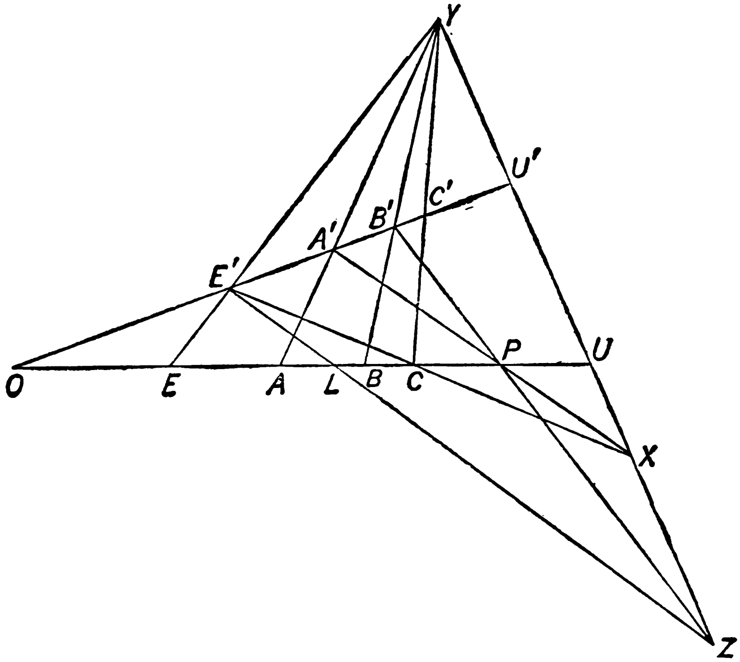


These follow by applying Desargues' theorem.



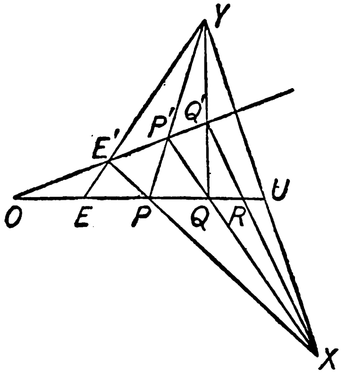
Ex. 2. The construction of the

point $O+abU$, from the points $O, U, E, O+aU, O+bU$, above given, may be stated thus: Let $OE'A'$ be any line through O meeting YE, YA, respectively, in E' and A', where Y itself is arbitrary; let UE' meet YB in R, and OR meet UA' in K; then YK gives the required point P. To reduce this to what is given above we apply Desargues' theorem to the two triads E', R, B and A', K, P.



Ex. 3. If O, U, E, A, B, C represent O, U, $O+U$, $O+aU$, $O+bU$, $O+cU$, and an arbitrary line through O meet the joins of the arbitrary point, Y, to E, A, B, C, respectively, in E', A', B', C', also $E'C$ meet YU in X, and XA' meet OU in P, and $B'P$ meet YU in Z, and lastly ZE' meet OU in L, shew that the symbol of this point is $O+b^{-1}acU$. Then construct a geometrical representation of the equation

$$b^{-1}ac+b^{-1}dc=b^{-1}(a+d)c.$$



Ex. 4. If relatively to the points O, U, the points

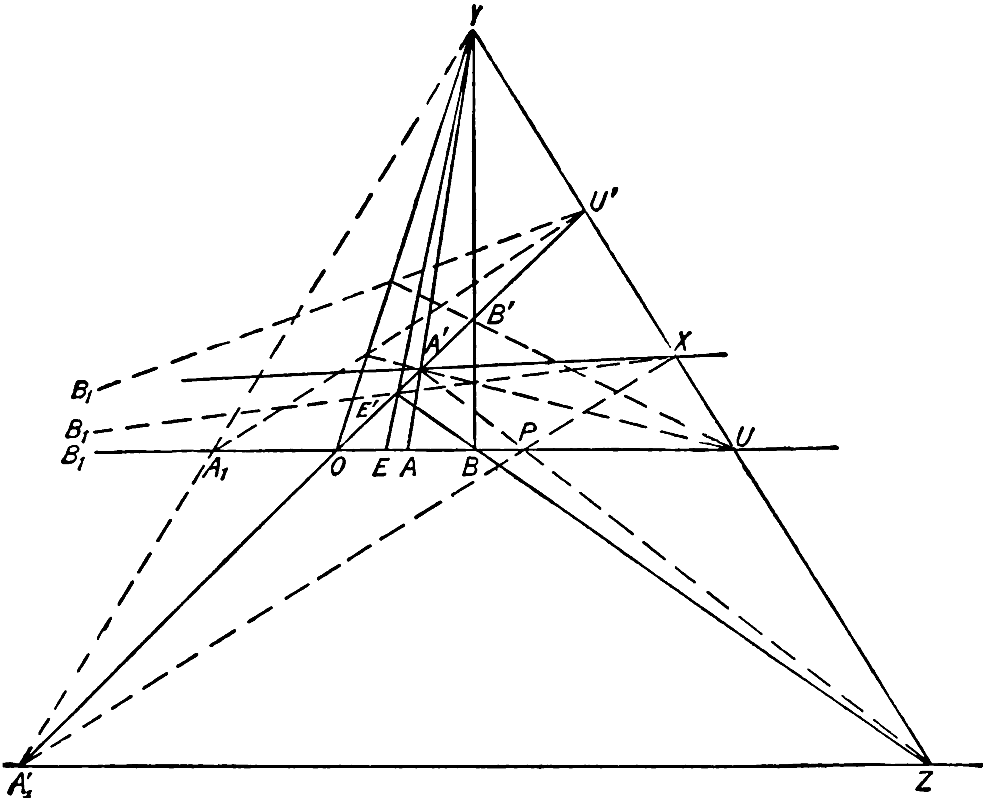
$$E=O+U, \quad P=O+aU$$

be given, and a^2, a^3, ... as usual denote $a.a$, $a^2.a$, ..., construct the points

$$Q=O+a^2U, \quad R=O+a^3U, \quad S=O+a^4U, \text{ etc.}$$

Ex. 5. To represent the equations

$$(-a)(-b) = ab,$$
$$(-a)b = a(-b) = -ab.$$



Let $\quad E = O + U, \quad A = O + aU, \quad B = O + bU.$

Let Y be any point, not on OU, and U' be any point on YU. Let the line OU' meet YE, YA, YB, respectively, in E', A', B'; let $E'B$ meet YU in Z, and $A'Z$ meet OU in P. Join the intersection of YO and UA' to U', the join meeting OU in A_1; join the intersection of YO and UB' to U', the join meeting OU in B_1. Let YA_1 meet OU' in A_1'; let B_1E' meet YU in X.

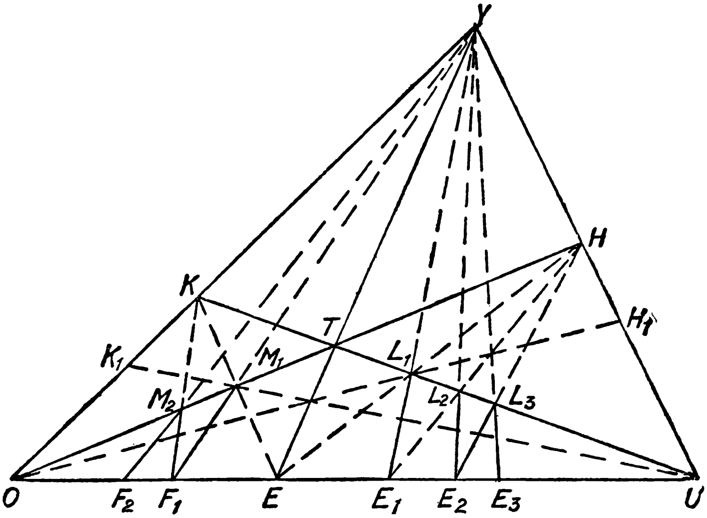
Prove that $A_1'X$ passes through P, and hence represent the equation $(-a)(-b) = ab$. Also prove that $A'X$ and $A_1'Z$ meet on OU, and hence represent the equation $(-a)b = a(-b)$.

Ex. 6. The iterative symbols.

(*a*) Let m, n be any iterative symbols formed, respectively, by $\bar{m}$ and by $\bar{n}$ repetitions of the symbol 1, where $\bar{m}$, $\bar{n}$ are positive integers of ordinary arithmetic. In the first place, given the points O, U

and $E = O + U$, we seek to represent the points $O + mU$, $mO + U$, $mO + nU$.

Take any point Y, not on OU, and any point T on YE, for which the symbol, with proper choice of the symbol Y, may be written $T = Y + E$. Let OT meet YU in H, and UT meet OY in K; the point H, on the join of U and Y, and on the join of O and $Y+O+U$, has symbol $H = U + Y$; so the point K has symbol $K = O + Y$. Let HE and KE meet UT and OT, respectively, in L_1 and M_1; and let YL_1 and YM_1 meet OU in E_1 and F_1, respectively. Let HE_1 and KF_1 meet UT and OT, respectively, in L_2 and M_2; and let YL_2 and YM_2 meet OU in E_2 and F_2, respectively. And let the process thus suggested be continued indefinitely.



It can then be seen at once that the symbols for $E_1, E_2, \ldots$ are $E_1 = O + 2U$, $E_2 = O + 3U$, ..., in general $E_{r-1} = O + rU$. So we have $F_1 = 2O + U$, $F_2 = 3O + U$, ... Then, if the join of O and L_1 meet YU in H_1, the join of K to H_1 meets OU in $O - 2U$; and, in general, if the join of O and L_{r-1} meet YU in H_{r-1}, the join of K to H_{r-1} meets OU in $O - rU$. Similarly if the join of U and M_1 meet YO in K_1, the join of H to K_1 meets OU in $2O - U$, and so on. We thus easily construct all the points $O + mU$, $mO + U$, $O - mU$, $mO - U$.

In fact these points arise by successive applications of the construction for determining the harmonic conjugate of a point P in regard to a given pair of points A, B. For, denoting this fourth harmonic point by $(A, B)/P$, we clearly have

$$E_1 = (E, U)/O, \quad E_2 = (E_1, U)/E, \ldots,$$

$$F_1 = (E, O)/U, \quad F_2 = (F_1, O)/E, \ldots,$$

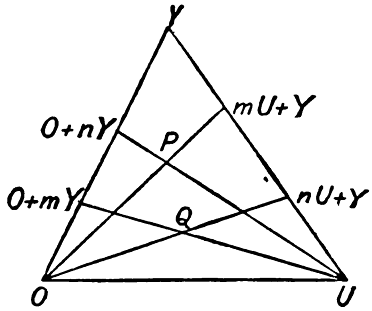
and in general

$$E_{r+1} = (E_r, U)/E_{r-1}, \quad F_{r+1} = (F_r, O)/F_{r-1}.$$

More generally, recalling that the harmonic conjugate of a point $A + \lambda B$, in regard to the points A, B, is $A - \lambda B$, we see that the point $mO + nU$, being $mO + (n-1)U + U$, is the harmonic conjugate of $mO + (n-2)U$ in regard to $mO + (n-1)U$ and U, or, say,

$$E_{m,n} = (E_{m,n-1}, U)/E_{m,n-2}.$$

Every one of the points $mO \pm nU$ can thus be constructed by a succession of processes of finding the fourth harmonic, given the points O, U, $O + U$; the m, n being any iterative symbols.



Consider next the representation of the equation $nm = mn$; we shew that this is independent of Pappus' theorem. In the first place, from the points O, U, E, and Y, we can construct, as above, the points $mU + Y$, $nU + Y$, upon the line YU, and the points $O + nY$, $O + mY$, upon the line YO. The intersection, P, of the joins of O to $mU + Y$ and of U to $O + nY$, evidently has a symbol

$$P = O + nmU + nY,$$

and the intersection, Q, of the joins of O to $nU + Y$ and of U to $O + mY$, evidently has a symbol

$$Q = O + mnU + mY.$$

A representation of $nm = mn$ is, thus, that the points P, Q are in line with Y.

If, however, we consider the three points, P, so arising, for the same value of n, and the three values $m - 1$, m, $m + 1$ in place of m, and at the same time the three points, Q, so arising, for this value of n and the same values for m, the harmonic relation shews that the assumption that two of the three triads Y, P, Q so arising are in line, involves that the third is also. For the initial values of m, n the theorem that Y, P, Q are in line is easy to prove. Thus, by induction, it is always true.

Ex. 7. As examples of the fact that a geometrical result obtainable without Pappus' theorem should be representable symbolically without use of the commutative law for multiplication, and the converse statement that, if we can obtain a symbolical deduction of a result without use of this commutative law, we should be able to construct a geometrical proof from the Propositions of Incidence only, we may reconsider some of the work of Section I.

A simple result was that, if p, a, b, c, d be five arbitrary lines of which no two intersect, lying in three dimensions, and, from an arbitrary point, P, of the first line, p, there be drawn the transversal to two of these lines, say a and c, to meet these respectively in A and C, and also from P the transversal to the other two given lines b, d meeting these in B and D, then the intersection, say Q, of the joins of A and C respectively to B and D, that is of AB and CD, also describes a line. The intersection of AD and CB equally describes a line.

Choosing the symbols of A, B, C, D suitably, we can suppose, these points being in a plane, that the syzygy connecting them is $A+B+C+D=0$. Then the point represented equally by $A+C$ and $-(B+D)$ is necessarily P; and, similarly, for the symbol of Q, we have either $A+B$ or $-(C+D)$. If a second point, P', be taken on the line p, and a similar construction, with a like convention, be made, we can equally write

$$P' = A' + C', \; = -(B' + D'), \quad Q' = A' + B', \; = -(C' + D').$$

A third point, P'', upon p, has a symbol given by

$$\begin{aligned} P'' = mP + m'P' &= m(A + C) + m'(A' + C') \\ &= mA + m'A' + mC + m'C', \end{aligned}$$

which is also $\qquad -(mB + m'B') - (mD + m'D')$.

Thus P'' is in line with the points $mA + m'A'$ and $mC + m'C'$, lying, respectively, on the lines a and c, and is also in line with the points $mB + m'B'$ and $mD + m'D'$, lying, respectively, on the lines b and d. These are then the points obtainable by the original construction from the point P''. If we denote them, respectively, by A'', C'', B'', D'', we see that

$$A'' + B'' = m(A + B) + m'(A' + B') = -(C'' + D''),$$

and the lines $A''B''$, $C''D''$ meet in a point, Q'', given by

$$Q'' = mQ + m'Q',$$

lying on the join of Q and Q'. This establishes the result stated.

Ex. 8. In Section I (p. 27) we gave a construction, when four, or more, non-intersecting lines AA', BB', CC', DD' ..., have three common transversals $ABCD$ ..., $A'B'C'D'$ ..., $PQRS$ ..., for finding a fourth transversal $P'Q'R'S'$.... The point P' was the harmonic conjugate of P in regard to A and A', say $P' = (A, A')/P$; and, similarly, $Q' = (B, B')/Q$, $R' = (C, C')/R$, ...

The figure being in three dimensions we can regard A, A', B, B' as fundamental points, and write the symbols of P and Q in the forms $P = A + \rho A'$ and $Q = B + \sigma B'$; and then the symbols of R and S in the forms $R = P + mQ$ and $S = P + nQ$. By properly choosing the symbols A', B', Q, we can, if we wish, suppose $\rho = 1$,

$\sigma = 1$, $m = 1$; but n is not 1, save for a particular position of S. The form of R,

$$R = A + \rho A' + m(B + \sigma B'),$$

enables us then to infer, for the symbols of C and C', respectively,

$$C = A + mB, \quad C' = \rho A' + m\sigma B';$$

also if, as before (p. 27 above), the transversal from R to the cross lines AB' and $A'B$ meet these respectively in F, F', this form of R shews that, for F and F',

$$F = A + m\sigma B', \quad F' = \rho A' + mB.$$

Hence

$$C - F' = A - \rho A', \; = F - C', \text{ and } C - F = m(B - \sigma B'), \; = F' - C';$$

thus the lines $F'C$ and FC' meet in a point P' of AA' given by $P' = A - \rho A'$, and the lines FC and $F'C'$ meet in a point Q' of BB' given by $Q' = B - \sigma B'$. These are such that $P' = (A, A')/P$ and $Q' = (B, B')/Q$; and they give

$$P' + mQ' = C - C',$$

which shews that $P'Q'$ meets CC' in a point R' such that $R' = (C, C')/R$. From $S = P + nQ$, we can, similarly, shew that the line $P'Q'$ contains a point S' for which $S' = (D, D')/S$.

Ex. 9. Another construction was given (p. 29 above) for a fourth transversal of the four lines AA', BB', CC', DD' of Ex. 8. If BR, $B'S$ meet in X, and XP meets BB' in Q'; while $A'R$, AS meet in Y, and YQ meets AA' in P', then $P'Q'$, which meets AA' and BB', also meets CC' and DD'. Further XA, YB', CC', $P'Q'$ meet in a point, as do YB, XA', DD', $P'Q'$.

With the notation of the previous Example we find, in fact,

$$X = R - (m - n)B = S + (m - n)\sigma B' = A + \rho A' + nB + m\sigma B', \; = D + C',$$

so that
$$nQ' = nB + m\sigma B', \quad Q' = B + n^{-1}m\sigma B'.$$

Also we find

$$Y = m^{-1}R - (m^{-1} - n^{-1})\rho A' = n^{-1}S + (m^{-1} - n^{-1})A$$
$$= m^{-1}A + n^{-1}\rho A' + B + \sigma B', \; = m^{-1}C + n^{-1}D',$$

so that
$$m^{-1}P' = m^{-1}A + n^{-1}\rho A', \quad P' = A + mn^{-1}\rho A'.$$

These lead to

$$P' + mQ' = C + mn^{-1}C', \quad P' + nQ' = D + mn^{-1}D',$$

which we may denote, respectively, by R' and S'. These express that $P'Q'$ meets CC' and DD'.

But, unless we assume a relation, in multiplication, for the symbols, which would follow if we assumed Pappus' theorem, this line $P'Q'$ does not meet any other general transversal of the three given lines $ABCD$, $A'B'C'D'$, $PQRS$. For if, from a point, T, of the line *PQRS*, there be drawn *ETE'* to meet *ABCD* and *A'B'C'D'*, respectively, in *E* and *E'*, taking the symbol of *T* to be $T = P + tQ$, we obtain

$$E = A + tB, \quad E' = \rho A' + t\sigma B',$$

and $$P' + tQ' = E + mn^{-1}E' + mn^{-1}(nm^{-1}tn^{-1}m - t)\,\sigma B'.$$

This shews that the point in which the line *P'Q'* meets the plane *EE'B'* is not on *EE'*, that is, that *P'Q'* does not intersect *EE'*, unless

$$tn^{-1}m = mn^{-1}t,$$

a relation which, if we suppose $m = 1$, as we may do without loss of generality, becomes $tn^{-1} = n^{-1}t$, or $nt = tn$. It would follow if Pappus' theorem were assumed, and it is true, independently of this, when t, m, n are iterative symbols.

The other results stated to be true follow from the syzygies

$$X + (nm^{-1} - 1)\,A = nY + (m - n)\,\sigma B' = nm^{-1}C + C' = nQ' + nm^{-1}P',$$

these combinations being all identical with

$$nm^{-1}A + \rho A' + nB + m\sigma B',$$

together with

$$mY - (m - n)\,B = X + (mn^{-1} - 1)\,\rho A' = D + mn^{-1}D' = nQ' + P',$$

these being all equivalent with

$$A + mn^{-1}\rho A' + nB + m\sigma B'.$$

The facts that *X* lies on *C'D*, and *Y* lies on *CD'*, have already appeared.

If we proceed by the same rule to deduce *P''*, *Q''* from *P'*, *Q'*, just as we have deduced *P'*, *Q'* from *P*, *Q*, we find that

$$P'' = A + mn^{-1}mn^{-1}\rho A', \quad Q'' = B + n^{-1}mn^{-1}m\sigma B',$$
$$P'' + mQ'' = C + mn^{-1}mn^{-1}C', \quad P'' + nQ'' = D + mn^{-1}mn^{-1}D',$$

the last two giving the points where *P''Q''* meets *CC'* and *DD'*.

Ex. 10. We gave in Section I (p. 26) a process for finding a transversal, in the figure of Ex. 8, for the two pairs of cross joins *AB'*, *A'B*, *CD'*, *C'D*. With the notation of Ex. 8, we find for the symbols of the points *M*, *M'*, *N*, *N'*, occurring in that construction

$$M = A + m\sigma B', \quad M' = \rho A' + mB, \quad N = A + n\sigma B', \quad N' = \rho A' + nB,$$

from which follow

$$M + N' = C' + D, \quad M + mn^{-1}N' = C + mn^{-1}D',$$
$$M' + N = C + D', \quad M' + mn^{-1}N = C' + mn^{-1}D,$$

in agreement with the fact that the lines *MN'* and *M'N* both meet the lines *CD'* and *C'D*.

Ex. 11. Consider now the question of two Moebius tetrads of points, *A*, *B*, *C*, *D* and *A'*, *B'*, *C'*, *D'*, such that each of the latter is on the plane determined by three of the former, and each of the former on the corresponding plane determined by three of the latter. We have, in Section I (p. 61), deduced the possibility of this by help of Pappus' theorem; it is to be expected that the symbolical representation will require the commutative restriction.

The points A, B, C, D being regarded as fundamental, the fact that the points D', A', B', C' are, respectively, on the planes ABC, BCD, CAD, ABD, can be expressed, by properly choosing the symbols of A, B, C, A', B', C', by the syzygies

$$D' = A + B + C,$$
$$A' = mC - n_1B - D, \quad B' = nA - l_1C - D, \quad C' = lB - m_1A - D,$$

as is easily seen, l, m, n and l_1, m_1, n_1 being suitable algebraic symbols. Then, the line $B'C'$ meets the plane ABC in a point which is represented by either of the symbols

$$B' - C', \quad (m_1 + n) A - lB - l_1C;$$

if this point is to be on the line $D'A$, remembering $D' = A + B + C$, this point must be a derivative of A and $B + C$. Thus $l = l_1$. So we get $m = m_1$ and $n = n_1$.

In order that D should lie on the plane $A'B'C'$, there must be a derivative of A', B', C', say $aA' + bB' + cC'$, which coincides with D. As A, B, C, D are not in a plane, there must then be a syzygy

$$a(mC - nB) + b(nA - lC) + c(lB - mA) = 0,$$

which, since A, B, C are independent, requires the equations

$$bn = cm, \quad cl = an, \quad am = bl;$$

these however require

$$a = cln^{-1}, \quad a = blm^{-1}, \quad cm \,.\, m^{-1}ln^{-1} = bn \,.\, n^{-1}lm^{-1},$$

and hence $m^{-1}ln^{-1} = n^{-1}lm^{-1}$, or $lm^{-1}n = nm^{-1}l$. By a slight change of symbols we can, without loss of generality, suppose $m = 1$. The condition for the Moebius tetrads is thus $ln = nl$, as was anticipated.

Ex. 12. Obtain the symbolical representation of the result given in Section I (p. 37) that, in four dimensions, an infinite number of planes can be drawn through an arbitrary point to meet three given lines of general position, and that these planes all meet three other lines.

Also of the result that, in five dimensions, an infinite number of planes can be drawn to meet four given lines of general position, and that these planes all meet six other lines.

Ex. 13. It has been seen that a plane meeting four lines, of general position, in four dimensions, can be drawn to meet two of these lines in assigned points (Section I, p. 37). Denote the four lines by a, b, c, d. Let the transversal of b, c, d meet b, c and d, respectively, in C, B' and P'; the transversal of c, a, d meet c, a, d respectively, in A, C' and Q'; and the transversal of a, b, d meet

a, b, d, respectively, in B, A' and R'. The points P', Q', R' are, thence, respectively, derivatives of B' and C, of C' and A, and of A' and B; they are however in syzygy among themselves, as being on the line d. This syzygy leads then to a syzygy connecting the symbols of A, A', B, B', C, C'; by a proper choice of the symbols this last syzygy may be taken to be $A+A'+B+B'+C+C'=0$, any five of these points remaining independent, and sufficient to determine the fourfold space. Hence the symbol of P', a derivative of B' and C, may be taken to be $P'=B'+C$, with, similarly, $Q'=C'+A$, and $R'=A'+B$. The line a is determined by the points B and C', the line b by the points C and A', and the line c by the points A and B'.

If then we take the plane containing the three points

$$B+\rho C', \quad C+\sigma A', \quad A+\rho^{-1}\sigma^{-1}B',$$

we see, because

$$B+\rho C'+\sigma^{-1}(C+\sigma A')+\rho(A+\rho^{-1}\sigma^{-1}B')$$

is the same as

$$\sigma^{-1}(B'+C)+\rho(C'+A)+A'+B,$$

that this plane meets the line d. As ρ, σ may be taken arbitrarily we thus have the general plane meeting the four lines a, b, c and d, spoken of at starting.

With the four given lines is associated another line; it is that, namely, which contains the three points whose symbols are $A+A'$, $B+B'$, $C+C'$, these being in syzygy in virtue of the syzygy connecting the six points A, A', B, B', C, C'. This line may be defined geometrically; if a' be the common transversal of b, c, d; b' of c, a, d; c' of a, b, d and d' of a, b, c, it is the intersection of the four threefolds $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$, $\{d, d'\}$, which meet in a line. For brevity we omit the geometrical proof of this, which will arise later; it is independent of Pappus' theorem.

But now we remark that

$$\rho^{-1}(B+\rho C')+C+\sigma A'+\sigma(A+\rho^{-1}\sigma^{-1}B')$$

is the same as

$$\sigma(A+A')+\rho^{-1}B+\sigma\rho^{-1}\sigma^{-1}B'+C+C'.$$

The point of which this is the symbol will lie on the line containing the three points $A+A'$, $B+B'$, $C+C'$, only if $\rho^{-1}=\sigma\rho^{-1}\sigma^{-1}$, or $\rho\sigma=\sigma\rho$. Hence we forecast a geometrical theorem: *If, and only if, Pappus' theorem be assumed, all planes meeting four lines of general position in four dimensions are met by another line.*

A proof of this theorem, apart from the symbolism, will be given later (in Volume IV).

Ex. 14. Finally we make a brief remark in regard to von Staudt's introduction of numbers in Geometry. This depends (*Beiträge zur*

Geometrie der Lage, Zweites Heft, No. 256, p. 166) upon considering two sets of four points, on two different lines, each set taken in a definite order, as being equivalent, when they are in perspective with one another. In effect such a set of four points, (or *Wurf*), is regarded as determining what we have called an algebraical symbol, and rules for the sum and product of two such symbols, or, in general, for computation with these symbols, are developed from geometrical theorems previously established on geometrical grounds.

It is impossible in a few words to make a comparison, between two methods of arranging the fundamental logical ideas, which shall not be open to objection. But it would seem that von Staudt's scheme, when approached from the point of view we have taken up, assumes Pappus' theorem in its initial definition. For, as will appear more fully below, for four points of a line which have, with the suppositions here made, the respective symbols A, B, $A+\lambda B$, $A+\mu B$, the symbol of von Staudt corresponding to one arrangement of these points, is effectively $\lambda\mu^{-1}$. We have shewn (p. 25 above) that these points can be placed in perspective with the points B, A, $A+\mu B$, $A+\lambda B$; in accordance with our conventions these last may equally be written $B, A, B+\mu^{-1}A, B+\lambda^{-1}A$. The symbol associated with these four, by the rule suggested, would be $\mu^{-1}\lambda$. The equivalence of the symbols $\lambda\mu^{-1}$ and $\mu^{-1}\lambda$ involves however $\mu\lambda=\lambda\mu$. We have shewn that, unless λ, μ be iterative symbols, that is (p. 86 above) unless the points $A+\lambda B$, $A+\mu B$ are limited to such as are obtainable, from the points A, B and another point of the line taken to be $A+B$, by a finite number of harmonic constructions, the equation $\mu\lambda=\lambda\mu$ in general requires Pappus' theorem.

It is very interesting however to notice that von Staudt recognises the possibility that the symbols which he introduces may not be commutative in multiplication. He gives a proof that the set of four points by which he defines the product of two of his symbols, each defined by a set of four points, is independent of the order in which these two sets are taken (*loc. cit.*, § 20, No. 268, p. 171).

The relation of Pappus' theorem with commutativity of multiplication, under certain hypotheses, is considered by F. Schur, *Analy. Geom.*, *Introd.*, 1898, and by Hilbert, *Grundlagen der Geometrie*, 1899, § 31, p. 71.

CHAPTER II

REAL GEOMETRY

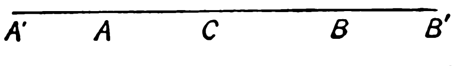
SECTION I. THE PROPOSITIONS OF INCIDENCE. INTRODUCTION OF A PLANE, AND OF A SPACE

Preliminary remarks. The point of view to which we have sought to guide the reader by the preceding discussion is that which we finally adopt as basis of the theory; it appears to possess a simplicity which justifies its being taken first. But it is general, and therefore abstract; as, for instance, in its use of the word *line* in such a sense that every two lines of a plane intersect one another. And it may appear to be artificial; as, for instance, in its adoption of Pappus' theorem. Moreover, it gives no recognition to at least two notions which are, probably, inseparable from any conception of space founded directly on experience. One of these notions is, that space consists of a limited part which is *accessible*, surrounded by an unlimited part which is inaccessible. The other notion, intimately connected with the former, is, that, when a point is given upon a line, there is thereby effected a distinction, between the points of the line on one side of the given point, and those on the other; or, when a line is given in a given plane, there is thereby effected a distinction between the points of the plane, according as they lie on one side, or the other, of the given line; or, when a plane is given in space, a similar separation is thereby made. This notion, applied to the points of a line, leads to the further notion, of the *order* of a set of points on the line. It will be found that the development of this notion of order suggests a view of what points are possible upon a line which, if it were adopted, would make Pappus' theorem inevitable. Though we regard this view as, finally, inadequate, we follow the usual procedure, of allowing the suggestion of a simpler case to become a starting point of the theory of the more general case.

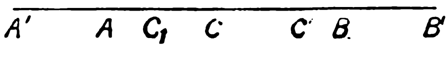
We proceed, then, to examine the fundamental conceptions in a more concrete way than has hitherto been done. Logically, this examination should precede the foregoing formulation; but it will readily be seen that the examination, in proportion as it is complete, is prolix. Moreover, it will appear that the real geometry, which is built up on the basis of this examination, may be regarded as included in the geometry so far dealt with.

Frankly recognising, now, the possibility of inaccessible points, we can assume no general proposition that every two lines in a plane intersect one another, or that any two planes intersect one another; and, requiring, now, that all constructions made shall utilise only accessible points, we need assumptions as to what points are accessible when others are known to be so. These we obtain, for points of a line, by recognising the separation effected by the assignment of a point of the line; this leads to the notion of the points of the line which are *between* two given points, all of which are regarded as being equally accessible. A similar notion dominates our recognition of the points of a plane which are to be regarded as accessible when three points are given as being so; and similarly for the points of (three dimensioned) space when four points are given.

Betweenness for points of a line. Segments. We assume that when two points, A and B, are given, there exists a point, C, which is *between* A and B; and then, also, a point, B', such that B is between A and B'; and also a point, A', such that A is between A' and B.



Thus, between C and B there exists a point, C'; and between A and C there exists a point, C_1.



We assume that the point, C', between C and B, is between A and B, and the point, C_1, between A and C, is equally between A and B. And we assume, conversely, that every point between A and B, not coinciding with C, is either between A and C or between C and B. Also that B is not between A and C, nor A between C and B. Thence it follows that the point C, not being between C' and B, is between A and C'; and C is, similarly, between C_1 and B.

There is then, for example, a point between C and C', and this point is between C and B, and, therefore, is between A and B. This argument can be continued. There is thus an infinite aggregate of points between A and B; and every point between A and B, not coinciding with C_1, or C, or C', is between A and C_1, or between C_1 and C, or between C and C', or between C' and B.

Again, as B is between A and B', and not between A and C, it is between C and B'. In this statement, C may be any point between A and B. Similarly the point A is between A' and any point which is between A and B.

It appears then, similarly, that there is an infinite aggregate of points B', and an infinite aggregate of points A'.

And, as B is between A and B', the point A is not between B and B'; thus a point A' cannot be the same as a point B'.

Any point, P, between A' and B', and not between A and B, if not coincident with A or with B, is then between A' and A or between B and B'. But there exist points which are not between A' and B', in infinite number, namely points, Q', such that B' is between A' and Q', and also points, P', such that A' is between P' and B'.

We regard the aggregate, of all the points spoken of, as being determined when the points A and B are given. We assume that the same aggregate is determined when any two points of this aggregate are the given points.

When the points A, B are given, and the point C is between them, and B' is a point such that B is between C and B', we say that the points $ACBB'$ are *in order*. If C' is a point between C and B, it is also between A and B, and B is between C' and B', as we have remarked. Thus $AC'BB'$ are in order. These orders are regarded as identical. Again C is between A and C', as we have said, and C' between C and B. Thus $ACC'B$ are in order. This order we again regard as identical with the former. Similarly, A' being, as before, a point such that A is between A' and C, the points $A'ACB$ are in order; and we regard this as identical with the preceding. In this order we then say that B is subsequent to, or follows A, or is superior to A; and that A is anterior to, or precedes B, or is inferior to B. And for greater clearness, we sometimes write this in the form $A < B$. Thus B' follows B, and A' precedes A. Similarly C' follows C, but precedes B.

The order described is determined by the points A, C, B. A similar account may be given of an order determined by the three points B, C, A; this we regard as inverse to the former, speaking of A as, in this order, *following* B, and of A' as following A, of B' as preceding B, and so on.

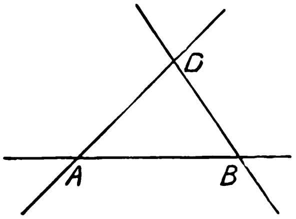
The aggregate of points C between A and B, together with A and B, we speak of as lying upon the *segment* AB, of which A, B are the end points. The complete aggregate of points determined by A and B, that is, of points C, and of points B' which follow B, and of points A' which precede A, we speak of as lying upon a *line*.

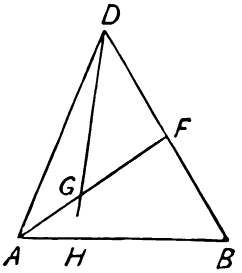
A more minute disentanglement of the ideas involved, than in the foregoing, may be found in the authorities quoted in the Bibliography at the end of this volume.

Points determined by three points. Intersections of lines determined therewith. We now suppose that, beside the points of the line determined by the points A, B, there is given a point D. Each of the three pairs of points, A and B, A and D, B and D, will then be the end points of a segment, and each of these three segments will be part of a line. The point A will not be on the line DB, nor B on DA.

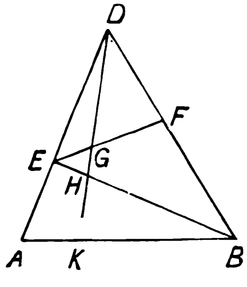
An important step in the argument is then the following: let F

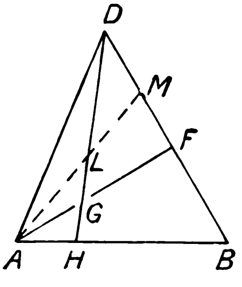
be a point of the segment DB, and G a point of the segment AF: we *assume that there is a point, H, of the segment AB, such that G is a point of the segment DH.* In particular when F is *between* D and B, and G is between A and F, then H is between A and B. For clearness of reference we may often refer to this assumption as Peano's axiom.





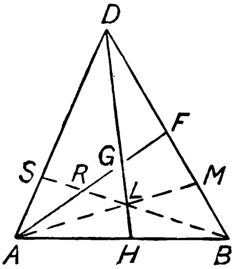
From this it follows that, if F be between D and B, and E be between D and A, and G be taken between E and F, then there is a point K between A and B such that G is between D and K. For, applying the axiom to D, E, B, we infer a point H, between E and B, such that G is between D and H; and then, considering D, A, B, from H, between E and B, we infer K, such that H is between D and K. Then as G is between D and H, it is also between D and K.



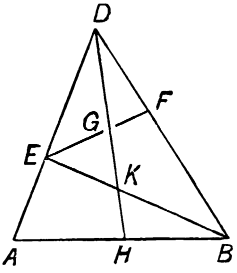


And it also follows that, if F be between D and B, and H be between A and B, there is a point G which is between A and F and is also between D and H. For, if L be a point between D and H, it follows, from the axiom, that there is a point, M, between D and B, such that L is between A and M. If M be at F we may take for G the point L. If M be not at F, it is between D and F, or between F and B. Suppose, first, that M is between D and F; then the existence of L between A and M, involves, by the axiom, the existence of a point G_1, between A and F, such that L is between D and G_1; and the existence of G_1 between A and F again involves a point H_1 between A and B, such that G_1 is between D and H_1. Then L, between D and G_1, is between D and H_1. But L

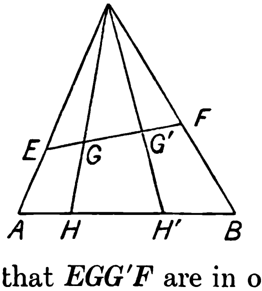
is also between D and H. Thus either H and H_1 coincide, or there is a line determined by H and H_1 which contains both L and D; in the latter case D would be on the line AB. Therefore H_1 coincides with H, and thence G_1, between D and H, and between A and F, is the point G required. Next, suppose that M is between F and B; then, considering A, B, F, the point L, between A and M, involves, as before, a point R, between A and F, such that L is between B and R; and, considering D, A, B, the point R, between A and F, involves S, between A and D, such that R is between B and S; while, as L is between B and R, the point R cannot be between L and B, nor coincide with L, and is between S and L. Then, from A, L, D, the point R, between S and L, involves a point G_1, between D and L, with R between A and G_1. However, R was between A and F. Thus, considering the line determined by A and R, if we assume that G_1 is between A and F, as it is between D and L and therefore between D and H, it is such a point G as was required. And, indeed, considering D, A, M, the point G_1, between D and L, involves a point, F_1, between D and M, such that G_1 is between A and F_1; then F and F_1 are both on the line AR, and, as A is not on the line DB, the point F_1 coincides with F, and so G_1 is between A and F.



Whence, further, if E be between D and A, and F be between D and B, and H be between A and B, there is a point G between E and F which is also between D and H. For, from D, A, B, there follows K, between E and B and between D and H. And then, from D, E, B, from K, between E and B, and F, between D and B, there follows G, between D and K and between E and F. And this G, between D and K, is between D and H.

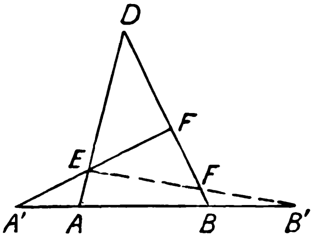


Also, still with E given between D and A, and F given between D and B, if G, G' be between E and F, with G between E and G' and G' between G and F, so that $EGG'F$ are in order, and if H, H' be between A and B such that G is between D and H and G' is between D and H', then also the points $AHH'B$ are in order. For, from D, A, H', as G is between E and G', so also is H between A and H'; while, similarly, from DHB, the point H' is between H and B. And conversely, given $AHH'B$ in order, we can infer that $EGG'F$ are in order.

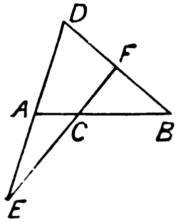


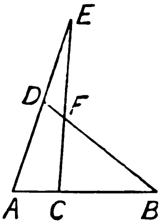
The statement that when H is between A and B then G is between D and H is the same as that, unless E be at A or F be at B, there is no point of the segment EF which is also a point of the segment AB. Consider now, taking E on the line DA, and F on the line DB, the question whether there are points of the *line* EF which are also points of the *line* AB.

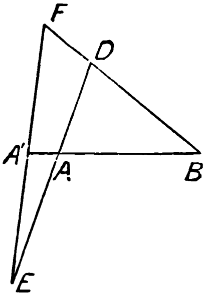
If a point A' be such that A is between A' and B, it follows, by Peano's axiom, considering D, A', B, that, when E is between D and A, there is F, between D and B, such that E is between A' and F; and, also, from what we have seen, if B' be such that B is between A and B', there is an F', between E and B', which is between D and B. Thus, when E, F are, respectively, between D and A and between D and B, while there is no point of the segment EF which is a point of the segment AB, there *may* be a point, A', or B', of the *line* AB which is a point of the *line* EF.

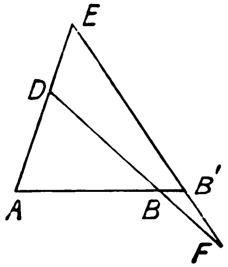


We may examine the various combinations of the possible positions of E on the line DA, with those of F on DB, putting together those in which such a common point of the two lines AB, EF necessarily follows from Peano's axiom, and then those for which this does not follow. There are evidently nine possible combinations.



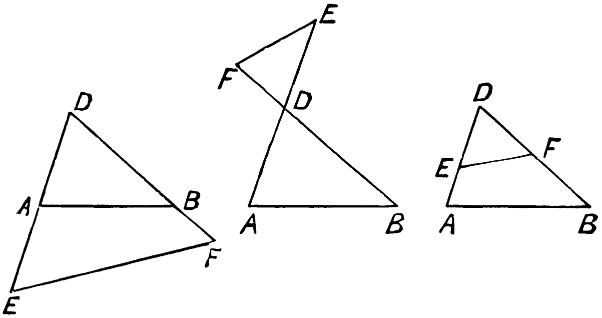






When F is between D and B and E not between D and A, whether A be between D and E, or D be between A and E, there is necessarily a point, C, of the line EF which is a point between A and B. And similarly when E is between D and A. This accounts for four combinations. When F is not between D and B, it may be such that D is between F and B; then if E be such that A is between E and D, there is a point A' between E and F such that A is between A' and B; while, if F be such that B is between D and F, and E be such that D is between A and E, there is a point B'

between E and F such that B is between A and B'. This accounts for two other combinations. For the three remaining combinations no common point of the two lines is certain, the last of these being the case considered above. In these three cases, E, F are either both interior points of the segments DA, DB, or neither is an interior point. In none of these three cases can there be a common point of the segments AB and EF, as we easily prove from the axiom.



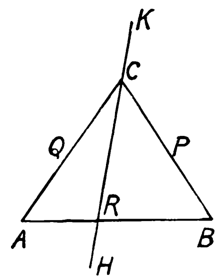
In any case, the *total number of points of the line EF which lie within one of the three segments AB, DA, DB, is zero or two.*

Example. If E, F be points such that D is between A and E, and D is between B and F, and G be any point between E and F, prove that there is a point C between A and B such that D is between C and G.

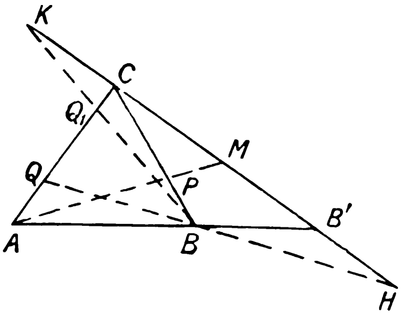
Considering E, F, B, there follows a point, L, of DE, between B and G, and, then, considering A, B, G, there follows C.

Points of a plane system. Let A, B, C be three given points, not belonging to one line. A line is determined by the point A and any point, P, of the *segment* BC; so, a line is determined by the point B and any point, Q, of the segment CA, and a line is determined by the point C and any point, R, of the segment AB. The system of points lying upon the aggregate of all such lines, of the three kinds, is called a plane system, and the points are said to lie in a plane. All points of the lines BC, CA, AB belong to this system. The system contains points lying upon a line of all the three kinds, AP, BQ and CR; but, also, it contains points lying upon a line of one kind only.

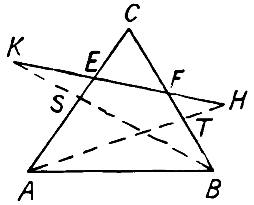
Consider a line of the kind CR, R being between A and B, and, thereon, points, H and K, such that R is between H and C, and C is between R and K. A point between C and R is equally upon a line AP, and upon a line BQ, by what we have shewn; and, therefore, neither H nor K is upon



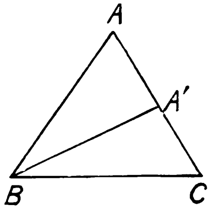
a line of the kind AP or the kind BQ. If B' be a point of the line AB such that B is between A and B', a point, M, between C and B' is on a line AP; a point, H, such that B' is between C and H, is upon a line BQ, as also is a point, K, such that C is between K and B'. A similar remark can be made, starting from a point A' such that A is between A' and B. The plane system thus contains the points of all lines determined by C and a point of the *line* AB; but to say that the system consisted only of such points would be to assume that every line, in the plane, drawn through C, had a point of intersection with the line AB.



Let E be a point between C and A, and F a point between C and B; and let H, K be points of the line EF, the point F being between E and H and the point E between K and F. It is clear from what has been said that a point of this line between E and F lies on a line CR, where R is between A and B. Consider C, A, H; as E is between C and A, and F is between E and H, there is a point, T, between A and H, such that F is between C and T; thus H lies on a line through A containing a point of the line CB. We remarked that the plane system contains all points of a line determined by C and a point of the line AB. By similar reasoning it contains all points of a line determined by A and a point of the line CB. Thus H is a point of the system. Again consider C, K, B; as F is between C and B, and E is between K and F, there is a point, S, between K and B, such that E is between C and S; thus K lies on a line through B containing a point of the line CA. It is thus shewn that every point of the line EF belongs to the plane system. Therefore a line, containing points of any two of the segments BC, CA, AB, belongs to the plane.



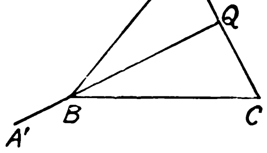
Now let A', B', C' be any three points of the plane system not lying on one line. We shew that these may be used, instead of A, B, C, to determine the plane system. For this it is sufficient to prove that if H be a point such that at least one of the three possibilities, that the line HA' contains a point of the segment $B'C'$, or the line HB' contains a point of the segment $C'A'$, or the line HC' contains a point of the segment $A'B'$, is a fact, then at least one of the corresponding possibilities, for H in regard to A, B, C, is also a fact, and to prove that the converse is also true. This proof will follow if we shew that, when A' is any point of the plane system (A, B, C), the plane systems, (A', B, C) and (A, B, C), are the same.

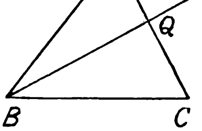


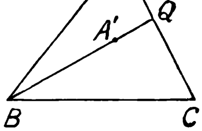
First, if A' be between A and C, then a point, H, for which the line HA' contains a point between B and C, is on a line containing points of two of the segments AC, CB, BA; so that such point H belongs to the plane system (A, B, C), by what was proved above. The same follows if the line HA' contains B, or C. If the line HB contains a point of the segment $A'C$, it contains a point of the segment AC; and if the line HC contains a point of the segment BA', it contains a point of the segment AB; in either case it belongs to the plane system (A, B, C). Conversely, if the line HA contains a point of the segment BC, other than B, it contains a point of the segment BA', and so belongs to the plane system (A', B, C), by what was proved above; and the same follows when H is on the line AB. Again, if the line HB contains a point of the segment CA, it contains a point of the line CA', and belongs to the system (A', B, C); while, if the line HC contains a point of the segment AB, it contains a point of the segment $A'B$, and then, also, belongs to (A', B, C). A point between any two of the points A, B, C may then be used instead of either of these to determine the plane system. The plane systems so obtained are the same. But from this it follows that any one of the points, A, may be replaced by any point except B and C on either of the *lines*, AB, AC, containing this point.

Next, let A' be any point of the plane system (A, B, C), not lying on the line BC: then, either

(*a*) the line BA' contains a point of the segment CA, say the

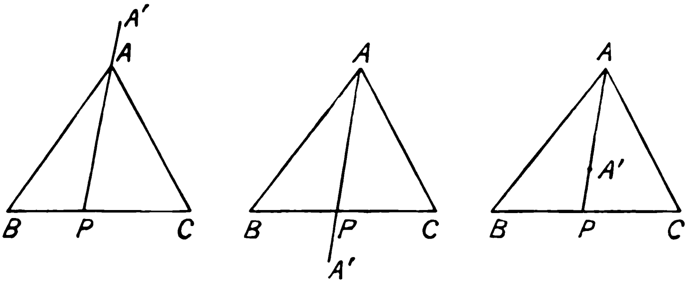






point Q; then the plane system (A', B, C) is the same as (Q, B, C) —each being the same as (Q, A', C)—and (Q, B, C) is the same as (A, B, C), or

(*b*) the line CA' contains a point of the segment BA, in which case a similar argument shews that the systems, (A', B, C), (A, B, C), are the same, or, lastly, (*c*) the line AA' contains a point of the segment BC, say the



point P; then the system (A', B, C) is the same as (A', B, P), and this as (A, B, P), and this as (A, B, C).

It is thus established, whatever be the point A' of the plane system (A, B, C), not lying on the line BC, that the system (A', B, C) is the same as (A, B, C). Hence it is easy to deduce that the system (A, B, C) may be defined from any three points (A', B', C') of itself which are not in line.

An incidental consequence is that all the points of the line determined by any two points of a plane system lie also in the system.

Separation of the points of a plane effected by a line therein. Let us say that two points of a plane system belong to the same class when the segment of which they are the end points contains no point of a certain line lying in the plane, but that two points belong to different classes in the contrary case. Then the results we have developed shew that every point of the plane belongs to one of two classes, save that the points of the chosen line are exceptional. For let A, B be any two chosen points of different class, the segment AB containing a point of the line; let C be any other point of the system not lying on the chosen line: we have shewn that, of points of the three segments AB, BC, AC, there are two or none lying on a line of the plane: the chosen line therefore contains a point of the segment AC, or a point of the segment BC. The point C thus belongs either to the same class as A or to the same class as B. Moreover, in this description, A may be replaced by any point of its own class, and B by any point of its class. For if A' and C be of the same class as A, the line not containing a point of either segment AA' or AC, then the line does not contain a point of the segment $A'C$, and C belongs to the same class as A'. Similarly if B' be of the same class as B, but C of a different class from B, then C, B' are of different class.

As soon as we are in a position to shew that we can do so in a self-consistent manner, we shall discard the distinction between the accessible and the inaccessible points of a plane, upon which our theory at present rests; and then the separation of the points of a plane, effected by a line, will disappear, unless reintroduced by a convention as to which points are to be regarded as accessible. But for the present this separation is most important. If it were introduced at the start, as an axiom, and three lines were taken in the plane, not having a common point; and the two classes of points of the plane with reference to one line were denoted, say, by α and α', the two classes with reference to another of these lines by β and β', and the two classes with reference to the remaining line by γ and γ', then the points of the plane, not lying on any of the lines, would be each of one of seven categories (α, β, γ), (α', β, γ), (α, β', γ), (α, β, γ'), $(\alpha, \beta', \gamma')$, $(\alpha', \beta, \gamma')$, $(\alpha', \beta', \gamma)$. Peano's axiom would then arise in a very natural way.

Example. Given two lines, AB, AC, with a common point, A, it may be proposed to define a plane system as the aggregate of points through which a line can be drawn having a point, other than A, common with each of the two given lines. Examine whether this system is coextensive with that, adopted above, defined by the points A, B, C.

Space systems, defined by four points not lying in a plane. As a preliminary remark, before proceeding to a formal definition, suppose that we have a system of points, not lying in a plane, of such character and multiplicity that it contains an infinite number of plane systems, while those of the system not lying on a specified plane are, by means of this plane, divided into two classes. Then, if A, B, C, D be four points of the system not lying in a plane, the plane determined by A, B, C distinguishes two classes, say δ and δ'; similarly the plane BCD distinguishes two classes, α and α'; the plane CAD two classes, β and β'; and the plane ABD two classes, γ and γ'. The points of the system are then, by these four planes, distinguished into $2^4 - 1$, or fifteen, categories, such as $(\alpha, \beta, \gamma, \delta)$, or $(\alpha', \beta, \gamma, \delta)$, etc., exception made of points on one of the four planes. Suppose further that two points of the system are of the same class, with respect to one of the planes, or of different class, according as the segment of which these are the end points does not, or does, contain a point of this plane, it being understood that, with reference to the plane ABC, the point D is of class δ, while A is of class α with reference to the plane BCD, and B of class β with reference to the plane CAD, and similarly for C. Then, if a point of the system, P, be of the category $(\alpha, \beta, \gamma, \delta')$, the segment PD, whose end points are respectively of classes δ' and δ, contains a point of the plane ABC; this will, however, be of category (α, β, γ) with reference to the other three planes, this being so both for P and D. A point P of the category $(\alpha, \beta, \gamma, \delta')$ may therefore be said to be such that the line PD contains a point of the *face* ABC; this statement is then equally true for a point of the category (α', β', γ', δ). But if a point, P, be of either of the categories (α, β', γ', δ) or (α', β, γ, δ'), then a line can be drawn through P to contain both a point of the segment BC and a point of the segment AD. And similarly in other cases. We therefore say:

Let A, B, C, D be four points not lying in a plane; let the points, Q, of the plane ABC which are such that, simultaneously, the line AQ contains a point of the segment BC, the line BQ contains a point of the segment CA and the line CQ contains a point of the segment AB, be said to belong to the *face* ABC; with a similar definition of the faces BCD, CAD, ABD. Then a (threefold) space system, or, briefly, a space, is the aggregate of all possible points, P, which are such that, either the line PD contains a point of the *face* ABC, or the line PA of the face BCD, or PB of CAD, or PC of ABD, or all of these, together with points, P, such that a line is possible containing P which contains a point of each of the segments BC and AD, or a line, containing P, which contains a point of each of the segments CA and BD, or a line, containing P, which contains a point of each of the segments AB and CD. To this space belong all the points of the four planes ABC, BCD, CAD, ABD.

In a manner analogous to that employed for plane systems it can then be deduced

(*a*) That, if A', B', C', D' be any four points of the space system which do not lie in a plane, the system can equally be determined by these. In particular the points of the line determined by any two points of the system, or the points of the plane determined by any three points of the system not lying in line, all belong to the space system.

(*b*) That, when a plane of the space is given, the points of the space system are divided into two classes, two points, E, F, being of the same or of different classes, according as there is no point of the segment EF, or there is a point, which belongs to the plane.

(*c*) That any two planes of the space which have a point in common have all the points of a line in common.

(*d*) That, if P, Q, R be three points of the space system which are not in line, there are none, or two, points of any plane, belonging to the space, which are points of the segments QR, RP, PQ.

Of these, (*a*) is proved by first proving that D can be replaced by any point on any of the lines DA, DB, DC, and continuing this process; then (*b*), from the definition of the space system, is seen to be true for any one of the planes ABC, BCD, CAD, ABD; after (*a*) it is then true for any plane. For (*c*), if α, β denote the two planes, and O their common point, take two points P, P' in the plane α such that O is between them. As O belongs to the plane β, the points P, P' are of different class with respect to β. Now let

Q be another point of the plane α, not lying on β; of the two points P, P', let P be that one which is of the same class as Q with respect to the plane β, so that P', Q are of different class. The segment $P'Q$ thus contains a point of the plane β, say O'; this point, lying on the segment $P'Q$, is also on the plane α. Thus both the planes contain the two points O and O', and the line determined by these is, therefore, common to the two planes. The statement (d) then follows by remarking that, in virtue of (c), if a plane contain a point of one of the segments QR, RP, PQ, it meets the plane PQR in a line. For we have shewn that in a plane (PQR) a line contains none, or two, points of the segments QR, RP, PQ.

Extension to fourfold and higher space. If now it be allowed that five points A, B, C, D, E can exist, of which A, B, C, D are not in a plane, such that E does not belong to the threefold space system determined by A, B, C, D, we can regard these five points as determining a fourfold space, specifying, in a manner analogous to that followed above for twofold and for threefold space, what points are to be regarded as belonging thereto. Such a space will contain lines, and planes, and also threefold space systems. It will not be true, in such a fourfold system, to take one illustrative example, that two planes which have one point in common, necessarily have in common the points of a line; for it will now be by a threefold space, and not by a plane, that the points of the fourfold system are separated into two classes. But a similar proof to that followed above will shew that two threefold spaces of the fourfold system which have a point in common, have also a plane in common; let α, β denote the threefold spaces, O the assumed common point, and P, P' two points of α having O between them; as before a point Q is possible in α such that the segment QP' contains a point, O', of β which also lies in α, so that the line OO' is common to α and β. But now, beside Q, a point R is possible, in α, which does not lie in the plane QPO; and this gives a further point, O'', common to α and β, which does not belong to the line OO'. The points O, O', O'' determine a plane common to α and β.

We regard such a fourfold space, or spaces of higher multiplicity, as entirely possible; and experience has shewn the great usefulness and interest of taking them into consideration. It seems, however, unnecessary, at this stage, to enter into further detail.

SECTION II. THE GENERALISATION OF THE REAL GEOMETRY BY THE INTRODUCTION OF POSTULATED POINTS

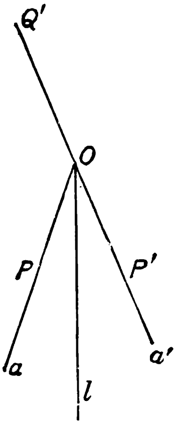
Purpose of the Section. In comparison with the geometry of Chapter I, the Real Geometry we have dealt with in the preceding Section is inconvenient to apply, owing to such circumstances as that two lines in a plane may not have a common point, and two planes in a threefold space may not have a common line. The common point of two lines, in Chapter I, is, however, only of use because, as a point, it can serve as one of two points determining a line; in other words, the proposition of Chapter I is that, if two lines be given in a plane, there is thereby determined through an arbitrary point which does not lie on either of them, a definite line having no common point with either of them, save perhaps where they have a common point. Now, in this form, the proposition is equally true for the Real Geometry. When this is made clear it will be convenient to speak of the two given lines as having in common a so-called *postulated point*, leaving open the question whether this point is accessible or not. Similar remarks apply to the other cases in which the Real Geometry appears incomplete in comparison with that dealt with in Chapter I. The result of the whole examination now to be made is that, we may regard the space of the Real Geometry as part of a space in which the Propositions of Incidence formulated in Chapter I are completely valid, provided that postulated points, lines and planes are allowed an existence, whether they be accessible or not.

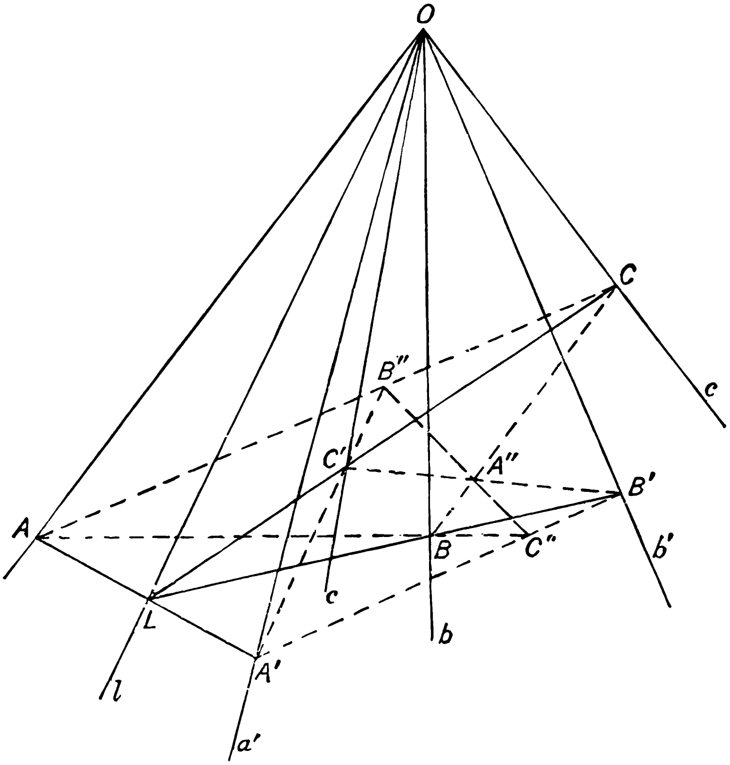
Extended form of Desargues' Theorem, for lines and planes having a common point. Let O be a point, and a, b, c, a', b', c' be six lines passing through this point, no three of which lie in a plane. The pairs, a, a', and b, b', and c, c', of these lines, will determine three planes containing the point O; any two of these planes will, therefore, have a line in common, passing through O. Suppose the three lines so arising coincide with one another, so that the three planes all contain one line, l, passing through O. Consider six other planes determined by the given lines, of which two are the planes containing respectively the pairs b and c, and b' and c'; these two planes, which we may respectively call $[b, c]$ and $[b', c']$, meet in a line through O. There are then two other lines through O, given, respectively, by the planes $[c, a]$, $[c', a']$ and $[a, b]$, $[a', b']$. With the hypothesis made, that the three planes $[a, a']$, $[b, b']$, $[c, c']$ have a line, l, in common, we prove that the three lines which are the intersections of the respective pairs of planes

$$[b, c], [b', c']; \quad [c, a], [c', a']; \quad [a, b], [a', b'],$$

lie in one plane. This plane will pass through O.

To prove this, remark, first, that if three lines a, a', l in a plane have a point O in common, and points P', Q' be taken on a' so that O is between P' and Q', the line l must have a point in common either with PP' or PQ'. For, as we saw in Section I, the line l has none or two points, in all, lying on the segments PP', $P'Q'$, $Q'P$; and it has the point O on the segment $P'Q'$. It is thus possible to take P' and P (or Q' and P), respectively on a' and a, so as to lie on opposite sides of the line l.





It follows that, in our figure, points A, A' can be taken, respectively on the lines a and a', such that there is a point, L, of the line l, lying between them. By a similar argument, a point, C, can be taken, on the line c, so that there is a point, C', of the line c', lying between L and C. Then it follows, from Peano's axiom, that, in the plane $AA'C$, there is a point, B'', lying between A and C, such that C' is between A' and B''. Again, a point, B', can be taken, on the line b', so that there is a point, B, of the line b, lying between L and B'; then it follows, from Peano's axiom, that, in the plane $AB'A'$, there is a point, C'', between A' and B', such that B is between A and C''. It follows further, by the axiom, that, in the plane CLB', there is a point, A'', which is between C' and B' and also between C and B.

It is at once seen that the points A'', B'', C'' are upon the plane ABC and upon the plane $A'B'C'$. Thus A'', B'', C'' are in line. But the line OA'' is the line of intersection of the planes $[b, c]$ and $[b', c']$; similarly OB'' is the line of intersection of the planes $[c, a]$ and $[c', a']$, and OC'' is the line of intersection of the planes $[a, b]$ and $[a', b']$. These three lines of intersection are therefore in one plane; as we desired to prove.

Consider now the converse proposition. As before consider six lines a, b, c, a', b', c', passing through the point O, of which no three are in one plane; suppose that the lines of intersection of the respective pairs of planes, $[b, c]$ and $[b', c']$, $[c, a]$ and $[c', a']$, $[a, b]$ and $[a', b']$, are in one plane. It can be shewn that the planes $[a, a']$, $[b, b']$, $[c, c']$ have a line in common.

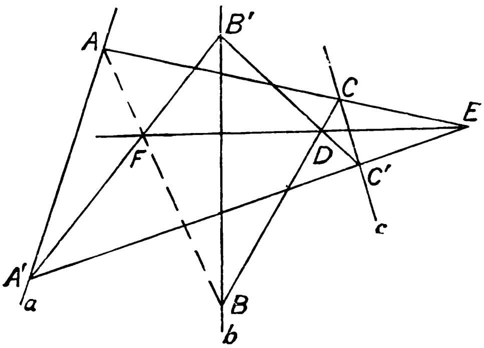
For, let the planes $[a, a']$, $[b, b']$, which meet in O, meet in the line l. Let the line of intersection of the plane $[l, c]$ with the plane $[a', c']$ be the line c_1, passing through O; so that the planes $[a, a']$, $[b, b']$, $[c, c_1]$ have the line l in common. Then, by the preceding, it follows that the lines of intersection of the three pairs of planes $[b, c]$ and $[b', c_1]$, $[c, a]$ and $[c_1, a']$, $[a, b]$ and $[a', b']$, lie in one plane. By construction, however, the plane $[c_1, a']$ is the same as the plane $[c', a']$; thus, of the three lines which are in one plane, two are the intersections, respectively, of $[c, a]$ and $[c', a']$ and of $[a, b]$ and $[a', b']$; and it was to be supposed that the plane of these contains the line of intersection of the planes $[b, c]$ and $[b', c']$. The line of intersection of the plane of these, with the plane $[b, c]$, thus lies in both the planes $[b', c_1]$ and $[b', c']$; and, therefore, these last planes, both containing the line b', coincide with one another. Wherefore the lines c_1, c' are the same, or the plane containing them contains the line b'. By construction, however, the plane of c_1 and c' contains the line a'; and, it was assumed that the lines a', b', c' are not in one plane. Thus it follows that the lines c_1 and c' coincide; and, therefore, the planes $[a, a']$, $[b, b']$, $[c, c']$ meet in a line, as was to be proved.

For the proposition here given, and its application, we may refer to Reyes-y-Prosper, "Sur les propriétés graphiques des figures centriques," *Math. Annal.* XXXII, 1888, 157; Pasch, *Math. Annal.* XXXII, 1888, 159; F. Schur, *Grundlagen der Geometrie*, 1909, § 2.

Fundamental theorem in regard to the correspondence of the lines and planes of two central figures, or stars. A figure consisting of lines and planes passing through a point, such as that considered above, may be called a *central figure*, or a *star*; the point itself being called the centre. Suppose now we have two centres, O_1 and O_2. Let α be a plane, not passing through O_1 or O_2.

To every line through the point O_1 which has a point in common with the plane α, we can evidently make correspond a line through the point O_2, namely that which meets the plane α in the same point as does the former; and either of these lines determines the other. To establish such a correspondence it is not necessary, however, to assume an actual, or accessible, intersection of the former line with the plane α. We can shew that, *to every line through the centre* O_1, *there corresponds a definite line through the centre* O_2, *and conversely, without utilising an actual, or accessible, point common to the plane* α *and the lines. And this in such a way that to three lines through* O_1 *which lie in a plane, there shall correspond three lines through* O_2 *which also lie in a plane—these planes coinciding when the former plane, through* O_1, *passes through* O_2.

First consider the statement in regard to single lines through O_1 and O_2. Let l_1 be any line through O_1; we make no use of any intersection it may have with the plane α. Upon the plane α take three arbitrary points; through the line l_1 pass three planes, each containing one of these points; any one of these planes, having a point common with the plane α, will have a line of intersection with it. Denote these lines by a, b, c, respectively. If the line l_1 has an actual point of intersection with the plane α, each of the lines a, b, c will pass through this point; conversely a point, common to two of the lines a, b, c, lies upon two planes passing through the line l_1, and therefore lies upon l_1, as well as upon the plane α. Thus, to refrain from using any possible intersection of the line l_1 with the plane α, we must avoid the assumption of an accessible intersection of any two of the lines a, b, c. On the line a, take two arbitrary points, A and A'. By what has been seen above in regard to the separation of the points of a plane effected by a line of that plane, it is possible to take a point, E, of the plane α, such that each of the segments EA and EA' contains a point of the line c. Say these points are, respectively, C and C'. On the line b take an arbitrary point, B; it is then possible to take a further point, B', upon the line b, such that there is a point, D, of the segment BC lying between B' and C'. Then, considering the points A', C', B', we see that there is a point F between A' and B', such that D is between F and E.



We can now prove, by the theorem of the foregoing article, that the points A, F, B are in line. For, let the lines joining the point O_1, respectively to the points A, B, C, A', B', C', be called $a_1, b_1, c_1, a_1', b_1', c_1'$; so that the planes $[a_1, a_1']$, $[b_1, b_1']$, $[c_1, c_1']$ are, respectively, the planes through O_1 containing the lines a, b, c. By construction these planes have a line in common, the line l_1. Therefore, by the theorem referred to, the lines of intersection of the three pairs of planes

$$[b_1, c_1] \text{ and } [b_1', c_1'],\quad [c_1, a_1] \text{ and } [c_1', a_1'],\quad [a_1, b_1] \text{ and } [a_1', b_1']$$

lie all in one plane; of these lines, however, the two former are respectively the lines O_1D and O_1E; and the last line meets the plane α in a point lying upon both the lines AB and $A'B'$. Thus the point F must lie on the line AB.

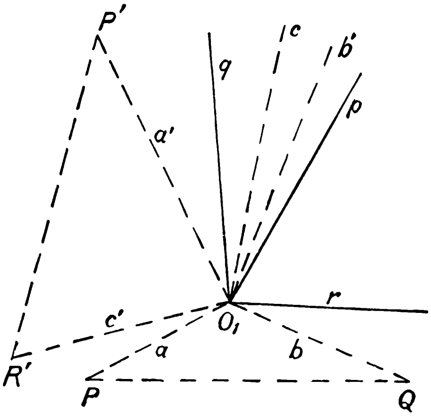
By the converse part of the theorem of the foregoing article, we can thence infer that the planes through O_2 which contain, respectively, the lines a, b, c, also meet in a line, which we shall call l_2. For, if the lines joining the point O_2 to the points A, B, C, A', B', C' be, respectively, called $a_2, b_2, c_2, a_2', b_2', c_2'$, it follows from the fact that the lines which are the intersections of the three pairs of planes

$$[b_2, c_2] \text{ and } [b_2', c_2'],\quad [c_2, a_2] \text{ and } [c_2', a_2'],\quad [a_2, b_2] \text{ and } [a_2', b_2'],$$

lie in a plane, namely the plane through O_2 which contains the line DEF, that the three planes $[a_2, a_2']$, $[b_2, b_2']$, $[c_2, c_2']$ meet in a line. These are the planes joining O_2 respectively to the lines a, b, c.

It is thus proved that, given the points O_1 and O_2, and a plane α not containing O_1 or O_2, if any line l_1 be drawn through O_1, and, through this, be taken three planes intersecting the plane α respectively in the actual lines a, b, c, then the planes through O_2 containing these lines a, b, c have in common another line, l_2, which will pass through O_2. There is thus determined a line through O_2 corresponding to any given line passing through O_1; and, if the line through O_2 be given, that through O_1 is determined by the same construction. The lines a, b, c were determined by planes containing l_1, drawn, respectively, through three arbitrarily taken points of the plane α; it is easy to see that the line l_2 is the same whatever these points may be. For, if two of these points remain the same, say those which lead to the lines b and c, the third may vary; the line l_2 will still be determined by the intersection of the planes joining O_2 to the lines b and c. By varying, thus, one point at a time, all can be varied. The construction given has not used any intersection of the plane α with the line l_1; when such a point is given the line l_2 is that joining O_2 to this point.

Next consider three lines, passing through O_1, which lie on a plane; we desire to prove that the lines through O_2, corresponding to these respectively, by the above rule, also lie in a plane. Let the given lines through O_1 be p, q, r. Take a point P of the plane α; the plane, Pr, containing the line r and the point P of the plane α, will meet α in a line; let Q be a point of this line other than P. Let the lines O_1P and O_1Q be called, respectively, a and b. The planes Pq and Qp, that is the planes joining the points P and Q respectively to the lines q and p, have the point O_1 in common; they therefore have a line in common, which we call c. Similarly, take another point P' of the plane α, and then, in the line in which the plane $P'q$ meets the plane α, take another point R'; denote the lines O_1P' and O_1R', respectively, by a' and c', and denote the line of intersection of the planes $P'r$ and $R'p$ by b'. Then



(a) the plane Qp contains the lines p, b and c; the plane $R'p$ contains the lines p, b' and c'; thus the planes $[b, c]$ and $[b', c']$ meet in the line p;

(b) the plane Pq contains the lines q, c and a; the plane containing q and $P'R'$ contains the lines q, c' and a'; thus the planes $[c, a]$ and $[c', a']$ meet in the line q;

(c) the plane $P'r$ contains the lines r, a' and b'; the plane containing r and PQ contains the lines r, a and b; thus the planes $[a, b]$ and $[a', b']$ meet in the line r.

It was said, however, that the lines p, q, r are in one plane. Thus, by the theorem of the last section, we are able to infer that the planes $[a, a']$, $[b, b']$, $[c, c']$ have a line in common. Call this line l_1. Further these planes have each a line common with the plane α, having, respectively, thereon, the point P (upon a), the point Q (on b) and the point R' (on c'). Whence, by what has immediately preceded, the planes joining the point O_2, respectively to these lines of the plane α, have also a line in common, say l_2. And, also, to the six lines a, b, c, a', b', c' through O_1, will correspond, respectively, lines through O_2, say a_2, b_2, c_2, a_2', b_2', c_2', there being through O_2, corresponding to the planes $[a, a']$, $[b, b']$, $[c, c']$, the planes $[a_2, a_2']$, $[b_2, b_2']$, $[c_2, c_2']$. As these meet in a line, l_2, the lines of intersection of the three pairs of planes

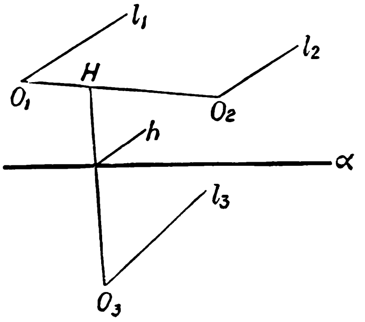
$$[b_2, c_2] \text{ and } [b_2', c_2'],\quad [c_2, a_2] \text{ and } [c_2', a_2'],\quad [a_2, b_2] \text{ and } [a_2', b_2'],$$

are three lines, through O_2, lying in a plane. These are the lines, say p_2, q_2, r_2, through O_2, which correspond, respectively, to p, q, r. We have thus proved what we desired to prove. We have not utilised any intersections of p, q, r with the plane α; if such points exist, they will lie, respectively, on p_2, q_2, r_2.

Lastly, we prove that, in this manner, to a plane, ϖ, through O_1 which contains O_2, corresponds, as plane through O_2, this plane itself. The result of course is obvious if the plane ϖ meets the plane α, any line through O_1 in the plane ϖ corresponding then to a line through O_2 in this plane meeting the plane α in the same point as the former; but, as before, we can avoid using the possible intersection of the plane ϖ with the plane α.

When there is a point of the plane α between O_1 and O_2, any plane through O_1O_2 will intersect the plane α in a line, and any line through O_1 drawn in such plane to contain a point of the plane α on this line, will correspond to a line through O_2, lying in such plane, meeting the plane α in the same point as does the line to which it corresponds. In this case the argument can be simplified; it will then be sufficient to suppose that there is no point of the plane α lying between O_1 and O_2. We can then take a point O_3 so situated that the plane α contains a point of both the segments O_1O_3 and O_2O_3; the simplifying considerations are then effective, for example, for O_2 and O_3.

To any line, l_1, drawn through O_1, we can make correspond, as above, with respect to the plane α, a line, l_2, through O_2, and, also, a line, l_3, through O_3. Take a point, H, between O_1 and O_2, so that the plane Hl_3, through H and l_3, containing a point of the plane α, meets α in a line, say h. We can then prove that the line l_2 lies in both the planes O_2l_3 and O_2h; it lies in the plane O_2l_3, because this plane meets α in a line, of which any point, joined to O_2 and O_3, gives corresponding lines, while, as we have proved, lines through O_2 lying in a plane correspond to lines through O_3 also lying in a plane; it lies in the plane O_2h, because it must lie in the plane through O_2 corresponding to the plane Hl_3 through O_3, which contains l_3, and this latter plane meets the plane α in the line h.



Now consider correspondence, not with respect to the plane α, but with respect to the plane Hl_3; it will be seen that the line through O_2 corresponding to l_1 in this way, is also l_2. For just as l_2 was shewn to lie in the plane O_2h, so it follows that l_1 lies in the plane O_1h. The line through O_2 corresponding to l_1, with respect to the plane Hl_3, must therefore lie in the plane O_2h, the line h being in the plane Hl_3; for the same reason it must be in the plane O_2l_3. These two facts shew that this line is l_2.

But then, as l_1, l_2 correspond to one another with respect to the plane Hl_3, which contains a point H between O_1 and O_2, they lie in a plane through O_1 and O_2. And, then, as l_1 is an arbitrary line through O_1, there follows what was desired, that, to any plane through O_1 containing O_1O_2, there corresponds, with respect to the plane α, as plane through O_2, the same plane.

Postulated points. Consider any two lines in a plane, of which the intersection, if it exists, is not given. With the help of the foregoing considerations, we can shew that it is possible to make constructions which are of the same effect as if the point of intersection were known to exist and were accessible and given; and this in a unique manner. This will enable us to speak of the lines as having what we may call a *postulated point* of intersection.

Let the lines be a, b, lying in the plane α. Let O be any point, not lying in this plane. The two planes Oa, Ob, with the common point O, have a line in common, passing through O, which we call l. Let c be a third line of the plane α, such that the plane Oc also contains l. We desire to find justification for saying that the lines a, b have a postulated point of intersection, and that the line c passes through this. For this to be legitimate it is sufficient, first, that the statement shall not be dependent upon the position of the point O; the preceding article shews that it is not; and, second, that if the lines a, b have an actual point of intersection, the line c shall contain this, which is obvious.

Then, if C be an arbitrary point of the plane α, the line from C to the postulated point of intersection of the lines a and b, would, by the definition, be the line of intersection, with the plane α, of the plane Cl. As the preceding article shews, the construction for this line may be made in the plane α alone; we have only to take two arbitrary points, A, A', on the line a, an arbitrary point, B, on b, then a point, B', on b such that a point, F, between A and B, is between A' and B'; then, further a point, E, so that the given point C is between A and E, so that, from the points A, B, E, there is a point, D, between B and C which is also between F and E, and there is also, from the points A', B', E, a point, C', between A' and E, such that D is between B' and C'. The line CC' is then the line required. The line which we regard as determined by the postulated point, and a point O not lying in the plane of a, b, is the line, l, found as above, from the planes Oa and Ob.

The postulated point has then, like an actual point, the capability of determining a line, when taken with another point. It is likewise sufficient, when taken with two other points, O and O', to determine a plane; for we have seen, regarding the postulated point as arising from two given lines a, b, lying in a plane α, that these, with the point O, outside this plane, give rise to a line, l, passing through O; with the point O', they also give rise to a line, l', passing through O'; and we have seen that these corresponding lines, through O and O', lie in a plane. This plane, in virtue of its construction, may be said to contain the postulated point.

If we have two pairs of lines in the plane α, namely a, b and a', b', but no actual intersection of a, b, or of a', b', we have, thereby, two postulated points. With a point O, not lying in the plane α, these two points determine a plane; for they determine, by the preceding, two lines through the point O, say the lines l and l'. The plane of these is the plane in question.

When we have three postulated points of the plane α, it may happen that the lines l, l', l'' similarly arising, one from each of these points, through a point O, not in the plane α, lie in one plane. By the preceding article it follows, that, if this is so for one position of the point O, it is so for another position. We shall then be justified in speaking of the three postulated points as being in line. This leads to a new conception, that of the *postulated line*, which is defined in the first instance by two postulated points; if the postulated points be actual, the postulated line which joins them is also actual, and subject to the construction given.

The postulated point may equally be defined by means of a plane, α, and a line, l, not lying in this plane; the line joining any given point, P, of the plane α, to the postulated point, being the intersection of the plane Pl with the plane α, this definition accords with the former. And, similarly, the postulated line joining two postulated points of the plane α, is, by the preceding, in fact determined by two planes, the plane α and another. Conversely, if two planes, say α and ϖ, be given, but no intersection of these, we can find postulated points by the join of which a postulated line is determined which we may regard as determined by the planes α and ϖ; for take on the plane α the point A, and on the plane ϖ the point O; then, an arbitrary plane through the line AO meets the plane ϖ in a line, say l, and meets the plane α in a line, say a; if B be another point of the plane α, the plane through l and B meets the plane α in another line b; these lines a, b, in the plane α, define a postulated point of this plane which is equally, in accordance with what precedes, to be regarded as being on the plane ϖ. It is also clear incidentally that, a postulated line determines, with an actual point, an actual plane.

Summing up the results so far obtained, we may then say:

1. Two actual lines in a plane have a point of intersection, actual or postulated.

2. A line containing an actual point is an actual line; for two planes having an actual point common have an actual line common. But a point lying on an actual line may quite well be a postulated point.

3. An actual line and a postulated line, in the same plane, have a point in common, which is a postulated point. If the postulated line be that joining two postulated points defined respectively by the pairs of actual lines a, b and a', b', lying all in a plane, the actual line being the line, c, of the same plane, and we take a point, O, external to this plane, the planes Oa, Ob intersect in a line l, and the planes Oa', Ob' intersect in a line l'; and the point common to the line, c, and the postulated line is that defined by c and the plane (l, l').

4. Similarly it may be seen that two postulated lines in a plane have a point common, a postulated point. More generally, from an actual point an actual line can be drawn to meet in a point, actual or postulated, each of two skew lines; for an actual plane can be drawn through the point to contain a line, actual or postulated, and two such planes through the point have an actual line in common.

5. We have seen that an actual line and an actual plane not containing this line, define a point, actual or postulated. More generally a postulated line and an actual plane define a point, unless the line lie in the plane; for, from an arbitrary point of the plane, there passes a line to each of two points taken on the line, and these two lines determine a plane meeting the given plane in a line; this line with the postulated line determine the point in question.

6. Thus, also, any three actual planes meet in a point, actual or postulated.

7. The duality between actual lines and points, in a plane, continues to hold for postulated points and lines, being a consequence of the reciprocity existing for a star; not only is it true that two lines through the centre of the star determine a plane, but, also, two planes through the centre of the star meet in a line. In three dimensions, a line and a plane determine a point, which may be a postulated point; the reciprocal theorem would be that a line and a point determine a plane. If the point, or the line, be only postulated, we may still regard them as determining a plane; but in case both the point and line are postulated elements this is a *postulated plane*. It must then be proved that the line determined by two points of a postulated plane lies entirely in that plane; and, when the notion of a postulated plane is acquired, it is to be shewn that two postulated planes determine a line, which will be a postulated line because every plane through an actual line is an actual plane; and, further, that three postulated planes determine a (postulated) point, and that a line and a plane in all cases determine a point. It seems unnecessary to develop these results in detail; the above account of the course of the argument will make its character clear.

Remark on the theory of postulated elements. A particular case of the foregoing theory is that view of Euclid's Geometry wherein two parallel lines are said to have a point in common, lying at infinity. This view is at least as old as Kepler (*Ad Vitellionem paralipomena, de Coni sectionibus*, Francofurti, 1604), who deduced it independently of notions of perspective, and Desargues (*Brouillon project.*, Paris, 1639; *Oeuvres de Desargues, par M. Poudra*, Paris, 1864; t. I, pp. 104, 243. See Ch. Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881, pp. lvi–lxxii). In this view every line has a single postulated point, and two lines are parallel when they intersect therein; if two other lines both pass through the postulated point of a line, this is the postulated point of all. In the more general theory outlined above, two lines, b, c, may determine different postulated points of a line a but intersect one another in an actual point. We arrive at such a theory, in a plane, by confining our attention to a limited portion of the plane, regarding as actual, points in this limited portion, and, as postulated, points without this. The theory involves the definite discovery that we can make constructions affecting the points exterior to the limited portion though only using *as data* the points interior to it. The limited portion of the plane which is considered must be such that the segment, whose end points are any two points of this, contains only points belonging to the limited portion; we have assumed that every point between two actual points is accessible. A point which arises by postulation may very well be actual and accessible.

Application of the theory of postulated elements. Separation. If we now allow to the postulated points, lines and planes, assuming that the theory of these has been shewn to be self-consistent, as real an existence as that which we attribute to the accessible elements, the system of real geometry which results is in accord with that considered in Chapter I, so far as the Propositions of Incidence are concerned. But care must be taken in regard to the modifications thereby introduced in the notions of a segment, and of the order of points on a line, which were fundamental in the development of the Real Geometry of the present Chapter. When postulated points are allowed an equal standing with given points, not only is it true, given a line l and a point O, that every point of this line determines a line through O, but, conversely, every line through O, in the plane Ol, determines a point of the line l; thus notions of separation, and order, for the points of the line l, can only retain their validity so far as similar notions can be established for the lines drawn through O, in this plane. Thus it appears that, without some convention as to which are inaccessible points of a line, two points of the line must be regarded as determining not one segment, but two segments, which can be distinguished from one another only by naming a point contained in the segment referred to, in addition to the two given points. Thus, if A, B be the given points, we may have either the segment ACB, or the segment ADB, where C and D are two points which we speak of as being *separated* by A and B; in this case A and B are equally separated by C and D. An *order*, among points of the line, is now determined by three points; and the orders ACB, CBA, BAC are the same, being the same as BDA, or DAB, or ABD. Thus we can speak of four points of the line as being in order. And to any order there is a reverse, or opposite. When points of a line, $A, C, B, D, E, \ldots$, are in order, the points C and D being separated by A and B, if $a, c, b, d, e, \ldots$ be, respectively, the lines joining these points to a point, O, not on the line, then these lines are also in order, and the lines c and d are separated by a and b. Thus, if these lines are met by another line, respectively in the points $A', C', B', D', E', \ldots$, then these points are also in order, and the points C' and D' are separated by A' and B'. It is important to notice, however, that *we may not speak of $A'C'B'D'$ as being the same order as $ACBD$.* For if this were justified, the distinction between the order $ACBD$ and its reverse, given by $CADB$, would persist after any number of perspectivities. We have shewn however in Chapter I (p. 25 above) that, by a succession of perspectivities, we can relate the points A, C, B, D respectively to C, A, D, B.

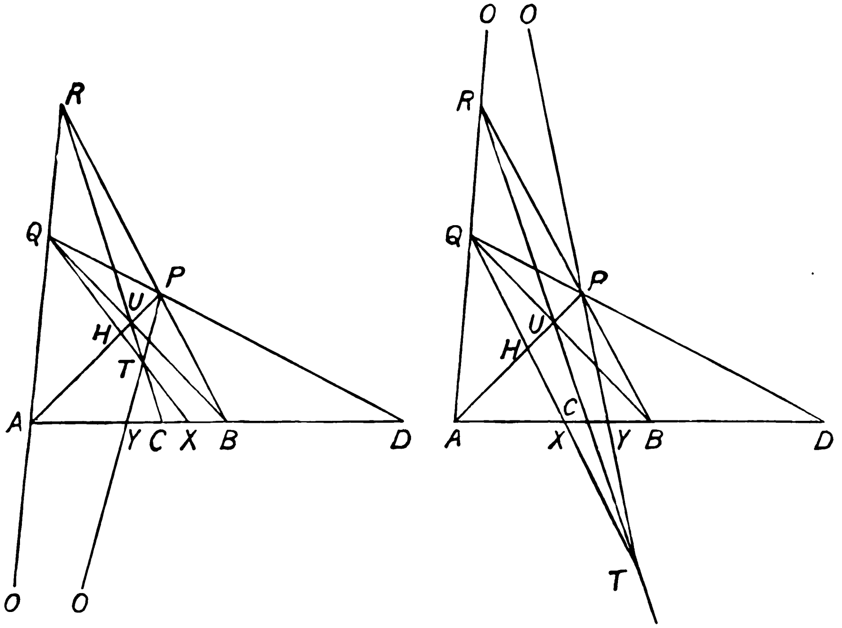
This remark shews that the ideas involved might well be submitted to a more minute analysis than we have entered upon.

The question at issue is intimately related with the fact that, if all points of a line be regarded as accessible, not only does a point of a line cease to separate the other points into two classes, but a line in a plane ceases to separate the points of the plane, and a plane in threefold space ceases to separate the points of the space, and so on. For if l be a line in a plane, and A, B be two points, not on this line, such that one of the segments AB contains a point of the line l, the other of the segments determined by A and B will not contain a point of this line; and a similar statement holds when l is replaced by a plane in space, and so on. The statement is sometimes made that, under the hypotheses adopted in Chapter I, the plane is a *onesided* surface; but an examination of this requires other explanations. (See Note III, at the end of the Volume.)

Separation for four points in harmonic relation. An important result which serves well to illustrate the ideas just discussed is that, if C, D be two points of a line which are harmonic conjugates of one another in regard to two points A, B of the line, then these points C, D are separated by A and B. (Cf. pp. 4 and 14.)

Join the points A, B to a point R; on RC take U; let AU, BU

meet *RB* and *RA*, respectively, in *P* and *Q*, and let *QP* meet *AB* in *D*.



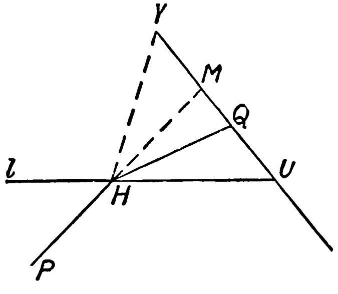
Now, on *AR* take *O*, separated from *Q* by *A* and *R*; let *OP* meet *RC* in *T* and *TQ* meet *AP* in *H*; let *TP*, *TQ* meet *AB* in *Y* and *X*, respectively.

Then, as *OAQR* are in order, so are *PAHU*, by perspective from *T*, and, thence, so are *DAXB*, by perspective of these from *Q*. Again, as *QAOR* are in order, so are *DAYB*, by perspective from *P*. Finally, as *AQRO* are in order, so are *AXCY*, by perspective from *T*.

From the orders *DAXB* and *DAYB*, wherein *D*, *A*, *B* are common and determine the order, it follows that either *DAXYB*, or *DAYXB*, are in order. Either of these contains *A*, *X*, *Y*, the former as *AXY*, the latter as *AYX*; while both *AXCY* and its reverse, *AYCX*, are in order, as we have seen. Thus either *DAXCYB*, or *DAYCXB*, are in order. In either case *C* and *D* are separated by *A* and *B* (as well as by *X* and *Y*).

This argument, which is found in F. Enriques, *Proj. Geom.* (Deutsch by Fleischer, Leipzig, 1903, p. 53), assumes that an order remains an order after perspective, and, also, as stated here, a rule for obtaining an aggregate order from two orders, specified by two sets of points containing, both, the same set of three points occurring in the same order. If we retain the point of view from which the present Chapter began, taking *C* between *A* and *B*, the point D may be only a postulated point; but the application of Peano's axiom to the theorem in question is immediate.

The impossibility of Desargues' theorem from plane Propositions of Incidence. In Chapter I we gave a proof of Desargues' theorem for two triangles in the same plane, based on a construction in three dimensions, assuming only the Propositions of Incidence. In the present Section we have given a construction for postulated points, based on Desargues' theorem for a star, which equally involves (beside Peano's axiom) the consideration of three dimensions. We now give an account of a formal proof that it is impossible to prove Desargues' theorem for the plane from the Propositions of Incidence which apply to figures in a plane, utilising for this the notions of betweenness which formed the basis of this Chapter. The reader may compare Moulton, "A simple non-Desarguesian plane geometry," *Trans. Amer. Math. Soc.*, III, 1902; Peano, "Sui fondamenti...," *Rivista di Mat.*, IV (1894), 73; and Hilbert, *Grundlagen der Geom.*, Göttingen, 1899, § 23.



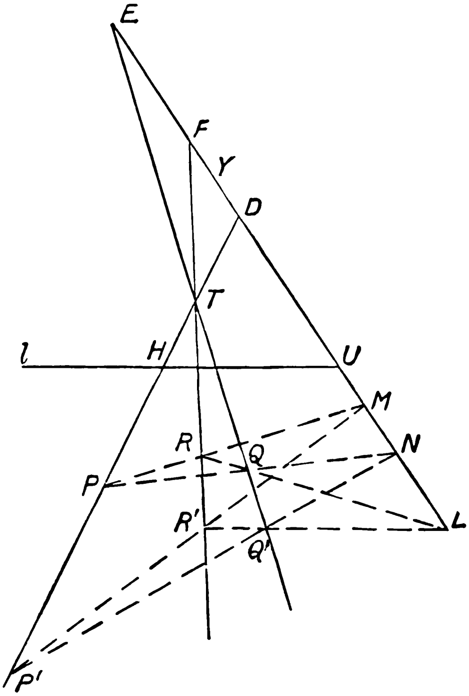
Assume two lines l, m, to be given, dividing the plane into two regions α, β (each representable diagrammatically by two opposite sectors connected through inaccessible points); let U be the intersection of these lines, and YU a given segment of the line m. Any natural line of the plane, meeting the line l in H and the line m in M, contains a segment, a, lying in the region α, and a segment, b, lying in the region β, the points H, M being considered as belonging to the segment α. If now M be in the segment YU, determine Q, in the segment MU, so that $YMQU$ is a harmonic range (or $Q=(U, M)/Y$; see p. 11); then the natural line gives rise to a *conventional* line, obtained by replacing the interior points of the segment b by the interior points of the segment HQ. This conventional line then meets the line m in M; one of its segments, a or HQ, is indefinitely continued. If the line joining Y to any point, M', of the segment a, meet l in U', the line of the segment HQ passes through the point, Q', of $M'U'$, for which $YM'Q'U'$ is a harmonic range; this remark is useful in constructing the conventional line through a point, M', of the region α, and a point of the region β. A natural line not containing a point of the segment YU, as also l or m, gives rise to a conventional line coinciding with itself.

It can now be shewn, with the convention that all points of the lines l, m belong to the region α, and with proper regard to the regions in which the points lie, that any two points of the plane lie on one conventional line, and that any two conventional lines have one point in common.

But it can at once be seen that, with these conventional lines, Desargues' theorem is not always true.

For consider two triads of points, P, Q, R, and P', Q', R', lying suitably in the region we have called α, whose corresponding joins QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$, meet, respectively, in points L, M, N, lying on the line YU, but not in the segment YU; while the joins of corresponding points, PP', QQ' and RR', which meet in a point T, meet the line YU, respectively in points D, E and F, of which only D is between Y and U. Such a figure is clearly possible.

Then of the nine lines in this figure, consisting of QR, RP, PQ, $Q'R'$, $R'P'$, $P'Q'$, with the joins PP', QQ', RR', there is only one which is not a conventional line by the previous definition, namely the line PP'; if this meet the line l in H, it would need to be modified by the replacement of HD by a line HK, to a point K of YU given by $K=(U, D)/Y$; the conventional line so obtained does not pass through the intersection, T, of QQ' and RR'.



Thus the Propositions of Incidence in a plane do not logically involve Desargues' theorem.

SECTION III. A DEDUCTION OF PAPPUS' THEOREM FROM THE NOTION OF ORDER AMONG THE POINTS OF A LINE, IN THE CASE OF THE REAL GEOMETRY AS AMPLIFIED BY THE ADJUNCTION OF THE POSTULATED POINTS

The abstract notion of order. We consider order in the abstract, independently of the objects to which this may be applied. These may, for example, be instants of time. To indicate this we speak of *positions*, of this order.

We conceive that the following must be granted, as necessarily inherent in the notion of such an abstract order:

(1) If two positions, say A and B, of this order, be considered, one of these, say B, is *in advance of* the other, A; and then A is *behind* B. We may denote this by $B > A$, or $A < B$; or even, as ordinary language is somewhat ambiguous, we may be allowed to say that B is to the right of A, and A to the left of B. If, then, Q is a third position which is in advance of B, it is assumed that Q is also in advance of A; and, if P is a position which is behind A, it is assumed that P is also behind B; while there may also be positions, R, which are both in advance of A and also behind B, these being said to be between A and B. In symbols, $A < B$, $B < Q$, together, involve $A < Q$; $A < B$, $P < A$, together, involve $P < B$; and there may be positions R for which $A < R < B$. It is assumed that every position, other than A and B, is such as Q, or P, or R. We thus build up the notion of an order of succession of any finite number of positions, determined by three of these.

(2) Between any two positions of the abstract order, there is another, and, therefore, by reapplication of the assumption, an unlimited number of other positions.

(3) Let A and B be any two positions of the order, for which, suppose, $A < B$. Consider the positions which are between A and B. Suppose that a rule has been devised, or exists, whereby everyone of these positions between A and B falls under one of two categories, (α), or (β); further, that every position falling under category (α) is seen, or known, to be behind every position falling under category (β), so that we may say that every (α) is less than every (β). Then it is in the nature of an abstract order, as we understand it, that, there is a position of the order, say C, lying between A and B, such that every position of the order which is between A and C falls under category (α), and every position of the order which is between C and B falls under category (β). The position C itself falls under category (α) or category (β). Thus, conversely, every position (α), between A and B, if not C itself, must be between A and C; for, if it were between C and B, it would be a position (β); similarly, every position (β), if not C itself, must be between C and B.

In cases, we may, conventionally, extend the result, regarding C as coincident with B when no positions (β) exist.

Remarks in regard to the preceding formulation. When it is said that any set of objects are in order, it is often meant, merely, that they satisfy no. (1) of the preceding descriptions. For instance the integer numbers 1, 2, 3, ... satisfy (1); they do not satisfy (2) or (3). If we take all positive rational fractions, such as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, arranged in ascending order of magnitude, they satisfy (1) and (2); they do not satisfy (3). We may for instance, among these fractions, consider, as falling under category (α), those whose square is less than $\frac{1}{2}$, and, as falling under category (β), those whose square is greater than $\frac{1}{2}$; there is, then, no rational fraction occupying the position required for C in the abstract order, that is, no rational fraction whose square is $\frac{1}{2}$; and, any fraction whose square is less than $\frac{1}{2}$, has, in advance of it, others whose squares are yet less than $\frac{1}{2}$. We may arrange these fractions, say those only which are less than 1, in another way, so that they satisfy (1), but do not satisfy (2), by supposing them to follow one another in batches, of which all those in a batch have the sum of their numerator and denominator the same; the position of each fraction may then be given by one of the numbers 2, 3, 4, ... From these, and other examples, it is clear that a set of concrete objects, infinite in number, may be such as to correspond uniquely, each to one position of such an abstract order as we have explained, though yet not containing objects corresponding to every position of such order. We may give a concrete example of this, which is capable of general statement; it assumes some acquaintance with the elements of Euclidean Geometry. Take a circle, of centre O, and radius OA; let the points of the plane interior to this circle be placed in order by the following rule: through any such point, P, draw a circle, also with centre O, which intersects OA in the point A_1; let the points on the circumference of this circle be placed in order, beginning with A_1, by regarding a point Q of this circumference as being in advance of the point P of this circumference ($Q > P$), when the positive angle A_1OQ, which is to be taken greater than or equal to zero, but *less* than four right angles, is greater than the positive angle A_1OP. Then, let all the points of the circumference of another circle of centre O, of radius OA_2 greater than OA_1 but less than OA, the first of these points being A_2, be regarded as being $>$ all the points of the circumference of the circle of radius OA_1, the points on the circumference of this second circle being, among themselves, in order by the rule above stated for the circle of radius OA_1. With such provision for any two circles of centre O within the original circle, all the points interior to this circle are arranged in order, and the above conditions, (1), and (2), are satisfied. Consider condition (3); regarding A_1 as a definite point between O and A, let the category (α) be that of all points on the circumference through A_1 and of all points on circumferences of radius less than OA_1, and let the category (β) be that of all points on circumferences of radius greater than OA_1 (but less than OA). Then every point interior to the circle of radius OA falls under one of these categories, and every point of category (α) is $<$ every point of category (β). But there is no point, C, such that every point $< C$ belongs to the category (α), and every point $> C$ belongs to the category (β). If we take any point, Q, of the circumference through A_1, there are always points,

Q', of this circumference, which are $> Q$; and then there are points, Q'', of this circumference, which are $>$ any such points Q', and so on continually; the process will not have an end unless either the point A_1 is allowed as $>$ all the various points $Q, Q', Q'', \ldots$, which, however, is not the case, since $A_1 < Q < Q' < Q'' \ldots$, or some point is allowed $>$ all the various points $Q, Q', Q'' \ldots$ which is a point of a circumference of radius greater than OA_1; and this, again, does not provide the point C in question, since all such points are of category (β). Thus, while the construction furnishes points arranged in such an abstract order as we have described, it does not give points corresponding to every position of such an order.

A further remark may be made: The notion of an abstract order furnishes no means for saying that the positions of the order between two positions A and B form an aggregate of the same *length*, or *magnitude*, or *power*, as the aggregate of positions between two other positions C and D. Such a notion of *congruence* must be supplied, if at all, by additional conditions. And only when this is supplied can there be question of the fulfilment of what is known as the axiom of Archimedes: the axiom namely, for an aggregate of elements arranged in order wherein the notion of equality of intervals between pairs of elements has been established, which states that, if A, B be two elements, and $P_1, P_2, \ldots$ be elements, such that $A < B$ and $A < P_1 < P_2 < \ldots$, and also such that the intervals $AP_1, P_1P_2, P_2P_3, \ldots$ are all *equal*, then there is a number n for which $B < P_n$. We may, as before, be allowed, in this case a very simple and concrete illustration, for the purpose of helping to fix the ideas; though, like a diagram used to assist the conception of a geometrical figure, the illustration may suggest implications which are not intended: In the case of time we measure intervals by observing the succession of natural phenomena, as for instance by heart-beats, or by a clock. But if all the clocks went twice as fast between what we now call 12 o'clock and 12.30 o'clock, as they now do, so that they indicated 1 o'clock at the time we now call 12.30, and went two-thirds as fast as they now do between the times we now call 12.30 and 2 o'clock, so that they marked 2 o'clock as they now do, we should, if we had agreed to regard as standard time that given by the clocks, reckon the interval between our present times 12 and 12.30 as being equal to the interval between 12.30 and 2. Though this would lead to modifications in our description of the behaviour of natural bodies, as, for instance, the contractions of the heart, it would not affect our notion of the succession of time, or of the order of events. The questions suggested in this connexion, recently also in vogue as arising in discussions of the Principle of Relativity in Physical Science, are inherent in the consideration of the Foundations of Geometry. The Theory of Proportion in Euclid's Fifth Book is evidence of their antiquity. In modern literature, the analysis leading to condition (3) above, relating to the existence of a dividing position there called C, was suggested by R. Dedekind, *Stetigkeit und irrationale Zahlen*, Braunschweig, 1872. Also for the relation of number and the measurement of quantity an article by O. Hölder, *Die Axiome der Quantität*, *Leipzig. Berichte*, LIII, 1901, may be consulted; and for this, and other fundamental questions, see F. Enriques, *Enzyklopädie der Math. Wiss.*, III, 1. 1, *Prinzipien der Geometrie*, where many references are given.

A theorem in regard to corresponding positions of an abstract order. Let us now suppose that we have some means of establishing a correspondence between two positions, P and Q, of such an abstract order, so that (i) to any position P there corresponds a position Q, and conversely; and (ii) so that the positions Q', Q'' corresponding to two positions P', P'', for which $P'' > P'$, are also such that $Q'' > Q'$. Let us suppose, further, that there are two positions of P, say A and B, for each of which the corresponding position of Q coincides with that of P; and suppose that the position of Q does not coincide with that of P for every position of P between A and B. When P has a position between A and B, the position of Q, if not coincident with P, relatively to P, may agree with or differ from that of B relatively to A; it will be sufficient to, and we shall, suppose $A < B$, and the two possibilities are, then, $Q > P$ and $Q < P$, the former being that when Q is in advance of P, and the latter that in which Q is behind P. It may be that Q, which coincides with P when P is at A, and coincides with P when P is at B, is, otherwise, for every position of P between A and B, in advance of P; or is, for every position of P between A and B, behind P. In either case we may say that Q *preserves the same relation to* P when P is anywhere within AB. More generally, if Q coincides with P when P is at each of two positions, P_1 and P_2, between A and B, and Q preserves the same relation to P when P is anywhere within P_1P_2, being either in advance of P for every position of P between P_1 and P_2, or behind P for every position of P between P_1 and P_2, it will be convenient to speak of P_1P_2 as a *complete interval.* We desire, then, to prove, assuming, as we have said, that Q does not coincide with P for every position of P between A and B, that, if AB is not a complete interval, it contains complete intervals, the end positions of these, whereat Q coincides with P, being definite. For this, consider a position, P_0, of P, between A and B, for which the corresponding position, Q_0, of Q, is not coincident with P_0. For definiteness suppose that Q_0 is in advance of P_0; a similar argument is applicable when Q_0 is behind P_0; it may be that, as P takes positions between P_0 and B, Q preserves that relation to P (of being in advance of P) which it has when P is at P_0, coinciding with P only when P is at B. In any case we prove that there is a definite position, V, between P_0 and B or identical with B, such that Q is in advance of P for every position of P between P_0 and V, but coincides with P when P is at V. And a similar argument will establish the existence of a definite position, U, between A and P_0 or identical with A, such that Q is in advance of P for every position of P between U and P_0, but coincides with P when P is at U. The segment UV will then be complete, in the sense adopted. For greater definiteness, let any position P between P_0 and B be said to be *like* P_0, when the corresponding position of Q is in advance of that position of P, and be said to be unlike P_0, when the corresponding position of Q either coincides with that position of P or is behind it. In particular B is unlike P_0. We may then place every position, F, between P_0 and B, under one of two categories, (α) or (β), agreeing that F is of category (α) when every position between P_0 and F, and F itself, is like P_0, but that F is of category (β) when there is one or more position between P_0 and F, or at F, which is unlike P_0. By condition (ii), the positions P between P_0 and Q_0 have their corresponding Q in advance of Q_0 and are of category (α); and B may be taken of category (β), since Q there coincides with P. A position of category (β) cannot coincide with a position of category (α); nor can any position of category (β) lie between P_0 and a position of category (α). In other words every position of category (α) is behind every position of category (β). Thus, recalling the condition (3) attached to the notion of an abstract order, there exists a position, V, between P_0 and B or at B, such that all positions of P in P_0B which are behind V are of category (α), and all positions in advance of V of category (β). This is equivalent to saying that every position, X, of P_0B, which is behind V, is like P_0, and every position of P_0B which is in advance of V, is either unlike P_0 or has positions unlike P_0 lying behind it; thus every position, Y, of P_0B, in advance of V, has positions unlike P_0 lying behind it; for every position between V and Y, being in advance of V, is either itself unlike P_0 or has such positions lying behind it.

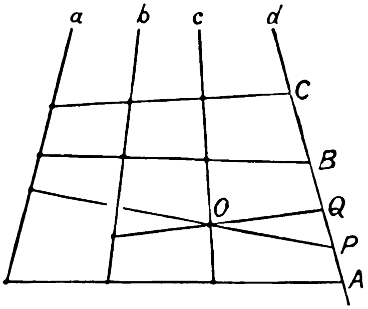
Hence, any position, X, of P_0B, which is behind V, is such that, when P is at X, the position Q is in advance of X; and every position Y, of P_0B, which is in advance of V, is such that, between V and Y, there is a position of P for which the corresponding Q is behind P, or a position P for which the corresponding Q coincides with it.

Now use again the condition (ii) above, that, if Q_1, Q_2 be the positions of Q corresponding to two positions P_1, P_2 of P for which P_2 is in advance of P_1, then Q_2 is in advance of Q_1. Then we can shew that, when P is at V, so also is Q. For if, when P is at V, the corresponding Q were at Q', behind V, and we take a position of P between Q' and V, and, therefore, behind V, the corresponding Q would, by the property of V, be in advance of P, and, therefore, in advance of Q', though the P to which this corresponds is not in advance of the P to which Q' corresponds, which is at V. Or, again, if, when P is at V, the corresponding Q were in advance of V, say at Q'', and we take a position of P between V and Q'', and, therefore, in advance of V, the corresponding Q would, by the property of V, either be at this P or behind it, and in either case behind Q'', though the P to which this corresponds is not behind the P to which Q'' corresponds, this being at V.

Thus, it is proved that, when P is at V, the corresponding Q coincides with it, there being between P_0 and V no position of P for which Q is not in the same relation to P as when P is at P_0. And, in particular, V may be B.

The proof of the existence of U is similar. Suppose that $Q_0 > P_0$, and let positions of P, between A and P_0, for which Q is in advance of P, be said to be like P_0, positions for which Q coincides with, or is behind P, being said to be unlike P_0; so that A is unlike P_0. Every position, F, between A and P_0 or at A, is either, (α), such that every position between F and P_0, and F itself, is like P_0, or, (β), such that there is one or more position between F and P_0, or at F, which is unlike P_0. By condition (ii), the P_1 whose Q is at P_0 is behind P_0, and all positions between P_1 and P_0 are of category (α). Every (α) is in advance of every (β); and, by the condition (3) for an abstract order, there exists a position, U, between A and P_0, or at A, separating positions (α) from positions (β). Thus, every position, X, between U and P_0, is like P_0, and every position between A and U is either unlike P_0 or has such positions lying in advance of it; from which, every position, Y, between A and U, has positions unlike P_0 lying in advance of it. Or, when P is at X, the corresponding position Q is in advance of it, while between Y and U there are positions of P for which Q, if not coincident with P, is behind P. And, as before, we can shew that when P is at U, so is Q. For, if the position U for Q arose from a position, P', of P, lying between U and P_0, a position Q, between U and P', would, by the property of U, correspond to a position P behind Q, and, therefore, behind P', and we should have the results $P' > P$, $U < Q$, which, by hypothesis, are inconsistent; while, if the position U for Q corresponded to a position P'' between A and U, then a position Q between P'' and U would correspond to a position which, if not coincident with Q, is in advance of Q, and is, therefore, in advance of P''; then we should have the inconsistent results $P'' < P$ and $U > Q$.

segments AB, consider that one which does not contain the point C. Let P be any point of the line d; from P draw the transversal to meet the lines a and c, its point of meeting with c being O; from O draw the transversal to meet the lines b and d, its intersection with d being Q. Then, to any position of P on the line d corresponds a definite position of Q, and, conversely, to every Q a definite P. The various positions of P correspond, each to one of an axial pencil of planes, having the line a for axis, the points P being the intersections of the planes of this pencil with the line d; three of these planes, in particular, are those containing, respectively, the points A, B and C. The planes of this pencil, respectively, are also intersected in points by the line c, the point O of this being that which corresponds to P. But, again, the various positions of O, give rise, with the line b for axis, to an axial pencil of planes, of which every plane has also a point on the line d, the point Q being that so arising from O. Therefore, four positions of P which are in order, give rise to four positions of O which are in order, and these, in their turn, to four positions of Q which are in order. In particular, Q coincides with P when P is at A, or at B, or at C. Thus, if P_1, P_2 be two positions of P such that A, P_1, P_2, B, C are in order, and if Q_1, Q_2 be the positions of Q respectively corresponding to P_1 and P_2, then, also, A, Q_1, Q_2, B, C are in order. We may regard the points ABC as defining, for the points of the line d, such an order as was contemplated in the discussion of an abstract order above, which is also such as was considered at the beginning of the present chapter (p. 96, above), where the inaccessible points of a line in fact play the part here played by the point C. We may then say, limiting the position of P, now, to the segment AB which does not contain C, that: *To any position of P between A and B there corresponds a definite position of Q, also between A and B, and conversely; and, if Q_1, Q_2 be the positions of Q corresponding to the positions P_1, P_2, of P, of which P_2 is in advance of P_1, then, also, Q_2 is in advance of Q_1.*



It therefore follows, from the preceding theory, that, unless, for all positions of P between A and B, the position of Q is coincident therewith, there can be found two positions, U and V, between A and B, or coinciding, respectively, with A and with B, such that, when P is at either U or V, then Q coincides with P, while, for a position of P between U and V, the position of Q preserves the same relation to that of P, and never coincides with it. This last alternative is, however, untenable: For, if we take the harmonic conjugate of the point C in regard to U and V, we obtain a point, C', lying between A and B, as has been proved (p. 119, above). By the construction given, four positions of P which are in harmonic relation, give rise to four positions of Q also in harmonic relation, as follows from the properties of this relation (Chapter I, p. 13, above). Therefore, as the positions of Q coincide with those of P when P is at U, or V, or C, the position of Q coincides with that of P when P is at C'. The remaining alternative must then be taken: *The positions of P and Q are coincident for all positions of P between A and B; and hence for all positions of P on the line d; and the transversal drawn from any point of d to meet two of the lines a, b, c, also meets the third.*

As has been said, it follows from the theory given in Section II of Chapter I (p. 49, above), that this amounts to a deduction o Pappus' theorem, under the assumptions made above as to the points that are possible upon a line. It proves also, with the same assumptions, that if, of two related ranges on two lines, possibly coincident, there be given three points, A', B', C', of one range, to correspond, respectively, to three points, A, B, C, of the other, then the point, D', of the former range, which corresponds to an arbitrary point D, of the latter, has a unique position (see pp. 45 ff., above). In particular, if, of two related ranges on the same line, there be three common points, each self-corresponding, then the ranges are identical.

Similar statements then follow in regard to flat pencils, of lines in a plane all passing through the same point, related to one another; and in regard to axial pencils, of planes in space all passing through the same line, related to one another.

Alternative deduction of related ranges, based on a harmonic net. In what has preceded in this volume, two ranges have been related to one another by a sequence of perspectivities. In a perspectivity four points of a line which are in harmonic relation lead to four points of another line which are also in harmonic relation. Conversely, suppose that we can set up a correspondence between the points of two lines, of such character that to any point of either line there belongs a definite point of the other, in such a way that any four points of either line which are in harmonic relation, say A, B, C, D, of which B, D are harmonic conjugates of one another in regard to A and C, correspond to four points of the other also in harmonic relation, say, respectively, A', B', C', D', of which B', D' are harmonic conjugates of one another in regard to A' and C'; suppose further that the order $A'B'C'$ is the same as, or different from, a definite order previously established on the second line, according as the order ABC is the same as, or different from, a definite order previously established on the first line; then it may well appear probable that the two ranges are related, in the sense previously employed.

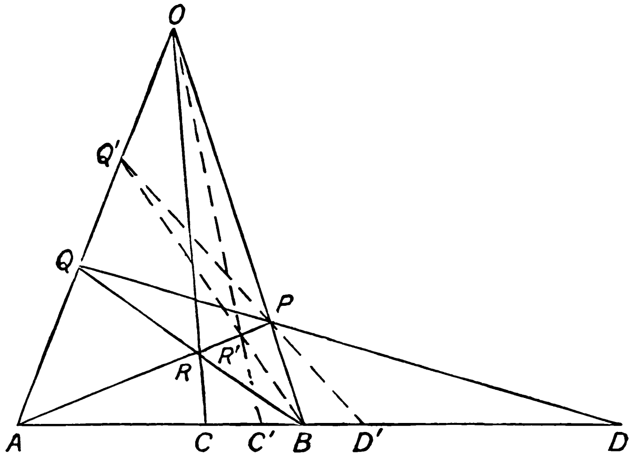
We do not enter fully into the analysis of this suggestion. But there is one result which would be necessary for this purpose, to which we should refer; it has an interest and importance by itself, and its deduction is by reasoning cognate to that immediately preceding. Suppose that, upon any line, are taken three points, A, B, and C; and, then, the harmonic conjugate of each of these in regard to the other two, and, then, further, the harmonic conjugates of each of the new points in regard to every other two, and so on continually. The points so obtained will have the property that, in each of the two segments determined by any two of the points, there are other points of the set; for beside these two points there are certainly others, and one such, lying in one of these segments, necessitates one, the harmonic conjugate of the other, lying in the other segment. This property, however, by itself, does not secure that, if two *arbitrary* points be taken on the line, not assumed to be points of the constructed set, there is a point of the set lying in a specified one of the two segments determined by these arbitrary points, though, if not, such points are in the other segment. In virtue of the special character of the harmonic relation it is however the case that this is so, and it is this fact which we now prove.

The definition of related ranges here referred to, founded on the condition that four harmonic points always correspond to four harmonic points, is that which was adopted by von Staudt (*Geometrie der Lage*, 1847, § 9, p. 49), who assumed without proof the result we have stated—that a point of a harmonic net of points constructed as above is found in every arbitrary segment. This omission was remarked by F. Klein, who gives (*Math. Annal.* VII, 1874; *Ges. Abh.* I, p. 344) the following demonstration, as due to Zeuthen, stating it to be equivalent to another simultaneously communicated by Lüroth. See also Darboux, *Math. Annal.* XVII (1880), p. 52.

Two assumptions are made, into a detailed analysis of which we need not enter:

(1) If, in regard to two points A, B, the harmonic conjugates of two points C, C' be respectively D and D', then the orders $CC'B$ and $DD'B$ are different, as also are the orders ACC' and ADD'. In other words, if $ACC'B$ be in order, then $ABDD'$ are not in order; but if $ABDD'$ are in order, then $ACC'B$ are not in order.

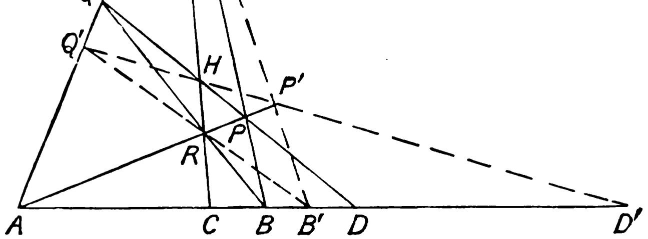
For let A, C, B be in perspective, from a point O, with A, R, P, respectively, and A, R, P be in perspective, from B, with A, Q, O, respectively, and, lastly, A, Q, O be in perspective, from P, with A, D, B, respectively. If then C' be in the segment CB which does not contain A, it gives rise, by perspective from O, to a point, R', in the segment RP which does not contain A; and R', by perspective from B, gives rise to a point, Q', in the segment QO which does not contain A; and, then, Q', by perspective from P, gives rise to



the point, D', in the segment DB which does not contain A; and D' is the harmonic conjugate of C' in regard to A and B.

The assumption may be expressed by saying that points which are harmonic conjugates of one another, in regard to two fixed points, move in opposite directions.

(2) If of the four points $ACBD$, in harmonic relation, we modify,



not C and, consequently, D, but, keeping C the same, we modify B, and, consequently, D, then, when $ACBB'$ are in order, so are $ABDD'$, and conversely; the orders CBB' and CDD' are the same, as are ABB' and ADD'.

For, with the same construction for D, from C, as in (1), let a point, B', of AB, be joined to R and O by lines meeting AQ and

AP respectively in Q' and P', and $Q'P'$ meet *AB* in D', so that D' is the harmonic conjugate of *C* in regard to *A* and B'; then, if B' be in the segment *AB* which does not contain *C*, it follows, by perspective from *O*, that P' is in the segment *AP* which does not contain *R*, and, by perspective from *R*, that Q' is in the segment *AQ* which does not contain *O*. Hence, remarking that *OR*, *PQ*, $P'Q'$ meet in a point, *H*, as follows from the triads *P*, P', *O* and *Q*, Q', *R*, it is seen that D' is in the segment *DA* which does not contain *B*.

In (1) two separated points, of four in harmonic relation, were kept unaltered; in (2) two contiguous points are to be kept unaltered. Then, we may say, the other two, contiguous, points move in the same direction.

These conditions being understood, consider such a set of points on a line as was described, containing the harmonic conjugate of every one of them in regard to every other two. We may speak of such a set as a harmonic net; and may associate with it the name of Moebius. Suppose, if possible, there are two points, *F*, *G*, of the line, such that there are no points of the harmonic net on one of the two segments determined by *F* and *G*. Points of this segment may be spoken of as lying between *F* and *G*; and, by taking a further point of the line to define, with *F* and *G*, an order, we may, without loss of generality, speak of *G* as being in advance of *F*. We first extend this segment *FG* as much as possible, seeking to replace *F* by a point *X* behind *F*, and replace *G* by a point *Y* in advance of *G*. Precisely, we shew that there are two definite points, *X*, *Y*, upon the line, of which *X* either coincides with *F* or is behind it, and *Y* either coincides with *G* or is in advance of it, so that *XFGY* are in order, having the properties, (*a*), that there is no point of the harmonic net within the segment *XY*, which contains *F* and *G*, (*b*), that either *X* is a point of the harmonic net, or there are points of the harmonic net lying between *X* and any *arbitrary* point taken behind *X*, (*c*), that either *Y* is a point of the harmonic net, or there are points of the harmonic net lying between *Y* and any arbitrary point taken in advance of *Y*.

To prove the existence of the point *Y*, we remark, first, that if *G* be a point of the harmonic net we may take *Y* at *G*. Suppose, that *G* is not a point of the harmonic net; there must, then, be points of the harmonic net in advance of *G*, or behind *F*, since otherwise, there being none between *F* and *G*, there would be none anywhere. In our conventions of order, however, there is, finally, no distinction between points said to be in advance of *G*, and points said to be behind *F*. We shall therefore say that there is a point, *N*, of the harmonic net which is in advance of *G*. Consider now the segment *GN*, containing points in advance of *G*, but not containing F. We apply the formal argument introduced in our discussion of an abstract order, to shew that there is such a point as Y within this segment. The points of the segment GN, other than G, are of one of two categories: (α), Points, R, such that there is no point of the harmonic net between G and R, or at R; it is assumed that points (α) exist; but, if not, we may take Y at G; (β), Points, S, such that there is a point of the harmonic net between G and S, or at S. It is clear that every point (α) is behind every point (β). Therefore, by the condition named (3) in the discussion of an abstract order, there is a point, Y, between G and N, or at N, such that all points of the segment GN which are behind Y are points (α), and all points of the segment YN are points (β), the point Y, itself, being of one of these categories. Every such point (β) of the segment YN, say S, is then either, itself, a point of the harmonic net, or has such points, lying in GN, behind it. If behind S there be no such harmonic points, so that, in particular, Y is not a harmonic point, and if, further, there are no harmonic points between Y and S, in which case S itself is a harmonic point, then there are points (α) in the segment YS, which is contrary to the definition of Y. Wherefore, every point, S, of category (β), in advance of Y, has points of the harmonic net behind it, and these, by the construction, are not behind Y. Thus, either Y is a harmonic point, or such harmonic points lie between Y and T, whatever point of the segment YN the point T may be. Thus Y is such a point as was to be shewn to exist. The existence of X can be shewn in a similar manner.

Having thus deduced, from the hypothesis of the existence of the segment FG containing no points of the harmonic net, the existence of X and Y, with no harmonic points between them, we proceed, with use of the precise definitions of X and Y, to give a construction for finding a point of the harmonic net which does actually lie between X and Y. We shall then be able, finally, to conclude that such a segment as FG is not possible.

For this purpose, suppose, first, that neither X nor Y is a point of the harmonic net. And, for the sake of clearness, denote points of the harmonic net which arise in the construction by A_1, B_1, C_1, D_1, points not assumed to belong to the harmonic net being denoted by letters without suffix. Then make the following construction:

(1) Take any point, A_1, of the harmonic net, not in the segment XY containing the segment FG first considered, the point A_1 being, say, behind X. From A_1 find the harmonic conjugate in regard to X and Y, say H, represented by

$$H = (X, Y)/A_1,$$

so that H is within the segment XY.

(2) Take also the point J given by

$$J = (A_1, Y)/X,$$

so that J is in advance of Y.

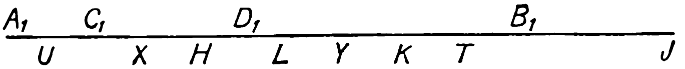
(3) Remembering the property of the point Y, take a point, B_1, of the harmonic net, in advance of Y but behind J, and then a point, T, in advance of Y but behind B_1, and thence find the point, U, such that

$$U = (A_1, H)/T.$$

This shews that U is in advance of A_1 but behind H; while, as we also have

$$X = (A_1, H)/Y,$$

we can infer, from the assumption (1) above, that the order A_1XU is the opposite of the order A_1YT, namely that U is behind X, and it is in advance of A_1.



(4) By the property of the point X, there is a point of the harmonic net in advance of U and behind X; take such a point, C_1. From this find K so that

$$K = (A_1, H)/C_1;$$

then, as UC_1XY are in order, and A_1, U, H, T and A_1, X, H, Y, as well as A_1, C_1, H, K, are all sets in harmonic relation, it follows, also from assumption (1) above, that $TKYX$ are in order, or that K is in advance of Y and behind T.

(5) Next take L so that

$$L = (C_1, J)/A_1.$$

Then, comparing the facts that A_1, J, L, C_1 and A_1, J, Y, X are both sets in harmonic relation, we infer, from assumption (2) above, that A_1, L, Y are in the same order as A_1, C_1, X, or that L is behind Y. While, comparing the facts that A_1, C_1, H, K and A_1, C_1, L, J are also both sets in harmonic relation, we similarly infer that A_1, H, L are in the same order as A_1, K, J, or that L is in advance of H.

(6) Lastly take D_1, belonging to the harmonic net, such that

$$D_1 = (C_1, B_1)/A_1.$$

Then, from the harmonic sets, A_1, C_1, D_1, B_1 and A_1, C_1, L, J, we similarly infer that C_1, D_1, L are in the same order as C_1, B_1, J, or, that D_1 is behind L. And, from the harmonic sets, A_1, C_1, H, K and A_1, C_1, D_1, B_1, we infer that the order A_1KB_1 is the same as A_1HD_1, so that D_1 is in advance of H.

We may then say that D_1 is between H and L. But, similarly, H is between X and Y, and L is between H and Y. Thus D_1 is between X and Y; and is a point of the harmonic net. Wherefore, in the case when neither X nor Y is a point of the harmonic net, we are able to infer that the hypothesis of the segment FG is untenable.

Next suppose that X is a point of the harmonic net, but, as before, that Y is not such a point. Then the point C_1, which was any point of the harmonic net between U and X, can be taken, instead, to be at X. Hence, by comparison of (3) and (4), the point K will be at Y; and, by comparison of (2) and (5), the point L will be at Y. But D_1 will still be between X and Y; and the same conclusion will follow.

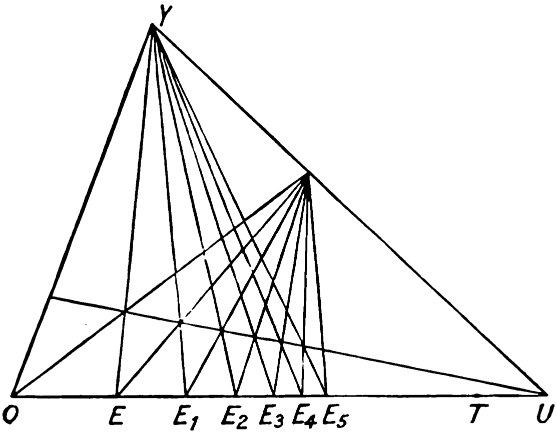
If Y, but not X, be a point of the harmonic net, we can interchange the parts played by X and Y in the preceding case, with the same inference.

While, finally, if both X and Y belong to the harmonic net, the harmonic conjugate of A_1 in regard to X and Y is between X and Y, and, in this case, is a point of the harmonic net.

The conclusion, then, is, that every segment of the line contains one, and, therefore, an infinite number of points of the harmonic net.

Consequences of the equable distribution of points of the harmonic net. It follows from the preceding theorem that any point, P, of a line upon which a harmonic net has been constructed, may be regarded as lying in advance of an indefinitely continued sequence of points of the harmonic net, say $A_1, A_2, A_3, \ldots$, where $A_1A_2A_3\ldots P$ are in order, and also as being behind another such sequence, $B_1, B_2, B_3, \ldots$, where $P\ldots B_3B_2B_1$ are in order. We have only to take two arbitrary points, A, B, upon the line, such that APB are in order; there is, then, a harmonic point, A_1, between A and P, and a harmonic point, B_1, between P and B. There is, also, a harmonic point, A_2, between A_1 and P, and another, B_2, between P and B_1; and so on, indefinitely. Conversely, if we have an indefinitely continued sequence of points of the harmonic net, in order, say $A_1, A_2, A_3, \ldots$, and, also, another indefinitely continued sequence of harmonic points $\ldots, B_3, B_2, B_1$, in the same order, and if every point of the former set lies behind every point of the latter, there is, by the condition (3) introduced in the discussion of an abstract order, a point P which has the specified relation to the two given sets of harmonic points. This point P may be, or may not be, a point of the harmonic net.

Another result, involving similar ideas, should be referred to here. If O, E, U be any three points of a line, and we carry out in succession the operations of finding the harmonic conjugate, E_1, of O in regard to E and U; then the harmonic conjugate of E in regard to E_1 and U, say E_2; then the harmonic conjugate of E_1 in regard to E_2 and U, say E_3, and so on, indefinitely, the points $OEE_1E_2E_3\ldots U$ will be in order. And, if T be any point of that

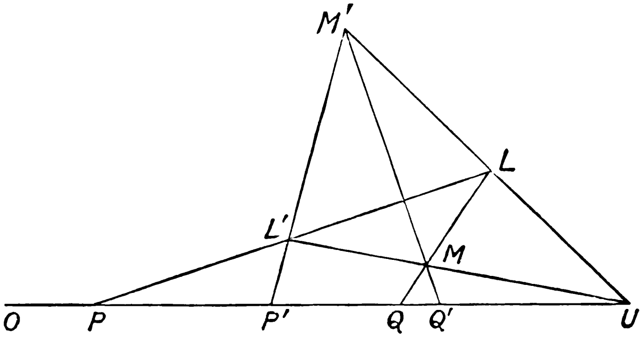


segment EU which does not contain O, there will be found points of the set E_1, E_2, E_3, ..., in infinite number, in that segment TU which does not contain E, *and this wherever T be taken.* This result is like one referred to previously (p. 124, above), applicable to measurable quantities, known as Archimedes' axiom, which would state that, if EE_1, E_1E_2, E_2E_3, ... were *equal lengths*, and T arbitrary, there is an n such that OE_n is greater than OT. But the fundamental ideas are somewhat different: the result here stated is a theorem of 'betweenness,' when all the points of the line are supposed accessible; Archimedes' axiom is a theorem of that which is beyond what is given.

Introduction of symbols, in particular of the iterative symbols. We have, in this chapter, avoided reference, so far, to the algebraic symbolism described in Chapter I, and to the numbers of ordinary arithmetic. But the effect of the hypothesis introduced, for the points of a line, to obtain a deduction of Pappus' theorem, that the real points existing on a line are such, and only such, that condition (3) for an abstract order (p. 122, above) is satisfied, is to set up a correspondence between the points of a line and the numbers of arithmetic. In this correspondence, the numbers are used only to specify order of succession, not magnitude, and there is one postulated number, denoted by ∞, which is to be regarded as coming not only after all positive numbers but also before all negative numbers; this being understood, the numbers associated with three arbitrary points of the line may be taken arbitrarily, and the correspondence is then definite.

Taking three arbitrary points of the line O, E, U, and supposing the symbols belonging to these to be such that $E=O+U$, as was explained in Chapter I, we have obtained an interpretation, by a point of the line, of the symbols $nO+U$, $O+mU$, $nO+mU$, $nO-mU$, wherein m, n are iterative symbols; and it is easy to establish, by induction, from the assumptions of the present chapter, that, if the positive numbers of arithmetic, $\bar{n}$, $\bar{m}$, $\bar{n}'$, $\bar{m}'$, with which the symbols n, m, n', m' are formed, be such that $\bar{m}'/\bar{n}'$ is greater than $\bar{m}/\bar{n}$, then the points of which the symbols are, respectively, O, $nO+mU$, $n'O+m'U$, U, are in order; with a similar conclusion for points of symbol $nO-mU$. By this, a correspondence is established between the points of a harmonic net upon the line, built up from the points, O, E, U and the rational numbers of arithmetic, taken in order in both cases, the points O, E, U being associated respectively with the numbers 0, 1, ∞. Then, by the proposition of the equable distribution of the points of the harmonic net, proved above, the points of the line not belonging to the harmonic net are made to correspond, uniquely, each to an irrational number. It will be seen, immediately, below, that any other numbers than 0, 1, ∞, might have been attached to the points O, E, U.

It is not, however, thereby asserted, as has several times been



remarked, that, if two points, P, Q, of the line be thus associated with two symbols p, q, respectively, and two other points, P', Q', with two symbols p', q', and if $q'-p'=q-p$, then the *length* $P'Q'$ is equal to the length PQ. Such an assertion would be meaningless in the absence of a definition of length. As we have seen, the points P, Q, ... being $O+pU$, $O+qU$, ..., and therefore, the points P', Q, Q' being, respectively, $P+(p'-p)U$, $P+(q-p)U$, $P+(q'-p)U$, the equation $q'-p'=q-p$, or $q'-q=p'-p$, expresses that, if arbitrary lines be drawn through P, P', Q, respectively, in one plane containing the line, that through P meeting those through P' and Q respectively in L' and L, and UL, UL' meet the lines through P' and Q respectively in M' and M, then the line $M'M$ meets the original line in the point Q'. Evidently, when P, P' and Q are assigned, the position of Q' depends on the position of U.

Thus the following question arises: If we take three arbitrary points of the line, O, $O+U$, U, and thereby associate every point of the line, of symbol $O+xU$, with a number, which we shall denote by x; and also take three other arbitrary points, denoting these by O', $O'+U'$, U', and thereby associate every point, $O'+x'U'$, with a number x'; what is the relation of the numbers x', x, when the symbols $O+xU$, $O'+x'U'$ refer to the same point? By denoting an arbitrary point by $O'+U'$, we have fixed the multiplier of the symbol U'; we have, therefore, syzygies

$$O' = mO + aU, \quad U' = bO + cU,$$

where m is arbitrary, and may be replaced by 1 unless O' coincides with U, but $m^{-1}a$, b, and c, are definite. Thence

$$O' + x'U' = mO + aU + x'(bO + cU), \ = (m + x'b)\,O + (a + x'c)\,U;$$

identifying this with $O+xU$, we have a relation connecting x and x'. We saw, in Chapter I, that, without assumption of Pappus' theorem, the iterative symbols, corresponding to the rational numbers of Arithmetic, are commutative in multiplication; the hypotheses of the present chapter involve Pappus' theorem, and therefore that the symbols of all the real points of the line recognised by these hypotheses, are also commutative in multiplication. We may, therefore, write

$$x = \frac{a + cx'}{m + bx'}, \quad x' = \frac{mx - a}{c - bx}.$$

In other words, the use of the fundamental points O', $O'+U'$, U', instead of O, $O+U$, U, leads to the substitution of x by a general linear (fractional) function of x, involving three coefficients.

Consider, now, two related ranges upon the line. By what we have seen, two such ranges are obtained in the most general way by taking three points O, E, U, of one range, to correspond, respectively, to three points, O', E', U', of the other. In particular, any set of four points, of the first range, which are in harmonic relation, gives rise to four points of the second range in similar relation. Thus, every point of the harmonic net constructed from O, E, U, corresponds to a point, constructed, with precisely similar steps, from O', E', U', belonging to the harmonic net built up from these. By using the symbols U, U' with proper multipliers, we can suppose the symbols E, E' to be such that $E = O + U$, and $E' = O' + U'$. Then the point $P = mO + nU$ corresponds to the point $P' = mO' + nU'$, where m, n are iterative symbols; and, hence, $P = O + xU$ corresponds to $P' = O' + xU'$, where x is the symbol for any real point of the line. When $O' = mO + aU$, $U' = bO + cU$, as above, this last is $P' = (m + xb)\,O + (a + xc)\,U$. Thus: *In order that a range of points* $P, = O + xU$, *for varying* x, *should be related to a range of points* P', $= O + yU$, *the symbols* x, y *must be connected by an equation of the form* $y = (cx + a)/(bx + m)$. By proper choice of c, a, b, m, it is possible, in this formula, to make three arbitrary points of the first range correspond, respectively, to three arbitrary points of the second; the formula appears thus to cover the case of every two possible related ranges.

Ex. 1. In the construction above given to prove the equable distribution of the points of a harmonic net, prove that the symbols of the various points may be supposed to be of the following forms, respectively,

$$X = A_1 + Y,\quad H = A_1 + 2Y,\quad J = A_1 - Y,\quad B_1 = A_1 - bY,\quad T = A_1 - tY,$$
$$U = (t+1)\,A_1 + tY,\quad C_1 = (c+1)\,A_1 + cY,\quad K = A_1 - cY,\quad L = A_1 + 2cY$$

leading to

$$\begin{aligned} D_1 &= (bc - c + b)\,A_1 + 2bcY \\ &= (b-1)\,c^2H + (c-b)\,L, \end{aligned}$$

the numbers of arithmetic belonging to the symbols c, t, b, 1, being such that

$$c > t > b > 1,$$

shewing that, then, the points are in the order required in the demonstration.

Ex. 2. In the Ex. 4, p. 84, of Section III of Chapter I, shew that it is impossible, consistently with the assumptions, in regard to order, adopted in this chapter, for the point $Q = O + a^2U$ to coincide with the point $O - U$, obtained as $(O, U)/E$. Give constructions for which $O + a^3U$ coincides with $O + U$, or $O - U$.

Ex. 3. If the points $O + x'U$, $O + xU$, be harmonic conjugates in regard to the points $O + aU$, $O + bU$, prove that

$$x' = [x\,(a+b) - 2ab] \div [2x - a - b].$$

Using the same letters for the associated numbers, deduce that

$$\frac{\partial x'}{\partial x} = -\frac{(a-b)^2}{(2x-a-b)^2},\quad \frac{\partial x'}{\partial b} = 2\,\frac{(x-a)^2}{(2x-a-b)^2};$$

these are in accordance with the two assumptions made for the proof of the equable distribution of the points of a harmonic net (above, p. 132).

Ex. 4. In the correspondence of p. 125, prove that the conditions (i), (ii) involve that if Q_0, corresponding to P_0, is in advance of P_0, and P_1 is the P-position whose corresponding Q is at P_0, then P_0 is in advance of P_1; and that, for every P between P_1 and Q_0, the corresponding Q is in advance of P.

CHAPTER III

ABSTRACT GEOMETRY, RESUMED

RELATED SPACES; JUSTIFICATION OF THE SYMBOLS. GEOMETRICAL ASSUMPTION FOR IMAGINARY ELEMENTS; REPLACEMENT OF IMAGINARY ELEMENTS BY SERIES OF REAL ELEMENTS

General discussion of the problems of this chapter. The preceding chapter has shewn how, by the recognition of the order of the real points of a line, and by adjunction of postulated points defined by constructions in a limited accessible space, we can build up a geometry in which, with a definite assumption as to the points existing upon a line, Pappus' theorem can be proved. The Real Geometry so arising may then be regarded as a particular case of the Abstract Geometry considered in Chapter I; it was seen that the algebraic symbols which arise in this Real Geometry are precisely similar, in their mutual relations, to the numbers of real arithmetic.

The present chapter is a continuation of Chapter I. It was there shewn that upon the Propositions of Incidence, and the assumption of Pappus' theorem, a theory of related ranges can be built, the correspondence of two such ranges being without ambiguity when three points of one are assigned to correspond to three points of the other. This establishes a corresponding theory for related flat pencils of lines in a plane, all passing through the same point, and for related axial pencils of planes in space, all passing through the same line. And it is, here, further, shewn that there follows from this a theory of two related planes, in which every point of either plane corresponds to a point of the other plane, and every line of either plane, with any range thereon, corresponds to a line of the other plane, with a range thereon which is related to the former; this correspondence, of the points of one plane to those of the other, is determined when *four* points of one plane, of which no three are in line, are assigned to correspond to four arbitrary points, of similar generality, in the other plane. Then, further still, it is shewn that there follows the possibility of a similar unique correspondence of the points of two threefold spaces, in which not only every point of either space corresponds to a definite point of the other, and every range of either space corresponds to a related range of the other space, but also every plane of either space, with the points and ranges thereon, corresponds to a definite plane of the other space which is related to the former plane, as regards its points and ranges, in the manner just described for two related planes; this correspondence is determined when *five* points of one of the two spaces, of which no four are in a plane, and, therefore, no three are in line, are assigned to correspond to five arbitrary points of similar generality in the other space. A similar theory of the correspondence of two n-fold spaces can be deduced, which is determined when $(n+2)$ points of one space, of general position, are assigned to correspond to $(n+2)$ points, of general position, of the other space.

There is, as we shall point out, an exact similarity between the condition necessary to fix the correspondence of the points of two spaces, and the condition necessary to fix the correspondence between the points of either of these spaces and the algebraic symbols by which these points are represented. To fix the correspondence between the points of a line and the symbols by which these points are represented, it is necessary to assign the symbols of three points of the line. This is a consequence of our initial assumption whereby a point by itself is represented as well by a symbol P as by a symbol mP, in which m is an arbitrary algebraic symbol; in what precedes we have frequently employed the remark, having denoted three arbitrary points of a line, respectively, by symbols A, B and $A+B$, thus fixing the multiplier of the symbol B with respect to that of A, and also the multiplier of the symbol of the third point. To fix the correspondence between the points of a plane and the symbols by which these points are represented, it is similarly necessary to assign the symbols of four points of the plane, the points being such that no three of them are in line, and the symbols, therefore, such that no three of them are in syzygy. It is the same for higher cases.

The symbolism is, then, in exact correspondence with the geometry, provided that, (1), to every geometric operation there corresponds a definite operation with the symbols; (2), to every operation with the symbols there corresponds a definite geometrical operation. That the former condition is satisfied follows from the initial descriptions of the use of the symbols; and the latter condition is satisfied for the fundamental laws of operation of the symbols (above, p. 74), each of which has been interpreted geometrically. The Abstract Geometry is not opposed to, but includes, as part of itself, the Real Geometry discussed in the last chapter, the symbols then appropriate being those iterative symbols, and those deduced from them, which correspond to the numbers of ordinary arithmetic. But now the way seems open to us, still further to generalise the Abstract Geometry, with the help of suggestions arising from the symbols themselves, using the words point, line, etc., in a proper sense consistent therewith.

We have in fact remarked (p. 66, above) the existence of a system of algebraic symbols, each formed with two numbers of ordinary arithmetic, which obey the laws of operation set out above, and are subject to the commutative law of multiplication; and these contain among themselves symbols corresponding to the numbers of ordinary arithmetic. These symbols, then, are a system appropriate to the Abstract Geometry, as so far developed, assuming Pappus' theorem. This system of symbols has, however, a possibility which we have not alluded to as yet: *to every symbol c, of the system, there corresponds a symbol z, such that $z^2=c$.* In fact, denoting any of the symbols of this system by (x, y), where x, y are numbers of ordinary arithmetic, we have $[(0, 1)]^2=[-1, 0]$, and, more generally, if $r=(x^2+y^2)^{\frac{1}{2}}$, $p=\frac{1}{2}(r+x)$, $q=\frac{1}{2}(r-x)$, then

$$(p^{\frac{1}{2}}, q^{\frac{1}{2}})^2=(x, y);$$

while, since $(-z)(-z)=z^2=(z)(z)$, beside any z satisfying $z^2=c$, there is also $-z$. Thus, in the geometry to which the symbols of this system are appropriate, the geometrical operation, previously given (p. 76), by which we pass from a point $O+zU$ to a point $O+z^2U$, must be capable of reversion in all cases; we must assign to the Abstract Geometry such points that, when $O+cU$ is a point, there is also a point $O+zU$, in which $z^2=c$. It will be shewn that a geometrical fact corresponding to this is that, if A, B, C be given points, and a, b, c be given lines, in a plane containing A, B, C, then it is possible to draw, in this plane, a line p through the point A, a line q through the point B, and a line r through the point C, so that the point (q, r), where the lines q and r intersect, lies on the line a, and, similarly, the point (r, p) lies on the line b, and the point (p, q) lies on the line c. Another geometrical fact corresponding to the algebraic condition, which, therefore, is geometrically reducible to the former, requires the notion of related ranges, which we have developed on the basis of Pappus' theorem. This theorem has been used to shew that, if two related ranges on the same line have *three* corresponding points in common, then they coincide entirely. In the Real Geometry two such related ranges may, however, have *two* corresponding points in common; for instance, it may be shewn, by the assumptions employed in the last chapter, that a *sufficient* condition for this, when the points A, B, C of one range correspond respectively to the points A', B', C' of the other, is that the order ABC should be opposite to the order $A'B'C'$. The geometrical fact now referred to, for the case of the Abstract Geometry represented by the system of symbols now being considered, is that two related ranges on the same line, which do not coincide entirely, *always have two corresponding points in common*, which, however, may coincide with one another in particular cases. We shall give the geometrical deduction of this geometrical fact from the former, relating to three given points and three given lines. It is clear from what has been proved (cf. pp. 18, 129), that a still further way of stating a geometrical fact equivalent to the above, would be to assume that four lines in threefold space, of which no two intersect, have always two common transversals. Though this is clear by the intervention of the theory of related ranges, it will be proper to reduce it directly to the previously stated theorem for three given points and three given lines. This last theorem, though not considered by Steiner in this general way, we shall refer to, for the sake of brevity, as Steiner's theorem. (See Steiner's *Gesamm. Werke*, Vol. I, pp. 303—305, where reference is made to papers by Servois, Gergonne and L'huilier, in Vol. II, 1811, 1812, of the *Annales de Mathématiques.*)

When we extend the Abstract Geometry in accordance with the suggestions of the system of algebraic symbols above referred to, or in some other way that may prove possible, two questions naturally arise: (1), Is there any geometrical utility in this extension? (2), Is it legitimate to use the postulated properties of the abstract points, lines, etc., in order to prove relations existing among the real points, lines, etc., that is, relations which can be stated without any reference to the abstract elements? The former question is analogous to the question: Is there any utility, for the purposes of the calculus of real numbers, in the introduction of complex numbers? And, as in that case, it may be said, briefly, that experience has amply shewn that the gain in the generality of the statements of geometrical fact, and the increased power of recognising the properties of a geometrical figure, enormously outweigh the initial feeling of artificiality and abstractness[1]. The second question may perhaps be made clearer by reference to two simple examples: (1) By the use of complex numbers it is easy to prove that a polynomial of the form $x^{2n} - 2x^n \cos n\theta + 1$ is the product of quadratic factors of the form $x^2 - 2x \cos (\theta + 2k\pi/n) + 1$, where n is a positive integer and k has one of the values $0, 1, \ldots, (n-1)$. In this theorem, x may be a real number, and θ a real angle whose cosine is determined by constructions made with only real points. Is then this result, for real numbers only, legitimately proved by the use of complex numbers? And, if so, does this proof involve that it must be possible to prove the theorem with the use of real numbers only? (as in fact it is). This example, though referring to matters outside our plan, is given as having been of interest to eminent mathematicians. (2) Another example is of directly geometrical importance; but involves some anticipation of succeeding work. As we shall see, for the curve which is called a conic section, if A, B, C, A', B', C' be any six points of the curve, the three point intersections, of pairs of joins of these points, represented, respectively, by $(BC', B'C)$, $(CA', C'A)$ and $(AB', A'B)$, lie on a line. Of this theorem, in fact, Pappus' theorem is a particular case, namely when the conic section degenerates into two lines containing, respectively, the points A, B, C and A', B', C'. It is possible to establish, for the points of a conic section, a theory of related ranges precisely like the theory of related ranges on a line; and, as in that case, three points of a range assigned as corresponding to three given points of another range, suffice to determine the correspondence of other pairs of two points, belonging to these ranges, respectively. Suppose then we establish two related ranges upon the conic, in which the points A, B, C, of one range, correspond, respectively, to the points A', B', C' of the other. Then it may, or may not, happen, that upon the conic, regarded as a locus of real points, there are two points, U, V, each of which, regarded as belonging to one range, corresponds to itself, regarded as belonging to the other range. When these points U, V are existent, as real points, the theorem above stated can be proved at once; the line of the theorem is then, in fact, the join of the points U and V. The question, of which the theorem is to serve as an example, becomes then: Is the theorem then necessarily true, without further proof, when the points U, V do not exist as real points? and can a proof be devised for this case which shall only employ real points? In the Abstract Geometry, the points U, V always exist, and a proof of the theorem may be given identical in form with that which is valid in the real geometry: does this suffice to prove the theorem for the Real Geometry when the points U, V do not exist?

---

[1] Cayley's remarks may be quoted (Presidential Address, Brit. Assoc., 1883, Collected Papers, xi, 434): "The notion which is really the fundamental one (and I cannot too strongly emphasize the assertion), underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis, and of imaginary space (or of space as a *locus in quo* of imaginary points and figures) in geometry."

The opening words of the brief *Vorwort* of von Staudt's *Beiträge zur Geometrie der Lage* (Erlangen, 1856) may also be quoted: "Indem die Mathematik darnach strebt, Ausnahmen von Regeln zu beseitigen und verschiedene Sätze aus einem Gesichtspunkte aufzufassen, wird sie häufig genöthigt, Begriffe zu erweitern oder neue Begriffe aufzustellen, was beinahe immer einen Fortschritt in der Wissenschaft bezeichnet. Dahin gehört namentlich die Einführung von imaginären Grössen in der Analysis und die Einführung von imaginären Elementen in der Geometrie."

Some of the objections raised to the original introduction of complex numbers in Geometry and Analysis are very interesting to read. Many references to the early literature are given by Cayley, *On multiple algebra*, Coll. Papers, xii, 466. A single reference additional to these is Caspar Wessel, *Essai sur la représentation analytique de la direction*, Kopenhagen, 1799.

The question is an old one. The reader may compare the discussion of Kepler's ideas, and of the Principle of Continuity, in Ch. Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. lviii, etc.

In a less concrete form, the matter may be stated thus: We have a set of entities, $a, b, c, \ldots$, subject to certain laws of combination; we have also another set of entities, $A, B, C, \ldots, A', B', C', \ldots$, subject to certain laws of combination; these laws will include laws for the combination of $A, B, C, \ldots$ among themselves alone, as well as laws for the combination of the whole aggregate $A, B, C, \ldots, A', B', C', \ldots$, when some of $A', B', C', \ldots$ enter. Suppose now that to every law for the combination of $A, B, C, \ldots$ among themselves alone, there corresponds a precisely similar law for the combination of $a, b, c, \ldots$ among themselves, *and conversely*. Then any theorem concerning $A, B, C, \ldots$ alone, which is obtainable logically by the application of the laws of combination which relate to $A, B, C, \ldots$ alone, is equally true for $a, b, c, \ldots$, being obtainable from the laws of combination of $a, b, c, \ldots$ by exactly similar logical processes. Now it may very well happen that a theorem relating to $A, B, C, \ldots$ alone, is easier to prove when regarded as a theorem for particular entities of the whole aggregate $A, B, C, \ldots, A', B', C', \ldots$, the natural method of proof being one which employs not only the laws of combination of $A, B, C, \ldots$ among themselves alone, but also the laws relating to the whole aggregate, using, in its course, one at least of the entities $A', B', C', \ldots$. The question then is, does such a method of proof establish the corresponding theorem for $a, b, c, \ldots$? And can a method of proof be, therefore, devised, employing only the laws of combination of these alone?

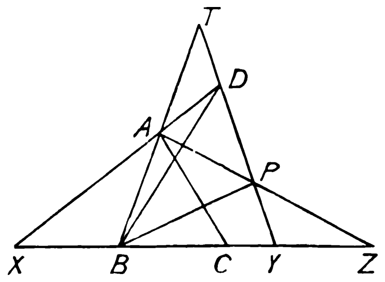
In the former of the two examples quoted above, the entities $a, b, c, \ldots$ are the real numbers; the entities $A, B, C, \ldots, A', B', C', \ldots$ are the complex numbers; each of these, we know, is representable as a couple, (x, y), of two real numbers, and among these are the couples $(x, 0)$ whose laws of combination among themselves are precisely similar to those of the real numbers. These couples $(x, 0)$, whose second element is zero, are the entities $A, B, C, \ldots$ of our abstract formulation of the question; the entities $A', B', C', \ldots$ are those complex numbers, (x, y), for which y is not zero, among which is the number $(0, 1)$ whose square is equal to $(-1, 0)$. In the latter of the two examples discussed above, the entities $a, b, c, \ldots$ are the real points, lines, etc., of a Real Geometry, widened by the introduction of the postulated points, lines, etc. The entities $A, B, C, \ldots, A', B', C', \ldots$ are the elements called points, lines, etc. in the Abstract Geometry, among which are elements $A, B, C, \ldots$ (associable with the iterative symbols and those derived from them, which behave like the real numbers of ordinary arithmetic) whose laws of combination and mutual relations (as to order, etc.) are precisely similar to those of the elements of the Real Geometry.

When stated in this abstract form, the question at issue would seem to evaporate. To say that a theorem, which may have been proved with mention of elements $A', B', C', \ldots$, relates to $A, B, C, \ldots$ alone, must mean that the theorem is a logical consequence of the laws of combination of these among themselves alone; it must therefore be capable of proof with only the use of these laws, provided sufficient ingenuity be forthcoming. That such a constructive faculty of ingenuity is indispensable to any fruitful combination of the preliminary formal laws of combination of the entities of mathematics, is a familiar experience of the study of mathematics; and an explicit recognition of this fact should not seem out of place in discussing such a question as that before us. When however the proof is found which can be stated in terms of $A, B, C, \ldots$, and their laws of combination, alone, an identical proof is possible for the corresponding theorem relating to $a, b, c, \ldots$. That the theorem may have been originally found by considerations in which the elements $A', B', C', \ldots$, were involved, as well as $A, B, C, \ldots$, thus does not affect the truth of the theorem for $a, b, c, \ldots$. But the introduction of the elements $A', B', C', \ldots$ may well have assisted the constructive faculty to which reference has been made; that this may happen is, indeed, one of the discoveries of the history of reasoning. The argument is not invalidated by the fact that the elements $A', B', C', \ldots$, in their first conception, may be creations of a judicious imagination; so regarded, they may be called abstract elements, and then the elements $A, B, C, \ldots$, as forming part of an aggregate containing $A', B', C', \ldots$, may equally be regarded as abstract, though framed to have the same relations with one another as have the elements $a, b, c, \ldots$.

The importance, and wide bearing, of the matter, seem to call for the indications, which have been given, of the argument by which the attitude we adopt is to be justified. There is however another way in which the use of so-called imaginary elements in geometry may be justified. As complex numbers may be introduced in arithmetic, each as a pair of real numbers, so aggregates of real elements may be introduced in the Real Geometry—if such aggregates can be discovered—whose laws of combination among themselves, and with the existing real elements, shall be those which we ascribe to the imaginary elements. Then such aggregates may be used in place of the imaginary elements; any proposition which we should otherwise state for the imaginary elements, becomes then a proposition for aggregates of real elements, capable of proof by real geometry only. And no doubt arises that a result thereby found, relating to the original real elements only, may not be true. It must always remain a striking monument to Karl Georg Christian von Staudt, that he should have elaborated, on a descriptive basis, a theory of aggregates, of real geometrical elements, with the necessary properties (with which should be compared the, frankly metrical and less complete, work of Chasles, *Géométrie supérieure*, esp. Ch. v and Ch. xxxiii). Such a theory does not raise the same logical difficulties as that spoken of above; but, setting aside the necessary prolixity of shewing that the system itself is logical, it should not be out of place to recognise explicitly that the proof of any property of these real aggregates, by the methods of real geometry, is often hard to find; it is often easier to use the language, if not the associated symbolism, of the abstract geometry. When such a theory as von Staudt's has been established this is logically allowable, if it is preferred to such a frank recognition of abstract elements as is suggested above. However this may be, the properties of such aggregates of real elements are interesting in themselves; and we give some account of them, choosing for this purpose a scheme elaborated since von Staudt's time.

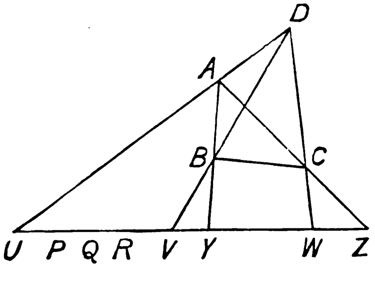
Theory of two related plane systems. Suppose two planes, ϖ, ϖ', to be given; and, in one of these, four points, A, B, C, D, which do not lie in line, there being also, in the other plane, four points, A', B', C', D', given, not lying in line. As has been indicated, we agree to say that two flat pencils, each consisting of lines through a point lying in a plane, are related to one another, when the range of points, determined by the former pencil of lines upon an arbitrary transversal, is related to the range similarly determined by the latter pencil upon any transversal. Here, as in what follows, unless the contrary be said, we are assuming Pappus' theorem. If, then, in the former plane, ϖ, any line, l, be drawn through D, a definite line, l', in the plane ϖ', passing through D', is determined, corresponding to l, by the condition that the flat pencil of four lines, $D'A'$, $D'B'$, $D'C'$, l', shall be related to the pencil of four lines, DA, DB, DC, l. In the same way, to any line drawn, in the plane ϖ, through the point A, we can make correspond a definite line, drawn, in the plane ϖ', through the point A'; and similarly for lines drawn through B and B', and for lines drawn through C and C'.

Hence, if P be any point of the plane ϖ, and we draw the lines DP, AP, and then, in the plane ϖ', we draw the lines, through D' and A', corresponding, respectively, to the lines DP, AP, in the sense just explained, we obtain, by the intersection of these, a point, P', of the plane ϖ'. It can be shewn that, then, the lines $B'P'$ and $C'P'$ also correspond, respectively, to BP and CP, in the sense just explained. For, in the plane ϖ, let the lines DA, DP, AP meet the line BC, in the points X, Y, Z, respectively, and let DP meet AB in the point T. The pencil of lines, with D as centre, passing, respectively, to B, C, A, P, is in perspective with the range B, C, X, Y. As this pencil of lines is related to the corresponding pencil of the plane ϖ', it follows that the range B, C, X, Y is related to the range obtained by similar construction in the plane ϖ'. The same may be said of the range B, C, X, Z, which lies on lines from A passing, respectively, to the points B, C, D, P. We have shewn that, if we assume Pappus' theorem, as we do, here, and in the following work, two ranges are uniquely related by the assignment of three points of one to correspond to three points of the other. Thus, as B, C, X, Y and B, C, X, Z have the three points B, C, X in common, it follows that the range B, C, X, Y, Z is related to the similarly constructed range of the plane ϖ'. However, the points of this range, B, Y, X, Z, by perspective from A, give rise to the points T, Y, D, P, respectively, and these lie on the lines from B, respectively, to the points A,C,D,P; the pencil $B\,(A, C, D, P)$ is, thus, related to the similar pencil $B'\,(A', C', D', P')$ of the plane ϖ'. A similar proof shews that the pencil $C'\,(A', B', D', P')$ is related to the pencil $C\,(A, B, D, P)$.



It can, next, be shewn that to any range of points, lying on a line of the plane ϖ, there corresponds, by the construction used to determine P' from P, a range of points lying on a line of the plane ϖ', and that this range is related to the former. For this, we shew that, if P, Q, R be three points of the plane ϖ, lying in line, and if P', Q' be the points of the plane ϖ' which correspond, respectively, to P and Q, then there is, upon the line $P'Q'$ a point, R', similarly corresponding to R. Let the points where the line PQ is met by the lines DA, DB, DC, respectively, be called U, V and W; and, similarly, the points where $P'Q'$ is met by $D'A'$, $D'B'$, $D'C'$ be called U', V' and W'. Then, by what has been said, the range (U', V', W', P') is related to the range (U, V, W, P), and the range (U', V', W', Q') is related to (U, V, W, Q); hence the range (U', V', W', P', Q') is related to (U, V, W, P, Q). And to the point R, of the line PQ, regarded as a point of the range (U, V, W, P, Q, R), will correspond a point, R_1', of the line $P'Q'$, such that the range $(U', V', W', P', Q', R_1')$ is related to the former; we are to shew that this point R_1' is the same as the point, R', constructed, with the help of A', B', C', D', to correspond to R. Let AB, AC meet the line PQ, respectively, in Y and Z, and $A'B'$, $A'C'$ meet $P'Q'$ in

Y', Z'. Then, by the original construction, by perspective from A, the range (U', P', Q', Y', Z') is related to (U, P, Q, Y, Z); and, therefore, as three points suffice to identify a range which is related to another, the range $(U', V', W', P', Q', R_1', Y', Z')$ is related to (U, V, W, P, Q, R, Y, Z). Thus the point R_1' satisfies the condition which is to be satisfied by R' with regard to pencils of lines whose centre is A', and also with regard to pencils of lines whose centre is D'. These two are sufficient to identify R_1' with R'. There is thus a range $P'Q'R'$... in the plane ϖ' which is related to the arbitrary range PQR... in the plane ϖ, as we desired to prove. From this it is easy to deduce that there is a pencil of lines, passing through a point, in the plane ϖ', which is related to any such pencil of the plane ϖ.

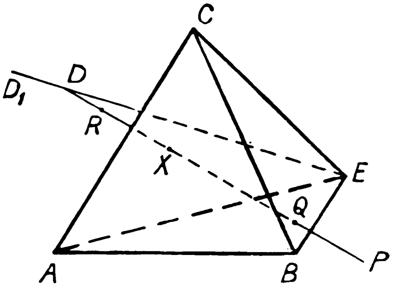


The plane systems, in ϖ and ϖ', so constructed, will be said to be *related*. The assumption made that, given DA, DB, DC, l, in one plane, and $D'A'$, $D'B'$, $D'C'$, in the other, then l' is determinate, is to be noticed.

Two related threefold systems, or spaces. A similar correspondence may be set up between the points, lines and planes of two three-dimensioned spaces, Σ and Σ', by taking any five points A', B', C', D', E' of the space Σ', of which no four lie in a plane, to correspond, respectively, to five arbitrary points A, B, C, D, E, of similar generality, in the space Σ.

There is a preliminary point to which a word may be given. In establishing a correspondence between two planes, we have presupposed a construction possible by which ranges on two lines, in these planes respectively, may be related to one another; this is certainly so when the planes lie in the same threefold space. But in comparing two threefold spaces, the presupposition must be a definite one; in the absence of a briefer general method, we may suppose the two threefold spaces to be both contained in a higher space, and ranges lying therein to be related by a chain of perspectivities, as in the foregoing theory, in Chapter I. When ranges belonging to the two spaces have been related, plane pencils of lines, and axial pencils of planes, can be related, by the ranges they determine on a transversal. In particular, however, the two threefold spaces may be the same.

This being understood, let ϖ be any plane through the line BC of the space Σ; with the planes joining BC to the points A, D, E, we then have an axial pencil of four planes. The planes, in the space Σ', which join $B'C'$ to A', D', E' respectively, taken with a plane, ϖ', through $B'C'$, in this space, form another axial pencil; and the plane ϖ' is determined to correspond to ϖ when we suppose it taken so that these two axial pencils of planes are related. Now let the line DE meet the plane ABC in the point D_1, and, similarly, the line $D'E'$ meet the plane $A'B'C'$ in the point D_1'; then, suppose the planes ABC and $A'B'C'$ related to one another, as in the preceding article, by the condition that the points A, B, C, D_1 shall correspond, respectively, to A', B', C', D_1'; and let P' be the point of the plane $A'B'C'$ so corresponding to a point P of the plane ABC. Next, let the planes BCE and $B'C'E'$, in the spaces Σ and Σ', meet DP and $D'P'$, respectively, in Q and Q'; and, corresponding to any point R taken upon the line DP, let R' be the point of the line $D'P'$ determined by the condition that the ranges $P\,Q'R'D'$, $PQRD$ shall be related. Then R' is such that the two axial pencils, each of four planes, in the spaces Σ' and Σ, represented, respectively, by $B'C'$ (A', E', R', D') and BC (A, E, R, D), with axes $B'C'$ and BC, are related to one another. The axial pencils represented, respectively, by $A'D'$ (B', C', E', R') and AD (B, C, E, R) are also related, since, on the planes $A'B'C'$ and ABC, these give, respectively, the flat pencils A' (B', C', D_1', P') and A (B, C, D_1, P), of centres A' and A; and, by construction, the plane systems, (A, B, C, D_1, P) and (A', B', C', D_1', P'), are related. The five given points A, B, C, D, E lead to ten lines joining pairs of these; by passing through each of these joining lines a plane, to the other three of the five given points, and to the point R, we obtain eight axial pencils, each of four planes, beside the two we have considered having BC and AD for axes. It can be shewn that every one of these is related to the axial pencil similarly constructed in the space Σ'. Consider, for instance, the axial pencil CA (B, D, E, R), of axis CA. The point P is on the plane ABC, and the point Q is on the plane BCE; let the line DP meet the plane ACE in the point X. The axial pencil CA (B, D, E, R) gives, on the line DP, the range (P, D, X, R). The line ED meets the plane CAB in the point D_1. Thus, the range (P, D, X, Q), being a section of the axial pencil CE (P, D, A, B), of which a plane section is the flat pencil C (P, D_1, A, B), is related to this flat pencil; and this, by construction, is related to the corresponding flat pencil C' (P', D_1', A', B') of the space Σ'. Wherefore the range (P, D, X, Q) is related to the corresponding range $(P', D'\ X', Q')$ of the space Σ'. By definition of R', however, the



range (P, D, Q, R) is related to the corresponding range (P', D', Q', R'); as (P, D, X, Q) and (P, D, Q, R) have the three points P, D, Q in common, we infer that the range (P, D, X, Q, R) is related to the corresponding range (P', D', X', Q', R'). We have seen, however, that the range (P, D, X, R) is a section of the axial pencil CA (B, D, E, R). This axial pencil is thus related to the corresponding axial pencil $C'A'$ (B', D', E', R'), of the space Σ', as we desired to shew. The range (D, X, Q, R) is also a section of the axial pencil EC (D, A, B, R), which is, therefore, related to the corresponding axial pencil $E'C'$ (D', A', B', R').

The point R is any point of the line DP, and P is any point of the plane ABC; thus the point R is any point of the threefold space A, B, C, D, E. With the proof of the theorem above stated, that the ten axial pencils of planes whose axes are the joins of pairs of the points A, B, C, D, E, each pencil containing a plane passing through R, are related, respectively, to the corresponding axial pencils in the space Σ', there is thus established a correspondence between the points of the spaces Σ and Σ'. It can then be shewn, in a manner analogous to that adopted for the case of two planes, that a range of four points lying in line in the space Σ corresponds to a related range of four points lying in line in the space Σ'; and then that a set of five points lying in a plane in the space Σ gives rise to a set of five points lying in a plane in the space Σ', in such a way that the pencil of lines joining one of the five points to the other four, in Σ, is related to the corresponding pencil in Σ'.

Case of two spaces both of n dimensions. And the argument can, it appears, be extended, by the method of induction, to two spaces of any number of dimensions, it being necessary and sufficient, in order that two n-fold spaces should be related, that $(n+2)$ points of one space, of general position, be assigned to correspond to $(n+2)$ points of the other, also of general position; and also a range in one space related to a range in the other. See Note I, at the end of the Volume.

Comparison of the correspondence between two spaces with the correspondence between either space and the symbols used to represent it. The correspondence in the three foregoing articles is established by purely geometrical considerations, based on the Propositions of Incidence, and the assumption of Pappus' theorem. When two spaces are so related to one another, either may be regarded as representative of the other, there being a geometrical incidence, or theorem, in either, corresponding to any geometrical incidence, or theorem, in the other. Conversely two spaces, in which are given two related ranges, one in each, which are such that, to every incidence of elements in one space corresponds a precisely similar incidence of elements in the other, are necessarily related, as we see by recalling the construction of two related ranges. Now compare, with this correspondence between two spaces, the correspondence we have sought to set up between one of these spaces, and the realm of the symbolism we have introduced to represent this space (Chap. I, Section III). To every element, or incidence of elements, in the space, corresponds a symbol, or agreement of two symbols, in the algebraic realm; and, conversely, to every fundamental identity (law of operation) between the symbols, and, hence, to every derived identity, corresponds a coincidence of points in the space, as we have shewn in detail. Moreover the *freedom* in the establishment of the correspondence between the related spaces, that $(n+2)$ points of one space may be chosen arbitrarily to correspond to $(n+2)$ points of the other, agrees with the circumstance that, to fix the symbol of every point of the space, the symbols of $(n+2)$ of its points must be assigned, although these $(n+2)$ symbols will be in syzygy. This condition is, in part, to be compared with the fundamental proviso that the dimensions of the two spaces must be the same; but is vital in order that the points referred to, in the statement that every identity of symbols corresponds to a coincidence of points, shall be without ambiguity; this however requires also, in the case of the abstract points not corresponding to iterative symbols, that the correspondence of the symbols and points be supposed assigned for all the points of one line of the space. See Note I, at the end of the Volume.

For the points of a line, in order to be able to assign a definite symbol to every point, it is necessary that the symbols belonging to three points of the line should be assigned. If A, B be two points of the line, the former of these may, by the rules given for the symbols, be represented either by a symbol A, or by a symbol mA, where m is any one of the algebraic symbols. Thus the symbol for a general point of the line, $A+xB$, is incapable of identifying a particular point, since it might equally have arisen in the form $mA+xnB$, where m, n are any two of the algebraic symbols; this is the same as $A+m^{-1}nxB$, since, with Pappus' theorem, the algebraic symbols are commutative in multiplication. If, however, the symbol be given which is to belong to a third definite point of the line, say $A+x_0B$, with a definite symbol for x_0, this suffices to fix the previously unspecified factor $m^{-1}n$. Then every point of the line has a definite symbol; and, conversely, we suppose there exists on the line the point corresponding to every symbol $A+xB$ which can arise by the operations to which these algebraic symbols are liable. That the condition secures the same kind of definiteness as is involved in distinguishing between two related ranges, may be well illustrated by proving the following result, of which frequent use is made subsequently: *The condition that four points of a line represented, respectively, by symbols A, B, $A + xB$, $A + yB$, should be a range related to that of four points, on the same or on another line, which have symbols P, Q, $P + pQ$, $P + qQ$, is that* $yx^{-1} = qp^{-1}$. In accordance with our conventions, this is the same, writing B for xB, and Q for pQ, as saying that the condition that the range of four points represented, respectively, by A, B, $A + B$, $A + \xi B$, should be related to the range of four points represented, respectively, by P, Q, $P + Q$, $P + \sigma Q$, is that $\xi = \sigma$. This is in accordance with the comparison we have made between the condition that two ranges should be related, and the condition that the points of a range should be represented by the symbols. To prove the result, when the two ranges are on non-intersecting lines in three dimensions, let the four joins of the corresponding points, of A to P, of B to Q, of $A + B$ to $P + Q$, and of $A + \xi B$ to $P + \sigma Q$, be met by another line, respectively, in points with symbols $A + \lambda P$, $B + \mu Q$, U and V. As the point U is then a derivative of $A + B$ and $P + Q$, as well as of $A + \lambda P$ and $B + \mu Q$, we infer $\lambda = \mu$, and $U = A + B + \lambda (P + Q)$. As V is a derivative as well of $A + \xi B$ and $P + \sigma Q$ as of $A + \lambda P$ and $B + \lambda Q$, the symbol $A + \xi B + \kappa (P + \sigma Q)$ must, for proper κ, be the same as $A + \lambda P + \xi (B + \lambda Q)$, so that $\kappa = \lambda$, and hence $\lambda\sigma = \xi\lambda$. This gives[1], as stated, $\sigma = \xi$. The symbols are assumed commutative in multiplication, without which, as was shewn in Chapter I, there is not definiteness in the condition that two ranges be related.

For the points of a plane, to render unique the correspondence between the symbols and the points, it is necessary to distinguish between the symbols A, B, C of three fundamental points of the plane, and, respectively, the symbols of the forms lA, mB, nC, which represent the same points. And this can be done by providing that in the symbol $xA + yB + zC$, belonging to a chosen definite point of the plane, the symbols $x^{-1}y$, $x^{-1}z$ shall have stated values. Very often the most convenient way of doing this is, to select the point to which the symbol $A + B + C$ is to be attached. As will be seen later, in dealing with the theory of linear transformations, the plane system of which the points are represented by symbols $xlA + ymB + znC$, wherein l, m, n are the same for all points, is related to the plane system of which the points are represented by symbols $xA + yB + zC$, wherein x, y, z vary from point to point.

---

[1] So that, as was remarked (above, p. 93), when approached from this point of view, von Staudt's introduction of numbers, to represent Würfen, or ranges of four points on a line, involves the assumption of Pappus' Theorem. For, without this Theorem, we have shewn that the range B, A, $B + \mu^{-1} A$, $B + \lambda^{-1} A$ is related to the range A, B, $A + \lambda B$, $A + \mu B$; and $\mu\lambda^{-1} = \lambda^{-1}\mu$ requires $\lambda\mu = \mu\lambda$.

Similar remarks evidently apply whatever be the number of dimensions of the space considered.

A geometrical existence theorem for the solution of an algebraic equation. The preceding article is concerned with the justification of the use of algebraic symbols of any generality which are subject to the laws of combination set out in detail in Section III of Chapter I, together with that of commutative multiplication. As indicated in the discussion at the beginning of this chapter, there is a quite definite system of symbols with these properties which has the further property of containing, corresponding to any one of its members, c, another member, z, which is such that $z^2 = c$. It is desirable then to examine the geometrical meaning of this. If this system of symbols be that which is adopted as guide in formulating the properties of the *points*, *lines*, etc., which we finally adopt as the elements of the Abstract Geometry, it is convenient to summarise these properties, if possible, by enunciating a geometrical construction which can be carried out with the abstract elements, or by stating a theorem which holds when these elements are allowed.

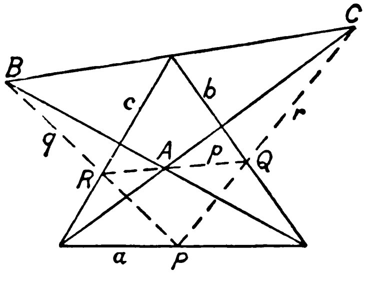
With this object we turn now to a geometrical problem, which we consider, in the first instance, in an elementary way; shewing how it is related to other problems whose fundamental importance may be, at first sight, more obvious:

In a plane, we are given three points, A, B, C, and three lines, a, b, c. It is required to draw three lines, p, q, r, passing, respectively, through the points A, B and C, whose points of intersection in pairs, namely P, of q and r, Q, of r and p, and R of p and q, shall lie, respectively, on the lines a, b, and c. In this case, the lines joining the pairs of points, namely p, of Q to R, q, of R to P, and r, of P to Q, pass, respectively, through the points A, B, and C. Thus the problem is one of a self-dual character.

We arrange our considerations under nine headings:

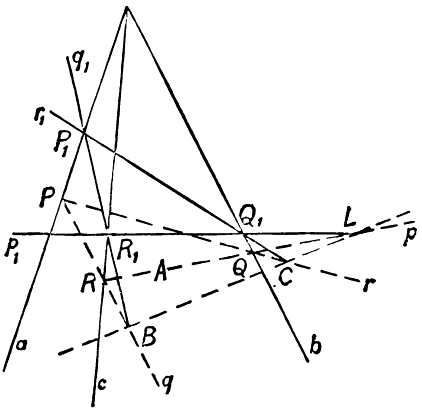
(i) There is an infinite number of solutions of the problem when the lines BC, CA, and AB, pass, respectively, through the points (b, c), (c, a), and (a, b). This is, in fact, another statement of Pappus' theorem.

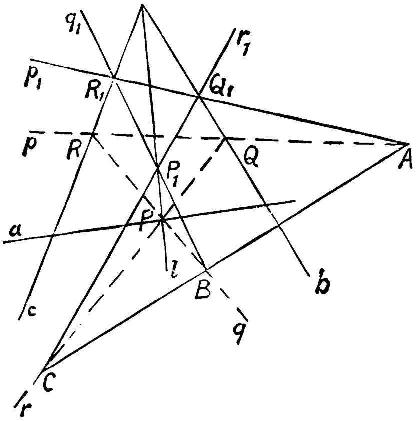
It can be shewn, when no one of the points A, B, C lies on any of the lines a, b, c, and A, B, C are not in line, that this is the only case in which there is an infinite number of solutions of the problem.



(ii) When the points A, B, C are in line, a degenerate solution of the problem is that in which the lines p, q, r all coincide in this line; in this case, another solution can be directly constructed. And dually.

In the first case, when the points A, B, C are in line, draw through A any line, p_1, meeting the lines b and c, respectively, in Q_1 and R_1, and let the lines CQ_1 and BR_1 intersect in the point P_1. Make a similar construction with a line, p_2, drawn through A, so obtaining a point P_2. Then, let the line P_1P_2, or l, which contains (b, c), meet the line a in the point P. It will be seen that, if the lines CP, BP be now drawn to meet the lines b and c, respectively, in Q and R, then the line QR passes through the point A, and we



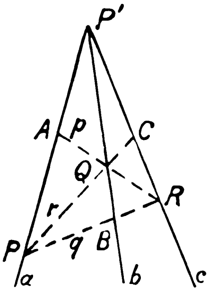


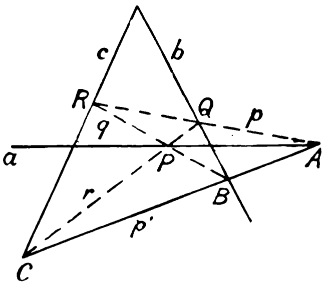
have a solution of the problem. A similar construction is valid for the reciprocal case, starting with points, P_1 and P_2, taken anywhere on the line a; the lines, p_1 and p_2, reciprocal to P_1 and P_2, intersect in a point L, lying on BC; the required line, p, is the line AL.

The proof of the statements made is immediate from Desargues' theorem. In the former case, the corresponding joins of points of the two triads P_1, Q_1, R_1 and P_2, Q_2, R_2, meet in the points A, B, C, respectively, which are in line. Hence the join P_1P_2 passes through the point (b, c), which is the intersection of Q_1Q_2 and R_1R_2. And then, as the lines, PP_1, QQ_1 and RR_1, meet in the point (b, c), it follows, from the triads P, Q, R and P_1, Q_1, R_1, that the lines QR and Q_1R_1 meet on the line BC, in the point A. The proof in the latter case is the reciprocal of this.

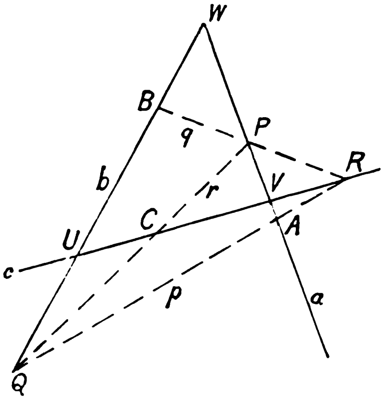
When A, B, C are in line, and, also, a, b, c meet in a point, the degenerate solutions are the only ones, unless, under a certain condition, there is an infinite number of solutions.

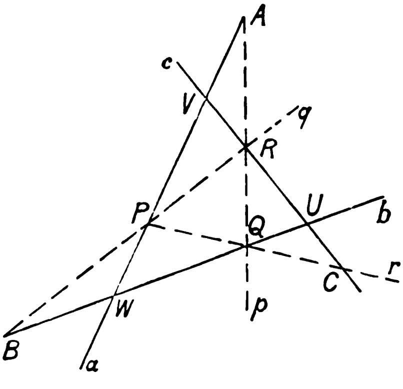
A particular case of the figure arises when the points A, B and C lie, respectively, on the lines a, b and c. Then it is easy to see that the points P, Q, R are harmonic conjugates, respectively, of A, B, C, in regard to the respective couples of lines b and c, c and a, a and b. If the line ABC be denoted by p', then the lines p and p' are harmonic conjugates in regard to the line a and the line joining the points A and (b, c), etc.





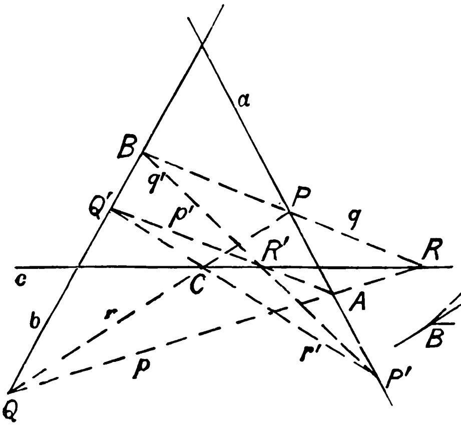
(iii) The simplest, and, as will be seen, the most fundamental of the cases in which the lines p, q, r cannot be directly constructed, is that, in which the points A, B, C lie, respectively, on the lines a, b, c, but are not in line. When this is so, it can be seen, by considerations of betweenness, such as are used in Chapter II, that, in the Real Geometry, there exist lines p, q, r provided the points A, B, C have suitably limited positions. If the points (b, c), (c, a) and (a, b) be denoted, respectively, by U, V and W, this is so when A, B and C are outside the respective segments VW, WU and UV; or when one of them is outside and the other two are inside these segments, respectively.

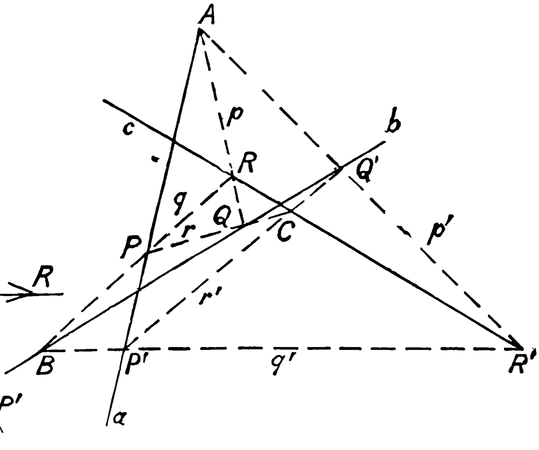




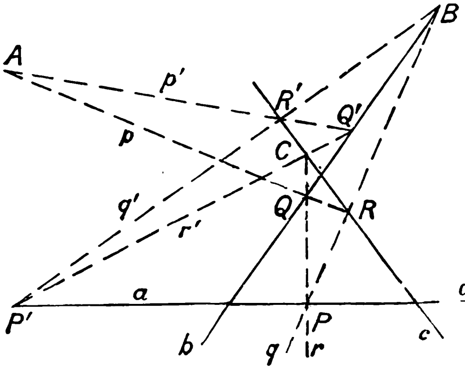
For in the former case, if two lines, p_1 and p_2, be drawn, through the point A, to meet the line b, respectively, in the points Q_1 and Q_2, and these meet the line c, respectively, in R_1 and R_2, in such a way that UR_1R_2V are in order, then also UQ_1Q_2W will be in order. If then BR_1 and CQ_1 meet the line a, respectively, in H_1 and K_1, and BR_2, CQ_2 meet a, respectively, in H_2, K_2, it follows that WH_1H_2V are in order, and that WK_2K_1V are in order. By use of the Dedekind suggestion, as in Chapter II, it can thence be shewn that there is a position of R_2 for which H_2 and K_2 coincide, say in P_2. A similar argument is applicable to the case, for instance, in which B is within the segment WU, and C within the segment UV, but A is without the segment VW, and to the other two such cases. The reciprocal figure, it is easily seen, leads, essentially, to the same results.

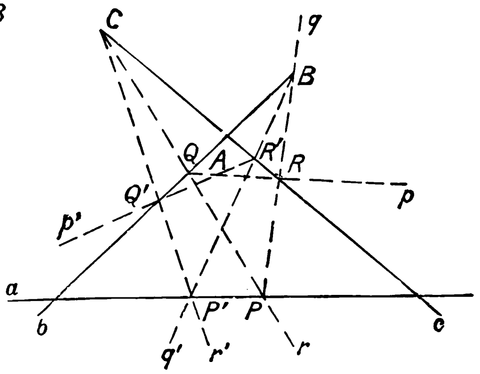
When we have one possible set of positions for P, Q, R, solving the problem, another set can be deduced at once; namely, by taking, in place of P, the point, P', which is the harmonic conjugate of P in regard to V and W, or, say, $P' = (V, W)/P$, and taking, at the same time, $Q' = (W, U)/Q$ and $R' = (U, V)/R$.





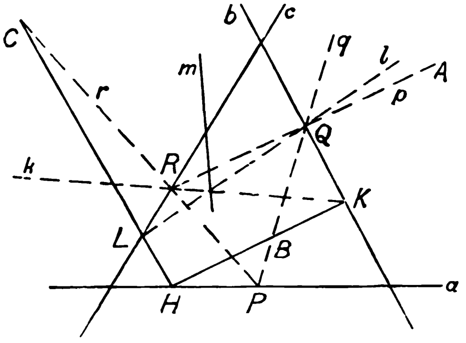
(iv) Another case which is worthy of special mention is that in which the points B and C lie on the lines b and c, respectively, but A does not lie on a.





It will be seen below that the solution in this case can be constructed from that considered under (iii).

(v) The case in which the points A, B, C have arbitrary positions can be reduced to that preceding, in which two of the points, B and C, lie on their respectively associated lines, b and c.



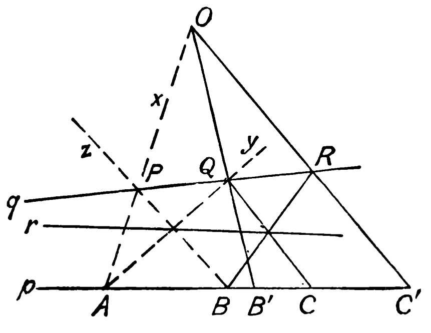
The given lines a, b, c, and the given points, B, C, define a line, m, as follows: taking arbitrary points H, H', H'', ..., on the line a, let the joins of these to the point B, namely the lines HB, $H'B$, $H''B$, ..., meet the line b in the respective points K, K', K'', ..., and the joins of these to the point C, namely the lines HC, $H'C$, $H''C$, ..., meet the line c in the respective points L, L', L'', ...; the two ranges, K, K', K'', ..., and L, L', L'', ..., are then related. Therefore, as was proved in Chapter I, the points of intersection of the various pairs of cross joins of these ranges, namely, the points $(KL', K'L)$, $(KL'', K''L)$, $(K'L'', K''L')$, ..., all lie on a line which we denote by m.

Take two corresponding points, K and L, of these two related ranges, lying, respectively, on the lines b and c, and, with these, the point A. Take the lines b, c and m. By the solution of the previous case, (iv), we can draw through K, L and A, respectively, the lines k, l and p, so that their intersections in pairs shall lie, respectively, on the lines b, c and m, namely the point (l, p) on b, the point (p, k) on c and the point (k, l) on m. Hence, if the points (l, p) and (p, k) be, respectively, Q and R, these will be points of the ranges $(K, K', ...)$ and $(L, L', ...)$ above described, and the lines BQ, CR, which we may call, respectively, q and r, will meet in a point, P, of the line a. The lines p, q, r are then such lines as are required for the solution of the general case of the problem in hand, which is then reduced to the case when B and C lie on b and c respectively, here considered under (iv).

(vi) There is another problem of fundamental importance which is also reducible to the case considered under (iv), that, namely, of the construction of the common corresponding points of two related ranges lying on the same line.

Let $(B, C, ...)$ and $(B', C', ...)$ be two ranges on a line, p, of which the latter is related to the former, according to the definition we have adopted in Chapter I (above, p. 25), by the fact that $(B', C', ...)$ is in perspective, from a point O, with a range, $(Q, R, ...)$, of another line, q, which is related to the range $(B, C, ...)$. The line

q will intersect p. Thus, as was proved for two related ranges on different lines in the same plane, the cross joins BR, CQ meet in a point lying on a definite line, r, whatever pair of corresponding points of these ranges is substituted for C and R.



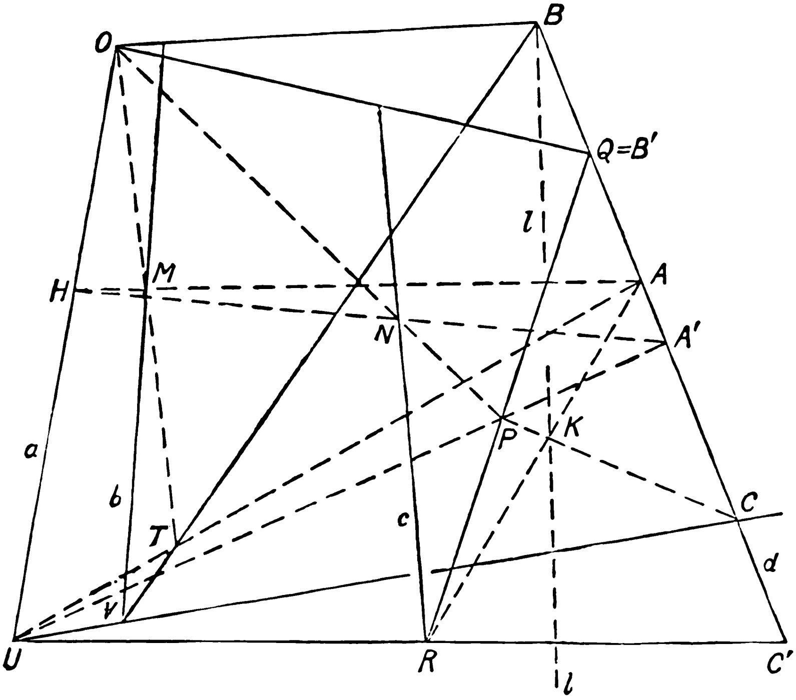
A point, A, of the range (B, C, ...), coincides with the corresponding point, A', of the range (B', C', ...), if, and only if, the corresponding point, P, of the related range on q, lie with A on a straight line through O. Thus, the problem of finding A is that of drawing through the point O, and through two, arbitrarily taken but corresponding, points Q and B, lying, respectively, on the lines q and p, the respective lines x, y and z, so that y and z may meet on r, z and x may meet on q, and x and y may meet on p.

This reduces the problem of finding the common corresponding points to that considered under (iv). For this problem, reference may also be made to a solution of entirely different character, Chasles, *Géom. supér.* (1880), p. 175.

(vii) The problem of finding common transversals of four lines in three dimensions, of which no two intersect, is also reducible to that considered under (iv).

Let a, b, c, d be four lines, in three dimensions, of which no two intersect. Take any two arbitrary fixed points, O and U, upon the line a. Draw from O the transversal to the lines b and d, meeting d in B; also from O the transversal to c and d, meeting d in Q. Draw from U the transversal to the lines b and d, meeting these, respectively, in V and C; also from U the transversal to c and d, meeting these, respectively, in R and C'. The plane VUR contains the line d; the plane from O containing the line b meets the plane VUR in the line VB, and the plane from O containing the line c meets the plane VUR in the line RQ. Now take any point, H, of the line a: from H draw the transversal to the lines b and d, meeting these, respectively, in M and A, and also the transversal to the lines c and d, meeting these, respectively, in N and A'. The plane from M containing the line a will meet the plane VUR in the line UA; this line will, therefore, intersect the line VB, say in the point T, and the line TM will intersect the line a; the line TM lies, however, in the plane MVB, which meets the line a in the point O; thus, the points O, M, T are in line. The plane from N containing the line a will meet the plane VUR in the line UA'; this line will, therefore, intersect the line RQ, say in the point P, and the line

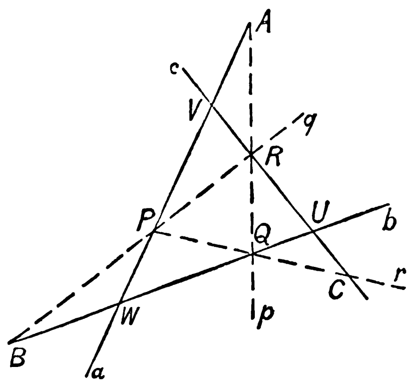
PN will intersect the line *a*; the line *PN* lies, however, in the plane *NRQ*, which meets the line *a* in the point *O*; thus, the points *O*, *N*, *P* are in line.



As the point *H* takes different positions on the line *a*, the points *A′*, *P*, *N*, *M*, *T*, *A* will take different positions, each on a line independent of *H*, constituting ranges which are all related. In particular, when *H* is at *O*, the points *A′*, *P*, *A* will be, respectively, at *B′*, *Q*, *B*, on the line *d*, the points *B′* and *Q* being the same; and, when *H* is at *U*, the same points *A′*, *P*, *A* will be, respectively, at *C′*, *R*, *C*. Considering the two related ranges, in one plane, whose points are the corresponding positions of *P* and *A*, respectively, of which two corresponding points are, respectively, *R* and *C*, arising when *H* is at *U*, the cross joins, *PC* and *AR*, meet in a point, *K*, which describes a definite line, *l*; and this line contains the intersection of the cross joins, *RB* and *CQ*, which is the point *B*; for *Q* and *B* are simultaneous positions of *P* and *A*, arising when *H* is at *O*.

In order, now, to draw a common transversal of *a*, *b*, *c*, *d*, it is necessary to find such a position for *H* that *A* and *A′* may coincide. For this it is necessary, and sufficient, to draw from the given points *U*, *R* and *C*, of the plane *VUR*, three lines *UPA′*, *RA*, *CP*, respectively, lying in this plane, so that the intersections of pairs of these, (*RA*, *CP*), (*CP*, *UPA'*), (*UPA'*, *RA*), shall lie, respectively, on the given lines l, RQ, $C'Q$, of this plane. Of the given points and lines, the points R and C lie, respectively, on RQ and $C'Q$. The problem of drawing the common transversals of the four given lines a, b, c, d, is, thus, reduced to the problem above considered, under (iv). For this problem, the reader may also consult Steiner, *Ges. Werke*, I, p. 403, where reference is made to Gergonne's *Ann. de Math.* XVII (1826, 7), p. 83.

(viii) Consider now the problem above referred to as (iii), in connexion with the symbolism which we have used.



Take the points (b, c), (c, a) and (a, b) as fundamental, the lines a, b, c being supposed not to meet in a point. Denoting these points, respectively, by U, V, and W, we may choose the symbols V and W, in relation to U, so that

$$C = U - V, \quad B = U - W,$$

and then take, for the symbol of the point A,

$$A = a^2V - W,$$

where a^2 is not 1, the points A, B, C not being in line. If then the symbol of the point P be taken as

$$P = \lambda V + W,$$

this involves in turn, for the symbols of the points Q, R and A, respectively

$$Q = \lambda U + W, \quad R = U + \lambda V, \quad A = \lambda^2 V - W.$$

Identifying this form for A with that given, we infer $\lambda^2 - a^2 = 0$, or $(\lambda - a)(\lambda + a) = 0$, and it was definitely assumed that the system of symbols used should have only 0 as singular symbol. Wherefore λ is a or $-a$. Thus, when the given points A, B, C are, respectively, written as $a^2V - W$, $U - W$ and $U - V$, one set of positions, for P, Q, R, is that given by

$$P = aV + W, \quad Q = aU + W, \quad R = U + aV,$$

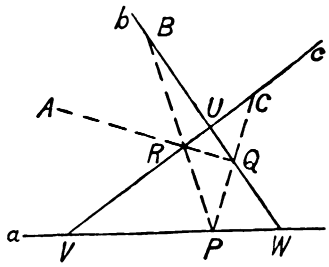
another, and the only other, set being obtained from this by putting $-a$ in place of a.

It is seen without difficulty that the figure employed here agrees with the construction given in Chapter I (above, p. 84) for determining the point $O + a^2U$ when O, U, $O + U$ and $O + aU$ are given.

(ix) Take now the case, considered in (iv), in which the given points B and C lie respectively on the lines b and c.

As before, let the points (b, c), (c, a) and (a, b) be, respectively, denoted by U, V and W, and suppose neither B nor C to be at U. We can, then, suppose the symbols of A, B and C to be, respectively, of the forms

$$B = U - 2W, \quad C = U - 2V,$$
$$A = U + mV + nW.$$



If then the symbol of the point P be taken as

$$P = mV + \lambda W,$$

those of Q and R will, respectively, be

$$Q = mU + 2\lambda W, \quad R = U + 2\lambda^{-1}mV,$$

and, in order that the line QR should pass through the point A, there must exist two symbols, q and r, securing the syzygy

$$q(mU + 2\lambda W) + r(U + 2m\lambda^{-1}V) = U + mV + nW.$$

This requires the equations

$$qm + r = 1, \quad 2rm\lambda^{-1} = m, \quad 2q\lambda = n;$$

we are assuming Pappus' theorem, so that the symbols are commutative in multiplication; hence, eliminating q and r, we deduce

$$mn\lambda^{-1} + \lambda = 2, \quad \lambda^2 - 2\lambda + mn = 0.$$

Thence, if θ be a symbol such that

$$\theta^2 = 1 - mn,$$

we have $(\lambda - 1)^2 = \theta^2$, and hence λ has one of the two forms

$$\lambda = 1 + \theta, \quad \lambda = 1 - \theta.$$

Now, by the constructions given in Chapter I (pp. 74, 76), when the points A, B, C, and therefore the symbols m and n, are given, we can construct the point whose symbol is

$$mV + (1 - mn)W,$$

or $mV + \theta^2 W$. From this, by the construction under (viii), preceding, we can find the two points with symbol

$$mV + \theta W,$$

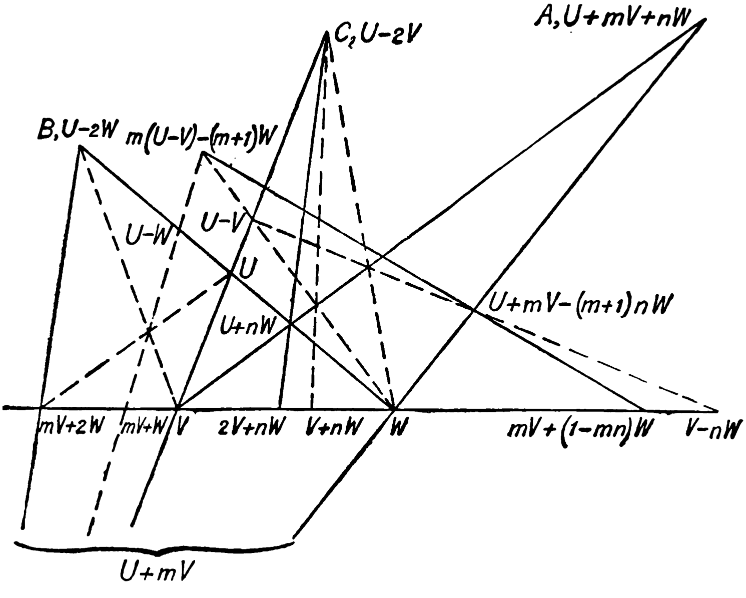
and hence the points of symbol

$$mV + (1 + \theta)W.$$

The general problem now under consideration can, therefore, be solved by direct construction if that under (iii) or (viii) can always be solved.

Remark. The reader may like to see these constructions carried out, in part. The given points are U, V, W, which are the points (b, c), (c, a), (a, b), together with $A = U + mV + nW$, $B = U - 2W$, $C = U - 2V$. A construction for

$mV+(1-mn)W$ is as follows: By joining $U+mV+nW$ to V and W we obtain, respectively, $U+nW$ and $U+mV$, on the lines UW and UV. By joining these, respectively, to $U-2V$ and $U-2W$, we obtain $2V+nW$ and $mV+2W$, on VW. From the former of these we construct $V+nW$, the harmonic conjugate of V in regard to $2V+nW$ and W; from the latter we construct $mV+W$, the harmonic conjugate of W in regard to $mV+2W$ and V. From $V+nW$, by taking its harmonic conjugate in regard to V and W, we obtain $V-nW$. Then, having $V+m^{-1}W$ and $V-nW$, we find $V+(m^{-1}-n)W$, or $mV+(1-mn)W$, by the rule for constructing a point specified by the sum of two algebraic symbols which was given in Chapter I. Namely, two lines through $mV+W$ and $V-nW$, respectively, are met by a line through V in the points $U+mV$ and $U-V$, and lines through W to these points meet the two lines first drawn, alternatively, in $U+mV-(m+1)nW$ and $m(U-V)-(m+1)W$; the join of these gives the point $mV+(1-mn)W$.



This is the point $mV+\theta^2W$, on the line VW; and we have $U-W$ on the line UW, the harmonic conjugate of W in regard to U and $U-2W$, and $mV+U$ on the line UV. Hence, if the result of (viii), preceding, be assumed, we can find $mV+\theta W$, on the line VW. Then, having $mV+W$, we can construct $mV+(1+\theta)W$.

Statement of the further Axiom henceforth assumed in the Abstract Geometry. In accordance with the view here adopted, of the possibility of extending the conception of the geometrical entities discussed, we shall suppose, in future, that, in the Abstract Geometry, the meanings attached to the words *point*, *line*, *plane*, etc., are such that the system of algebraic symbols appropriate to their discussion, beside obeying the formal laws of combination set forth in Chapter I, and the commutative law in multiplication, is also such that, to any symbol, c, there exists another, z, such that $z^2=c$. From the preceding discussion it follows that this is equivalent to supposing that the geometrical entities, beside being subject to the Propositions of Incidence, and such that Pappus' theorem is true, are also such that a certain geometrical problem is always capable, if not of an infinite number, then, of *two*, solutions, which may coincide. We have given three alternative forms for this problem, either of which may be taken, in virtue of the other assumed properties of the entities. Either we may say that, if A, B, C be points given anywhere on three lines a, b, c, lying in a plane, it is possible to draw three lines p, q, r, respectively through A, B, C, such that the points of intersection (q, r), (r, p), (p, q) lie, respectively, on a, b, c. From this it follows that the same is true for quite arbitrary positions of A, B, C, not lying on the lines a, b, c. Or we may say that two related ranges of points on the same line, not entirely identical, have two common corresponding points, which may coincide. Or, finally, we may say that, four lines in space of three dimensions, of which no two intersect, are met, if not by an infinite number, then, by *two* transversals, which may however coincide.

It may be said that, later on, it will be assumed that the geometrical entities are such that, in the symbolism appropriate thereto, equations of general character, not merely $z^2 = c$, are capable of solution. It would be proper to discuss now whether this is an additional condition, or contained in what has been adopted. But we do not enter into this at present.

Clearly, from the preceding discussion, the problem of which we postulate a solution in the Abstract Geometry, has not, in all cases, a solution in the Real Geometry. When the theory is supposed to deal only with a real geometry, it is, therefore, usual to speak of *imaginary* elements. As we have said, we may, however, introduce into the Real Geometry certain aggregates of real elements, whose relations with the primary real elements, and with one another, are formally the same as those of the so-called imaginary elements. We now pass to a brief account of aggregates of this kind which may be taken, if it be desired to maintain the point of view of a Real Geometry.

Aggregates of real elements with the properties of imaginary elements. We desire to avoid the use of the words *imaginary point*, *imaginary line*, etc., when establishing the theory, at least. We shall, therefore, use a phraseology which is, confessedly, somewhat cumbrous. In addition to points, we consider *Point-Sets*, each of three points on a line: Any three points on a line, A, B, C, are regarded as taken in one of two definite orders, the sets B, C, A and C, A, B being regarded as equivalent with the set A, B, C; and the sets C, B, A and A, C, B as equivalent with the set B, A, C. Then, further, the set A, B, C is regarded as equivalent with any other set, A', B', C', of three points of the same line, when the three ranges (A', A, B, C), (B', B, C, A), (C', C, A, B) are related. Thus, Pappus' theorem being assumed, a set, A', B', C', can be found equivalent with the set A, B, C, of which one point, say A', is any arbitrary point of the line, the points B', C', associated therewith, being then determinate. It is easy to prove that the ranges (A, A', B', C'), (B, B', C', A'), (C, C', A', B') are related, so that if the set A', B', C' is equivalent to the set A, B, C, then this latter is equivalent to the former. It can be further shewn that two sets, A', B', C' and A'', B'', C'', which are both equivalent to A, B, C, are equivalent to one another.

The whole aggregate of equivalent point sets of the line is the entity called a *point series*. From three points of the line, taken in different orders, two such point series can be formed; these may be called *conjugate* point series. When two of the points of a set A, B, C, of a point series, are coincident, say A with B, the point sets, equivalent with this, are those formed with A, B and any further arbitrary point of the line, and one particular set, of these, consists of the point A taken three times over. In the Real Geometry we have, upon the line, points; in the scheme we are explaining we have, *general point series*, each determined by a set of three points of which no two coincide, and *the degenerate point series*, each of which is determined by three coincident points; while a general point series lies on a definite line, an infinite number of lines contain the three coincident points of a degenerate point series. The laws of manipulation of the general point series will lead to laws for the degenerate point series which are exactly similar to those of ordinary real points[1]. We may say that the number of point series upon a line is ∞^2; for of a point set determining the series, one point may be at any position chosen beforehand, the same for all series, and any particular series then depends on the positions of the other two points of a set determining the series; each of these two points is capable of positions on the line which we usually say to be ∞^1 in number.

Reciprocal to point sets, each of three points on a line, we consider also plane sets, each of three planes passing through a line. A set of three such planes, α, β, γ, is regarded as equivalent to either of the two sets β, γ, α or γ, α, β, but distinct from the equivalent sets, γ, β, α, or β, α, γ, or α, γ, β. And a set of three other planes, α', β', γ', passing through the same line, is regarded as equivalent to α, β, γ when the axial pencils $(\alpha', \alpha, \beta, \gamma)$, $(\beta', \beta, \gamma, \alpha)$, $(\gamma', \gamma, \alpha, \beta)$ are related. The aggregate of all equivalent plane sets is called a *plane series*; and there are degenerate plane series in which all the equivalent plane sets have two coincident planes in common.

---

[1] The reader may compare the general logical formulation given above, p. 146.

A point series is then said to *lie upon* a plane series, when the points of any one of the equivalent sets of the point series, taken with regard to order, lie, respectively, upon the corresponding planes of any one of the equivalent sets of the plane series. In particular the condition is satisfied if the points A, B, C lie, respectively, upon the planes β, γ, α, as well as if they lie respectively upon α, β, γ. In general the line containing the point series, and the axis of the plane series, are different; but the condition is satisfied of itself when these lines coincide. A point series, (A, B, C), lies on a degenerate plane series, (α, α, γ), when the base line of (A, B, C) lies on the plane α; reciprocally, a degenerate point series (A, A, C) lies on a plane series (α, β, γ), when the axis of the plane series contains the point A. When both the point series and the plane series are degenerate, being (A, A, C) and (α, α, γ), the condition is merely that the point A shall lie on the plane α.

Two point series, whose base lines are not in one plane, determine what we call a *skew linear series*, consisting of sets of three lines of which no two intersect; such a set of three lines is obtained by joining the points, A, B, C, of any set of one of the point series, respectively to the points, A', B', C', of any set of the other point series. From any set of three lines, of which no two intersect, another set may be obtained, by taking any two transversals of these three given lines, meeting them, respectively, in A, B, C and in A', B', C'; then, determining upon the line ABC any point set, (P, Q, R), equivalent with (A, B, C), and upon the line $A'B'C'$ any point set, (P', Q', R'), equivalent with (A', B', C'); and, then, joining P and P', Q and Q', R and R'. The lines PP', QQ', RR' are then equivalent, as a line set, with the set of three given lines. It is clear that if, in this way, a line set (l', m', n') be equivalent with a line set (l, m, n), then the latter is equivalent with the former. It is necessary, however, if the word equivalence is to preserve its usual connotation, to shew, further, geometrically, that no logical inconsistency is involved in regarding two line sets, (l', m', n'), (l'', m'', n''), which are both equivalent, in the above sense, with a line set (l, m, n), as being equivalent with one another. In regard to this it may be remarked that the two common transversals of the six lines l, m, n, l', m', n', which enter into the definition of the equivalence of the line sets (l, m, n) and (l', m', n'), are not necessarily the same as the two common transversals of l, m, n, l'', m'', n'', which enter for the equivalence of (l, m, n) and (l'', m'', n''); it is therefore not part of the definition of the suggested extension of the meaning of the equivalence of two line sets, that the six lines $l', m', n', l'', m'', n''$ should have two common transversals. It is necessary, for our purpose, to retain for the present the point of view of the descriptive Real Geometry; but it can be seen (see Note II, at the end of the Volume), that there is, in fact, no logical difficulty. After what has been said in regard to degenerate point series, it is unnecessary to define degenerate skew linear series.

Reciprocally, a skew linear series may be defined from two plane series, whose axes have no point in common. A set of such a series consists of the intersections of the planes α, β, γ, of any set of one plane series, respectively, with the planes, α', β', γ', of any set of another plane series. If the lines forming a set so defined meet the axes of the two plane series, respectively, in A, B, C and A', B', C', the line set is equally determined by the two point sets A, B, C and A', B', C'. Take any three planes, λ, μ, ν, passing, respectively, through the lines of the line set, having a common axis, and, also, any other three planes, λ', μ', ν', with the same description; let ρ, σ, τ be a plane set equivalent with λ, μ, ν, and ρ', σ', τ' a plane set equivalent with λ', μ', ν'. If we regard the three lines, (ρ, ρ'), (σ, σ'), (τ, τ'), as equivalent with the original set, we shall only be giving, in a different form, the previous definition of equivalent line sets; for two equivalent sets of three planes, with the same axis, meet any base line in equivalent point sets.

A skew linear series is said to contain a point series when one of the equivalent line sets, of the former, contains one of the equivalent point sets of the latter. And, reciprocally, for the condition that a skew linear series should lie on a plane series. Thus, any two point series lying on a skew linear series serve to define the skew linear series (just as any two points of a line serve to define the line); and the skew linear series, which contains two point series lying on the same plane series, lies entirely on the plane series (as the line joining two points of a plane lies entirely on that plane).

It is clear that, if any line set of three lines, belonging to a particular skew linear series, be taken, and then any line, p, be drawn to meet any two transversals of the three lines taken, there is a set, of the skew linear series, of which the line p is one line. If l, m, n be three lines of a set of the series, and p a line, such that two transversals meet these four lines, respectively, in A, B, C, P and in A', B', C', P', then a point set, P, Q, R, can be found, on the former transversal, equivalent to A, B, C, and a point set, P', Q', R', on the latter, equivalent to A', B', C'. Denoting the lines QQ' and RR', respectively, by q and r, the three lines p, q, r are a set of the skew linear series. Hence it follows that, *of a given skew linear series, there is one set of which one line is an arbitrary line.* For, let O, O' be any two points of the arbitrary line; let ABC and $A'B'C'$ be two transversals of one set of the skew linear series; from O draw the line, OPP', meeting these two transversals, respectively, in P and P'. Denoting this line, OPP', by p, there is a set of the skew linear series consisting of p and two other lines, q and r. Now, from O, draw the line, OMN, meeting the lines q and r, respectively, in M and N. Then O, M, N is a set of a point series lying on the skew linear series. Another such set can, similarly, be found from the point O', say O', M', N'. The three lines, OO', MM', NN', are then a set of the skew linear series, of which the given line OO' is one line. For simplicity, the description has supposed all the lines involved to be in general positions relatively to one another.

The skew linear series, as has been indicated, generalises the properties of a line. There is however another aggregate of lines which also generalises the properties of a line. This we shall call a *line series*; it can coexist with, and must be distinguished from, a *skew linear series*. This last has been defined by means of two point series whose base lines do not lie in the same plane. If the base lines of two general point series, (A, B, C) and (A', B', C'), meet in a point, O, we may take, as sets of the two series, respectively, points, O, P, Q, on the former base line, and points, O, P', Q', on the latter base line. If then the lines PP' and QQ' meet in the point H, the three lines, HO, HP, HQ, are a set of three lines which meet in a point and lie in a plane. Taking any three such lines, l, m, n, we may define sets of lines equivalent with this set in a manner analogous to that followed above for point sets and plane sets; namely, we may regard the set l, m, n as equivalent with m, n, l or n, l, m, and as equivalent with any set, l' m', n', of three lines, all passing through the same point H and lying in the same plane, as l, m, n, when the three flat pencils (l', l, m, n), (m', m, n, l), (n', n, l, m) are related. The aggregate of all such equivalent line sets, of three lines in a plane passing through a point, constitutes what we call a *line series*. It is distinguished from a skew linear series by lying in a definite plane and passing through a definite point.

As before, we may then say that such a line series contains a certain point series, when the lines of one set of the line series contain, respectively, with proper regard to order, the points of one set of the point series. For this it is necessary that the base line of the point series lie in the plane of the line series. Reciprocally such a line series is contained in a certain plane series, when the lines of a set of the line series lie, respectively, in the planes of a set of the plane series. For this, the axis of the plane series must contain the point of the line series.

Such a line series may be determined by a general point series and a degenerate point series, of which one set consists of three coincident points. One set of the line series then consists of the lines joining this triple point to the points of a set of the general point series. Such a line series is also determined by a general plane series and a degenerate plane series, of which one set consists of three coincident planes. And, as has been indicated, the fundamental definition of a line series may be by two general plane series of which the axes lie in a plane.

Now consider three kinds of entities. The first kind consists of point series, and of points, the latter being regarded as degenerate point series. The second kind consists of skew linear series, and of line series, and of lines; the lines are here degenerate forms of either of the others. The third kind consists of plane series, and of planes, regarded as degenerate plane series. With appropriate definitions, we can then shew that, as the Propositions of Incidence, with Desargues' theorem, and Pappus' theorem, hold for points, lines and planes, so theorems corresponding thereto hold for the three kinds of entities. It is in the nature of the definitions of these entities that the Principle of Duality continues to hold also.

The detailed verification of the statement will only be sketched: the three kinds of entities will be referred to as point entities, line entities and plane entities:

(*a*) Two point entities define a line entity, which is equally determined by any two point entities lying thereon. Two plane entities similarly determine a line entity. This is clear from what has been said.

(*b*) A point entity and a line entity determine a plane entity; a plane entity and a line entity determine a point entity.

It will be sufficient to consider the former statement. A point and a line determine a plane. A point and a line series determine a plane series, of which one plane set consists of the planes joining the point to the lines of a line set. A point, O, and a skew linear series, determine a plane series; for if u and v be two transversals of a set of three lines in the skew linear series, and p be the line drawn from O to meet u and v, the skew linear series, as we have remarked above, may be determined from a line set consisting of the line p and two other lines, q and r; and, if x be the line drawn from O to meet q and r, the planes xp, xq, xr are a set of a plane series containing both O and the skew linear series. It is interesting to verify that the line x so found is unique. The result involves that, through every point can be drawn a line to contain some point series lying on a given skew linear series (in another phraseology, through every point can be drawn a line to contain an imaginary point lying on a given imaginary line of the second kind; cf. von Staudt, *Geom. der Lage*, *Beiträge*, I, 1856, p. 81, no. 123).

Next, a point series, and a line, determine a plane series, whose axis is the line. A point series and a line series determine a plane when the base line of the point series is in the plane of the line series, namely this plane itself. When this is not so, let *O* be the point in which the plane of the line series is met by the base line of the point series; on this base line take the point set *O*, *P*, *Q* of the point series. If the point of the line series be at *O*, a plane series containing both entities is at once clear. When this is not so, let *H* be the point of the line series, and take a line set of the line series consisting of the line *HO* and two other lines, *p*, *q*, through *H*, lying in the plane of the line series. The planes, *Pp* and *Qq*, will meet in a line, say *l*, passing through *H*; these two planes together with the plane *lO*, are then a plane set containing both the point set *O*, *P*, *Q* and the line set *HO*, *p*, *q*; and they determine a plane series containing both the given point series and the given line series. Now consider a point series and a skew linear series. Let the point series be determined from a point set *P*, *Q*, *R*, and the skew linear series be determined from the two point series (*A*, *B*, *C*) and (*A′*, *B′*, *C′*). A plane series containing the last two series will contain the skew linear series, as we have remarked above; the problem, then, is, to determine a plane series containing three given point series (*P*, *Q*, *R*), (*A*, *B*, *C*), (*A′*, *B′*, *C′*). If (*U*, *V*, *W*) be any point series upon the skew linear series determined by (*P*, *Q*, *R*) and (*A*, *B*, *C*), then a plane series containing (*U*, *V*, *W*) and (*P*, *Q*, *R*) will contain (*A*, *B*, *C*); similarly, if (*U′*, *V′*, *W′*) be any point series upon the skew linear series determined by (*P*, *Q*, *R*) and (*A′*, *B′*, *C′*), then a plane series containing (*U′*, *V′*, *W′*) and (*P*, *Q*, *R*) will contain (*A′*, *B′*, *C′*). It will then be enough to find a plane series containing the point series (*U*, *V*, *W*), (*U′*, *V′*, *W′*) and (*P*, *Q*, *R*). We can, however, so choose (*U*, *V*, *W*) and (*U′*, *V′*, *W′*) that these lie on a line series, as we shall see; any plane series containing this line series will contain (*U*, *V*, *W*) and (*U′*, *V′*, *W′*). It will then be sufficient to find a plane series containing this line series and containing the point series (*P*, *Q*, *R*); which is the problem last considered. There remains then, only, to find the point series (*U*, *V*, *W*) and (*U′*, *V′*, *W′*) in the appropriate way: (*U*, *V*, *W*) is to be a point series lying on the skew linear series determined by (*P*, *Q*, *R*) and (*A*, *B*, *C*), and (*U′*, *V′*, *W′*) is to be a point series on the skew linear series determined by (*P*, *Q*, *R*) and (*A′*, *B′*, *C′*), and the point series (*U*, *V*, *W*), (*U′*, *V′*, *W′*) are to be on a line series. We can suppose that the point *P* is not on the line *AA′*; for if *P*, *Q*, *R* be all, respectively, upon the lines *AA′*, *BB′*, *CC′*, the point series (*P*, *Q*, *R*) lies on the skew linear series determined by (*A*, *B*, *C*) and (*A′*, *B′*, *C′*). The points, *P*, *A*, *A′*, then, determine a plane; let *QB*, *RC* meet this plane in *V* and *W*, respectively, and let the line *PA* meet the line *VW* in *U*;

again, let QB', RC' meet this plane, respectively, in V' and W', and let the line PA' meet the line $V'W'$ in U'. The two point sets U, V, W and U', V', W', lying in a plane, determine two point series, and these lie on a line series in this plane, determined by a construction given above.

Lastly, consider two line entities. If these have a point entity in common, they both lie on a plane entity; reciprocally, if they both lie on a plane entity, they have a point entity in common. It will be sufficient to consider the former statement. A line and a line series having a point series in common, evidently, lie on a plane. A line and a skew linear series having a point series in common, lie in a plane series having the line as axis. Two line series of which the centre points are the same point, but the planes different, these meeting in a line p, contain respectively two line sets, p, q, r and p, q', r'; the planes (qq'), (rr'), together with the plane joining p to the line of intersection of the planes (qq') and (rr'), constitute a plane set containing a set of each of the given line series, and define a plane series containing these line series. Two line series, whose centre points are different as well as their planes, which have a point series in common, lie on a plane series whose axis is the line joining the two centre points. A line series and a skew linear series, which have a point series in common, lie on a plane series containing the line series and any other point series of the skew linear series. Finally, two skew linear series having a point series in common, lie on the plane series containing this point series and two other point series, one on each of the skew linear series.

(*d*) It follows from the preceding that three point entities, not on the same line entity, determine a plane entity, and that three general plane entities determine a point entity. Further, Desargues' theorem, as a consequence of the above Propositions of Incidence, is true of two triads of point entities. And Pappus' theorem, which now becomes a theorem that three point entities lie on a line entity, can also be proved (Ex. 9, below).

The preceding aggregates in the general Abstract Geometry. In addition to Pappus' theorem in its general form, there are many interesting geometrical questions raised by the preceding constructions. And, when these have been investigated, there is the question whether the entities serve the main purpose for which they are introduced; whether, for example, given four line entities, of which no two have a point entity in common, there exist two line entities having each a point entity common with each of the four given line entities. Such questions are not only of interest in themselves, but are of theoretical logical importance if it be desired to refrain from the extension of the Abstract Geometry indicated earlier in this chapter; the extension, namely, to be quite precise, in which a system of symbols is appropriate for which the equation $z^2 = c$ is soluble for all symbols c. But, if this extension be allowed, an account can be given of the preceding point series, line series and plane series, which not only sets at rest the logical questions, but, further, removes the apparent artificiality of the theory; such an account, by introducing clearness of conception, renders the geometrical discussion, of questions not considered above, much easier. We do not enter into this, in detail. Some indications, stated as the following examples, may, however, be desirable. See Note II, at the end of the Volume.

Ex. 1. For the cubic form

$$f = (x - ay)(x - by)(x - cy),$$

shew that the Hessian form, defined as

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2,$$

is, save for a numerical factor,

$$(x - ay)^2 (b - c)^2 + (x - by)^2 (c - a)^2 + (x - cy)^2 (a - b)^2.$$

Ex. 2. Shew that if two triads of points on a line be given by

$$A = O + aU, \quad B = O + bU, \quad C = O + cU,$$
$$A' = O + a'U, \quad B' = O + b'U, \quad C' = O + c'U,$$

the necessary and sufficient condition that the ranges (A', A, B, C), (B', B, C, A), (C', C, A, B) should be related, is that the two cubic forms

$$f = (x - ay)(x - by)(x - cy), \quad f' = (x - a'y)(x - b'y)(x - c'y),$$

should have Hessian forms whose (three) coefficients are proportional. The result can be obtained from what was proved above (p. 154), that the condition, for two ranges, $(P, Q, P + \rho Q, P + \sigma Q)$ and $(P, Q, P + \rho' Q, P + \sigma' Q)$, to be related, is that $\rho^{-1}\sigma = \rho'^{-1}\sigma'$.

Ex. 3. If ω be a symbol for which $(2\omega + 1)^2 = -3$, and the Hessian form of Ex. 1 be brought to the form $(x - ty)(x - uy)$, save for a factor independent of x and y, prove that, with proper distinction of t and u,

$$(t - a)(b - c)\,\omega^{-2} = (t - b)(c - a)\,\omega^{-1} = (t - c)(a - b),$$
$$(u - a)(b - c)\,\omega^{-1} = (u - b)(c - a)\,\omega^{-2} = (u - c)(a - b).$$

Ex. 4. In the preceding example, when ω is a particular one of the two symbols for which $(2\omega + 1)^2 = -3$, agreed upon, there is a distinction between t and u. And the point series of which a set consists of the points represented by $O + aU$, $O + bU$, $O + cU$, taken in a certain order, can thus be associated with a particular one of the two points $O + tU$, $O + uU$, which are determined when a, b, c are given; the other series, formed from the points of the same set differently arranged, will then be associated with the other of these two Hessian points. A similar remark is then applicable to a plane series, or to a line series. For a skew linear series an adequate statement requires the theory of quadric surfaces. But see Note II, at the end of the Volume.

Ex. 5. It has already been remarked, in the introductory article of this chapter, in anticipation of subsequent work, that, upon the plane curve called a conic section, a theory of related ranges can be set up; and the points of the conic can be put into unambiguous correspondence with the points of a line. We may then have point series upon such a conic, defined by equivalent sets each of three points, just as upon a line; and the two series, arising from the three points of a set taken in different orders, will be associated, respectively, with the two Hessian points of the cubic form belonging to any one of the equivalent sets. Although an anticipation, the geometrical form of this is so simple, that it may be mentioned: If the tangent lines to the conic at three points, A, B, C, be drawn, and meet the lines BC, CA and AB, respectively, in P, Q and R; then these points P, Q, R lie upon a line; and this line meets the conic in the two points, say U and V, which are the Hessian points of the three points A, B and C. Conversely, to an arbitrary line UV, there corresponds a series of sets A, B, C. With a further anticipation, it may be added, that when the line is what is sometimes called the line at infinity, and the conic is a 'circle,' the points A, B, C are the angular points of an arbitrary 'equilateral' triangle inscribed in the circle.

The points U, V are the common corresponding points of two ranges related by the fact that A, B, C, of one range, correspond, respectively, to B, C, A, of the other.

Ex. 6. If the theory of point series, etc., above given, is competent to represent all possibilities (as it is), then statements must be true which can be described, in a usual phraseology, not adopted here, as follows: An imaginary point lies on one real line; an imaginary plane contains one real line; there are imaginary lines of such kind that each contains a real point and lies in a real plane; and there are imaginary lines of another kind, upon which is no real point, through which passes no real plane. For the present we do not enter further into these questions; the precise discrimination of what is to be understood, in the Abstract Geometry, by a *real* point, and by an *imaginary* point, comes more naturally later.

Ex. 7. In the above account of point series, etc., regarded as aggregates of elements of the Real Geometry when this is extended by the adjunction of the postulated points, etc., we have remarked that three point series, of general position, determine a plane series. This is equivalent to saying that, if P, Q, R be three points of one line, and O, U, W be three points of another line, and O', U', W' be three points of a third line, all in general position, then three points, O_1, U_1, W_1, can be found upon the second line, equivalent to O, U, W in the sense explained, and three points O_1', U_1', W_1', can be found upon the third line, equivalent to O', U', W' in the sense explained, such that the three planes, PO_1O_1', QU_1U_1', and RW_1W_1', meet in a line. We can regard the four points O, U, O', U' as fundamental for the threefold space of the figure; and, without loss of generality, we can choose the symbols for the points so that

$$W=O+U, \quad W'=O'+U', \quad R=P+Q;$$

if then the points P, Q be determined from O, U, O', U' by the syzygies

$$P=m_1O+n_1U+m_1'O'+n_1'U',$$
$$Q=m_2O+n_2U+m_2'O'+n_2'U',$$

it can be shewn that the line of intersection of the three planes, the axis of the plane series, contains the two points

$$pO+rU+p'O'+r'U',$$
$$-rO+qU-r'O'+q'U',$$

where

$$p=m_1+m_2-n_2, \quad q=-n_1-n_2+m_1, \quad r=m_2+n_1,$$
$$p'=m_1'+m_2'-n_2', \quad q'=-n_1'-n_2'+m_1', \quad r'=m_2'+n_1'.$$

Ex. 8. It can be shewn that the general set of three points, O', U', W', equivalent to a set, O, U, W, upon the same line, in the sense explained above, the symbols of these being chosen so that $W=O+U$, $W'=O'+U'$, is found by taking

$$(O', U')=\xi(O, U),$$

where ξ denotes
$$\xi=\begin{pmatrix} a, & a-b \\ b-a, & b \end{pmatrix},$$

this notation meaning that

$$O'=aO+(a-b)\,U, \quad U'=(b-a)\,O+bU.$$

Here a, b are arbitrary symbols subject to $ab=ba$. Thus, if ξ' be a matrix of the same form constructed from a' and b', we easily find, from the law of multiplication explained above (p. 67), that $\xi\xi'=\xi'\xi$. We can write ξ in either of the two forms

$$\xi=\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a-b)\,\epsilon, \quad \xi=b+(a-b)\,\tau,$$

where
$$\epsilon=\begin{pmatrix} 1, & 2 \\ -2, & -1 \end{pmatrix}, \quad \epsilon^2=-3,$$

$$\tau=\begin{pmatrix} 1, & 1 \\ -1, & 0 \end{pmatrix}, \quad \tau^2-\tau{}^{+}1 \quad 0.$$

Hence shew that the general point series contained on a skew linear series defined by the two (not equivalent) sets, O, U, $O+U$ and O', U', $O'+U'$, consists of sets of points, P, Q, $P+Q$, where P, Q are given by

$$(P, Q)=\xi(O, U)+\xi'(O', U'),$$

ξ and ξ' being any two matrices of the form above. Thus

$$\begin{aligned} P&=aO+(a-b)\,U+a'O'+(a'-b')\,U', \\ Q&=(b-a)\,O+bU+(b'-a')\,O'+b'U', \\ R&=bO+aU+b'O'+a'U'. \end{aligned}$$

This is a generalisation of the symbolic expression of any point on a line determined by two points.

Ex. 9. Let l, m, n be three skew lines, and p a line met by all transversals of l, m, n; let l', m', n' be three other skew lines such that all transversals of these equally meet p. From any point, P, of p, draw the transversal of l, m, n, meeting these, respectively, in A, B, C; and from P draw the transversal $A'B'C'$ of l', m', n'. Let U, V, W denote, respectively, the points $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$, which, by Pappus' theorem, are in line. Prove (pp. 20, 21, above) that the locus of each of U, V, W is a line. And, by taking three positions of P upon p, hence prove Pappus' theorem, for two triads of imaginary points lying respectively on two intersecting imaginary lines of the most general kind, in terms of the representation of this Chapter.

This representation assumes Pappus' theorem for real points; the theorem of pp. 20, 21 is independent of this.

BIBLIOGRAPHICAL

It is possible to distinguish seven main sources of the theory of this volume: (*a*) The writings of the Greek geometers, and in particular the Collection of Pappus. Beside the particular result, here called Pappus' theorem, which is made to play an important part in the theory, the notion of the fourth harmonic point, the notion of six points in involution, developed by Desargues, and, essentially, what in the hands of Chasles and Steiner became the anharmonic ratio, here replaced by the theory of related ranges, are all to be found there, dealt with on a metrical basis; (*b*) The writings of Kepler and Desargues, who, both, from different points of view, regarded points at infinity as particular cases of finite points, and so made (or adopted from others) a momentous advance from the point of view of the Greeks; (*c*) The unpretentious but imperishable volumes of von Staudt. Though Desargues used the method of projection to prove the theorem we have here called by his name in the case of two triangles not in the same plane, he had no scruple in employing the metric methods of the Greeks for the case of two triangles in the same plane; not forgetting the inspiration of the great work of Poncelet, and his wonderful discovery of the imaginary elements of metric space, nor the far reaching character of the work of Chasles and Steiner, it is still to be remarked that for none of these did there exist any doubt of the obviousness of the notion of distance. It was von Staudt to whom the elimination of the ideas of distance and congruence was a conscious aim, if, also, the recognition of the importance of this might have been much delayed save for the work of Cayley and Klein upon the projective theory of distance. Generalised, and combined with the subsequent Dissertation of Riemann, v. Staudt's volumes must be held to be the foundation of what, on its geometrical side, the Theory of Relativity, in Physics, may yet become; (*d*) To von Staudt, however, the conceptions of the theory of aggregates, due to G. Cantor and others, were not clear; and his definition of related ranges, by means of harmonic nets, assumes that points of such a net are found everywhere on the line, as was pointed out by F. Klein. Thus it is necessary either to introduce into his scheme ideas which seem alien to his point of view, as indicated here in Chapter II, or frankly to recognise the need of an assumption. The discovery of the geometrical significance of this necessity is contained in an incidental remark of H. Wiener; it is this remark which has led to the recognition of the importance to be attached to the assumption of Pappus' theorem, or some equivalent assumption. It is a hazardous, but it seems a natural forecast, that wide developments still await the detailed investigation of the geometry in which Pappus' theorem is not assumed, in which the appropriate algebraical symbols are not commutative in multiplication. The definition of related ranges given in this volume is not the same as that formally adopted by von Staudt, but seems more in harmony with his general point of view; and it is mentioned by him, *Beiträge*, II, p. 136, No. 203; for the possibility of the detailed development of this definition, the volume owes much to the work of F. Schur; (*e*) The recognition that geometry does not deal with space in itself but with definite figures, constructed from points assumed to be given, in accordance with postulated laws of construction, and that these constructions may be made in a limited accessible region. It would seem right to associate these conceptions with the names of F. Klein, Pasch and Peano; (*f*) To C. Segre and the writers of the Italian school, is due the insistence upon the logical necessity and practical utility of not limiting our outlook to space of three dimensions. Though significant papers of Cayley, Riemann and Clifford are to be named, the conviction of the increased geometrical insight obtainable by the widened outlook is to be won only by experience of the many striking successes in simplification and grasp which have been obtained in this way. (*g*) Finally, the introduction by Descartes of the equation to a curve, and the epoch-making work of Poncelet (and of Laguerre) in regard to the imaginary elements, though mainly of importance in subsequent volumes, are responsible, ultimately, for the scope here suggested for the Abstract Geometry. This extension is of quite vital importance for the development of geometry.

Some detailed references to these sources may be of interest:

(*a*) Exhaustive references to the Greek geometry must be sought elsewhere. We may name M. Cantor, *Geschichte der Mathematik*; the smaller *History of Mathematics* by F. Cajori; the German *Enzykl. d. Mathem. Wiss.*; Chasles, *Aperçu historique*, and Zeuthen, *Die Lehre von den Kegelschnitten in Alterthum* (Kopenhagen, 1886). The English reader will find the Prolegomena in Charles Taylor, *Ancient and Modern Geometry of Conics*, Cambridge, 1881, especially instructive (pp. xvii—lv deal with the Greek geometry), and will also consult T. L. Heath, *The thirteen books of Euclid's Elements, translated from the text of Heiberg with Introduction and Commentary*, 3 vols., Cambridge, 1908, also *Apollonius of Perga, Conic Sections*, Cambridge, 1896, and a *History of Greek Mathematics*, 2 vols., Oxford, 1921. The actual text of the theorem we have called Pappus' theorem may be seen in Hultsch (Fridericus), *Pappi Alexandrini Collectionis quae supersunt*, Volumen II, Berolini, 1877, and is as follows, the Latin version being by Hultsch:

pp. 884, 885 *ἐὰν παράλληλοι ὦσιν αἱ* ΑΒ ΓΔ, *καὶ εἰς αὐτὰς ἐμπίπτωσιν εὐθεῖαί τινες αἱ* ΑΔ ΑΖ ΒΓ ΒΖ, *καὶ ἐπιζευχθῶσιν αἱ* ΕΔ ΕΓ, *γίνεται εὐθεῖα ἡ διὰ τῶν* Η Μ Κ.

pp. 886, 887 *ἀλλὰ δὴ μὴ ἔστωσαν αἱ* ΑΒ ΓΔ *παράλληλοι, ἀλλὰ συμπιπτέτωσαν κατὰ τὸ* Ν *ὅτι πάλιν εὐθεῖά ἐστιν ἡ διὰ τῶν* Η Μ Κ.

si parallelae sint $\alpha\beta$ $\gamma\delta$, et in eas incidant quaedam rectae $\alpha\delta$ $\alpha\zeta$ $\beta\gamma$ $\beta\zeta$, quarum $\alpha\delta$ $\beta\gamma$ concurrant in μ, et a quovis rectae $\alpha\beta$ puncto inter α et β sumpto ducantur $\epsilon\gamma$ $\epsilon\delta$, quarum $\epsilon\gamma$ cum $\alpha\zeta$ concurrat in η et $\epsilon\delta$ cum $\beta\zeta$ in κ, rectam esse quae per η μ κ transit.

at ne sint parallelae $\alpha\beta$ $\gamma\delta$, sed convergant in puncto ν; dico rursus rectam esse quae per η μ κ transit.

(*b*) Kepler published a volume with the title *Ad Vitellionem Paralipomena, quibus Astronomiae pars Optica traditur* (Francofurti 1604, Cum privilegio S. C. Majestatis), of which the dedication to his patron, the emperor Rudolph II, is signed Joannes Keplerus, S. C. M$^{\text{tis}}$ Subjectissimus Mathematicus. In the four pages of this bearing the title *De coni sectionibus*, we find (pp. 93, 95) the two following passages, which indicate plainly enough how he was led to regard a point at infinity as a particular case of a finite point. The letters refer to a diagram given in his text.

Focus igitur in circulo unus est *A*, isque idem qui & centrum: in Ellipsi foci duo sunt *BC*, aequaliter à centro figurae remoti & plus in acutiore. In Parabole unus *D* est intra sectionem, alter vel extra vel intra sectionem in axe fingendus est infinito intervallo à priore remotus, adeò ut educta *HG* vel *IG* ex illo caeco foco in quodcunque punctum sectionis *G*, sit axi *DK* parallelos.

Oportet enim nobis servire voces Geometricas analogiae; plurimum namque amo analogias, fidelissimos meos magistros, omnium naturae arcanorum conscios: in Geometria praecipuè suspiciendos, dum infinitos casus interjectos inter sua extrema, mediumque, quantumvis absurdis locutionibus concludunt, totamque rei alicujus essentiam luculenter ponunt ob oculos.

Desargues' work with the title *Méthode universelle de mettre en perspective les objets donnés réellement ou en devis, ..., sans employer aucun point qui soit hors du champ de l'ouvrage* (à Paris 1636. *Oeuvres de Desargues, par M. Poudra*, Paris, 1864), reads (t. I, p. 82):

Quand les lignes suiet sont paralelles entr'elles et que la ligne de l'œil menée paralelle à icelles, n'est pas paralelle au tableau; les aparences des ces lignes suiet sont des lignes qui tendent toutes au poinct auquel cette ligne de l'œil rencontre le tableau....

His treatise on Conics with the title: *Brouillon proiect d'une* *atteinte aux éuénemens des rencontres d'un cone avec un plan* (Paris, 1639; *loc. cit.* t. I, p. 104) gives the definition:

Ordonnance des lignes droites.—Pour donner à entendre de plusieurs lignes droites, qu'elles sont toutes entr'elles ou bien paralleles, ou bien inclinées à mesme point, il est icy dit, que toutes ces droites sont d'une mesme ordonnance entr'elles; par ou l'on conceura des ces plusieurs droites, qu'en l'une aussi bien qu'en l'autre de ces deux espèces de position, elles tendent toutes à un mesme point.

(*c*) In the Vorwort to the *Geometrie der Lage*, by Dr Georg Karl Christian von Staudt, ord. Professor an der Univ. Erlangen (Nürnberg, 1847), we read:

Man hat in den neuern Zeiten wohl mit Recht die Geometrie der Lage von der Geometrie des Masses unterschieden, indessen gleichwohl Sätze, in welchen von keiner Grösse die Rede ist, gewöhnlich durch Betrachtung von Verhältnissen bewiesen. Ich habe in dieser Schrift versucht, die Geometrie der Lage zu einer selbstständigen Wissenschaft zu machen, welche des Messens nicht bedarf.

(*d*) H. Wiener, "Ueber Grundlagen und Aufbau der Geometrie" (*Jahresber. d. Deutsch. Math.-Verein.* I, 1892, p. 47), states:

Diese beiden Schliessungssätze (Der Satz von Desargues über perspective Dreiecke, Der auf das Geraden paar bezogene Pascal'sche Satz) aber genügen, um ohne weitere Stetigkeitsbetrachtungen oder unendliche Processe den Grundsatz der projectiven Geometrie zu beweisen, und damit die ganze lineare projective Geometrie der Ebene zu entwickeln.

v. Staudt's reference to the commutativity of multiplication of his Würfen is on p. 172, no. 268, of his *Beiträge zur Geom. der Lage* (Zweites Heft). A particular case of Pappus' theorem, considered with a view to the theory of similar triangles, is connected with commutativity of multiplication of the symbols by F. Schur, *Lehrbuch der Analytischen Geometrie* (Ers. Auf. 1898, Zw. Auf. 1912), Einleitung, p. 11. Cf. Hilbert, *Grundlagen der Geometrie*, 1899.

See also F. Schur, "Ueber den Fundamentalsatz der projectiven Geometrie," *Math. Ann.* LI (1899), pp. 401-409; "Ueber die Grundlagen der Geometrie," *Math. Ann.* LV (1902), pp. 265-292; and the volume by the same author, *Grundlagen der Geometrie*, Leipzig, 1909, pp. 1-192.

(*e*) For the many contributions of F. Klein to the foundations of geometry reference may be made to his forthcoming Collected Works. A general oversight may be found in the *Gutachten* pronounced for the first award of the Lobatschewsky prize, *Math. Ann.* L (1898), pp. 583-600.

Pasch's significant book has the title *Vorlesungen über neuere Geometrie* (Leipzig, 1882).

Peano's words may be quoted ("Sui fondamenti della geometria," *Rivista di Mat.* IV, 1894, p. 52): Ritenendo pertanto il concetto di spazio come fondamentale per la geometria, ne viene che non si potrebbe scrivere un trattato di questa scienza in una lingua che per avventura manchi di tali parole. Quindi non si potrebbe scrivere di Geometria nella lingua d'Euclide ed Archimede, ove appunto manca la parola correspondente al termine *spazio*, nel senso in cui lo si usa negli odierni trattati.

(*f*) The papers of Cayley and Clifford especially referred to are Cayley, "Chapters in the analytical geometry of *n* dimensions" (1843), *Coll. Papers*, I; Cayley, "A memoir on abstract geometry of *n* dimensions" (1869), *Coll. Papers*, VI; Clifford, "On the classification of loci" (1878), *Coll. Papers*, XXXIII; Riemann's paper, "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" (see *Ges. Werke*), is of date 1854.

For references to C. Segre, see Bertini, *Introduzione alla Geom. proiett. degli iperspazi*, Pisa, 1907, pp. 1–426.

(*g*) Poncelet's statement of what constitutes in effect the Principle of Continuity in Geometry is given on p. 66, t. I, of his great work *Traité des Propriétés Projectives*, 2^me^ Ed., Paris, 1865. For his discovery in regard to the imaginary (or ideal) points of circles, see pp. 48, 64, 380, etc., of that work, and, for the corresponding theorem in regard to spheres, p. 370. For Chasles' account of pairs of imaginary points reference may be made to Chap. V of his *Traité de Géométrie Supérieure*, 2^me^ Ed., Paris, 1880. (See also the *Discours*, pp. 547–585 of the same volume, and his *Aperçu historique*.) Von Staudt's systematic justification of the use of imaginary elements occupies the Erstes Heft of his *Beiträge zur Geometrie der Lage*, 1856, pp. 1–129. In the Vorwort to this we read: Dass eine Ellipse oder Hyperbel durch ihre Brennpunkte und eine Tangente bestimmt ist, war schon den ältern Geometern bekannt. Das aber die Curve aus dem Grunde bestimmt ist, weil von ihr eigentlich fünf Tangenten gegeben sind, und dass also der erwähnte Satz nur ein besonderer Fall von einem allgemeinern Satze ist, ergab sich erst aus der Betrachtung der imaginären Elemente. A comparison of this with preceding formulations makes clear a notable advance in conception.

We pass now from these general remarks to some detailed references in regard to various points treated in the preceding pages. Explicit reference should however first be made to the two Cambridge Mathematical Tracts by Dr A. N. Whitehead, *The Axioms of Projective Geometry*, 1906, and *The Axioms of Descriptive Geometry*, 1907; also to Veblen and Young, *Projective Geometry*, Vol. I, 1910; and also to the *Bibliography of Non-Euclidian Geometry*, by Dr D. M. Y. Sommerville, St Andrews, 1911, which is a most valuable collection of exact references to geometrical literature extending far beyond its title.

pp. 6, 7. Desargues' theorem is proved as in the text by von Staudt. Desargues' proof for the case of two triangles in one plane is metrical. Mr R. J. Lyons, of the University of Sydney, remarks that the converse section (beginning at line 8 from below, on p. 7) may be shortened, using the preceding, by remarking that the joins of corresponding points of the two triads MCC', NBB' meet in the point L; hence the points (MC, NB), (MC', NB'), (CC', BB') are in line; and the first two are A and A'.

p. 11. For the origin of the fourth harmonic point, see Ch. Taylor, *Geometry of Conics*, referred to above, p. liv.

p. 14. Let a, b, c, d, e, f be imagined capable of associative and commutative multiplication, and such that

$$a^2 = b^2 = \ldots = f^2 = 1, = abcdef;$$

and let any product of two different elements, of these six, denote a point, three points being in line when the product of their three symbols is 1, as, for instance, ab, cd, ef, or ab, ac, bc. Then we have a 'geometry' of only fifteen points, thirty-five lines, and fifteen planes, such that any two points determine a line (containing three points), and every three points not in line determine a plane (containing seven points), in which every two lines have a point in common. Such a set of seven coplanar points are those represented by bc, ca, ab, ad, bd, cd, ef; the joining lines of these form a quadrangle of which the three vertices are in line, and the construction for the fourth harmonic point, given on p. 14 of the text, fails. The whole configuration may be represented by the symbol

$$15\,(\cdot, 7, 7)\; 35\,(3, \cdot, 3)\; 15\,(7, 7, \cdot),$$

meaning that there are fifteen points, each lying in seven lines and on seven planes, that there are thirty-five lines, each containing three points and lying in three planes, and fifteen planes, each containing seven points and seven lines. The reader may compare Veblen and Bussey, *Trans. Amer. Math. Soc.* VII (1906), p. 241, and *ibid.* VIII (1907), p. 266.

p. 15. See the note to p. 60, below.

p. 20. Cf. Zeuthen, *Compt. rend.* CXXV (1897), pp. 639, 858; a reference I owe to the kindness of Prof. F. Schur.

pp. 22–24. See F. Schur, *Math. Annal.* LI, 1899, pp. 406–409.

pp. 28, 29. This result, arising from Ex. 2 (p. 26), was so obtained by C. V. Hanumanta Rao, now Professor of Mathematics at Lahore.

p. 30. See K. Th. Vahlen, *Abstrakte Geometrie*, Leipzig, 1905, p. 129.

p. 47. Pappus' theorem may also be stated by saying that, if two triads of points L, L', L'' and M, M', M'' be in perspective in two ways, from centres N, N', then they are also in perspective from a third centre, N''. The three triads of points are then symmetrically related. Cf. Steinitz, *Enzykl. Math. Wiss.* III, 1, p. 491; also Rosanes and Schroeter, *Math. Annal.* II (1870), pp. 549, 553; Schroeter, *Oberflächen Zw. Ordn.* Leipzig, 1880, p. 405, and C. V. H. Rao, *Camb. Phil. Soc. Proc.* XXIII (1927), p. 496.

pp. 48, 49. See F. Schur, *Math. Annal.* LI, 1899, p. 405; F. Schur, *Grundlagen der Geometrie*, 1909, p. 37.

p. 60. The involution upon an arbitrary line by the six joins of four points, to which this work ultimately leads, is found in Pappus; see Ch. Taylor, as above, p. lii. The theorem was extended by Desargues to the intersections of a line by conics through four given points. *Œuvres de Desargues*, t. I (*Traité des coniques*, 1639), p. 268. For Desargues the relation is one among rectangles of segments on the line.

p. 61. Moebius, *Gesamm. Werke*, I, Leipzig, 1885, p. 443 (*Crelle*, III, 1828, p. 273).

p. 66. The symbols first given as example have of course the laws of combination of the complex numbers of ordinary analysis.

p. 71. In effect the symbols are here so introduced as to have the same laws of manipulation (save for commutative multiplication) as those of Moebius, in his barycentric calculus (1827. See *Ges. Werke*, I, p. 176), founded on metrical and statical considerations. In fact they were used as in the text, as the most appropriate representation of the geometry, before this was realised.

pp. 74 f. For such diagrams cf. F. Schur, *Math. Annal.* LV (1902), p. 281, or *Grundlagen der Geometrie* (1909), p. 51; Hilbert, *Grundlagen* (1899); Vahlen, *Abstrakte Geometrie* (1905); Veblen and Young, *Proj. Geom.* (1910). See also Hessenberg, *Acta Math.* XXIX (1905), p. 1; Hölder, *Leipzig. Ber.* LXIII (1911), p. 65; Mehmke, *Punkt- u. Vectorenrechnung*, 1913; von Staudt, *Beiträge*, No. 276, p. 175, etc.

p. 86. Moebius, *Ges. Werke*, I (Leipzig, 1885), p. 247, constructs the point $\alpha A+\beta B+\gamma C$, given $D=A+B+C$.

p. 96. Cf. Pasch, *Vorlesungen über neuere Geometrie* (1882); Peano, *I principii di geometria logicamente esposti* (Torino, Bocca), 1889; Peano, "Sui fondamenti della Geometria," *Rivista di Mat.* IV, 1894, pp. 51–90; Pieri, "I principii della Geometria di posizione," *Mem. Acc. Torino*, XLVIII (1898); also, Pieri, "Un sistema di postulati per la geometria proiettiva," *Rev. mathém. Torino*, VI (1896), and *Atti Torino*, 1904, 1906; *Mem. Torino*, 1905; Fano, "Sui postulati fondamentali della geometria proiettiva," *Giorn. di mat.* XXX (1892), pp. 106–132; Hilbert, *Grundlagen der Geometrie* (1899); F. Schur, *Grundlagen der Geometrie* (1909); Moore, "The projective axioms of geometry," *Trans. Amer. Math. Soc.* III (1902); Veblen, "A system of axioms for geometry," *Trans. Amer. Math.*

Soc. v (1904); Veblen and Young, "A set of assumptions for projective geometry," *Amer. J. Math.* xxx (1908); Robb, *Mess. of Math.* XLII, 1912, p. 121; Enriques, the German *Enzykl. Math. Wiss.* III, *Prinzipien*; and *Questioni riguardanti le matematiche elementari* (2 vols., Bologna, 1912); and many other writings (see the Bibliography of Sommerville, above referred to).

pp. 108–113. Cf. F. Schur, *Grundlagen der Geometrie*, pp. 16–18.

p. 125. In the hope that the geometrical reader may be willing to examine the theory of an abstract order, given in the text, on its own merits, references to, and the technical terms of, the theory of the so-called arithmetical continuum, have been avoided. For these the reader may consult authorities on the Theory of Functions. The reference to Dedekind might quite well be accompanied by references to Weierstrass, G. Cantor, and others.

p. 137. For Archimedes' axiom see the writings of Stolz; Hölder; Enriques, "Prinzipien der Geom." *Enzykl. der Math. Wiss.*; and Veronese, *Fondamenti di geometria a più dimensioni ed a più spezie di unità rettilinee esposti in forma elementare*, Padova, 1891, pp. 1–628, which has given rise to an extensive literature. For the use of the result stated on p. 137, instead of the axiom (3) of p. 122, to prove Pappus' theorem, see Newman, *Proc. Camb. Phil. Soc.* XXII (1925), p. 919; also Hilbert, *Grundlagen der Geom.* § 14.

p. 153. The argument of the text was suggested by F. Enriques, *Proj. Geom.*, Leipzig, 1903, Anhang, pp. 349–354.

pp. 165–175. Von Staudt's systematic treatment of imaginary elements is contained in the *Beiträge zur Geometrie der Lage* (1856); see, in particular, p. 76. An exposition of von Staudt's theory was given by Ch. An. Scott, *Math. Gazette*, 1900, and *Bulletin of the American Math. Soc.* VI, 1900, p. 163. For this and the theory of the text the following may be consulted: Stolz, *Math. Annal.* IV (1871), p. 416; August, "Untersuch. ü. d. Imag. in Geom." (*Prog. d. Fr. Realschule in Berlin*, 1872); Klein, *Gött. Nachr.* 1872, p. 373, *Ges. Abh.* I, p. 404; Lüroth, *Math. Annal.* VIII (1875), p. 145; Sturm, *Math. Annal.* IX (1876), p. 333; Schroeter, *Math. Annal.* X (1876), p. 420; Harnack, *Zeitschr. f. Math. u. Phys.* XXII (1877), p. 38; Lüroth, *Math. Annal.* XI (1877), p. 84; Lüroth, *Math. Annal.* XIII (1878), p. 305; Klein, *Math. Annal.* XXII (1883), p. 242; Grünwald, *Zeitschr. f. Math. u. Phys.* XLV (1900), p. 10; and K. Th. Vahlen, *Abstrakte Geometrie*, Leipzig, 1905, pp. 163–169.

NOTE I, to pp. 148–152.

On the condition that two spaces be related; and on the assignment of the symbols to the points of an Abstract Space.

§ 1. It has been shewn in the text (pp. 148–152), in particular for $n=2, 3$, that if, of two n-fold spaces, there be given $(n+2)$ points of one, to correspond respectively to $(n+2)$ given points of the other, then the spaces may be regarded as *related*. But the proof assumes that if, of a range of points (or pencil of lines) in one space, four points (or rays) be given, and, of the corresponding range (or pencil) in the other space, three points correspond to three of the four points of the former range, then the fourth point of the latter range which is to correspond to the fourth point of the former, is determinate. In other words it is assumed, as stated in the text, that we have agreed on some method of knowing when a range of one space is related to a range of the other (as by a sequence of perspectivities, for example).

§ 2. The geometrical correspondence, between two spaces of n dimensions, set up by means of $(n+2)$ points given in each space, may be obtained by a sequence of $(n+1)$ successive perspectivities, if the two n-fold spaces be supposed to lie in the same space of $(2n+1)$ dimensions. Such a succession of $(n+1)$ perspectivities is in fact equivalent to a projection from a single n-fold space (cf. p. 43, above); and we state the theorem first in this way. Suppose we have two n-fold spaces, lying in a space of $(2n+1)$ dimensions, S_{2n+1}, in general relative positions, the former space, S_n, containing the $(n+2)$ points $O, A, B, \ldots, D$, and the latter space, S_n', containing the $(n+2)$ points $O', A', B', \ldots, D'$, no set of $(r+1)$ points of either of these sets lying in a space of less than r dimensions (for $r < n+1$). It has been shewn (above, p. 39) that, through an arbitrary point of the line OO', a definite space, Σ_n, of n dimensions, can be drawn to meet all the $(n+1)$ lines $AA', BB', \ldots, DD'$. Take one such space Σ_n. Any point, P, of the space S_n determines then, with Σ_n, a space of $(n+1)$ dimensions; and this Σ_{n+1} meets S_n', in the space S_{2n+1}, in a single point, P'; and, conversely, P is thus determinable from P'. The line PP' then meets Σ_n. There is thus a (1, 1) correspondence between the points of the spaces S_n, S_n', in which, in particular, $O, A, B, \ldots, D$ correspond, respectively, to $O', A', B', \ldots, D'$. The correspondence can be shewn to transform a line, plane, ... of one space into a line, plane, ... of the other space, and evidently transforms any incidence of two spaces in S_n into an incidence of the two corresponding spaces of S_n'—so that, in particular, when $n \geqq 2$, it transforms a harmonic range into a harmonic range. And, if we assume Pappus' theorem, it can be shewn that the correspondence is the same whatever be the space Σ_n chosen (of the ∞^1 such that exist), and the same as if some other construction be used, likewise of (1, 1) character, which transforms $O, A, B, \ldots, D$ respectively into $O', A', B', \ldots, D'$. The construction is the direct generalisation of that followed, for the case $n=1$, to define two related ranges upon two straight lines which do not intersect (p. 18, above).

Essentially, as a sequence of $(n+1)$ successive perspectivities, the construction is due to Grassmann (*Crelle*, XLIX (1855), p. 55, for the case $n=2$. See further references in Segre, *Mehrd. Räume*, *Enzykl. Math. Wiss.*, III C. 7, pp. 820–822). We may indicate briefly this mode of statement, for the case $n=3$, supposing that the first projection, from an arbitrary point of the line

OO', has already been made, upon an arbitrary sixfold space containing O, A, B, C, D. We then have two threefold spaces, O, A', B', C', D' and O, A, B, C, D, in this sixfold space, with the point O in common. If we denote the points by symbols, for brevity, and suppose, for these, that

$$O = A + B + C + D = A' + B' + C' + D',$$

it can be shewn that there is a definite plane meeting the four lines AA', BB', CC', DD'; the points of meeting are the points of symbols $A - A'$, $B - B'$, $C - C'$, $D - D'$. We may then first project the points O, A', B', C', D' from the point $(A - A')$ on to the threefold space O, A, B', C', obtaining thus, respectively, the points $O, A, B', C', (-A + A' + D')$. This operation we can denote by

$$(-A + A')\{O, A', B', C', D'\} = \{O, A, B', C', -A + A' + D'\}.$$

We may then project from the point $(B - B')$ upon the threefold space O, A, B, C', giving the operation expressed by

$$(-B + B')\{O, A, B', C', -A + A' + D'\} = \{O, A, B, C', -B + B' - A + A' + D'\}.$$

If now we project from $(C - C')$ upon the space O, A, B, C, we obtain

$$(-C + C')\{O, A, B, C', -B + B' - A + A' + D'\} = \{O, A, B, C, D_1\},$$

where D_1, arising as

$$-C + C' - B + B' - A + A' + D',$$

is in fact D. We have thus passed from O, A', B', C', D' to O, A, B, C, D, respectively.

§ 3. We have in the text (p. 153) justified the use of the symbols, for the points of a space, by comparing the process by which two spaces are related with the process by which the symbols are assigned to the points of a given space. It appears worth while to remark in more detail upon the step of the latter process which corresponds to the condition, in relating two spaces, that a definite range of one of these shall be supposed, by some construction, related point by point, to the corresponding range of the other space. The step in question consists in supposing that the symbol for every point of a single definite line of the space is assigned, and conversely that the point of the line represented by every appropriate symbol is likewise supposed given. The explanation serves to emphasise further the point of view here taken in regard to the so-called imaginary (or abstract) points.

It has been assumed (p. 71, etc.) that if A, B be the symbols of two points, made definite by the assignment of the symbol of some other point of the line joining these, for instance the point $A + B$, then every symbol of the form $mA + nB$ represents a point of the line, and conversely that every point of the line can be represented by such a symbol. When m, n are iterative symbols we have shewn how to construct the point represented by $mA + nB$, in certain cases (p. 86); the two assumptions may then need no detailed justification when we are considering only the points in which m and n are real symbols (that is, the iterative symbols and the symbols arising from these as the irrational numbers of arithmetic arise from the rational numbers). But our aim is to set up an Abstract Geometry, admitting the existence of points whose geometrical relations are suggested by the relations of symbols which are of the form $A + iB$, where i is an algebraic symbol such that $i^2 = -1$ (cf. p. 143). For such points the case is different, the point $A + iB$ not being in linear order with the points $mA + nB$, in which m, n are iterative symbols.

Consider the case of a plane. Say that a line is *determinate by construction* when two points of it are supposed given, and a point is likewise determinate when two lines through it are given. Suppose that the four points of the plane of which the symbols are A, B, C, $A+B+C$, are given. Suppose further that for one definite line of the plane we know the symbol, $U+\lambda V$, of every point existing on this line, and, for every symbol of the form $U+\mu V$, we know the point of the line of which this is the symbol. We wish to make it clear that then the symbol $xA+yB+zC$, of every point of the plane, is determined, and conversely that the point is determined which is represented by any given symbol of this form, in which x, y, z are any algebraic symbols of the system we are using. It is sufficient to suppose that the line UV, upon which the correspondence of points and symbols is known, is one of the lines joining two of the points A, B, C, $A+B+C$. We will suppose that it is the line BC. Then the correspondence upon any other line of the plane is determined by projection. For instance upon the line AB: any given point, Q, of AB, joined to O, gives a point, P, of the line BC, of which then, by hypothesis, the symbol $B+\lambda C$ is known; the symbol of Q is then

$$-(B+\lambda C)+\lambda(A+B+C),$$

or $\lambda A+(\lambda-1)B$. Conversely, any symbol $A+\mu B$, wherein μ is given, by taking $\lambda-1=\mu\lambda$, or $\lambda=(1-\mu)^{-1}$, determines a symbol $B+\lambda C$, or $(1-\mu)B+C$; by the hypothesis made this determines a definite point, P, upon BC, and, from this, the point, Q, of AB, given by $A+\mu B$, is determined by the intersection of PO with AB. Similarly for the line AC, or for the lines OB, OC (using A instead of O); and similarly for AO, points of which can be projected upon AB from C. For a point, P, of the plane which does not lie on any of the lines joining A, B, C, $A+B+C$, let the lines joining P to A and O meet BC in A' and O', respectively. Then if P is given, both A' and O' are determinate, and conversely. If P be $xA+yB+zC$, the points A', O' are, respectively, $yB+zC$ and $(y-x)B+(z-x)C$; whence, if P is given, the ratios y/z and $(y-x)/(z-x)$ are known, and hence the ratios of x, y, z. Conversely if these latter are given, so, by the hypothesis, are the points A' and O', and hence the point P.

It is sufficiently clear that the general proposition is that if, in an n-fold space, the symbols O, A, B, ..., D, of $(n+2)$ given points, be given, with the expression of O in terms of A, B, ..., D (say $O=A+B+\ldots+D$); and also the correspondence be given between the symbols and the points, including the abstract points, upon one of the lines, say AB, joining two of the $(n+2)$ given points, then the correspondence of points and symbols is determinate by construction for every point of the space, and every allowable symbol.

This assignment of the correspondence between the symbols and the points of one line serves the purpose of shewing what is involved in regarding the operations of the geometry as controlled by the behaviour of the symbols. But, by its merely theoretical character, it serves also to explain the sense in which we say that certain processes are not geometrically possible, by construction, except with the supposition that this assignment has been made. For instance, it is true in the Abstract Geometry that two related ranges on the same line have two common corresponding points; but as appears under (vi), p. 159, above, these are not always separately determinable by construction. Or, to take another illustration: later on (Vol. ii, Chap. iv) we identify the iterative symbols with the real numbers of arithmetic, utilising the notion of order in the real geometry for the points of a line. Then we are able to identify the algebraic symbols with the numbers of ordinary Analysis, regarding any symbol as being of the form $\lambda+i\mu$, with $i^2=-1$, in which λ, μ are real numbers, which are then known when the symbol is given. Thereby also the symbol $\lambda-i\mu$, and a point defined by this, are determinate, when the symbol $\lambda+i\mu$ is given. A set of points with coordinates of the forms $\lambda+i\mu$, determines then, in this sense, a corresponding set of points with coordinates of the forms $\lambda-i\mu$;

and every incidence of elements of the former set determines an incidence of the latter. But the two sets of points are not corresponding parts of two spaces which are *related*; nor are the points of the second set determinate by *construction* from the points of the former set, except with the supposition that the assignment of the correspondence of the points and symbols for a line, referred to above, has been made. Such a construction would in particular involve that a symbol $x-iy$ was an analytic function of $x+iy$, which is not the case. It would seem then to add to clearness to distinguish between (1, 1) correspondences of two spaces which are *arithmetical*, in the sense explained, and those which may be called *geometrical*. For the questions here discussed, cf. von Staudt, *Geom. d. Lage, Beiträge*, Nos. 225, 226, pp. 147, 148; also Nos. 276, 283, pp. 174, 177. See also Segre, *Un nuovo campo di ricerche geometriche, Atti...Torino*, xxv (1890), pp. 276, 430, and xxvi, pp. 35, 592.

NOTE II, to pp. 165–175.

On the representation of imaginary elements by sets of real elements.

The following remarks, though in logic belonging to a subsequent stage of the geometrical theory, may explain the meaning of and establish confidence in the representation given in the text.

If, in space of three dimensions, A, B, C, D be the symbols of four points, of general positions, regarded as real, an 'imaginary' point, with respect to these, is a point of symbol

$$(a+i\alpha)A+(b+i\beta)B+(c+i\gamma)C+(d+i\delta)D,$$

or $P+iQ$, where $i^2=-1$, and P, Q are the real points given respectively by

$$P=aA+bB+cC+dD, \quad Q=\alpha A+\beta B+\gamma C+\delta D.$$

Thus an imaginary point lies on a real line, joining two real points P, Q. The same point is represented, if l, m be real, by the symbol $(l+im)(P+iQ)$, or

$$lP-mQ+i(mP+lQ);$$

the pairs of real points $lP-mQ$, $mP+lQ$ belong to an involution upon the line, of which the double points are $P+iQ$ and $P-iQ$. These points coincide when P, Q represent the same point, in which case $P+iQ$ is a real point.

An imaginary line of the most general kind is that joining two general imaginary points. The points of such a line have then a symbol of the form

$$A+iB+(p+iq)(C+iD),$$

in which A, B, C, D represent real points, and p, q are real symbols which vary from point to point of the line. As an imaginary point depends upon six real symbols, it follows that a general imaginary line depends upon eight real symbols. The symbol of a general point of the line being

$$A+pC-qD+i(B+qC+pD),$$

it follows that the line does not contain a real point unless, for proper values of p and q, the symbols $A+pC-qD$, $B+qC+pD$ represent the same point. For this, the necessary and sufficient condition is that the four points A, B, C, D should lie in a plane. Then the line lies in this plane. Through the general point of a general imaginary line there passes a definite real line; that, namely, joining the points $A+pC-qD$, $B+qC+pD$; this line does not intersect the real line passing through another point of the imaginary line, or, again, the points A, B, C, D would be in a plane. All these real lines evidently meet also the imaginary line which joins the points $A-iB$, $C-iD$, the real line spoken of passing through the point

$$A-iB+(p-iq)(C-iD);$$

we may regard this last as the symbol of the general point of the second line, and speak of this line as the conjugate imaginary of the former. The real lines meeting a general imaginary line are thus, in a subsequent phraseology, the real lines of a linear congruence of lines. As stated in the text (p. 170), one of the real lines meeting a general imaginary line passes through an arbitrary real point, the real line through $xA+yB+zC+tD$ which meets the imaginary line being that through the point

$$(x-iy)(A+iB)+(z-it)(C+iD)$$

of the imaginary line. From the abstract point of view, if the given real point be joined by planes to the imaginary line and its conjugate, these conjugate imaginary planes meet in the real line in question.

Now take any three of the real lines meeting the given general imaginary line, say the lines a, a', a''; and consider the common transversals of the lines a, a', a'', among which are the original line and its conjugate imaginary. Let these latter meet the line a in the points I and J, respectively, and similarly meet a' in I', J', and meet a'' in I'', J''. Also, from a point P, in the line a, let the transversal to a' and a'' meet these in P' and P'', respectively; and, similarly, let Q, Q', Q'' be another transversal, R, R', R'' another, and so on. It can be shewn that the ranges $I, J, P, Q, R, \ldots$; $I', J', P', Q', R', \ldots$; $I'', J'', P'', Q'', R'', \ldots$ are related. We may suppose P, Q, R so taken on a that I, J are the common corresponding points of the two related ranges on a which are defined by the condition that P, Q, R of one range correspond respectively to Q, R, P of the other range; then exactly similar relations hold on a' and a''. The condition stated is the same as that I, J are the Hessian points of the triad P, Q, R (see Ex. 5, p. 174). If, further, another triad of real points X, Y, Z, be taken on a, such that the three ranges X, P, Q, R; Y, Q, R, P; Z, R, P, Q are related, then triads X', Y', Z' and X'', Y'', Z'' exist on a', a'', respectively, for which similar relations hold. Conversely, three real lines PP', QQ', RR', in which the real points P, Q, R are on a line a, and the real points P', Q', R' are on a line a', determine conjugate imaginary points I, J on a, and conjugate imaginary points I', J' on a' (cf. Ex. 1, p. 173), and so determine the conjugate imaginary lines II', JJ'; also ∞^1 real lines, such as a'', meeting PP', QQ', RR', are determinate, as well as sets of lines such as XX', YY', ZZ', all equally meeting the line a''. We may, however, regard as given, not the lines PP', QQ', RR', but, for example, the lines PX', QY', RZ'; then the same pair of imaginary lines II', JJ' will be determined by these as by the triad PP', QQ', RR', but the line a'' is not a transversal, in general, of PX', QY', RZ'; the real lines meeting PX', QY', RZ' will be another ∞^1 of real lines, than those meeting PP', QQ', RR', also belonging however to the ∞^2 real lines which meet II' and JJ'. In a phraseology employed in Vol. III, there are two quadric surfaces involved, which have four lines in common, namely, the two real lines a, a', and the two conjugate imaginary lines II', JJ'. More generally, varying the set X', Y', Z' upon the line a', we obtain ∞^1 real quadric surfaces, every one of which contains the lines II', JJ' as well as ∞^1 real lines which meet these. We may, in fact, verify that, with coordinates relative to the fundamental points A, B, C, D of our discussion, the equation

$$\begin{vmatrix} x^2+y^2, & xz+yt, & yz-xt, & z^2+t^2 \\ 1, & p, & q, & p^2+q^2 \\ 1, & p', & q', & p'^2+q'^2 \\ 1, & p'', & q'', & p''^2+q''^2 \end{vmatrix} = 0$$

represents the quadric surface containing all points, $(1, i, \theta, i\theta)$, of the imaginary line II', and all points, $(1, -i, \phi, -i\phi)$, of the line JJ', as well as all points, $(1, \sigma, p+q\sigma, p\sigma-q)$, of the line a, and, similarly, all points of the lines a', a''. A set of points such as P, Q, R (or X, Y, Z) above, corresponds to three values of σ satisfying an equation of the form

$$\sigma^3+3h\sigma^2-3\sigma-h=0,$$

where h is variable as we pass from set to set; the Hessian of this form is $(\sigma^2+1)(h^2+1)$, and the points I, J arise for $\sigma=i$ and $\sigma=-i$. If we choose P, Q, R corresponding to the three values of σ satisfying this equation for any given value of h, and choose P', Q', R' corresponding to the three values of σ arising when h' is put for h, then there are ∞^1 lines a'' for which the quadric surface, with the equation above, contains all the sets of lines equivalent to PP', QQ', RR'. These sets form one of the *skew linear series*, in the phraseology of the text, which, with a proper order for P, Q, R, represent the line II'.

The statement in the text (p. 168), that of a given skew linear series, there is one set of which one line is an arbitrary line, follows at once. For, from two real points taken on the arbitrary line l, draw the real lines which meet II'; these are such as the lines a, a' of the preceding discussion; and two lines can be taken meeting a and a', which, with l, form three lines whose intersections with a (or with a') have as their Hessian pair the intersections of a with II' and its conjugate imaginary.

Beside the most general imaginary line thus considered, there are imaginary lines each containing only one real point. If H be this point, the points of such a line have symbols of the form $A+iB+(p+iq)H$, where p, q are real variable symbols. Such a line lies in the real plane determined by the points A, B, H. The conjugate imaginary of a general point, $(1, i, p+iq, 0)$, of this line, is the point $A-iB+(p-iq)H$, which does not lie on the line. The *line series* of the text consists then of sets of three lines through the point H, one such set being, relatively to A, B, H, expressed by an equation

$$y^3+3hy^2x-3yx^2-hx^3=0,$$

with an appropriate value for h.

By von Staudt, the general imaginary line was said (*Beiträge*, pp. 76, 77, Nos. 116, 117) to be of the second kind, the lines containing one real point being of the first kind. A line containing two real points is real.

It is very usual to represent a line by means of the ratios of the coefficients in the equations of the planes which join the line to four fundamental points, which we shall consider as real; of these planes two are in general sufficient. If their equations be taken to be

$$l't+mz-ny=0=m't+nx-lz=n't+ly-mx=l'x+m'y+n'z,$$

with $ll'+mm'+nn'=0$, then l, m, n, l', m', n' are the homogeneous coordinates of the line. An imaginary line of the most general kind will then be given by such coordinates, with the condition that two at least have a ratio which is not real; or, if we write $l=l_1+il_2, \ldots, n'=n_1'+in_2'$, not all of $l_2, \ldots, n_2'$ are zero. The conditions that such a line should contain a real point are that two of the four planes above (and, therefore, the other two) should contain this point; and, for one of these planes to contain a real point, two conditions, in real symbols, must be satisfied. The elimination of the coordinates of the real point, from the four conditions so arising, leads to one condition as necessary among the line coordinates. For a general imaginary line we have the relations

$$l_1l_2'+l_2l_1'+m_1m_2'+m_2m_1'+n_1n_2'+n_2n_1'=0,$$
$$l_1l_1'+m_1m_1'+n_1n_1'=l_2l_2'+m_2m_2'+n_2n_2';$$

the condition that the line should contain a real point is found to be that both the last written expressions should vanish. If a general imaginary line be described, as above, as joining the two points $A+iB$, $C+iD$, its coordinates relatively to A, B, C, D are easily seen to be $(i, -1, 0; -i, 1, 0)$; the real lines meeting the imaginary line are then such that $l=l'$, $m=m'$.

Von Staudt's theory uses (*Beiträge*, § 15, p. 137) the notion of a *chain* of points upon a general imaginary line, determined by three arbitrary points thereon. If, as above, a, a', a'' be the real lines through these points, the chain consists of the points, U, of the line, such that the real line through U is met by all the transversals of a, a', a''. Thus, also, from what we have said, it consists of all points $A+iB+(p+iq)(C+iD)$, where p, q satisfy a relation of the form $a+bp+cq+d(p^2+q^2)=0$, in which a, b, c, d are real. Or, again, it consists of all points $P+n\omega Q$, where P, Q are the symbols of two points of the line, ω is a definite imaginary symbol, and n is a variable real symbol. From the theory of quadric surfaces it is clear that two chains with one common point have in general another common point. But also, if one quadric surface have the conjugate imaginary lines II', JJ' as generators, and have the real generator a, meeting these, as another generator; and if u be a real line meeting II', JJ', not lying on the quadric surface, another quadric surface can be constructed, having II', JJ', a and u as generators, to touch the former at every point of a; so that (von Staudt, *Beiträge*, p. 139, No. 208), there is one chain containing one given point of a given chain, and another assigned point of the line not belonging to the first chain, which contains no other point of the first chain. If the first chain be given by $A+iB+n\omega(C+iD)$, and the point U be $A+iB+\mu(C+iD)$, in which μ/ω is not real, the second chain is $\mu(A+iB)+p\omega U$, for variable real p; namely, is $A+iB+\omega\nu(C+iD)$, where $\nu^{-1}=p^{-1}+\omega/\mu$; and ν can be real for no real value of p other than $p=0$. (Cf. Veblen, *Projective Geometry*, II, p. 30, Assumption I.)

NOTE III, to p. 118.

On onesidedness.

The statement referred to, that the plane is a onesided surface, retains from the limited real geometry the notions, (i) of a segment of a line, to which a definite direction, or sense, is assigned; (ii) of a region of a plane bounded by segments; but uses also the notion, (iii) that two points of a line determine two segments thereon, the line being continued, *with its direction*, through the inaccessible point of the line. The property of onesidedness arose however for surfaces of which all the points are accessible, and may first be explained for such a case (given by Moebius, *Ges. Werke*, II, p. 482).

Suppose we have, in space of three dimensions, a set of triangular pieces of paper, joined together along sides, so that any side of any triangle is also a side of another triangle, such for instance as the four triangles bounding a tetrahedron. By assigning a direction to one side of one triangle we may establish a direction of circulation round the sides of the triangle, each of the other two sides acquiring thus also a direction. Taking now a particular one of the three sides of this triangle, and regarding it as a side of the other triangle of the figure to which it belongs, we may attach to it, in this other triangle, the reverse direction to that formerly assigned, and so establish a direction of circulation round the sides of the second triangle. The same process may now be applied to the other sides of the first and second triangles, and so on continually. It may then be the case that it is possible to assign a direction of circulation for all the triangles of the figure, in such a way that the two directions thus arising in any segment (which belongs by hypothesis to two triangles) are opposite. This is the case, for instance, for the four triangles of a tetrahedron whose vertices are A, B, C, D: we may take for ABC the circulation indicated by the order of the letters, with the other respective circulations DCB, DAC, DBA; then BC in ABC arises as CB in DCB, and DA in DAC arises as AD in DBA, and so on.

But such an assignment of circulations may be impossible. Take, for instance, five points, which we denote in order by 1, 2, 3, 4, 5; let the five segments 12, 23, 34, 45, 51 be called the sides, and the five segments 13, 24, 35, 46, 51 be called the diagonals; two consecutive sides such as 12, 23 together with the associated diagonal 13 form a triangle, and there are five such triangles; two of these which are consecutive have in common one of the five original sides of the figure, but we neglect the possibility of other common points of any two of these triangles. Now take an arbitrary sixth point, say 6; and let the two segments joining it to the two ends of a diagonal, together with that diagonal, be regarded as forming another triangle. Thereby we have five new triangles; and the whole ten triangles are such that any bounding side of one of them is equally a bounding side of another triangle. It can easily be seen that the figure now obtained is symmetrical in its description, whichever five of the six points be regarded as first taken; the description may be represented by

$$6(\cdot\,, 5, 5)\,15\,(2, \cdot\,, 2)\,10\,(3, 3, \cdot\,),$$

meaning that there are six points each an end point of five segments, or edges, and a corner of five triangles, or faces; that there are fifteen edges, each with two corners and lying in two faces; and that there are ten faces, each containing three corners and three edges. To avoid the possible intersections of the faces, save along the assigned edges, we may suppose the figure constructed in space of four dimensions. For this figure, now, it will be found impossible to assign such a circulation to every one of the ten triangles that the two directions thus arising in any edge, regarded as belonging in turn to the two faces of which it is a bounding edge, shall in all cases be opposite. In fact this condition applied to the five triangles 613, 624, ..., 651 necessitates the assignment of a uniform sense to the diagonals 13, 35, 52, 24, 41 (one of two senses); but the sense 31 of the diagonal edge of the face 123, involves the sense 23 in the side edge, while the sense 42 of the diagonal edge of the face 234 also involves the sense 23 in the side edge; and thus the two senses arising in the edge 23 are not opposite, as they should be. The figure may then be spoken of as a closed one-sided polyhedron.

Now consider, in a plane, four points A, B, C, D; let the pair of opposite joins BC, AD meet in P, and similarly CA, BD meet in Q, and AB, CD meet in R. On each of the six joining lines, regarded as divided into two segments by the two points of A, B, C, D which lie upon it, take the segment which does not contain the respective point, of P, Q, R, which lies on this line. Denote the segments so obtained, respectively on AB, BC, CD, DA, BD, AC by 1, 2, 3, 4, 5, 6. Thereby we may regard the whole plane as divided into three regions each bounded by four segments; one region by the segments 1, 2, 3, 4; another by the segments 2, 4, 5, 6; and the third by the segments 1, 3, 5, 6. In a phraseology which will be understood the connection of two of these regions is secured by using inaccessible points of the region. Each segment contains two points and is a boundary of two regions. It will be found then that it is impossible to assign a direction of circulation for every region, round the four segments bounding the region, in such a way that the directions so obtained, along any segment separating two regions, are opposite. It is understood, as has been stated, that for a segment containing the inaccessible point of the line on which the segment lies, the direction of the segment, regarded as part boundary of one of the regions, is continued through the inaccessible point.

The origin of the peculiarity evidently is that the points of the plane are obtainable from the lines passing through an arbitrary point not in the plane; the dissection of the plane which we have considered may be obtained by projecting the surface of, say, a cube, from its centre, upon a plane not passing through this; then two opposite points of the surface of the cube, upon the same line through the centre, give rise to the same point of the plane. The same peculiarity arises for any dissection of the complete plane; and for this reason the plane is said to be one sided.

It is interesting to remark that the extension of similar considerations to the complete space of three dimensions leads us to regard this as *two* sided. For brevity employ the description of the real metrical geometry. Consider a rectangular parallelepiped. Let the corners of one face be named in order A, B, C, D, and the diametrically opposite corners of the opposite face be named, respectively, A', B', C', D'. Consider the segment of the diagonal AA' which does not contain the centre of the figure; this may be regarded as beginning at A, passing through a point X, then through the inaccessible point and then to A' through a point X'. Similarly let the exterior segment of the diagonal BB' be $BYY'B'$, the inaccessible point being passed between Y and Y'; and we likewise consider the other two exterior diagonal segments, $CZZ'C'$ and $DTT'D'$. The whole threefold space may now be divided into four regions. One of these is within the original parallelepiped; another is bounded by the two faces $ABCD$, $A'B'C'D'$, the two faces $ABYY'B'A'X'X$, $CDTT'D'C'Z'Z$, and the two faces $DAXX'A'D'T'T$, $BCZZ'C'B'Y'Y$; and there are two other regions, each of which has, as two of its six boundary faces, two opposite faces of the original solid. There are thus, in all, twelve faces, each a boundary of two of the regions; and there are sixteen edge segments each lying in the boundary polygon of four faces. The whole figure may be represented by

$$8(\cdot, 4, 6, 4)\,16(2, \cdot, 3, 3)\,12(4, 4, \cdot, 2)\,4(8, 12, 6, \cdot),$$

meaning that there are eight corner points, each an end of four segments, lying in six faces and belonging to four of the regions; that there are sixteen edge segments, each containing two corners, lying in three faces and belonging to three of the regions; and so on. It can be verified, (i) that, considering any one of the regions, we can assign, for each of its faces, a direction of circulation of its edge segments, so that for every two of the faces the directions in their common edge are opposite, (ii) that this can be done so that the directions of circulation for any polygon face common to two regions, arising in this way for the two regions, are opposite. The complete threefold space may then be regarded as two sided.

We have said that the dissection of the plane we have employed may be regarded as obtained by projecting the faces of a cube from its centre. The centres of the faces of the cube define an octahedron. If this be projected from its centre we obtain a dissection of the plane into four triangular regions: let A, B, C be three points of the plane, let a, b, c denote, respectively, the segments BC, CA, AB which do not contain a point of some line drawn in the plane, and let a', b', c' be the complementary segments of the lines BC, CA, AB. The four regions are then bounded, respectively, by the segments

$$(a, b, c), \quad (a, b', c'), \quad (b, c', a'), \quad (c, a', b').$$

The numerical schemes which describe the former dissection and the latter respectively are

$$4(\cdot, 3, 3)\,6(2, \cdot, 2)\,3(4, 4, \cdot), \quad 3(\cdot, 4, 4)\,6(2, \cdot, 2)\,4(3, 3, \cdot)$$

in which the numbers come in the reverse order. The diagrams are in fact 'reciprocal' in the sense employed in the theory of force diagrams (cf. Maxwell, *Sc. Papers*, I, p. 514, II, p. 161 and II, p. 492).

In the same way for the subdivision of the complete space of three dimensions, instead of a cube we may employ a tetrahedron, so dividing the space into eight tetrahedra. It will be found that in the numerical scheme for the figure obtained the numbers come in the reverse order to that in the scheme obtained above. But the two sidedness is obtainable also by this figure; or indeed by any other subdivision.

In connection with this note the reader may consult Dénes König, *Archiv d. Math. u. Phys.* XIX (1912), p. 214, and the references there given; in particular, Möbius, *Ges. Werke*, II. A drawing of a closed onesided surface is given in Klein, *Ges. Abh.* II, p. 76.

INDEX

to the pages of the text (not including the Bibliography).

Abstract Order, 121
Algebraic symbols, 62–69; interpretation of laws of, 74–80; examples of, 83–92; iterative, 66, 86, 138; for extended Abstract Geometry, 143, 146
Apollonius, 70
Archimedes' axiom, 124, 137

Brianchon hexagon, 47
Betweenness, 95

Cayley, 144
Chasles, 160
Correlative, 5
Correspondence of the points of two lines, 44

Dedekind, 125
Dependent points, 33
Desargues' theorem, 6, 31; for a star, 107, 120; Desargues, 117
Duality, principle of, 5, 42, 55

Enriques, 119, 125
Equable distribution of points of harmonic net, 136

Fano, 14, 120

Gergonne, 144

Harmonic point, from three others, 10; separation in harmonic relation, 119
Heath, 47
Hilbert, 93, 120
Hölder, 125
Homographic, 150

Imaginary elements, 165–175
Incidence, Propositions of, in three dimensions, 4, 70; in higher space, 33
Involution, 62
Iterative symbols, 66, 86, 138

Kepler, 117, 146
Klein, 131

L'huilier, 144
Lüroth, 131

Moebius tetrads, 61, 91
Moulton, 120

Numbers, v. Staudt's introduction of, 93

Onesided surface, 118
Order of points of a line, 96, 118; abstract order, notion of, 121

Pappus' theorem, 47–52, 56, 60, 78, 81, 87–93, 128, 175
Pasch, 109
Peano, 97, 120; Peano's axiom, 97
Perspective, figures in, 10, 13
Perspectivities, a chain of, 17, 21
Plane, definition of, 100; plane systems, related, 148
Postulated points, 107, 114; elements, 116
Projection, 43; Projective, 150
Ptolemy, 70

Reciprocal, 5
Related Ranges, 18, 25, 41, 53, 55, 139, 140, 154, 160
Reyes-y-Prosper, 109

Schur, 93, 109
Segment, 95
Separation, effected by a line, 103; by a plane, 104; of two points by two, 118, 119
Servois, 144
Sixth point, constructed from five points in line, 15, 60, 77
Space, system of three dimensions, 104; of four and five dimensions, 37
Steiner, 144, 162; Steiner's existence theorem, 155
Symbols, algebraic, 62–69; element-symbols, 70; iterative, 66, 86, 138; argument for, 152
System, of points, 33, 100, 104, 148, 150; of symbols, 66, 143, 165
Syzygy, 33, 71

Taylor, 47, 62, 117, 146
Transversals of four lines having three transversals, 27; of any four lines, 161

Zeuthen, 131

CAMBRIDGE: PRINTED BY
W. LEWIS, M.A.,
AT THE UNIVERSITY PRESS

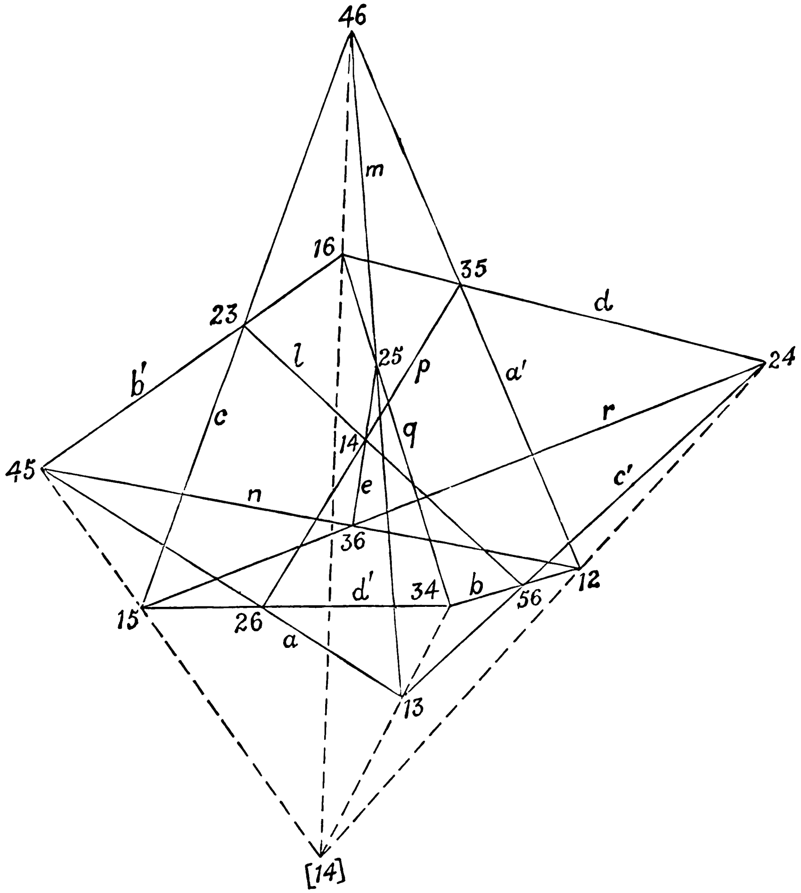
PRINCIPLES OF GEOMETRY

Cambridge University Press
Fetter Lane, London

New York
Bombay, Calcutta, Madras
Toronto
Macmillan

Tokyo
Maruzen Company, Ltd

All rights reserved


46
m
16
35
d
23
l
25
p
a'
24
b'
c
14
q
r
45
e
c'
n
36
d'
34
b
12
56
15
26
a
13
[14]

HEXAGRAMMUM MYSTICUM
(see p. 219)

PRINCIPLES OF GEOMETR

BY

H. F. BAKER, Sc.D., F.R.S.,

LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY, AND FELLOW OF
ST JOHN'S COLLEGE, IN THE UNIVERSITY OF CAMBRIDGE

VOLUME II

PLANE GEOMETRY

CONICS, CIRCLES, NON-EUCLIDEAN GEOMETRY

In minimis maxima

CAMBRIDGE
AT THE UNIVERSITY PRESS
1930

First Edition 1922
Second Edition 1930

PRINTED IN GREAT BRITAIN

PREFACE

THE present volume has in effect two aims: In the first place, in pursuance of the general purpose of the book, it seeks to put the reader in touch with the main preliminary theorems of plane geometry. Chapter I is devoted to a deduction, with synthetic methods, of the fundamental properties of conic sections; it is an introduction to what is usually called Projective Geometry, in the plane, in which, however, the notions of distance and congruence are not assumed. Chapter II, also without help of these notions, develops results that arise by considering conics in relation to two Absolute points, including, for instance, the properties of circles, and of confocal conics; the matter here contained is usually found in sequels to Euclid, books on Pure Geometry, and books on Geometrical Conics. Chapter III is designed to explain the application of the algebraic symbols to plane geometry; it contains methods and formulae found in works on Analytical Geometry of the Plane. Chapter IV is a brief consideration of some logical questions, and marks the recognition of a limitation in the symbols employed; it deals with the sense in which the words real and imaginary are used, and calls attention to the elements of Analysis assumed in the following chapter. Chapter V deals with the theory of measurement, of length and angle, with the help of an Absolute conic, shewing how the so-called non-Euclidean geometries may be regarded as included in our general formulation. It considers the metrical plane also as deduced from the geometry of a quadric surface, incidentally dealing with the fundamental properties of this surface and, in particular, with Spherical Trigonometry. As a corollary from this point of view, Riemann's space of constant curvature is seen not to require the assumption of absolute coordinates; and further, that form of the hyperbolic geometry in which lines are replaced by circles cutting a fixed circle at right angles (which, for instance, was an inspiration to Poincaré in his development of the theory of automorphic functions) is seen to arise naturally. Notes I and II deal with the theorems of incidence which were developed very gradually for the complete Pascal figure and appeared very intricate; from the point of view here explained they are natural, if particular, properties of a figure which arises otherwise, and will much concern us in a later volume. Note III gives some indications of the literature of non-Euclidean geometry. Note IV contains remarks and corrections for Volume I, for many of which I am indebted to friends. There is also an Index; but it is possible that the extensive Table of Contents may be more useful. No attempt is made to give a general Bibliography for the contents of the volume.

It will be seen that the volume deals with a wide range of theory; in other conditions than the present, a less condensed treatment might have been desirable. The order in which the ideas are taken has been chosen largely in view of the second aim of the volume; it will not be difficult, with the help of the Table of Contents, for the reader to modify this order. It is believed, however, that a large amount of the time usually spent, at present, in learning geometry, could be saved by following, from the beginning, after an extensive study of diagrams and models, the order of development here adopted; and such a plan would make much less demand upon the memory.

But the second aim of the volume may, I hope, appeal to attentive readers. It is an attempt, tempered indeed by practical considerations, to test the application in detail of the logical principles explained in Volume I. It seeks to bring to light the assumptions which underlie an extensive literature in which coordinates are freely used without attempt at justification. It suggests the question whether, in the case of distance, as in many other cases, we may not have derived from familiarity with physical experiences, a confidence which a more careful scrutiny can only regard as an illusion. When this view, which seems sure, shall win acceptance, the change in scientific thought will be rapid and momentous. As the first step in this sense was made in the development of the theory of our geometrical conceptions, it is proper that the matter should be dealt with here. It will be of importance if the reader come to see how deep lying are the questions involved in the use of coordinates, and the assumption of distance as a fundamental idea.

As in the case of the first volume, I desire to express my thanks to the Staff of the University Press for their care and courtesy, and to Mr J. B. Peace, M.A., for the great trouble he has taken with the numerous diagrams.

H. F. B.

2 *September* 1922

To the present Reprint are added, at the end of the Volume, various remarks and examples; these are referred to in the text by the abbreviation [Add.]. To many friends who have made suggestions I offer my best thanks.

H. F. B.

24 *January* 1930

TABLE OF CONTENTS

PRELIMINARY

PAGES

Related ranges on the same line 1

Involution . . . . 2—4

Symbolical expression of the preceding results . . . . 4—6

Examples of involution . . . . 6—8

A general abbreviated argument for relating two ranges . . 8, 9

Of the distinction between the so-called real and imaginary points 9

CHAPTER I. GENERAL PROPERTIES OF CONICS

Definition of a conic . . . . 10—12

Involution on a line by conics having four common points . . 12, 13

Maclaurin's definition of a conic. Pascal's theorem . . . 13, 14

Ranges of points on a conic 14

Involution of pairs of points of a conic 14, 15

Pascal's theorem and related ranges of points upon a conic . . 15, 16

Converse of Pascal's theorem 16

Introduction of the algebraic symbols . . . . 16—20

Geometrical theory resumed. Polar lines 20, 21

Duality in regard to a conic 21

Dual definition of a conic 22, 23

Examples of the application of the foregoing theorems (Exx. 1—36) 23—62

Ex. 1. *The self-polar triad of points for conics through four points; the self-polar triad of lines for conics touching four lines* . . 23—25

Ex. 2. *Dual of Pascal's theorem* 25, 26

Ex. 3. *The director conic of a given conic in respect of two arbitrary points* 26, 27

Ex. 4. *The polars of three points determine a triad of points in perspective with the original triad* 27

Ex. 5. *A construction for two points conjugate in regard to a conic* 27, 28

Ex. 6. *Range determined by a conic, and by the joins of three points of the conic, upon any line drawn through a fixed point of the conic* 28, 29

Ex. 7. *The six joins of two triads of points of a conic touch another conic* 29

Ex. 8. *The pedal line of three points of a conic in regard to three other points of the conic, taken in a particular order* . . . 29, 30

Ex. 9. *The Hessian line of three points of a conic. The Hessian point of three tangents of a conic* 30, 31

PAGES

Ex. 10. *Polar triads in regard to a conic. Hesse's theorem* . . 31—33

Ex. 11. *Outpolar conics* . . . . 33—35

Ex. 12. *Two self-polar triads of one conic are six points of another conic* 35, 36

Ex. 13. *Particular cases of outpolar, or of inpolar, conics* . . 36

Ex. 14. *The joins of four points of a conic and the intersections of the tangents at these points. The common points, and the common tangents, of two conics* 36, 37

Ex. 15. *The polar lines of any point in regard to the conics through four given points all pass through another point* . . . 37, 38

Ex. 16. *A point to point correspondence in regard to four given points* 38, 39

Ex. 17. *Four conics can be drawn through two given points to touch three given lines* . . . . 39—41

Ex. 18. *The poles of a line, in regard to conics with four common points, lie on a conic* 41, 42

Ex. 19. *Two conics of which the tangents at their common points meet, in fours, at two points* . . . . 42—44

Ex. 20. *A conic derived from two given conics* 44, 45

Ex. 21. *To construct a conic in regard to which two triads of points of a given conic shall both be self-polar* 45, 46

Ex. 22. *To construct a conic for which a given triad of points shall be a self-polar triad, and a given point and line shall be pole and polar* 46

Ex. 23. *To construct a conic for which a given triad of points shall be self-polar, to pass through two given points* 46, 47

Ex. 24. *The polar reciprocal of one conic in regard to another. A particular case* 47, 48

Ex. 25. *Gaskin's theorem for conics through two fixed points outpolar to a given conic* . . . . 48—50

Ex. 26. *Two related ranges upon a conic obtained by composition of two involutions* . . . . 50—52

Ex. 27. *Three triads of points of a conic are all in perspective with another triad of points of the conic* 52

Ex. 28. *The envelope of the line joining two corresponding points of two related ranges on a conic* 52, 53

Ex. 29. *The tangents from three points, A, B, C, to a conic, meet BC, CA, AB in six points of a conic* 53, 54

Ex. 30. *Two tangents of one conic meet two tangents of another in four points lying on a conic through the common points of the two conics, if the four points of contact are in line* 54, 55

Ex. 31. *Poncelet's theorem, for sets of points A, B, C, ... of one conic, of which the joins AB, BC, ... separately touch other fixed conics, all the conics having four points in common* . . . 55—58

Ex. 32. *Construction of a fourth tangent, with its point of contact, of a conic touching three lines at given points* 58, 59

PAGES

Ex. 33. *Hamilton's extension of Feuerbach's theorem* . . . 59, 60
Ex. 34. *Theorem in regard to the remaining common tangents of three conics, which touch three given lines* 60, 61
Ex. 35. *The four conics through two points of which each contains three of four other given points* 61
Ex. 36. *The polars of any point in regard to the six conics which can be drawn through the sets of five, from six arbitrary points of a plane, touch a conic* 61, 62

CHAPTER II. PROPERTIES RELATIVE TO TWO POINTS OF REFERENCE

Introductory 63
Middle points, Perpendicular lines, Centroid, Orthocentre . . 63, 64
Circles. Circles at right angles 65, 66
Coaxial circles 66, 67
Inversion in regard to a circle 67, 68
Examples of inversion in regard to a given circle . . . . 68—70
Examples in regard to circles (Exx. 1—9) . . . . . . 70—75
Miquel's theorem, pedal (Wallace) line, property of four circles . 70—72
Proof of Feuerbach's theorem by inversion . . . . . . 73—75
Angle properties for a circle . . . . 75—77
Foci and axes of a conic . . . . 77—79
Confocal conics 79
Common chords of a circle with a conic 79, 80
The director circle of a conic. The director circles of conics touching four lines are coaxial 80, 81
Parabolas. The directrix of a parabola 81
Examples in regard to a parabola (Exx. 1—5) . . . . . 81—83
Definition of a rectangular hyperbola 83, 84
Examples in regard to a rectangular hyperbola (Exx. 1—3) . . 84, 85
Auxiliary circle of a conic 85, 86
Examples in regard to auxiliary circle, and pedal circle (Exx. 1—5) 87, 88
The locus of the centres of rectangular hyperbolas passing through three given points . . . . 88—90
The polars of a point in regard to a system of confocal conics touch a parabola. Four normals can be drawn from a point to a conic 90, 91
A particular theorem for intersection of a conic with a rectangular hyperbola 91, 92
The normals of a conic which pass through a given point. The feet of three of these, and the opposite of the other foot, lie on a circle. Case of the parabola also. Examples (Exx. 1, 2) . 92—94
The dual of the director circle of a conic. Sets of four points of one conic whose joins in order touch another conic . . . 94—96

CHAPTER III. THE EQUATION OF A LINE, AND OF A CONIC

PAGES
Preliminary remarks 97
The coordinates of a point in a plane 97, 98
The equation of a line 98, 99
The coordinates of a line. Symbol of a line 99, 100
The equation of a point 100
The equation of a conic. The expression by a parameter . . 101—104
A particular form of the equation of a conic. Equation of tangent. Tangential equation of a conic. Conic referred to self-polar triad 104—107
Equation of a circle. Of circles of coaxial system. Condition that two lines or two circles should cut at right angles . . . 107—110
Examples of the algebraic treatment of the theory of conics (Exx. 1—29) . . . . 110—152
Ex. 1. *Condition that a pair of points should be harmonic conjugates in regard to another pair* 110, 111
Ex. 2. *A result in regard to involutions* 111
Ex. 3. *Representation of all the pairs of an involution* . . . 111
Ex. 4. *A general result for two pairs of points* 111
Ex. 5. *Condition that two circles, in general form, should cut at right angles* 111
Ex. 6. *The radical axis of two circles; the radical centre of three circles* 111, 112
Ex. 7. *The common tangents and centres of similitude of two circles* 112
Ex. 8. *Feuerbach's theorem. Apolar triads* . . . . . 112—114
Ex. 9. *Continuation of the preceding example* . . . . 114—116
Ex. 10. *Hamilton's modification of Feuerbach's theorem. Introductory results* 116, 117
Ex. 11. *Hamilton's modification of Feuerbach's theorem* . . 117, 118
Ex. 12. *Conic referred to its centre and axes* 118, 119
Ex. 13. *Confocal conics* . . . . 119—121
Ex. 14. *Conics derived from a conic intersecting the joins of three points. Bipunctual conics* . . . . 121—123
Ex. 15. *Two triads reciprocal in regard to a conic are in perspective* 123, 124
Ex. 16. *The Hessian line of three points of a conic* . . . 124
Ex. 17. *The director circle of a conic, and generalisations* . . 124—127
Ex. 18. *Envelope of a line joining points of two conics which are conjugate in regard to a third, with particular applications* . 127—134
Ex. 19. *Indication of a generalisation of the preceding results* . 134
Ex. 20. *Of correspondences, in particular of general involutions, upon a conic* . . . . 134—139
Ex. 21. *Reciprocation of one conic into another when they have a common self-polar triad* 139, 140

PAGES
Ex. 22. *The determination of the common self-polar triad of two conics in general* 140, 141
Ex. 23. *Fundamental invariants for two conics* 141, 142
Ex. 24. *The algebraical condition for one conic to be outpolar to another, or to be triangularly circumscribed to another* . . 142—146
Ex. 25. *Enumeration of reduced forms for the equations of any two conics* 146, 147
Ex. 26. *A particular involution of sets of three points. Apolar triads* 147—149
Ex. 27. *Three conics related in a particular way* . . . 149—151
Ex. 28. *The director circle of a conic touching three lines is cut at right angles by a circle, for which the three lines form a self-polar triad* 151
Ex. 29. *A transformation of conics through four points into conics having double contact* 152

CHAPTER IV. RESTRICTION OF THE ALGEBRAIC SYMBOLS. THE DISTINCTION OF REAL AND IMAGINARY ELEMENTS

Review of the development of the argument. Real and imaginary symbols . . . . 153—155
The distinction of positive and negative for the real symbols . 155, 156
Order of magnitude for the real symbols 156, 157
The roots of a polynomial 157
The exponential and logarithmic functions 157
A simple application to the double points of an involution. Reality of double points of involution defined by two pairs . . 157—160
General statement of the meaning of real and imaginary points in a plane 160, 161
Some applications of the definition. Imaginary points and lines . 161, 162
Real and imaginary conics . . . . 162—164
The interior and exterior points of a real conic 164
The reality of the common points, and of the common self-polar triad, of two conics 164, 165
Example. Construction of a conic given three points and two pairs of conjugate points 165

CHAPTER V. PROPERTIES RELATIVE TO AN ABSOLUTE CONIC. THE NOTION OF DISTANCE. NON-EUCLIDEAN GEOMETRY

The interval of two points of a line relatively to two given points of the line 166, 167
Angular interval in regard to two absolute points 167
Interval in regard to an absolute conic. Meaning of movement . 167—170

PAGES

Case of an imaginary conic as Absolute . . . . . 170—172
Application to a triangle, in particular a right-angled triangle . 172—177
The sum of the angular intervals for a triangle 177
Formulae for a general triangle 177, 178
The case of a real conic as Absolute . . . . 178—180
The sum of the angles of a triangle and a quadrangle of three right angles 180, 181
Comparison with the Lobatschewsky-Bolyai geometry resumed . 181—183
Case of degenerate conic as Absolute . . . . 183—186
The arbitrariness and comparison of the several methods of measurement. Extent between two lines, of a triangle, of a circle, in the elliptic geometry . . . . 186—189
Plane metrical geometry by projection from a quadric surface. Spherical geometry. Elementary properties of a quadric surface 189—195
Metrical geometry in regard to an absolute conic. Cayley's measure of distance deduced from Laguerre's . . . . . . 195—197
Representation of the original plane upon another plane. Riemann's space of constant curvature. Lobatschewsky lines as circles cutting a fixed line at right angles . . . . . . 197—204
Remark, comparison of the two cases considered 204
Examples of the preceding considerations (Exx. 1—9) . . . 204—211
Ex. 1. *Delambre's formulae in spherical trigonometry* . . . 204—206
Ex. 2. *Napier's analogies* 206
Ex. 3. *A particular triangle corresponding to the division of a sphere into four congruent triangles* . . . . . . 206—209
Ex. 4. *General form of the theorem for the sum of the focal distances of a point of a conic. Circular sections and focal lines of a cone* 209, 210
Exx. 5, 6. *Particular theorems of importance* 210
Exx. 7, 8. *Generalisation of result of Ex.* 4 . . . . . 210, 211
Ex. 9. *Elliptic integral for porism of two conics* 211

NOTE I. ON CERTAIN ELEMENTARY CONFIGURATIONS, AND ON THE COMPLETE FIGURE FOR PAPPUS' THEOREM

The simpler Desargues' figure 212, 213
The figure of three desmic tetrads of points in space of three dimensions 213, 214
The figure formed from three triads of points which are in perspective in pairs, with centres of perspective in line . . . 214, 215
The complete figure for Pappus' theorem 215, 216
The figure formed from six points in four dimensions . . . 216—218
Exx. 1—4. *The configuration in space of* n *dimensions, formed from* $(n+2)$ *points, or from* $(n+2)$ *spaces of* $(n-1)$ *dimensions; it is self-polar* 218

NOTE II. ON THE *HEXAGRAMMUM MYSTICUM* OF PASCAL

PAGES
Introduction, and arrangement of note 219, 220
Sylvester's synthemes from six elements. The combinatorial identity with the Pascal figure . . . . 220—224
The geometrical interpretation of the relations, in space of four dimensions (see also the Frontispiece of the volume). Statement and proof of the Pascal incidences . . . . . 224—229
Direct deduction of Pascal's figure from a cubic surface with a node 229—231
Examples of particular properties of the figure (Exx. 1—6) . . 231—235
Ex. 1. *The pairs of conjugate Steiner planes* 231, 232
Ex. 2. *The Steiner planes and Steiner-Plücker lines* . . . 232
Ex. 3. *The separation of the complete figure into six figures* . . 232
Ex. 4. *Tetrads of Steiner points each in threefold perspective with a tetrad of Kirkman points* 232, 233
Ex. 5. *Relation of the Cayley-Salmon lines and the Salmon planes* 233
Ex. 6. *Proof of the relations by plane geometry* . . . . . 233—235
References to the literature of the matter 235, 236

NOTE III. IN REGARD TO THE LITERATURE FOR NON-EUCLIDEAN GEOMETRY 237

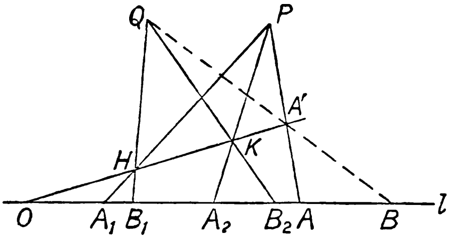
ADDITIONS 239

INDEX 254

PRELIMINARY

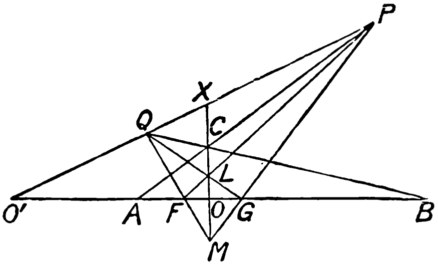
There are several matters, readily understood by the reader of Volume I, in regard to which we have not there entered into the detail which may be desirable for the purposes of the present volume.

Related ranges on the same line. With the purpose of avoiding the use of points whose existence could only be assumed after the consideration of the so-called imaginary points, we have (Vol. I, pp. 18, 25) defined two ranges on the same line as being *related* when, one of them is in perspective with a range on a second line which is related to the other range of the first line. From this definition we have shewn (Vol. I, p. 160) that in the abstract geometry two such related ranges on the same line have two corresponding points in common, though these may coincide. Assuming this, we may now formally prove that two such ranges also satisfy the general definition, namely that they are both in perspective with the same other range on another line, from different centres.



Let the ranges (a), (b), on the same line, l, be such that (a) is in perspective with a range (c), while (c) is related to (b). Let O be a point of the line l which corresponds to itself whether regarded as belonging to the range (a) or to the range (b); let A_1, A_2, A be other points of the range (a), respectively corresponding to the points B_1, B_2, B of the range (b). Let H, K be any two points in line with O; let A_1H, A_2K meet in P, and B_1H, B_2K meet in Q, and let PA meet the line OHK in A'. Then the range O, H, K, A' is in fact related to O, B_1, B_2, B. For the former is in perspective, from P, with the range O, A_1, A_2, A; this is, by hypothesis, in perspective with a range (c), which is itself related to O, B_1, B_2, B; so that the result follows from Vol. I, pp. 22–24. Thence, as the ranges, O, H, K, A' and O, B_1, B_2, B, have the point O in common, they are in perspective (Vol. I, p. 58, Ex. 2 (c)). Thus the line QB passes through A', and the two ranges (a), (b) are in perspective with the same range on the line OHK, respectively from P and Q. This is what we were to prove.

It is clear that the line PQ meets the line l in another common corresponding point of the two ranges (*a*), (*b*), which may however coincide with *O*.



Involution. Considering two related ranges upon the same line, it does not generally follow that, if *K* be the point of the second range corresponding to a point *H* of the first range, then to the point *K*, considered as belonging to the first range, there corresponds the point *H*, considered as belonging to the second. We proceed to shew, however, that, if this be true for one position, *F*, of *H*, and the corresponding position, *G*, of *K*, then it is true for every pair of corresponding points *H*, *K*.

For, in accordance with the preceding, let the ranges be in perspective, respectively from the points *P* and *Q*, with the same range; let *X* be the point of this range which lies on the line *PQ*, and *O′* the common corresponding point of the two ranges arising by perspective from *X*; let *L* be the point of this range which gives rise, respectively from *P* and *Q*, to the two particular corresponding points *F*, *G* of the two ranges, the points *P*, *L*, *F*, and also the points *Q*, *L*, *G*, being in line. Then, by hypothesis, the lines *PG*, *QF* meet in a point, *M*, of the line *XL*. Let *O* be the intersection of the line *XL* with the original line, so that *O* is also a common corresponding point of the two ranges.

It is clear from the construction that the points *O*, *O′* are harmonic conjugates in regard to *F* and *G*, and therefore do not coincide with one another; unless, indeed, they both coincide either with *F*, or with *G* (I, pp. 14, 119), in which case, as *O′* is a self-corresponding point, *G* would coincide with *F*, which we suppose not to be the case. Thus, when the two common corresponding points of the two given ranges are coincident, the case of two such different reciprocally related points *F* and *G* as we are now considering does not arise. Further, the points *O′*, *X* are harmonic conjugates in regard to *P* and *Q*.

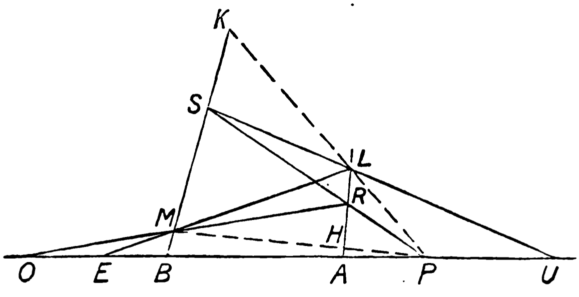
Now let *A*, *B* be any two other corresponding points, respectively of the two given ranges, arising from the point *C* of the line *XL*, by perspective from *P* and *Q*, respectively, so that *P*, *C*, *A*, and also *Q*, *C*, *B* are in line. Then, as *O′*, *X* are harmonic conjugates in regard to *P* and *Q*, it follows that *O′*, *O* are harmonic conjugates in regard to *A* and *B*. From this it follows that *PB* and *QA* meet on the line *XL*, and, therefore, that, to the point *B*, of the first range, corresponds the point *A*, of the second; as we desired to prove.

Conversely, any two points of the original line which are harmonic conjugates in regard to O' and O are a pair of reciprocally corresponding points of the two ranges. The aggregate of such pairs is called an *involution* of pairs of points; the points O', O are called the *double points* of the involution.

It is clear that if three pairs of points of a line, (A, B), (F, G) and (U, V), be pairs of an involution, then the range A, F, G, U, consisting of two points of one pair, and a point from each of the other two pairs, is related, point to point, to the range B, G, F, V, consisting of the respectively complementary points of the various pairs. For we can define two related ranges by the fact that the points A, F, U, of the one, correspond, respectively, to the points B, G, V, of the other; then, to the point G, of the former, corresponds the point F, of the latter. Conversely, if six points of a line be such that the range A, F, G, U is related, point to point, to the range B, G, F, V, then (A, B), (F, G) and (U, V) are three pairs of an involution. Also, an involution is established when two pairs are given; for, if these be (A, B) and (F, G), we have only to associate, to any point U, a point V for which the range B, G, F, V is related, point to point, to the range A, F, G, U.

Thus, further, a pair of points, O and O', exists, which are harmonic conjugates both in regard to one arbitrary pair of points, A, B, and also in regard to another arbitrary pair, F, G, which lie in the line AB. These points, O, O', are the double points of the involution determined by the pairs A, B and F, G.

From this it follows, also, that if there be two involutions of pairs of points upon the same line, there is a pair of points common to both involutions. This pair consists of the two points which are harmonic conjugates in regard to the double points of the first involution, and also harmonic conjugates in regard to the double points of the second involution.



The definition of an involution of pairs of points on a line may be approached differently, in connexion with a figure previously employed (Vol. I, pp. 76, 77).

Let A, B, O, U, E be arbitrary points of a line; in a plane through the line draw two lines AL, BM, met by a line through E respectively in L and M; let UL and BM meet in S, and OM meet AL in R. Let SR meet AB in P.

If, then, PL, PM, respectively, meet BM and AL in K and H, the range O, A, E, P is in perspective, from M, with the range R, A, L, H; this last is in perspective, from P, with the range S, B, K, M, and this, again, in perspective, from L, with the range U, B, P, E. As, then, the ranges O, A, E, P and U, B, P, E are proved to be related, it follows, from what is said above, that the pairs (O, U), (A, B) and (E, P) are in involution.

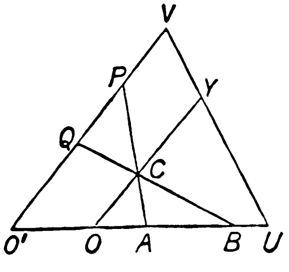
Thus the three pairs of joins of any four points of a plane (in this case M, R, L, S) meet an arbitrary line, of the plane, in three pairs of points of the same involution. As has been remarked in Vol. I (p. 181), the significance of the fact that five of these points determine the remaining one was noted by the Greeks.

Every result relating to pairs of points of a line corresponds, by the principle of duality, to a result relating to pairs of lines, in a plane, passing through a point. We may therefore consider pairs of lines, in a plane, through a point, forming a pencil in involution; these meet an arbitrary line of the plane, not passing through the centre of the pencil, in pairs of points of a range in involution. In particular, if four arbitrary lines be given in a plane, these, divided into two pairs in each of the three possible ways, determine, by the intersections of lines of a pair, three pairs of points. The lines which join an arbitrary point of the plane to these three pairs of points, are three pairs of lines belonging to the same involution.

Symbolical expression of the preceding results. We have in Vol. I (pp. 140, 154) reached the result that if a range of points represented by symbols $O + xU$, for different values of x, be related to a range of points represented by symbols $O + yU$, for different values of y, then there is a relation of the form

$$ayx + by + cx + d = 0,$$

wherein the symbols a, b, c, d are independent of x and y. We are, throughout, assuming Pappus' theorem, and there is no question of the commutativity of the multiplication of the symbols. It may be interesting, in the first place, to obtain this result by regarding the ranges as being in perspective, from different centres, with the same range on another line.



For this, let the ranges $(A\ldots)$, $(B\ldots)$, on a line $O'U$, be in perspective with a range $(C\ldots)$, on a line OY, respectively, from centres P and Q. Let the line PQ

meet $O'U$ in O'; let V be any point of this line, and let any line from V meet the line of the range $(C\ldots)$ in the point Y, and meet the line $O'U$ in the point U; let the line of the range $(C\ldots)$ meet the line $O'U$ in O. Then, regarding the points O', U, V as fundamental points of the plane, we may suppose, for the symbols of the points concerned, the following expressions

$$P = O' + pV, \quad Q = O' + qV, \quad O = O' + mU, \quad Y = U + V.$$

Thence, if we take for different points C,

$$C = O + cY,$$

with different values of c, that is

$$C = O' + mU + c(U + V), = O' + (m + c)U + cV,$$

we infer, for the points A, B, of the ranges on OU, which arise from C, respectively the symbols

$$(c^{-1} - p^{-1})O' + (c^{-1}m + 1)U, \quad (c^{-1} - q^{-1})O' + (c^{-1}m + 1)U;$$

if we write, then, $A = O' + xU$, $B = O' + yU$,

this gives $$\frac{1 + mp^{-1}}{1 - mx^{-1}} = 1 + mc^{-1} = \frac{1 + mq^{-1}}{1 - my^{-1}}$$

and, hence, $(p - q)xy + q(p + m)x - p(q + m)y = 0.$

This is of the form in question. It shews, putting $x = 0$, $y = 0$, that O' is a common corresponding point of the two ranges $(A\ldots)$ and $(B\ldots)$; and, putting $x = m$, $y = m$, that O is the other common corresponding point. It coincides with O' when $m = 0$.

Passing now to the condition for an involution: If, when A takes the present position of B, this takes the present position of A, we must have, beside the previous relation, also the relation

$$(p - q)xy + q(p + m)y - p(q + m)x = 0;$$

from these two relations we obtain, by subtraction,

$$[p(q + m) + q(p + m)][x - y] = 0,$$

and hence, A and B being any particular pair of corresponding points which do not coincide,

$$p(q + m) + q(p + m) = 0.$$

When this is satisfied the relation connecting the values of x and y, in general, is at once seen to reduce to

$$2xy - m(x + y) = 0.$$

We may, however, write

$$(x - y)O = (m - y)(O' + xU) - (m - x)(O' + yU),$$
$$= (m - y)A - (m - x)B;$$

thus the harmonic conjugate of O, in regard to A and B, is (Vol. I, p. 74) of symbol

$$(m - y)A + (m - x)B,$$

or $$(2m - x - y)O' + [m(x + y) - 2xy]U,$$

and coincides with O' whatever pair of corresponding points A and B may be. Thus O and O' must be different, and the general pair of points of the involution is a pair of points harmonically conjugate in regard to these.

Examples of involution. *Ex.* 1. The condition that the pair of points represented by $O'+xU$, $O'+yU$, should be harmonic conjugates of one another in regard to the pair of points represented by $O'+aU$, $O'+bU$, being the condition that, for proper symbols p, q,

$$x = (pa+qb)/(p+q), \quad y = (pa-qb)/(p-q),$$

is

$$(x-a)(y-b)+(x-b)(y-a)=0,$$

or

$$xy-\tfrac{1}{2}(x+y)(a+b)+ab=0.$$

This remark, (a), gives the pair of points which are harmonic conjugates of one another in regard to each of two other given pairs of points. In particular, the pair of points harmonically conjugate both in regard to O', U, and in regard to $O'+aU$, $O'+bU$, is given by $O'+kU$, $O'-kU$, where $k^2=ab$. It also gives, (b), the relation for a pair of points belonging to a given involution, and, (c), shews that two given involutions, on the same line, have a common pair of points.

Ex. 2. When $O'+xU$, $O'+yU$, are, as above, corresponding points of two related ranges, of which O' is a common corresponding point, the relation connecting x and y,

$$(p-q)xy+q(p+m)x-p(q+m)y=0,$$

is capable of one of the two following forms:

$$\frac{y-m}{x-m}=\sigma\frac{y}{x}, \qquad \frac{1}{y}=\frac{1}{x}+\frac{1}{\lambda},$$

where

$$\sigma=\frac{p(q+m)}{q(p+m)}, \qquad \frac{1}{\lambda}=\frac{1}{p}-\frac{1}{q},$$

according as m is not zero, or m is zero. In general, the relation

$$ayx+by+cx+d=0,$$

if the equation

$$ax^2+(b+c)x+d=0,$$

for which $y=x$, have two different roots α, β, is capable of the form

$$\frac{y-\beta}{x-\beta}=\sigma\frac{y-\alpha}{x-\alpha},$$

where

$$\frac{(\sigma+1)^2}{\sigma}=\frac{(b-c)^2}{ad-bc};$$

but, if $\beta=\alpha$, or $(b+c)^2=4ad$, the relation is capable of the form

$$\frac{1}{y-\alpha}=\frac{1}{x-\alpha}+\frac{1}{\lambda},$$

where

$$\lambda=\tfrac{1}{2}(b-c)/a.$$

The general condition for a pair of points, represented by $O' + xU$, $O' + yU$, to belong to an involution is of the form

$$axy + b(x+y) + c = 0.$$

If we choose U for one of the two double points, we have $a = 0$. If then, also, O' be the other double point, we have also $c = 0$. If O' and U be any pair of the involution, we have $b = 0$.

In general, the double points are given by the values of x for which

$$ax^2 + 2bx + c = 0;$$

if these coincide ($b^2 = ac$), the involution reduces to one fixed point, for which $x = -b/a$, taken in turn with every other point of the line. This very degenerate case has been excluded from consideration in what has preceded.

Ex. 3. If P, Q and P', Q' be any two pairs of points of a line, and P'' be the harmonic conjugate of P in regard to P' and Q', while Q'' is the harmonic conjugate of Q in regard to P' and Q', the pairs (P, Q), (P', Q'), (P'', Q'') are in involution.

For, by construction, P, P'' and Q, Q'' are pairs of an involution with P', Q' as double points, and the harmonic ranges P, P'', P', Q' and Q'', Q, P', Q' are related; the latter, and, therefore, also the former, is related to the range Q, Q'', Q', P' (Vol. I, p. 25, Ex. 1). This shews that (P, Q), (P'', Q''), (P', Q') are pairs of an involution.

If the points P, Q correspond to the roots of $ax^2 + 2hx + b = 0$, or, say, $F = 0$, being given by $O + x_1U$, $O + x_2U$, where x_1, x_2 are the roots of this equation, and, similarly, with the same points of reference O, U, the points P', Q' correspond to the roots of $a'x^2 + 2h'x + b' = 0$, or, say, $F' = 0$, it may be shewn that P'', Q'' correspond, similarly, to the roots of the equation

$$(a'b' - h'^2)F - (ab' + a'b - 2hh')F' = 0.$$

Ex. 4. If A, A'; B, B'; ... be pairs of points of a line, which are in involution, and we take the harmonic conjugate of every one of these in regard to two points, U and V, of the line, so obtaining the pairs P, P'; Q, Q'; ..., then these are also pairs in involution.

For any range P, Q ... is then related to the corresponding range A, B ...; this in turn is related to the range A', B' ...; and this to P', Q' ....

Ex. 5. Given any two pairs of points of a line A, B and A', B', let the pair of points which are harmonic both in regard to A, B and in regard to A', B', be denoted by $(AB, A'B')$; and, therefore, if A'' and B'' be two other points of the line, let $\{(AB, A'B'), A''B''\}$ denote the pair harmonic both in regard to the pair $(AB, A'B')$ and in regard to the pair A'', B''. Shew that the three pairs of points

$$\{(AB, A'B'), A''B''\}, \quad \{(A'B', A''B''), AB\}, \quad \{(A''B'', AB), A'B'\}$$

are in involution. If A, B be given by $f=0$, where

$$f = ax^2 + 2hxy + by^2,$$

and A', B' be given by $f'=0$, where $f' = a'x^2 + 2h'xy + b'y^2$, shew that $(AB, A'B')$ are given by

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a, & h, & b \\ a', & h', & b' \end{vmatrix} = 0,$$

and $\{(AB, A'B'), A''B''\}$ are given by $Hf - H'f' = 0$, where, if $A''B''$ be similarly given by $f''=0$, the function H is $a'b'' - 2h'h'' + b'a''$, while $H' = ab'' - 2hh'' + ba''$.

A general abbreviated argument for relating two ranges. Consider two lines, which may coincide, or, more generally, may lie in space of any number of dimensions. Let O, U be two points of the former line, and O', U' be two points of the latter line; if the lines coincide O' and U' will each be in syzygy with O and U, the symbols of O' and U' being each expressible by O and U; if the lines intersect there will be one syzygy connecting the four points. Now, suppose that we have a geometrical construction whereby there is determined a definite point P', of the second line, corresponding to every point P of the first line, and a construction whereby we may conversely pass back from P' to the same point P. The construction may be such as gives, when we start from P, other points $A', B', \ldots$, of the second line, beside P', provided these remain the same for every position of P; and similarly for the construction by which we pass back from P' to P.

For greater definiteness we must also add that the construction must be analogous to those considered in the Third Section of Chap. I of Vol. I (p. 74); it must be such that, if the point P have the symbol $O + xU$, and the point P' the symbol $O' + x'U'$, then x' is determined from x by those laws of combination of the symbols which do not involve the solution of any equation of the second or higher order, the solution of such an equation not being without ambiguity; and x must be determined from x' in a similar way. There is therefore a single algebraic equation, connecting x and x', of the form $\Sigma a x'^m x^n = 0$, containing only a finite number of terms, in which every coefficient a is quite definite, m and n being positive integers. Then, as, to every value of x, there belongs only one value of x', other than those independent of x, the equation may be expected to be such that the highest value of the exponent m is 1; and, similarly, such that the highest value of n is also 1. (Cf. Chap. IV, below.) The relation would then be of the form

$$ax'x + bx' + cx + d = 0.$$

When we have such a determinate construction we shall, sometimes, assume, for the sake of brevity, that it is possible to find a third range with which both the given ranges are in perspective, from appropriate centres; so that the given ranges are related.

Of the distinction between the so-called real and imaginary points. In the general discussion introductory to Chap. III of Vol. I (p. 141), and in some other cases, we have spoken of the distinction between real and imaginary points in a way which, if definite when we approach the matter from the point of view of the Real Geometry, is not so clear from the point of view to which we desire to reach. It might be proper then to enter now into more detail. But we do in fact regard the distinction as arising in connexion with an arbitrary limitation of the possibilities of the points which can exist in the space considered; for this reason we postpone this discussion until (in Chap. IV, below) we definitely agree to make this limitation. This limitation is represented by a restriction in the system of symbols appropriate to the geometrical results obtained. While we wish to leave the logical possibilities as open as we can, we desire to expound a system of geometry in harmony with what is commonly accepted. For this purpose we have suggested, in Vol. I, that we may take as a geometrical postulate the possibility of what we have called Steiner's construction (Vol. I, pp. 155 ff.). In a similar way we shall in the first three chapters of the present volume adopt as a postulate the theorem that *two* curves in a plane which are *conics*, in the sense to be immediately explained, *have four common points, of which two, or more, may coincide.* It will be seen below (in Chap. IV) how this would be proved when the limitation referred to is adopted. Cf. pp. 19, 157 below.

CHAPTER I

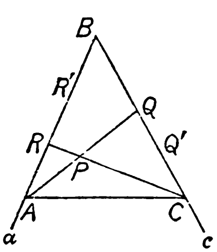
GENERAL PROPERTIES OF CONICS

Definition of a Conic. Consider two pencils of lines, in the same plane, of centres A and C, and suppose these are related to one another, in the sense explained in Vol. I; to any line, AP, of one pencil, there corresponds, then, a definite line, CP, of the other, and, conversely, to any line, CP, there corresponds one line, AP. The curve which is the locus of the intersection, P, of such corresponding rays AP, CP, is called a conic section, or, briefly, a conic. Conversely, it will be seen that any conic can be so obtained, from any two points, A, C, of itself.

By its definition the curve contains one point, P, beside A, upon any line drawn through A; though, when the line drawn through A is that which corresponds to the line CA drawn through C, the point P coincides with A. Similarly for a line drawn through C. Consider however any line not passing through A nor C. This line contains two points of the conic. For the related pencils, of centres A and C, determine upon this line two related ranges. These have two common corresponding points. If these be P_1 and P_2, the ray AP_1, of the one pencil, corresponds to the ray CP_1, of the other, and P_1 is on the locus; and it is the same for P_2.

There is however one case, which we regard as exceptional, to which reference should be made: It may be that, to the ray AC of the pencil (A), there corresponds the ray CA of the pencil (C). When this is so, the locus consists of a line of the plane, together with the line AC itself. For if P_1 and P_2 be, then, any two points of the locus, not lying on the line AC, and the line P_1P_2 meet the line AC in B, the two related pencils (A), (C) determine, on the line P_1P_2, two related ranges having three common corresponding points, namely P_1, P_2 and B. These two ranges thus coincide, and every point of the line P_1P_2 is a point of the locus. If there were any point, P, of the plane, not lying on the line P_1P_2 or the line AC, which was upon the locus, then the join of P to any point of P_1P_2, would, by what has been said, be a line of which every point was a point of the locus; and in that case every point of the plane would be a point of the locus. The complete locus thus consists of the line P_1P_2, and of the line AC, of which every point evidently satisfies the definition. Conversely, any two lines of the plane may be regarded as constituting a degenerate conic, determined by related pencils whose centres are any two points of one of the lines. Such a degenerate conic may be spoken of as a line pair. In Vol. I (p. 58), we have seen that two related ranges on two intersecting lines, which are such that the intersection of the lines is a common corresponding point, are in perspective. The aggregate of the lines joining corresponding points of the two ranges consists then of a pencil of lines, together with every line through the common point of the ranges. This is the result dually corresponding to that here remarked.

In what follows we shall, in general, suppose that, to the ray, AC, of the pencil (A), there corresponds a ray, CB, of the pencil (C), which does not coincide with CA. To the ray CA, of the pencil (C), will then correspond a ray, AB, of the pencil (A), meeting the former, say, in B. As the rays AC, CB meet in C, and the rays CA, AB meet in A, the locus contains the points A and C.

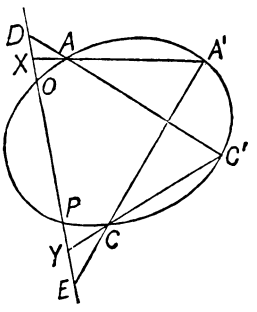


The lines CB, AB, say, respectively, c and a, are determined when the related pencils are given. Conversely, if these lines be given, and, upon them, respectively, the points C and A, and, also, a point, P, of the locus, be given, then the related pencils are determined; for, the three rays AC, AB, AP, of the one, then correspond, respectively, to the three rays CB, CA, CP, of the other. These lines c, a contain no other point of the locus beside, respectively, C and A. For, if AP, CP meet c and a, respectively, in Q and R, there are two related ranges on c and a, in which the points C, B, Q, of the one, correspond, respectively, to the points B, A, R, of the other; if then the point, P, of the locus, lie on c, the point R will be at B, and, therefore, Q, and hence also P, will be at C. Moreover if, for a moment, we limit ourselves to only real points, and take a further point, P', of the locus, for which the corresponding Q, say Q', is separated from B by Q and C, then the corresponding R, say R', will be separated from A by R and B. We therefore speak of the locus as having two *coincident* points at C; similarly, it has two coincident points at A. The lines c and a are called *tangents* of the conic, respectively at C and A. It will appear that, through every point of the conic, a line can be drawn containing only this point of the locus; this line will be called the tangent at the point.

We have remarked that, if the points A, C, and the tangents a, c at these points, and also a point, P, of the locus, be given, the related pencils, and, therefore, also, the conic, are determined. More generally, if the points A, C, and three points P_1, P_2, P_3 (of which we suppose at most one, but, in general, none, to lie on the line AC)

be given, the conic is determined; for we have only to make the rays CP_1, CP_2, CP_3, of the pencil (C), correspond, respectively, to the rays AP_1, AP_2, AP_3, of the pencil (A), whereby the pencils are related. Thus a conic can be described through five points of the plane, of general position; when three of these are in line, the conic becomes a line pair; when four of them are in line the conic is not determined.

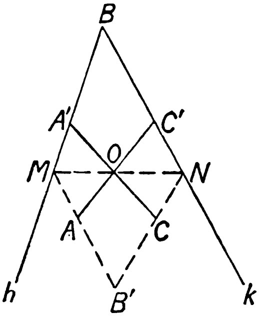
It has been stated that the points A, C, in the definition of the conic, may be replaced by any other two points of the locus, say A' and C'. This is easy to prove. Let O be any point of the conic, and let any line, drawn through O, meet the conic again in a point P. Let the lines AA', CC' meet the line OP, respectively, in X and Y, and the lines AC', $A'C$ meet this line, respectively, in D and E. Then, the pencil, of centre A, formed with lines joining this point to A', C', O, P, is, by hypothesis, related to the pencil, of centre C, formed with lines joining this point to the same four points. Wherefore, the range X, D, O, P is related to the range E, Y, O, P, and, therefore, also to the range Y, E, P, O. Thus (Preliminary, above, p. 3), the three pairs of points X, Y; D, E; O, P are in involution. To prove, now, that the conic defined with related pencils of centres A' and C' to pass through the points A, C, O, also passes through P, whatever point of the original conic P may be, we require only to shew that the pencil of lines joining A' to A, C, O, P is related to the pencil of lines joining C' to these same points, respectively. This will be so if the range X, E, O, P is related to the range D, Y, O, P; which will be so if the range X, E, O, P is related to the range Y, D, P, O. The condition for this, however, is the same as before, that the pairs of points X, Y; D, E; O, P should be in involution.



The result is therefore established. Thereby, there is also defined, at every point of the curve, a line, called the tangent, meeting the curve only at this point.

Involution on a line by conics having four common points. Incidentally there appears from the preceding proof a theorem which is of great importance: If four points (A, C, A', C') be taken on a conic, and two pairs of lines (AC', $A'C$ and AA', CC') of which each pair contains the four points, then an arbitrary line is met by these line pairs, and by any conic passing through the four points, in three pairs of points belonging to the same involution. We have seen that a conic can be drawn through five general points, so that an infinite number of conics can be drawn through four general points; we have also seen (Preliminary, above, p. 3) that an involution is determined by two of its pairs. Hence we have the result that, the pairs of points in which the conics, drawn through four general points, cut an arbitrary line, are pairs of the same involution. Any one of the three line pairs which can be drawn through these four points is a particular degenerate conic of this description.

Maclaurin's definition of a conic. Pascal's theorem. It has been shewn in Vol. I (pp. 21 ff.) that two related ranges, on different lines, a and c, are both in perspective with another range, on a line o, say, from proper centres, say H and K, respectively. The dually corresponding result is that, two related pencils, with different centres, A and C, must consist of lines joining A and C to the points, respectively, of two related ranges on different lines, say h and k, these ranges being in perspective with one another from some centre, say O.



Thus the definition of a conic which has been given is the same as, the locus of a point, B', which is the intersection of two lines MB', NB', passing, respectively, through two fixed points, A and C, where M and N are any two points on two given lines, h and k, such that MN passes through a third fixed point, O. Conversely, the ranges (M), (N) being in perspective, the pencils $A(B')$ and $C(B')$ are clearly related, so that the present definition leads back to the former.

If B be the intersection of the lines h and k, and OA, OC meet the lines k and h, respectively, in C' and A', the conic contains the points B, A', C'. For, first, when M and N coincide at B, the lines MA, NC intersect in B; and, next, when M is at A', and, therefore, N is at the point where OC meets k, the line NC contains A', which is then the intersection of NC and MA; similarly, when N is at C', so also is B'. As was remarked above, the locus contains also the points A and C. We have thus, at once, a construction, with lines only, for finding any number of points of a conic passing through five arbitrary given points B, A, C, A', C'; as was seen, such five points determine the conic. And, from the symmetry which was shewn to hold among the fundamental points, we can now infer that, if A, B', C, A', B, C' be any six points of a conic, the three points of intersection of the pairs of lines, BC', $B'C$; CA', $C'A$; AB', $A'B$, are in line. This result is the celebrated one known as Pascal's Theorem. From six points, we can form two sets of three in ten ways, and either of these sets can be arranged, relatively to the other, in six ways; thus, from six given points of a conic can be deduced sixty different lines, each containing three such points of intersection as those here remarked. When the conic consists of two lines, one containing the points A, B, C, the other the points A', B', C', Pascal's Theorem becomes that which, in Vol. I, we have called Pappus' theorem.

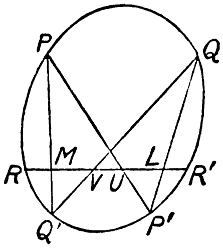
The construction at once gives the tangent of the conic at the point A: If the line AC meet the line k in N_0, and the line N_0O meet the line h in T, then AT is the tangent at A. So, if AC meet h in M_0, and M_0O meet k in U, then CU is the tangent at C. The tangent at A thus appears as the position of the line AB' when B' is at A, or, equally, as the ray of the pencil of centre A which corresponds to the ray CA of the pencil of centre C; and similarly for the tangent at C. To find the points, B', of the conic, which lie on an arbitrary line, it is necessary to solve the problem considered in Vol. I, pp. 155 ff., of finding lines through three given points whose intersections lie on three given lines.

Ranges of points on a conic. Since the pencils of four lines which join any four given points of a conic to other points of the conic are all related, the definition of a conic ascribes a definite character to four points of the curve in relation to any other four points: two such sets of four points of the locus may be said to be related to one another when the pencil of four lines, joining the first set of four to any point of the conic, is related to the pencil of four lines joining the second set of four points to any point of the conic. Thus, given any two triads of points of the conic, say A, B, C and A', B', C', we can, to any point, D, of the conic, determine a point, D', of the curve, such that, in the sense explained, the range A', B', C', D', of points of the conic, shall be related to the range A, B, C, D. There exists, therefore, a theory of related ranges of points of the conic precisely like that of related ranges of points of a line.

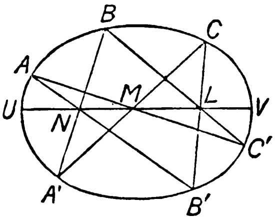
Involution of pairs of points of a conic. In particular, there is a theory of pairs of points of the conic which are in involution, one form of the condition that three pairs of points of the conic, P, P'; Q, Q'; R, R', be in involution, being, as in the case of points of a line (Preliminary, above, p. 3), that the range P, Q, R, R' should be related to the range P', Q', R', R. This condition, however, is capable of very simple geometrical interpretation: The necessary and sufficient condition that the three pairs of points of the conic, P, P'; Q, Q'; R, R', should be in involution, is that the lines PP', QQ', RR' should meet in a point.

This fact is easy to prove from what we have seen above. Take any three pairs of points of the conic, P, P'; Q, Q' and R, R'; let the line RR' meet the line PP' in U, and meet the line QQ' in V;

further, let the lines $P'Q$ and PQ' meet the line RR' in L and M. By what we have seen, on the line RR' the pairs of points R, R'; U, V and L, M are in involution; therefore, the range U, L, R, R' is related to the range V, M, R', R. The former range, U, L, R, R', is, however, the section, by the line RR', of the pencil of lines joining to P' the points P, Q, R, R' of the conic. The condition that the pairs of points of the conic, P, P'; Q, Q' and R, R', should be in involution, is that the pencil joining P, Q, R, R' to any point of the conic should be related to the pencil joining P', Q', R', R to any point of the conic. The lines joining, to P, these last four points, meet the line RR' in, respectively, U, M, R', R. The condition is, then, that the range U, M, R', R should be related to the range U, L, R, R'; this, however, we have remarked, is related to the range V, M, R', R. The condition, therefore, is, that the ranges, U, M, R', R and V, M, R', R, should be related, or that V should coincide with U. This is the condition that the lines, PP', QQ', RR', should meet in one point, as was stated.



Pascal's theorem and related ranges of points upon a conic. From the existence of related ranges upon a conic, we can at once deduce Pascal's theorem. And another proof of the preceding theorem, in regard to involutions of points upon the conic, offers itself immediately.



Let A, B, C, A', B', C' be any six points upon the conic; let L be the intersection of the lines BC', $B'C$, and, similarly, M the point $(CA', C'A)$ and N the point $(AB', A'B)$. We can, we have seen, determine two related ranges of points upon the conic by the condition that the points A', B', C', of one of these, correspond, respectively, to the points A, B, C of the other. These ranges will have two common corresponding points; let these be U and V, which, for the present, we suppose not to coincide. Then the pencil of lines joining the points U, V, A, B to any point of the conic is, by hypothesis, related to the pencil joining, respectively, U, V, A', B' to any point of the curve. If the former point be taken to be A', and the latter to be A, and we consider the sections of these pencils by the line UV, so obtaining two sets of four points of which three are the same in both, we infer at once that the point N is on the line UV. Similarly M, and L, lie on this line; thus L, M, N are in line. This is Pascal's theorem.

The condition that the ranges U, V, A, B and U, V, A', B' should be related is the same as that U, V, A, B and V, U, B', A' should be related. This, however, is the same as that the pairs U, V; A, B' and A', B, of points of the conic, should be in involution; for this, then, the lines UV, AB' and $A'B$, must meet in a point. Conversely, Pascal's theorem is equivalent to saying that, if A, B, C, A', B', C' be any six points of the conic, the three involutions determined, respectively, by the two pairs B, C' and B', C; by the two pairs C, A' and C', A; and by the two pairs A, B' and A', B, have a pair common to all three involutions (the pair U, V).

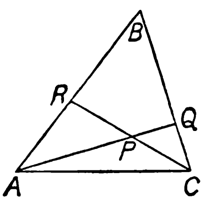
If H, K be any two points of the range $A, B, C, \ldots$, of points of the conic, and H', K' be the respectively corresponding points of the related range $A', B', C', \ldots$, the lines HK', $H'K$ can similarly be shewn to meet on the line UV. This line may be called the *axis of relation* of these two ranges, or the axis of homography; cf. Vol. I, p. 53.

Pascal's theorem remains equally true when the points U, V coincide. The line UV is then replaced by the tangent of the conic at U.

Converse of Pascal's theorem. It is easy to prove from the direct theorem that, if A, B, C, A', B', C' be such six points of the plane that the three intersections $(BC', B'C)$; $(CA', C'A)$ and $(AB', A'B)$ are in line, then the six points all lie on a conic.

Introduction of the algebraic symbols. We have thus obtained many of the fundamental theorems for a conic immediately from the definition. It is our purpose to make the geometrical theory complete in itself. But it may be interesting, at the same time, to consider the application of the algebraic symbols.

Returning to the definition, the point, P, of the locus, being the intersection of the corresponding rays AP, CP of two related pencils, of centres A and C, let AP meet in Q the tangent CB at the point C, and CP meet in R the tangent AB at the point A. Then Q, R describe related ranges on the lines CB, AB, of which the points C, B, of the former, correspond, respectively, to the points B, A, of the latter.



If then we represent the point Q by the symbol $C + \theta B$, where θ is an algebraic symbol, we can, absorbing a proper multiplier in the symbol A, represent R by the symbol $B + \theta A$. Thence the point P is given by the symbol

$$P = \theta^2 A + \theta B + C,$$

and all points of the conic are given by this, for different symbols θ. In particular, the points A and C arise, respectively, for $\theta^{-1} = 0$ and $\theta = 0$. We might equally, however, have represented the point Q by $C + m\theta B$, where m is an arbitrary algebraic symbol; to fix the symbol θ for every point of the conic, it is necessary to fix it for one point beside A and C; say, by specifying the point of the curve for which $\theta = 1$.

That a locus, of which every point is given by a symbol such as that here representing P, contains two points of an arbitrary line, is clear, by remarking that the point

$$x_1A + y_1B + z_1C + \lambda\,(x_2A + y_2B + z_2C),$$

of the line joining the given points $x_1A + y_1B + z_1C$, $x_2A + y_2B + z_2C$, will be a point of the conic if, and only if,

$$x_1 + \lambda x_2 = \theta\,(y_1 + \lambda y_2) = \theta^2\,(z_1 + \lambda z_2);$$

by elimination of λ from these two equations we obtain a quadratic equation for θ.

The result that the conics through four points, A, C, A', C', meet an arbitrary line in pairs of points in involution, which was proved in order to shew that the two points A, C, used in the definition of the conic, may be replaced by any other two points A', C', of the conic, may be obtained thus: there is, as was proved, one conic through the four points A, C, A', C' and an arbitrary point, P, of the line. This conic will meet the line in another point, say P'. Thus, to any position of P on the line, there corresponds one and only one position of P' on the line. Conversely, the conic through A, C, A', C' and P' meets the line in P. The determination of P' from P, or of P from P', is, beside, by a rational algebraic equation. Thus (see the remarks above, Preliminary, p. 8), (*a*) the ranges (P), (P'), on the line, are related, (*b*) when P assumes the position P', then P' assumes the position P. Hence the pairs P, P' belong to an involution.

The existence of related ranges of points on the conic follows from the expression of the points of the conic by means of the parameter θ. In fact, the line joining the arbitrary fixed point, H, of the conic, for which the symbol is $\alpha^2A + \alpha B + C$, to the point, P, of symbol $\theta^2A + \theta B + C$, contains the point $(\alpha\theta^2 - \theta\alpha^2)B + (\theta^2 - \alpha^2)\,C$, that is the point $\alpha\theta B + (\theta + \alpha)\,C$, of the line CB. Putting $M = B + \alpha^{-1}C$, for the symbol of a definite fixed point on the line CB, the point in question is $\theta M + C$. Similarly the line joining the fixed point, K, of symbol $\beta^2A + \beta B + C$, to the point P, meets the line CB in the point $\theta N + C$, where $N = B + \beta^{-1}C$, is the symbol of another fixed point of the line CB. The pencil of lines joining H to four points, P, is thus related to the pencil of four lines joining K to these same four points. Further, a range of various positions of points P of the conic, expressed by various values of the parameter θ, is related to a range of positions, Q, expressed by values of ϕ, when there is a condition of the form

$$a\theta\phi + b\theta + c\phi + d = 0,$$

in which a, b, c, d are independent of θ and ϕ. The pairs of corresponding positions, P, Q, then belong to an involution when, in this condition, $c=b$. In other words, points P, Q, of respective parameters θ, ϕ, form a pair of an involution of points on the conic, when θ, ϕ are connected by an equation

$$a\theta\phi+b(\theta+\phi)+d=0.$$

When this is so, we at once see that

$$(\phi a+b)(\theta^2A+\theta B+C)-(\theta a+b)(\phi^2A+\phi B+C)$$

is equal to $$(\phi-\theta)(dA-bB+aC),$$

so that the line PQ passes through the fixed point $dA-bB+aC$.

But the fact, that the lines drawn through an arbitrary point, O, of the plane, meet the conic in pairs of points, P, Q, of an involution, is clear, also, by remarking, (a), that, when P is given, Q is determined without ambiguity, by the line OP, which meets the conic again in Q; and that P is similarly determined when Q is given, the determination being by a rational algebraic equation connecting the parameters of P and Q; and (b) that when P takes the position Q, then Q takes the position P. The converse result, that, if P, P'; Q, Q' and R, R' be pairs of points of the conic which are in involution, the lines PP', QQ', RR' meet in a point, follows from this if we recall that an involution is determined by two of its pairs.

Pascal's theorem can also be proved at once with the symbols. For let A, B, C, A', B', C' be six points of the conic; and let U, V be the common corresponding points, on the conic, of the two related ranges A, B, C,... and A', B', C',.... In the first instance suppose U and V not to coincide. Let the tangents at U and V meet in W; the points of the conic can then be represented, as above, in terms of a parameter θ, by a formula $\theta^2U+\theta W+V$. Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the parameters for A, B, C, A', B', C' respectively. For corresponding points, with parameters θ, θ', of the two related ranges, there will be a condition of the form

$$p\theta\theta'+\theta'-q\theta+r=0,$$

where p, q, r are to be determined by the fact that this equation is satisfied by $\theta=\alpha$, $\theta'=\alpha'$; by $\theta=\beta$, $\theta'=\beta'$; and by $\theta=\gamma$, $\theta'=\gamma'$. By hypothesis, however, the equation

$$p\phi^2+(1-q)\phi+r=0,$$

which gives the common corresponding points of the ranges, is satisfied by $\phi=0$ (at the point V), and by $\phi^{-1}=0$ (at the point U). Thus $p=0$ and $r=0$. Therefore, for a proper value of q, we have

$$\alpha'=q\alpha,\ \beta'=q\beta,\ \gamma'=q\gamma.$$

These relations shew that the lines AB' and $A'B$ meet on the line

UV, namely in the point whose symbol is $q\alpha\beta U - V$. For this is $\alpha\beta' U - V$, or

$$[\alpha-\beta']^{-1}[\beta'(\alpha^2U+\alpha W+V)-\alpha(\beta'^2U+\beta' W+V)],$$

and this point, therefore, lies on the line AB'; and it similarly lies on $A'B$. By the same argument the lines BC', $B'C$ meet on the line UV, as do the lines CA', $C'A$. This establishes Pascal's theorem.

If the common corresponding points of the two related ranges, $A, B, C, \ldots$ and $A', B', C', \ldots$, on the conic, be coincident, say in U, we may take an arbitrary point, V, of the curve, and, as before, represent any point of the curve by a symbol $\theta^2U+\theta W+V$. The equation, $p\phi^2+(1-q)\phi+r=0$, for the common corresponding points of the two related ranges is, now, to be satisfied by $\phi^{-1}=0$ (at the point U), *twice over*; thus $p=0$ and $q=1$. The relation for the corresponding points of the two ranges is, therefore, $\theta'=\theta-r$, so that $\alpha'=\alpha-r$, $\beta'=\beta-r$, $\gamma'=\gamma-r$. The line AB' now contains the point, of the line UW, given by

$$\alpha^2U+\alpha W+V-(\beta'^2\,U+\beta' W+V);$$

this is the point given by $(\alpha+\beta')\,U+W$, or $(\alpha+\beta-r)\,U+W$. By the symmetry of this form, the line $A'B$ also passes through this point. Similarly the intersection of the lines BC', $B'C$, and the intersection of the lines CA', $C'A$, lie on this line UW, which is the tangent of the conic at U.

We have stated, in the Preliminary (above, p. 9), that, in the first three chapters of this volume, we propose to assume that two conics in the same plane have four common points. It may be desirable to remark on the representation of this fact in terms of the symbols. The expression for a point of a conic, $\theta^2A+\theta B+C$, is in terms of two points A, C, lying on the conic, and the intersection, B, of the tangents of the conic at these two points. If U, V, W be three arbitrary points of the plane, the points A, B, C will be expressible in terms of these by formulae $A=a_1U+a_2V+a_3W$, $B=b_1U+b_2V+b_3W$, $C=c_1U+c_2V+c_3W$ (Vol. I, p. 71). Thus in terms of U, V, W, a point of the conic will have a representation

$$(a_1\theta^2+b_1\theta+c_1)\,U+(a_2\theta^2+b_2\theta+c_2)\,V+(a_3\theta^2+b_3\theta+c_3)\,W.$$

If now there be another conic, the points of this may, similarly, be represented in terms of U, V, W, the expression being of the form

$$(l_1\phi^2+m_1\phi+n_1)\,U+(l_2\phi^2+m_2\phi+n_2)\,V+(l_3\phi^2+m_3\phi+n_3)\,W,$$

where ϕ is the parameter. At a point common to the two conics, the three functions

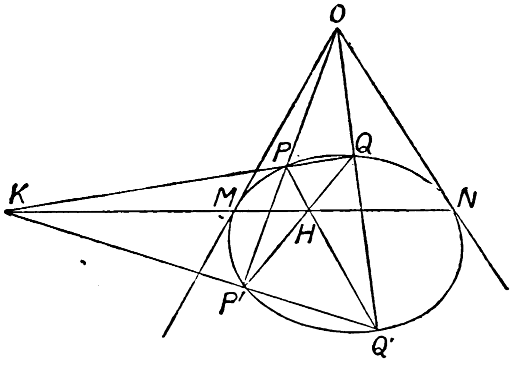
$$(a_r\theta^2+b_r\theta+c_r)/(l_r\phi^2+m_r\phi+n_r),$$

for $r=1, 2, 3$, must be equal to one another. From the two equations which express this fact we may eliminate ϕ; the result is at once seen to be a quartic equation to be satisfied by θ. The assumption to be made is then equivalent to the assumption that this equation has four roots. From any one of these, the corresponding value of ϕ can be found without ambiguity.

With these indications, we now leave the use of the symbols. They will be employed again in Chapter III.

Geometrical theory resumed. Polar lines. We saw that any line of the plane contains two points of the conic. We now shew that through any point, O, of the plane, not lying on the curve, two tangents of the curve pass. Let lines be drawn through O, each meeting the curve in a pair of points, say P and P'. The pairs P, P' have been shewn to belong to an involution on the curve; let M and N be the points of the curve which are the double points of this involution. The line OM will then meet the curve only at M, in two coincident points, and will be the tangent of the curve at M. So the line ON is the tangent at N. And there is no other tangent passing through O, since the involution has only two double points.

Moreover, an involution is, by definition, formed by the aggregate of two related ranges of points which are such that two corresponding points, one from each range, are interchangeable. Thus if, through the point O, be drawn various lines OPP', OQQ', ORR', ..., meeting the conic, respectively, in P and P', in Q and Q', in R and R', ..., the ranges, $P, Q, R, \ldots$ and $P', Q', R', \ldots$, are related, and M, N are the common corresponding points of these two ranges. Thus, by what has been proved, in particular by Pascal's theorem, the



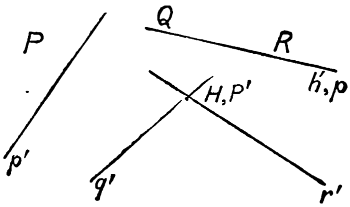
lines PQ', $P'Q$ meet on the line MN, say, in H, as do the lines PQ, $P'Q'$, say, in K. It follows that the line MN can be constructed when the four points P, P', Q, Q' are given; it also follows that the line PP' is met by MN in a point which is the harmonic conjugate of O in regard to P and P'. The line QQ' is similarly met by MN in a point which is the harmonic conjugate of O in regard to Q and

Q'; with a similar statement in regard to every one of the lines ORR', ... Thus, if lines be drawn from a point, O, not lying on the conic, and upon each be taken the harmonic conjugate of O in regard to the two points in which this meets the conic, the locus of this fourth harmonic point is a line. This line is called the *polar line*, or simply the *polar*, of O. From this there follows at once a property which is often applied, that, if the polar of one point O pass through a point O', then the polar of O' passes through O. And, therefore, if two points of the line MN be given, then O is determinate, namely, as the intersection of the polar lines of these two points.

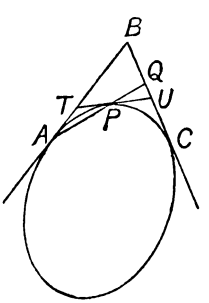
When O is on the conic, the harmonic conjugates of this in regard to the pairs P, P' and Q, Q', of which, say, P and Q coincide with O, also coincide with O. In this case we speak of the tangent of the curve at the point O as being the polar of O. Conversely, it may be shewn to follow from the character of the harmonic relation of four points that the polar of O can only pass through O when O is a point of the conic.

We may speak of the point O as the pole of the line MN; and in particular of a point of the conic as the pole of the tangent line of the conic at this point.

Duality in regard to a conic. From what we have just said, a conic, given in a plane, determines a particular duality therein. Any point, P, of the plane, regarded as belonging to one figure, determines a particular line, p', which we may regard as belonging to a derived figure, this line p' being here, the polar of P in regard to the given conic. Any line, p, of the first figure, joining two points, Q and R, of the first figure, gives rise to a point, P', of the derived figure, this being the intersection of the polars, respectively q' and r', of Q and R. To a point H, of the first figure, which coincides with P', corresponds the line, h', of the second figure, which is the polar line of P'; but, as P' lies on the polars of Q and R, these points, Q, R, lie on the polar of P', which is the line h'. Thus, the lines h' and p are the same. The correspondence is, therefore, of an involutory, or reciprocal, character, the line which arises from any point being the same whether the point is regarded as belonging to the first, or to the derived, figure; the point which arises from any line is therefore, also, capable of both derivations. The necessary and sufficient condition for a line to contain the point to which it corresponds is, that the point lie on the conic, and that the line be a tangent of the conic. We may, therefore, speak of the conic as the *curve*, or *envelope*, *of incidence*, of the two figures.

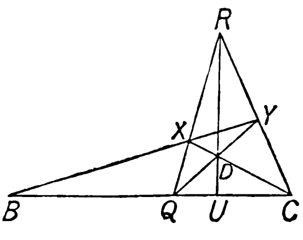


Dual definition of a conic. When one figure is the dual of another, it follows from the original definition of related ranges, by means of incidences, as developed in Vol. I, and the fact that Pappus' theorem involves the dual proposition which corresponds to it (Vol. I, chap. I, and p. 55), that related ranges give rise to related pencils, and related pencils give rise to related ranges. Hence, as we defined the conic as a locus of points, by the intersection of corresponding rays of two related pencils, so, equally well, we could have defined the aggregate of the tangents of the conic as the joins of corresponding points of two related ranges. In other words, if the tangent at any point, *P*, of the conic, meet the two tangents at the points *A* and *C*, respectively, in *T* and *U*, the range of points, *T*, upon *AB*, is related to the range of points, *U*, upon *BC*, the point *U* being at *B* when *T* is at *A*, and at *C* when *T* is at *B*.



In fact, the range of points, *U*, where the tangent, *CB*, at a fixed point, *C*, of the curve, is met by the tangent at a variable point *P*, is related to the pencil of lines joining the points *P* to any fixed point of the curve. To prove this, if *AP* meet the tangent *CB* in *Q*, the point *A* being any fixed point of the curve, it is sufficient to prove that the range of points *U* is related to the range of points *Q*.

That this is so follows by the abbreviated argument referred to above (Preliminary, p. 8). For when *U* is given, *P* is determined, without ambiguity, as the point of contact of the tangent, other than *UC*, which can be drawn from *U* to the conic; and when *P* is given, *Q* is found by the line *AP*. Conversely when *Q* is given, *P* is found, and, thence, *U* follows. As, however, this argument, as we have stated it, involves a tacit reference to the symbols, we may give an alternative one. In fact, the points *Q*, *C* are harmonic conjugates of one another in regard to *B* and *U*; that is, by perspective from *P*, if the point of intersection of *TU* and *AC* be *L*, the line *BP* meets *AC* in a point which is the harmonic conjugate of *L* in regard to *A* and *C*; in other words *BP* is the polar line of *L*; for as *L* lies on the tangent at *P*, and on the polar of *B*, the polar of *L* passes through *P*, and through *B*. We have then to shew that, if *Q*, *U* be variable points of the line joining two fixed points *B* and *C*, such that *B*, *U* are harmonic conjugates of one another in regard to *Q* and *C*, then the ranges (*Q*), (*U*) are related. For this, take an arbitrary

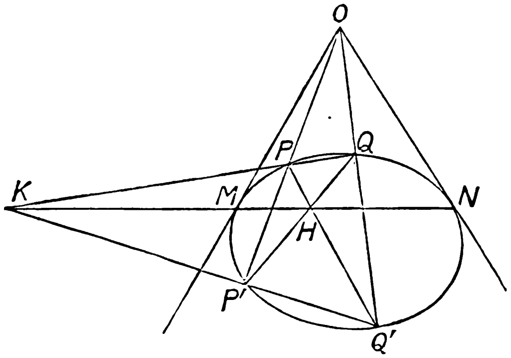


fixed point, R, and an arbitrary line, BXY, passing through B and meeting RC, RQ, respectively, in Y and X; thus Y is a fixed point; if then QY, CX meet in D, the line RD passes through U, and the line BD is a fixed line, meeting YC in the harmonic conjugate of R in regard to Y and C. The range (Q), by perspective from Y, is then related to the range (D) on this fixed line BD, and the range (D), by perspective from R, is related to the range (U). This makes the result clear.

Having proved that the range of points, U, on the tangent CB of the conic, is related to the pencil of lines joining A to the points, P, of the conic; and, then, similarly, that this pencil is related to the range of points, T, on the tangent AB, it follows that these ranges (U), (T) are related. Therein, to the point B, regarded as a point of AB, corresponds the point of contact, C, of the conic with BC, and to B, regarded as a point of BC, corresponds the point of contact, A, of the conic with BA. This is exactly the dual of what was the case when the points of the conic were determined by related pencils.

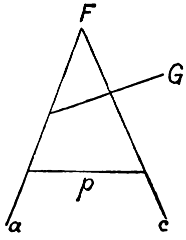
Examples of the application of the foregoing theorems. We may now regard ourselves as having given the indispensable fundamental propositions of a theory of conics. We proceed to make applications of these, in various directions.

Ex. 1. *The self-polar triad of points for conics through four points; the self-polar triad of lines for conics touching four lines.* It has appeared that if P, P', Q, Q', be four points of a conic, and the three pairs of lines which contain these points intersect respectively in the points O, H and K, then the line HK is the polar



of O; similarly the line OH is the polar of K; and, therefore, the line OK, containing the poles of HK and HO, is the polar of H. For this reason the triad O, H, K is called a self-polar triad, in regard to the conic. The same triad clearly arises for all conics which contain the four points P, P', Q, Q'. We saw that the conics through these four points meet an arbitrary line in pairs of points in involution; this involution has two double points. The conic through the four points and one of these, will not intersect the line again; this line will, therefore, be the tangent of the conic at this point. In other words, two conics can be drawn through four points to have a given line as tangent.

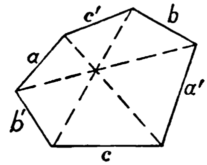
It was remarked that a tangent of the conic meets the curve in two coincident points. It will be found independently that, if this is understood, theorems in regard to four points of the curve continue to hold when these are not all distinct. For instance, if C, A, B be three points of the curve, and a line meet CA, CB respectively in L and M; meet the tangent at C, and the line AB, respectively in T and R; and meet the curve in X and Y, then it can be proved that the pairs L, M; T, R; X, Y belong to an involution. Or again, if A and C be two points of the curve, and a line meet AC in O; meet the tangents at A and C respectively in T and R; and meet the curve in X and Y; then O is one of the double points of the involution determined by the pairs T, R; X, Y. The other double point, U, is the harmonic conjugate of O in regard to T and R, or X and Y. In this latter case, only one conic can be drawn to pass through A and C and have at these points, for tangents, the given lines AT, CR, which shall touch the given line OU. It touches this at U. A degenerate conic having two coincident points at A, and at C, which also has two coincident points on OU, consists of the line OCA taken twice over.

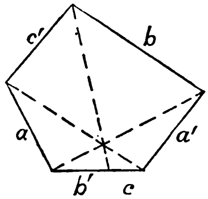


Dually, the aggregate of the tangents of a conic consists of lines, p, whose intersections with two given lines, a, c, form two related ranges thereon. The lines a, c are tangents of this conic. If the ranges have the point of intersection, F, of the lines a, c, for a self-corresponding point, they are in perspective, say, from a centre G. In this case we may regard the envelope as consisting of all lines through G, together with all lines through F; it is sometimes said to degenerate into a point pair, F and G. All this has been remarked above (p. 11). In the general case, such a conic envelope can be found having five arbitrary lines, of general position, for tangents. Let four tangents of such an envelope be p, p', q, q'; corresponding to the three ways of dividing these into two pairs, we have three pairs of points of intersection of the four lines. Let o, h, k be the joins of these pairs. Then, as the result dually corresponding to that above, any two of the lines o, h, k intersect in the pole of the remaining line, and form what we may call a self-polar triad of lines in regard to the envelope. There is an infinite number of conic envelopes touching the four lines p, p', q, q'; among these there are three point pairs, the pairs referred to above. The pairs of tangents that can be drawn, from an arbitrary point of the plane, one pair to each conic touching the four lines p, p', q, q', form a pencil in involution; of these pairs, three are those which join the arbitrary point to one of the point pairs referred to, namely, the pairs of points (p, p') and (q, q'); (p, q) and (p', q'); (p, q') and (p', q). This pencil in involution has two double rays; there are thus two conics which can be drawn, to touch the four lines p, p', q, q' and pass through an arbitrary point of the plane; the tangents of these conics at this point are the two double rays of the pencil in involution referred to.

In particular only one proper conic can be drawn to pass through an arbitrary point, O, and touch each of two given lines, which intersect in B, at given points, A, C. The tangent of this conic at O meets the line AC in the harmonic conjugate, with respect to A and C, of the point where the line BO meets AC. A degenerate conic with the definition given, consists of the point B taken twice over, that is of the aggregate of all lines through this point, taken twice over.

Ex. 2. *Dual of Pascal's theorem.* To Pascal's theorem there is a dually corresponding one, known as Brianchon's theorem. This may be stated by saying that, if a, b, c, a', b', c' be any six tangents of a conic, the three lines joining, the point (b, c') to the point (b', c), the point (c, a') to the point (c', a), and the point (a, b') to the point (a', b), are three lines which meet in a point. Taking the six lines in the order a, b', c, a', b, c', we may speak of the lines which are stated to meet in a point, as those joining opposite angular points of a hexagon, or as diagonals of this.





We saw that a tangent of a conic, regarded as a locus of points, may be regarded as meeting the conic in two coincident points. Dually, for a conic envelope, the point of contact of any tangent may be regarded as the intersection of two coincident tangents. Suppose then that, in the general figure for Brianchon's theorem, the two tangents b' and c coincide. We then infer that if a', b, c', a and b' or c, be five tangents of a conic, the point of contact of the last line lies on the line joining the point of intersection (b, c') to the point where the join of the points (a, b'), (a', b) meets the join of the points (a, c'), (a', c). We have given above, in connexion with Pascal's theorem, a construction for the tangent at any one of the five points by which a conic is determined; it may be seen that the present construction, for the point of contact, follows from this by the principle of duality. A particular case, when b and c' coincide, suggests the result that, if AB, BC, CD, DA be four tangents of a conic, the join of the points of contact with AB and CD passes through the intersection of AC and BD.

We remarked the existence, given six points of a conic, of sixty Pascal lines. There are similarly, given six tangents, sixty Brianchon points. The properties of the Pascal lines, which are extremely curious, have been much studied. It will be seen below, in Note II, pp. 219 ff., that these properties can be summarised and grasped most easily by regarding the figure as obtained by projection from a figure in space of three, or, better still, of four, dimensions.

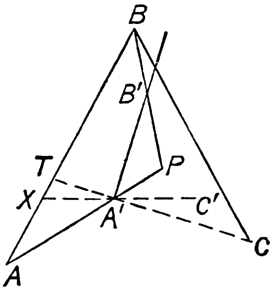
Ex. 3. *The director conic of a given conic in respect of two arbitrary points.* Let S be a given conic, and P, Q be two given points, not lying on the conic, in general. Let any line, l, be drawn through P, and, then, the line, m, be drawn through Q which contains the pole of the line l in regard to S. Thus the line l also contains the pole of m; and the lines l, m are a pair of lines, respectively through P and Q, each of which contains the pole of the other; two such lines are generally spoken of as *conjugate* to one another, in regard to the conic. We prove that the lines l, m intersect in a point which describes another conic; this conic contains P, and contains the points of contact of the two tangents which can be drawn from P to touch the given conic S; it also contains Q, and the two points of contact of the tangents from Q to the given conic. Further, the tangents at P and Q, of this conic, meet in the point, H, which is the pole of the line PQ in regard to the given conic. For reasons which will appear, this new conic may be called the *director* conic of the given conic, in regard to P and Q.

In fact, the poles, in regard to the conic S, of all lines drawn through P, lie upon a line, the polar line of P in regard to S; and they constitute a range upon this line which is related to the pencil of lines, spoken of, drawn through P. This fact is, at once, to be proved directly, by remarking that the pole of any line, l, through P lies on the polar of P, in regard to S, and that the harmonic conjugate of this pole, in regard to the two points where the polar of P cuts the conic S, lies on the line l; or the fact follows from what was remarked above, that, in two dually corresponding figures, a range of points, of one figure, is related to the corresponding pencil of lines, of the other figure. Therefore the lines m, through the point Q, each drawn to the pole of a line l, form a pencil related to the pencil of lines l; and thus the point of intersection of corresponding lines of these pencils describes a conic containing the points P and Q. Either of the tangents drawn to the conic S from the point P, is a particular line l, whose pole is its point of contact with S; thus the corresponding line m contains this point of contact, which is therefore on the conic described. Similarly for the other three points of contact of which we have spoken. If H be the pole of the line PQ, in regard to S, the pole of the line PH is on the line PQ; this line is then the line through Q which corresponds to the line PH through P; thus, by what was seen above, PH is the tangent of the new conic at the point P. So QH is the tangent at Q.

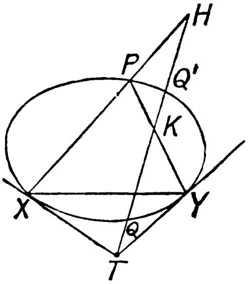
It is easy to see that if the line PQ touches the conic S, the director conic breaks up into PQ and another line.

Ex. 4. *The polars of three points determine a triad of points in perspective with the original triad.* As an application of the ideas involved in the definition of a conic, we may prove that, if A, B, C be three arbitrary points, whose polars, in regard to a conic are the lines a, b, c, respectively, and the points of intersection (b, c), (c, a), (a, b) be, respectively, called A', B', C', then the lines AA', BB', CC' meet in a point. The points A', B', C' are, respectively, the poles of the lines BC, CA, AB, and the figure may start from these.

Take the points B, C, A', C'; these fix the line BA, the polar of C', and fix the line $A'B'$, the polar of C. On the line BA, take various positions of A, to which correspond various positions of $C'B'$, the polar of A, and, hence, various positions of B', the pole of CA. Let BB' and AA' meet in P. The pencil formed by the various lines joining A' to P, or, say, the pencil $A'(P)$, is then related to the range (A); this range, determined by the pencil $C(A)$, of lines which are polars of the various positions of B', is related to the range (B'), on the fixed line $A'B'$. For in two dually corresponding figures, a range of points is related to the corresponding pencil of lines, as has been remarked. And the range (B') is determined by the pencil $B(P)$. The pencils $A'(P)$ and $B(P)$ are thus related, and the point P describes a conic. But, when A is at B, the point B' is at A', so that the ray $A'B$ of the former pencil corresponds to the ray BA' of the latter. The conic which is the locus of P, thus breaks up into the line $A'B$ and another line. If, however, the lines $C'A'$ and CA' meet AB in X and T, respectively, it is at once seen that, when A is at X, the point P is at C', and, when A is at T, the point P is at C. Thus P, C', C are in line. And it is this which it was desired to prove. Another proof of this result is given below, Ex. 10. [Add.]

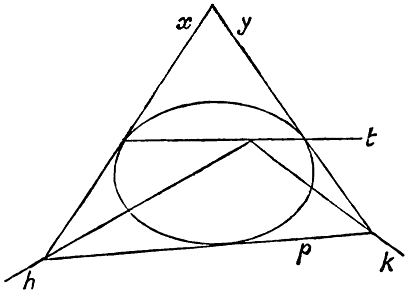


Ex. 5. *A construction for two points conjugate in regard to a conic.* Two points which are such that the polar of either contains the other are said to be conjugate in regard to the conic. The following simple property seems to deserve remark: If P, X, Y be three points of a conic, the lines PX, PY are met by any line drawn through the pole of the line XY in points which are conjugate in regard to the conic.



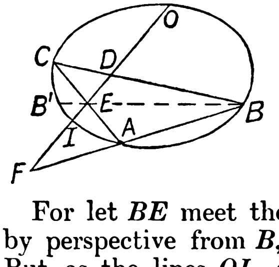
Let T be the intersection of the tangents of the conic at X and Y, the pole of XY; let any line drawn through T meet the conic in Q and Q', and meet PX, PY, respectively, in H and K. The pair of points Q, Q', on a line through T, is a pair of an involution of points of the conic, of which X and Y are the double points. Thus, in the pencil formed by lines joining the point P, of the conic, to X, Y, Q, Q', the rays PX, PY are harmonic conjugates in regard to the rays PQ, PQ'. Therefore on the line TH, the points H, K are harmonic conjugates in regard to Q and Q'. Wherefore the polar of H contains K, or H and K are conjugate points in regard to the conic.

The dually corresponding theorem is that, if x, y, p be three tangents of a conic, and any point of the line, t, which joins the points of contact of x and y, be joined, by lines h and k, to the points where p meets x and y, respectively, then the lines h, k are conjugate to one another in regard to the conic.



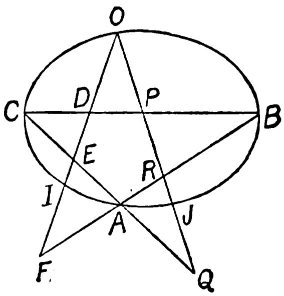
The result was remarked by von Staudt and, previously, by Seydewitz.

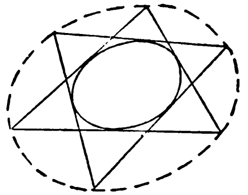
Ex. 6. *Range determined by a conic, and by the joins of three points of the conic, upon any line drawn through a fixed point of the conic.* Let A, B, C, O be four fixed points of a conic. Let any line drawn through O meet BC, CA, AB, respectively, in D, E and F, and meet the conic again in I. Then the range D, E, F, I is related to the pencil which the points A, B, C, O subtend at any point of the conic.



For let BE meet the conic again in B'. The range D, E, F, I, by perspective from B, gives the range C, B', A, I on the conic. But, as the lines OI, AC, BB' meet in E, the pairs of points OI, AC, BB' belong to an involution of points on the conic; therefore the range C, B', A, I, of points of the conic, is related to the range of points A, B, C, O, respectively paired with the former in this involution. Wherefore, the range D, E, F, I is related to the pencil of lines joining any point of the conic to the points A, B, C, O; as stated.

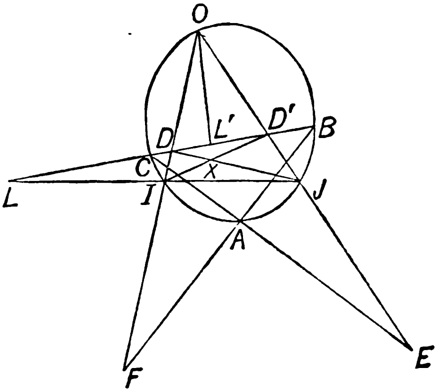
Ex. 7. *The six joins, of two triads of points of a conic, touch another conic.* Let O, I, J and A, B, C be any two triads of points of a conic; then the six joins consisting of BC, CA, AB and OI, OJ, IJ all touch another conic. For, as in the preceding example, let OI meet BC, CA, AB, respectively, in D, E, F; and, let OJ meet BC, CA, AB, respectively, in P, Q, R. The two ranges, D, E, F, I and P, Q, R, J, being, by the preceding example, both related to the points A, B, C, O of the conic, are related to one another. The two lines OI, OJ are thus met in related ranges by the four lines BC, CA, AB, IJ. Therefore, by the definition of a conic, in dual form, these six lines all touch a conic. (See also p. 142, Ex. 24.)





Dually we have the theorem that, if two triads of lines are tangents of a conic, and we take the three intersections in pairs of the lines of a triad, the six points so obtained lie on another conic. [Add.]

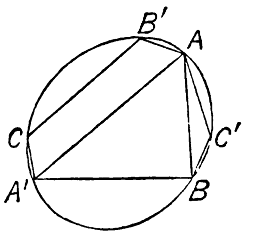
Ex. 8. *The pedal line of three points of a conic in regard to three other points of the conic, taken in a particular order.* The following result, though somewhat involved, may be proved here, because it depends on the same figure as that just employed. It is of interest later. Let A, B, C and O, I, J be two triads of points of a conic; let IJ meet BC, CA, AB, respectively, in L, M and N. From O are drawn three lines which are the harmonic conjugates, respectively, of OL, OM and ON, in each case in regard to OI and OJ. If these lines be, respectively, OL', OM', ON', then the pairs OL, OL'; OM, OM', and ON, ON' are in involution, of which the double



rays are OI, OJ. Supposing L', M', N' taken, respectively, upon BC, CA, AB, the theorem to be proved is, that L', M', N' are in line.

Let OI meet BC, CA, AB, respectively, in D, E, F, and OJ meet these, respectively, in D', E', F'. By what was seen in Ex. 6, the ranges D, E, F, I and D', E', F', J are related. Wherefore, the three points of intersection (ID', JD), (IE', JE), (IF', JF) are in line (Vol. I, p. 53); let these points be named, respectively, X, Y and Z. Of these, the point X is evidently on the line OL', being the harmonic conjugate of O, in regard to the point L' and the point where OL' meets IJ. Similarly, Y and Z are, respectively, on OM' and ON', and there is a corresponding harmonic relation. If then the line XYZ meet IJ in T, the points L', M', N' lie on a line through T, this being the harmonic conjugate of the line TIJ in regard to the lines TO and $TXYZ$. [Add.]

Ex. 9. *The Hessian line of three points of a conic. The Hessian point of three tangents of a conic.* If the tangents at the points A, B, C, of a conic, meet BC, CA, AB respectively in P, Q, R, then these points are in line.



By Pascal's theorem, if A, B', C, A', B, C' be six points of a conic, the three points of intersection $(AB', A'B)$, $(BC', B'C)$, $(CA', C'A)$ are in line. Suppose now that B' coincides with A, and A' with C, and C' with B. Then, instead of the lines AB', BC', CA', there will be the tangents of the conic at A, B and C, respectively; and, instead of the lines $A'B$, $B'C$, $C'A$, there will be the lines BC, CA, AB, respectively. The theorem in question thus expresses the fact that Pascal's theorem continues to hold when these coincidences occur. We can, in fact, prove it directly in the same way as Pascal's theorem. For, going back to the original definition of a tangent, we see that the theorem expresses that if two ranges of points of the conic be related to one another by the condition that the points A, B, C, of the one, correspond, respectively, to the points C, A, B, of the other, then the common corresponding points of these ranges are on a line containing the points, P, Q, R, where the tangents at A, B, C meet, respectively, the opposite joins (cf. Vol. I, p. 174, Ex. 5). We shall speak of the line PQR as the Hessian line of the triad A, B, C in regard to the conic.

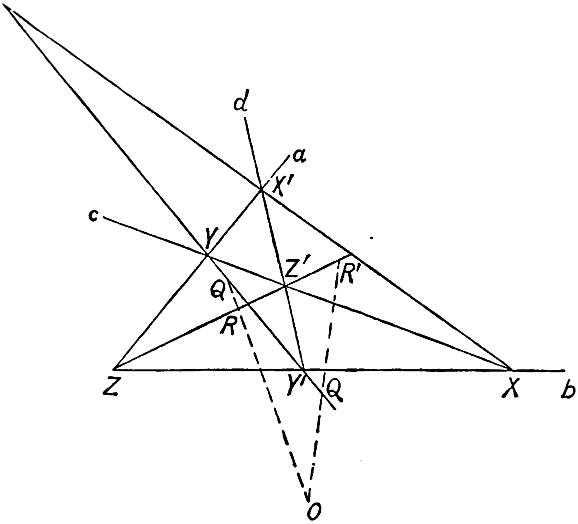
If the tangents of the conic at B, C meet in A', the line AA' is the polar of P. The three lines such as AA' thus meet in a point, which is the pole of the line PQR.

Dually, we have the result that, if a conic touch the joins VW, WU, UV, of any three points, U, V, W, respectively, in A, B, C, then the lines AU, BV, CW meet in a point. We shall call this the Hessian point of the three tangents in regard to the conic.

If BC meet VW in D, then D is the pole of UA. Thus the three points such as D are in line.

Ex. 10. *Polar triads in regard to a conic. Hesse's theorem.* The preceding result is a particular case of another, of which a proof has already been given (Ex. 4, above), which we may state as follows: The polar lines, in regard to a conic, of any three points A, B, C, meet the joins BC, CA, AB, respectively, in three points which are in line. If the polars of B and C meet in A', the polars of C and A meet in B', and the polars of A and B meet in C', the result will follow, by Desargues' theorem of perspective triads, if we prove that the lines AA', BB', CC' meet in a point. But if BB' and CC' meet in D, and we notice that B' is the pole of CA and C' the pole of AB, this is the same as saying that, if A, B, C, D be any four points such that one pair of joins of these, AC and BD, be conjugate lines in regard to a conic, and also another pair of joins of these, AB and CD, be conjugate lines in regard to this conic, then also the third pair of joins of these, BC and AD, are conjugate lines in regard to this conic. This result may be called Hesse's theorem; it is of importance in the sequel.

In dual form, this theorem is that, if any four lines a, b, c, d be divided into pairs in each of the three possible ways, and the points, Y, Y', obtained by the intersection of the pair a, c, and by the intersection of the pair b, d, be conjugate points in regard to a



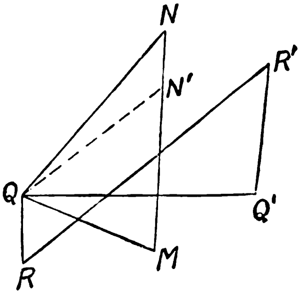
conic; and the same be true of the points, Z, Z', obtained by the intersection of the pair a, b, and by the intersection of the pair c, d: then the points, X, X', the intersections, respectively, of b and c, and of a and d, are also conjugate points in regard to the conic. We prove this by shewing that the points in which the conic meets the line XX' are harmonic conjugates of one another in regard to X and X'.

Let the conic meet the line YY' in Q and Q'; since the polar of Y passes through Y', the points Q, Q' are harmonic conjugates of one another in regard to Y and Y'. The points, R and R', in which the conic meets the line ZZ' are, similarly, harmonic conjugates of one another in regard to Z and Z'. Let QR and $Q'R'$ meet in O. As the conics through the four points Q, R, Q', R' meet the line XX' in pairs of points belonging to an involution, the given conic will be shewn to meet XX' in points harmonic in regard to X and X', if there be found two other conics, through the four points Q, R, Q', R', having this property; the points X, X' will then be the double points of the involution. By the original definition of the harmonic relation, the line pair QQ', RR', or YY' and ZZ', is one such conic. In fact the line pair OQR, $OQ'R'$ is another such conic. For the three line pairs joining O to the point pairs Y, Y'; Z, Z' and X, X' are in involution (p. 4, above), and the lines OQR, $OQ'R'$ are harmonic in regard to OY, OY', and also in regard to OZ, OZ', as we have seen; they are, therefore, the double rays of this pencil in involution of centre O, and are, thus, harmonic in regard to OX and OX'. Thus OQR, $OQ'R'$ are, as stated, a second conic through the four points Q, R, Q', R' which cuts XX' in points harmonic in regard to X and X'; the result required. [Add.]

We have seen, in Ex. 1, above, that the tangent pairs, from the point O, to conics which touch the four lines a, b, c, d, are a pencil in involution, of which OY, OY' are a pair of rays, as are OZ, OZ'; the lines OQR, $OQ'R'$ being, as we have just remarked, the double rays of this pencil, are then harmonic in regard to the tangents from O to any such conic. Now, two points, which are harmonic in regard to the two points in which their join meets a conic, are conjugate points in regard to the conic; and, dually, two lines, which are harmonic in regard to the tangents drawn to a conic from their intersection, are conjugate lines in regard to the conic. Thus, the lines OQR, $OQ'R'$ are conjugate lines in regard to any conic touching the four lines a, b, c, d. The lines YY' and ZZ' have also been seen (Ex. 1, above) to be conjugate to one another in regard to any such conic. The line pairs OQR, $OQ'R'$ and YY', ZZ' (or QQ', RR') are two of the three line pairs containing the points Q, R, Q', R'; by the dual of the theorem proved in this example, it follows, since two of these line pairs are conjugate lines in regard to any particular conic, Σ, touching a, b, c, d, that the third line pair, QR', $Q'R$, containing these points, are also conjugate in regard to Σ. We have thus proved a theorem which may be stated thus: Let a, b, c, d be any four tangents of a conic, Σ; let two (and, therefore, the third) of the point pairs of intersection of these tangents (Y, Y' and Z, Z') be conjugate points in regard to another conic, S. There is then a set of four points of the conic S (namely Q, R, Q', R') such that each line pair, containing these, is a pair of conjugate lines in regard to Σ.

Ex. 11. *Outpolar conics.* Let Q, R, Q', R' be four points of a conic, S, which are such that two (and, therefore, the third) of the line pairs containing these, say QR, $Q'R'$ and QQ', RR', are pairs of conjugate lines in regard to another conic, Σ. Such a figure is not possible for any two conics S, Σ, arbitrarily given. When it is possible, we shall speak of the conic S as being *outpolar* to Σ. Dually, when there are four tangents of Σ whose three point pairs of intersection are pairs of conjugate points in regard to S, we shall speak of Σ as being *inpolar* to S. The result obtained at the conclusion of the preceding example may then be stated by saying that when Σ is inpolar to S, then S is outpolar to Σ. By dual reasoning it follows that, when S is outpolar to Σ, then Σ is inpolar to S.

Proceeding now with the hypothesis made at the beginning of this example, that the line pairs QR, $Q'R'$ and QQ', RR', are conjugate lines in regard to Σ, take the polar line of Q, in regard to Σ, and let this meet the conic, S, containing the points Q, R, Q', R', in the points M and N. On MN take the pole, N', of QM, also in regard to Σ, so that M is the pole of QN'. The lines QM, $R'N'$ are then conjugate in regard to the conic Σ; and the lines QN', $R'M$ are also conjugate in regard to Σ. Now, when any lines are drawn through Q, their poles, in regard to Σ, lie on the polar line MN of Q in regard to Σ; and the lines joining R' to these poles form a pencil related to the range of these poles, and therefore related to the pencil of lines drawn through Q. Thus the pencil $Q(M, N', Q', R)$ is related to the pencil $R'(N', M, R, Q')$, and, hence, to the pencil $R'(M, N', Q', R)$, for the lines QQ', $R'R$ were given to be conjugate lines in regard to Σ, as were QR and $R'Q'$. It follows that the six points Q, Q', R, R', M, N' lie on a conic; this is, therefore, the same as the conic, S; and the point N', not coinciding with M in general, coincides with N. The points Q, M, N' were constructed to form a

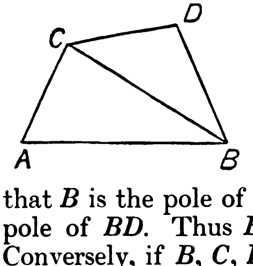


self-polar triad in regard to Σ; the points Q, M, N thus form such a triad. We have, therefore, shewn that when the conic S is outpolar to Σ, as containing four points whose line pairs of joins are conjugate pairs in regard to Σ, then a triad of points can be found on S which are a self-polar triad in regard to Σ. And one point of this triad is any one of the four given points of S. [Add.]

Conversely, let a triad of points of S, say Q, M, N, form a self-polar triad in regard to Σ; take, upon S, two other quite arbitrary points, R and R'; draw, also, the line QQ' through Q which is conjugate to the line RR' in regard to Σ, and draw the line $R'Q'$ through R' which is conjugate to QR in regard to Σ; let these lines meet in Q'. It can then be proved, as above, that the point Q' lies on the conic S. There are then four points, Q, R, Q', R', of the conic S, such that the lines QR, $Q'R'$ are conjugate in regard to Σ, as also are the lines QQ' and RR' (and, therefore, also QR' and $Q'R$). The conic S is thus outpolar to Σ. Therefore, by what was proved above, if the polar of R in regard to Σ be taken, this will meet S in two points which, with R, form a triad of points of S which is self-polar in regard to Σ. The point R was an arbitrary point of S; such triads exist, therefore, in infinite number, of which one point is arbitrary.

Hence, if two conics S, Σ be so related that there is one triad of points of S which are a self-polar triad in regard to Σ, there is an infinite number of such triads, one of the three points being arbitrary. There is then also an infinite number of sets of four points of S, which are such that each of the line pairs containing them is a pair of conjugate lines in regard to Σ; of such a set, three of the points may be taken arbitrarily upon S.

If two of such a set of four points upon S be conjugate to one another in regard to Σ, then the set consists of a triad which is self-polar in regard to Σ (consisting of these two points and another), together with a fourth point. For let A, B, C, D be such a set, the lines AB, CD being conjugate in regard to Σ, as also AC, BD, while B and C are conjugate in regard to Σ; and suppose A, B, C are not a self-polar triad in regard to Σ. Then the pole of AB is some point of CD, say C_1, and C_1 is not at C. The polar of C_1 being AB, and the polar of C passing through B, it follows that B is the pole of the line CC_1, or CD. In a similar way C is the pole of BD. Thus B, C, D are a self-polar triad in regard to Σ. Conversely, if B, C, D be points of S forming a self-polar triad in regard to Σ, and A be any other point of S, the four points A, B, C, D have the property that every line pair containing them consists of conjugate lines in regard to Σ.

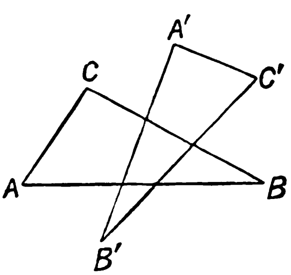


By what has been said, it follows, also, that if there are three points of S forming a self-polar triad in regard to Σ, then, the conic Σ being inpolar to S, there are sets of three tangents of Σ forming a self-polar triad of lines in regard to S.

An important consequence of what has been said is, that if S and S' be two conics which are both outpolar to a conic Σ, then the four common points of S and S' are such a set of four points as has been spoken of: every line pair which contains these four points consists of lines which are conjugate to one another in regard to Σ. And, thus, every conic through the four common points of S and S' is also outpolar to Σ.

For, let A, B, C be three points common to S and S'; if these form a self-polar triad in regard to Σ, then these, with the fourth intersection of S and S', form such a set of four points; and, every conic through A, B, C is outpolar to Σ. When A, B, C do not form such a self-polar triad, by drawing, from B, the line BD, conjugate to AC, in regard to Σ, and drawing, from C, the line CD, conjugate to AB, in regard to Σ, we determine a point D. By what has been proved above, this point D lies both on S and S'. There is a dually corresponding theorem in regard to the four common tangents of two conics Σ, Σ' which are both inpolar to a conic S.

Ex. 12. *Two self-polar triads of one conic are six points of another conic.* One incidental consequence of the theory of the preceding example is, that if two triads, A, B, C and A', B', C', be both self-polar in regard to a conic, Σ, then the six points A, B, C, A', B', C' lie on another conic. For the conic, S, which is drawn to contain A, B, C and A', B', is outpolar to Σ, as containing A, B and C. Therefore, by what has been shewn, the polar line of A', in regard to Σ, meets S in two points which, with A', are a self-polar triad in regard to Σ. One of these points is B'; the other is on $B'C'$, and is the pole of $A'B'$ in regard to Σ; it is therefore C'.



The direct proof, which is equivalent to a repetition, may be given. The lines AB, AC, AB', AC' are evidently conjugate, in regard to Σ, respectively, to $A'C$, $A'B$, $A'C'$, $A'B'$. Thus the pencil $A(B, C, B', C')$ is related to the pencil $A'(C, B, C', B')$, and hence, also, to the pencil $A'(B, C, B', C')$. Thus the six points lie on a conic.

By the dually corresponding proof, we can shew that the six lines BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ all touch the same conic.

We shall (Ex. 21) prove, conversely, that, if A, B, C, A', B', C'

be any six points of a conic, there exists another conic in regard to which both A, B, C and A', B', C' are self-polar triads. By combining this with the theorem here, we shall thus have another proof of a theorem given above (Ex. 7), that if A, B, C and A', B', C' be points of a conic, there is another conic which touches all of BC, CA, AB and $B'C'$, $C'A'$, $A'B'$.

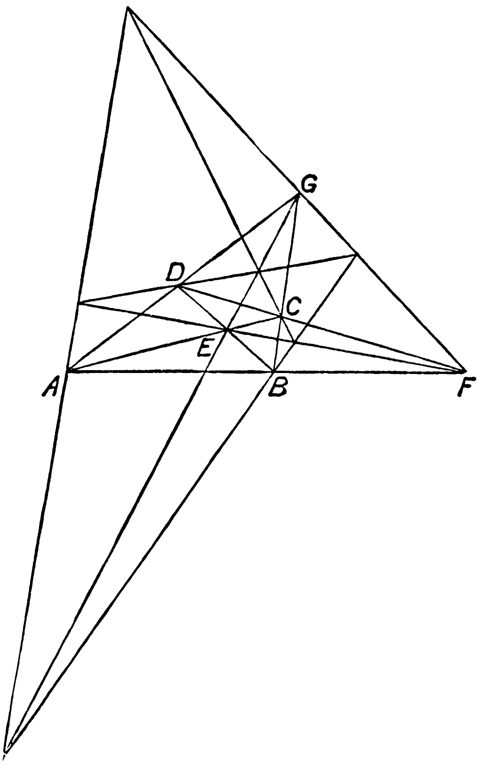
Ex. 13. *Particular cases of outpolar, or of inpolar, conics.* A particular case of a conic, S, outpolar to a proper conic, Σ, is a pair of lines which are conjugate in regard to Σ. A particular case of a conic, Σ, inpolar to a proper conic, S, is a pair of points which are conjugate in regard to S. In the general case, while the conic S is regarded as the locus of its points, the conic Σ is regarded as the envelope of its tangents; thus, a better statement of the latter case is, that the degenerate conic, Σ, consists of all lines through one of the two points conjugate in regard to S. A line pair may also be spoken of as inpolar to a conic, S, when the lines intersect at a point lying on S. But this point is to be regarded as a point pair of which the points coincide; the conic Σ consists then of all lines through this point. Similarly a line, taken twice over, may be spoken of as outpolar to a conic Σ when the line touches Σ.

These statements may be verified geometrically. Further confirmation will arise below when the matter is treated with the help of the algebraic symbols (Chap. III, Ex. 24).

Ex. 14. *The joins of four points of a conic and the intersections of the tangents at these points. The common points, and the common tangents, of two conics.* We have seen (Ex. 1) that the intersections of the three line pairs, which contain four points of a conic, form a self-polar triad in regard to the conic. Dually, the lines which join the three point pairs, through which four tangents of the conic pass, are a self-polar triad of lines in regard to the conic. When the four points are the points of contact of the four tangents considered, the triad of lines consists of the joins of the triad of points. In other words, the tangents at four points of a conic meet in pairs on the joins of the three points which form the common self-polar triad of all conics passing through the four points. We have seen, in Ex. 2, that, effectively, the result follows directly from Brianchon's theorem.

Let the line pairs containing the points A, B, C, D of a conic intersect, respectively, in E, F and G. The tangents of the conic at A and B meet in the pole of the line AB; as AB passes through one of the points E, F, G, say, through F, the pole of AB lies on the polar of F, that is, on the line EG. In the same way, the point of intersection of the tangents at any two of the points A, B, C, D is on one of the lines FG, GE, EF.

We have seen that, if we have two conics, the intersections of the line pairs passing through their intersections, form a self-polar triad of points for both conics. Dually, the joins of the point pairs, through which the common tangents of the two conics pass, form a self-polar triad of lines for the two conics (Ex. 1, above). These lines are in fact the joins of the points spoken of. In other words, through a point of intersection of two complementary chords common to two conics, there pass two lines each containing the intersection of two common tangents of the conics, and also the intersection of the other two common tangents; this line passes through the intersection of another pair of complementary chords common to the two conics.



To see that this is so, let us speak of two points P, P' as being harmonic images of one another in regard to a given point F, and a given line EG, when they are harmonic conjugates of one another, in regard to F, and the point where FP meets EG. Then either of P, P' determines the other; and if Q' be the harmonic image of another point Q, the lines $PQ, P'Q'$ intersect on EG. We may then regard a conic, for which E, G, F is a self-polar triad, as consisting of pairs of points, P, P', which are harmonic images of one another in regard to F and EG, the tangents of the conic at P and P' intersecting on EG. And, if E, G, F be the intersections of the line pairs through the common points of two conics, the harmonic image of a common tangent of the two conics is another common tangent of these conics, meeting the former on the line EG.

Ex. 15. *The polar lines of any point in regard to the conics through four given points all pass through another point.* For if the polars of a point, P, taken in regard to two of these conics, meet in a point P', the points P, P' are harmonic conjugates of one another in regard to the two points in which the line PP' meets either of these conics; they are, therefore, the double points of the involution determined on this line by the two pairs of intersections of this line with these two conics. As all the conics meet this line in pairs of points belonging to this involution, the points P, P', being harmonic conjugates in regard to any pair of the involution, are conjugate points in regard to any one of the conics. Thus the polar of P in regard to any one of the conics passes through P'.

In particular, if E be the intersection of one of the line pairs through the four points, the lines EP', EP are harmonic conjugates of one another in regard to this pair of lines. Thus, if F be the intersection of another of the line pairs through the four points, the point P' may be constructed from P by joining P to E and drawing a fourth harmonic line through E, and joining P to F and drawing another such fourth harmonic line through F. The point P' is the intersection of these. It lies on the similar fourth harmonic line which can be drawn through the intersection of the remaining line pair containing the four points.

Dually, the poles of any line in regard to all conics touching four given lines, lie on another line. This can be constructed by joining points determined as fourth harmonic points, on the lines containing the point pairs through which the four given lines pass.

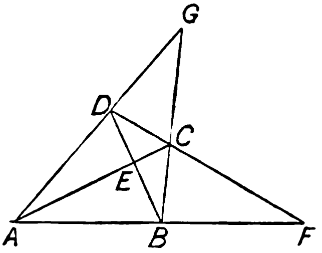
Ex. 16. *A point to point correspondence in regard to four given points.* Of the two points P, P' of the preceding example, either determines the other without ambiguity, when the four points A, B, C, D are given; and the relation is reciprocal or involutory, in the sense that, if P take the position of P', then P' takes the position of P. To this statement, however, there is an exception: Let E, F, G be the intersections of the line pairs containing A, B, C, D. Then when P is on the line FG, the point P' is at E; thus, to the position E, for P', the corresponding position of P is undetermined, being anywhere on the line FG. Similarly when P' is at F or G.

It can be shewn that, if P describes any line, then P' describes a conic passing through E, F, G; and, conversely, if P' describes any conic passing through E, F, G, then P describes a line. Evidently, conics passing through three fixed points E, F, G, are in some sense analogous to lines; in that any two such conics have one point of intersection, beside E, F, G, while such a conic can be drawn through two arbitrary points, other than E, F, G. The relation now in question enables us to pass from theorems relating to lines to theorems relating to such conics, or conversely.

To prove the statement made: when P describes any line, the pencils of lines joining E and F to P, or say $E(P)$ and $F(P)$, are related. As, however, the lines EP, EP' are harmonic in regard to the line pair intersecting in E, the pencils $E(P')$, $E(P)$ are related. Similarly, so are the pencils $F(P')$, $F(P)$. Thus the pencils $E(P')$, $F(P')$ are related. Thus, P' describes a conic containing the points E and F. This conic contains G; for this is the position of P' when P is on the line EF, at the point where this is met by the line on which P lies.

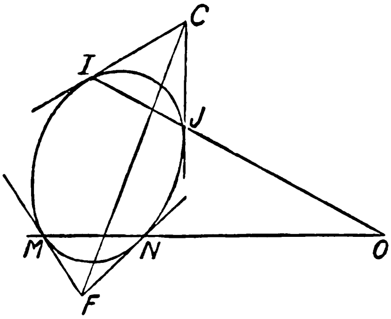
Now suppose that any points E, F, G are given, and also any other two points I, J. We shew that points A, B, C, D can be found so that, (*a*), the points E, F, G are the intersections of the line pairs containing A, B, C, D, (*b*), the points I, J are points corresponding to one another, in regard to A, B, C, D, in the sense just considered. The correspondence in question can then be regarded as determined by an arbitrary triad of points, E, F, G, and one given pair of corresponding points.

In fact two lines, FBA, FCD, are determined by the conditions that, they shall be harmonic conjugates of one another in regard to the given lines FE, FG, and shall also be harmonic conjugates of one another in regard to the given lines FI, FJ. And two lines, GDA, GCB, are determined by the conditions that, they shall be harmonic in regard to GE, GF, and also harmonic in regard to GI, GJ. These lines determine four points A, B, C, D. We have to shew that AC, BD meet in E, and are harmonic in regard to EI, EJ. In fact, AC and BD must meet on the line through G, which is the harmonic conjugate of GF in regard to the lines GCB and GDA; and they must, likewise, meet on the line through F which is the harmonic conjugate of FG in regard to the lines FBA and FCD; therefore AC, BD meet in E. And then it follows, by what was said above, that EC, EB are harmonic in regard to EI and EJ.



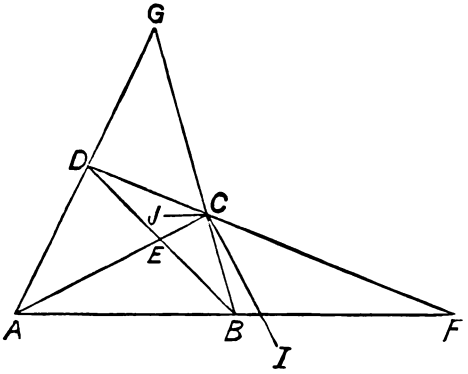
Ex. 17. *Four conics can be drawn through two given points to touch three given lines.* We have proved above that two conics can be drawn through a given point to touch four given lines. We consider now the conics that can be drawn through two given points to touch three given lines.

For this, notice, first, that, if FM, FN be the tangents drawn, from a point F, to a conic, of which I, J are two points, and C be the pole of the line IJ, then the involution determined by the two pairs of rays, FM, FN and FI, FJ, has FC for one double ray. In fact, if MN and IJ meet in O, this is the pole of the line FC, and its harmonic conjugates, both in regard to the pair I, J, and in regard to the pair M, N, lie on FC. Thus the lines FO, FC are harmonic, both in regard to FM, FN, and in regard to FI, FJ.



Take now the figure of the preceding example. We desire to find conics passing through I and J which shall touch the lines EF, FG, GE.

One conic can be drawn through I and J, to have CI, CJ as the tangents at these points, which shall touch FG. (Another degenerate conic which contains two coincident points of CI at I, two coincident points of CJ at J, and two coincident points of FG, consists of the line IJ, taken twice over. See Ex. 1, above.) This conic does, in fact, touch FE and GE, and is therefore such a conic as is desired. To prove that this conic, which touches FG, also touches FE, we use the remark made above, shewing that the involution which is determined by the pair of lines FI, FJ, and the double ray FC, contains the lines FE, FG as a pair. For this we have only to notice that FC, FB are harmonic in regard to FE, FG, and also harmonic in regard to FI, FJ (by the property of I, J in regard to A, B, C, D); the lines FC, FB are thus the double rays of an involution of which FI, FJ and FE, FG are pairs. This is then the involution spoken of. By a similar argument the conic described touches GE.

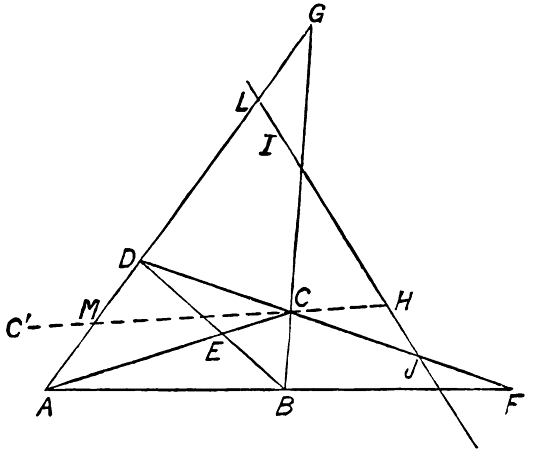


Another conic through I, J touching EF, FG, GE, is that described to touch DI, DJ, respectively, at I, J, and to touch FG. A third touches AI, AJ at I and J; and a fourth touches BI, BJ at I and J.

No other conics can be drawn through I, J to touch EF, FG, GE. For, by the remark made above, if FE, FG be the tangents from F to a conic which contains I and J, one of the double rays of the pencil in involution determined by the pairs, FI, FJ and FE, FG, must contain the pole of IJ in regard to this conic. The pole of IJ must then lie either on the line FAB or the line FCD. It must, similarly, lie either on GCB or on GDA. It must therefore be at one of A, B, C, D.

The dual of this proposition is that four conics can be drawn through three given points to touch both of two given lines. Of this a proof, which then also proves the direct proposition, follows at once from the correspondence explained in the preceding example (Ex. 16). This correspondence transformed conics passing through three given points into lines, and lines into conics passing through these three points. It therefore changes a conic which touches a line into a line touching a conic, the two common points of these being coincident in both cases. Thus a conic passing through the three given points and touching two given lines, becomes a line touching two conics which pass through the three given points. The result to be proved is, therefore, that, two conics have four common tangents. This we know, as the dual of the result assumed (Preliminary, above, p. 9) that two conics have four common points.

Ex. 18. *The poles of a line, in regard to conics with four common points, lie on a conic.* We have remarked that the poles of a line in regard to conics with four common tangents lie on a line. We may, however, consider the poles of a line in regard to conics with four common points. These lie on another conic, which is of importance. Of this conic eleven points can be specified at once, from the figure; it is therefore sometimes called the eleven-point conic.



Let A, B, C, D be the four common points of the conics considered. These determine, on the given line, an involution of pairs of points; let I, J be the double points of this involution. Let E, F, G be the intersections of the line pairs containing the four given points. A conic can be drawn through the five points E, F, *G*, *I*, *J*; we shew that this contains the pole of the given line in regard to every conic through the four given points.

For, *I*, *J* are harmonic conjugates in regard to the points in which any particular conic, *S*, through *A*, *B*, *C*, *D*, meets the line *IJ*; the polars of *I* and *J*, in regard to this particular conic *S*, thus pass, respectively, through *J* and *I*. Therefore, if these polars meet in *T*, the points *T*, *I*, *J* are a self-polar triad in regard to *S*. We have shewn that *E*, *F*, *G* are a self-polar triad in regard to any conic *S* (Ex. 1). We have also shewn that any two triads of points, which are both self-polar in regard to a conic, lie on another conic (Ex. 12). Thus *T* lies on the conic containing *I*, *J*, *E*, *F*, *G*. But *T* is the pole *IJ* in regard to the conic *S*. This proves the statement made.

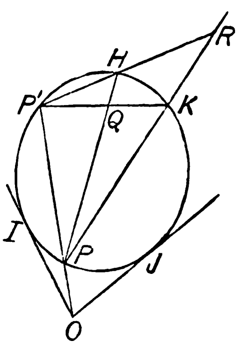
Now, let the line *IJ* meet *AD* in *L*; and let *M* be the harmonic conjugate of *L* in regard to *A* and *D*. Further, let *MC* meet the line *IJ* in *H*, and let *C′* be the harmonic conjugate of *C* in regard to *M* and *H*. If then we consider the particular conic through *A*, *B*, *C*, *D* which contains the point *C′*, the polar of *M* in regard to this conic contains both the point *L* and the point *H*. Thus *M* is the pole of the line *IJ* in regard to this particular conic. Wherefore *M* is on the conic obtained above, which is the locus of the poles of the line *IJ* in regard to all conics through *A*, *B*, *C*, *D*. By considering the intersections of the line *IJ* with the joins of every pair of the four points *A*, *B*, *C*, *D*, as we have just considered the intersection *L* with the join *AD*, and taking on each join the fourth harmonic in regard to the two points, as we have taken *M* on *AD*, we find in all six points of the conic in question. And we found that it contained the five points *E*, *F*, *G*, *I*, *J*.

Dually, the polars of a point, in regard to all conics touching four given lines, touch another conic, which will also be found to be of importance (p. 90, below).

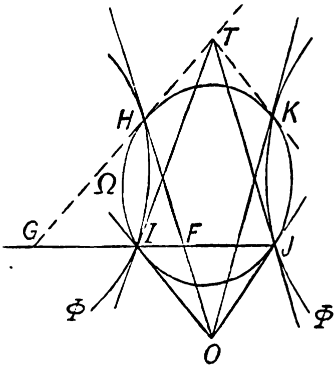
Ex. 19. *Two conics of which the tangents at their common points meet, in fours, at two points.* Suppose that the four points common to two conics are *I*, *J* and *H*, *K*; and that the tangent of one conic, at the point *H*, passes through the intersection of the tangents of the other conic at the points *I* and *J*. For distinctness, let the first conic be called Φ, and the second be called Ω, and let the tangents of Ω at *I* and *J* meet in *O*. It can then be shewn that the tangent of the first conic, Φ, at the point *K*, also passes through *O*, which is then a point through which four of the tangents of the conics pass; and that the tangents of the second conic, Ω, at the points *H*, *K*, as well as the tangents of the first conic, Φ, at *I* and *J*, all pass through another point.

Evidently, when the conic Ω is given, and the points *I*, *J*, *H*, *K* thereon, the conic Φ may be defined as that passing through the points I, J, H, K and having HO for its tangent at H. We obtain it somewhat differently.

Let Ω be any given conic, I, J, H, K any points thereon, the tangents at I, J meeting in O. Draw through O any line meeting Ω in P and P'; let PH, $P'K$ meet in Q, and PK, $P'H$ meet in R. We consider the locus of the point Q; which will be found to contain the point R. The pairs of points P, P' of the conic forming an involution thereon, the pencils of lines, $H\,(P)$, $K\,(P')$, with centres at H and K, are related. This shews that the locus of Q is a conic, say Φ, passing through H and K; and, as P, P' are interchangeable, this conic Φ contains R. When P' is at H, the line HP becomes the line HO, and the line KP' becomes KH; thus, HO is the tangent of the conic Φ at H. When P is at K, the line HP becomes the line HK, and the line KP' becomes the line KO, which is thus the tangent of the conic Φ at K. When P' is at I, so also is Q. The conic Φ thus passes through I; and, similarly, it passes through J.

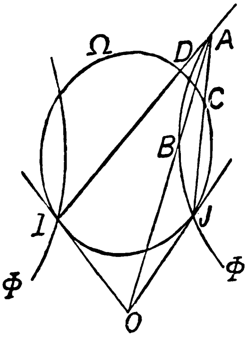


Now let T be the intersection of the tangents of the conic Φ at I and J. We shew that the tangents of Ω at H and K pass through T. Let HO meet IJ in F, and TH meet IJ in G. The point F is on the polar of T in regard to Φ, so that its polar contains T; the polar of F also passes through H, since the tangent of Φ at H passes through F. Thus the line THG is the polar of F in regard to Φ; and G is the harmonic conjugate of F in regard to I and J. If, however, the tangent of Ω at H meet IJ in G', the polar of G', in regard to Ω, passes through H, and passes through O, which is the pole of IJ in regard to Ω. Thus HO is the polar of G' in regard to Ω. Wherefore G and G' are the same, and TH is the tangent of Ω at H. So TK is the tangent of Ω at K.



It appears, thus, that the relation of the conics Φ, Ω is mutual, and either may be defined from the other. And the conic Φ may, equally, be defined by drawing variable lines through the point T to meet the conic Ω, say, in X and X', and, then, finding the locus of the intersection of the lines IX and JX'.

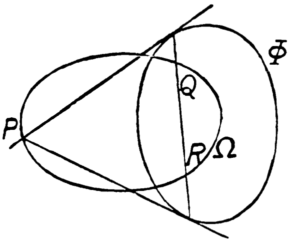
In the definition of the conic Φ, from Ω, the points Q, R of Φ are conjugate to one another in regard to all conics through the points P, P', H, K (Ex. 1, above), and the line QR, which is the polar of the intersection of the lines PP' and KH, contains the intersection, T, of the tangents of the conic Ω at H and K. The two conics, Ω, Φ, are, therefore, such that any line drawn through T meets one of the conics, say Φ, in two points which are harmonically conjugate in regard to the two points in which the line meets the other conic Ω. The same is therefore true of any line drawn through the point O.



If Ω be given, and, on any line drawn through the intersection, O, of the tangents of Ω, at any two points, I, J, of this, there be taken two points, A, B, conjugate to one another in regard to Ω, then any conic, Φ, through the four points I, J, A, B, is related to Ω as in the above. In fact, IB, JB pass through the further intersections, C, D, of Ω with AJ, AI, respectively, and CD passes through the intersection of the tangents of Φ at I and J. Or an independent proof may be given.

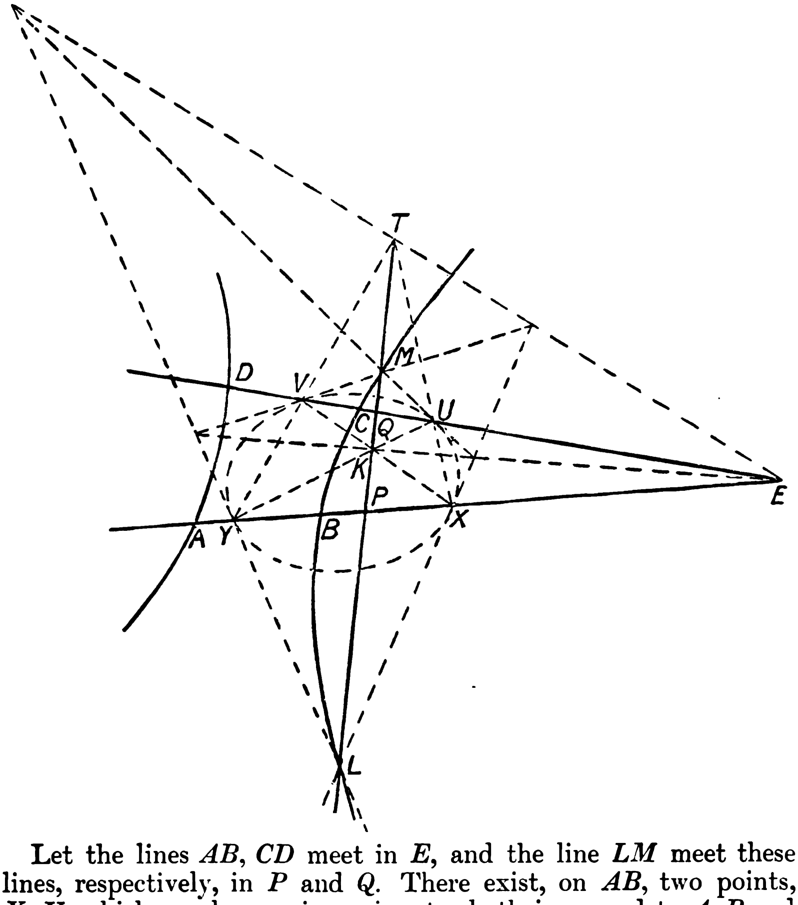
Ex. 20. *A conic derived from two given conics.* In connexion with the last example, the following remarks, though an anticipation of results given below (in Chap. III, p. 125), may be made.

Given any two conics Ω, Φ, we may consider all lines of the plane whose intersections with one of the conics are harmonic conjugates in regard to the intersections of the line with the other conic. Of such lines, two can be drawn through any point, P, of one of the conics, say Ω; those, namely, joining P to the points, Q, R, in which this conic, Ω, is met by the polar of P in regard to Φ. It may, therefore, be expected that the lines in question touch a conic; as will be found to be the case (pp. 125, 135, below). If this be assumed, it is at once clear that this third conic touches the eight lines which are the tangents of the given conics, Ω, Φ, at their four intersections. The two particular conics, Ω, Φ, considered in the preceding example, are so related that this third conic, regarded as an envelope, degenerates into a point pair, consisting of the points O and T. It can be shewn, further, that the points of contact of these particular conics Ω, Φ with their four common tangents, lie on two lines, four on each; and that the tangents, from any point of either of these lines, drawn to Φ, are harmonic conjugates in



regard to those drawn to Ω. In general, a point of a plane from which the tangents to two general conics form such a harmonic pencil, describes a conic passing through the eight points of contact of these given conics with their four common tangents. [Add.]

Ex. 21. *To construct a conic in regard to which two triads of points of a given conic shall both be self-polar.* Let A, B, L and C, D, M be any two triads of points lying on a conic.



Let the lines AB, CD meet in E, and the line LM meet these lines, respectively, in P and Q. There exist, on AB, two points, X, Y, which are harmonic conjugates both in regard to A, B and in regard to P, E. Any conic for which L, A, B is a self-polar triad must necessarily meet AB in points which are harmonic conjugates in regard to A and B; for any conic for which L, A, B and M, C, D are self-polar triads, the intersection, E, of AB and CD, must be the pole of LM; any such conic must, therefore, meet AB

in points which are harmonic conjugates in regard to P and E. The conic we seek to construct must then, if it exist, pass through X and Y. Two other points through which such a conic must pass are the points, U, V on CD, which are harmonic conjugates both in regard to C, D and in regard to Q, E. It follows from the fundamental properties of the harmonic relation that the lines XU, YV meet on the line LM, say, at T; and, also, that the lines XV, YU meet on LM, say, at K. Considering any four lines, we have seen (Ex. 10, above) that if, of the three point pairs through which all the four lines pass, two pairs consist each of points which are conjugate to one another in regard to a conic, so does the third pair. In the present case, for the four lines TUX, TVY, UKY, VKX, the three point pairs through which all the lines pass are X, Y; U, V and T, K; of these, the two former consist of conjugate points in regard to the given conic containing the two triads L, A, B; M, C, D; wherefore K and T are conjugate in regard to this conic. Thus T and K are harmonic conjugates in regard to L and M. The harmonic conjugate of T in regard to the points U and X is, however, the point in which UX is met by EK. Thus it follows that the lines MU, LX meet on EK; as do, for a similar reason, MV, LY. In the same way, the lines VM, LX meet on ET, as do UM and LY. Whence we see that a conic can be described through X, Y, U, V having MU, MV and LX, LY as tangents (Ex. 14, above).

For this conic, L, A, B and M, C, D are self-polar triads. It is therefore such a conic as we desired to construct. And there can be no other; for we have seen that such a conic must pass through X, Y, U, V; and, that L may be the pole of AB in regard to the conic, its tangents at X, Y must meet in L; its tangents at U, V must likewise meet in M.

Ex. 22. *To construct a conic for which a given triad of points shall be a self-polar triad, and a given point and line shall be pole and polar.* Let L, A, B be the triad, M the point and CD the line. Draw any conic through L, A, B, M; let it cut the line in points C, D. Construct, by the foregoing example, the conic having L, A, B and M, C, D as self-polar triads. This is such a conic as is desired. It is, in fact, the only one; for, though C, D may vary, they are pairs of the involution determined on the line CD by all conics through L, A, B, M; and the points U, V, of the conic described, are unique, being the double points of this involution.

Ex. 23. *To construct a conic for which a given triad of points shall be self-polar, to pass through two given points.* Let A, B, C be the triad, and I, J the two other points. There exists a point, D, such that the lines BC, AD, the lines CA, BD, and the lines AB, CD, meet the line IJ in three pairs of points of an involution of which

I, J are the double points. This is the particular case of Hesse's theorem (Ex. 10, above), which arises when the conic, there referred to, becomes a point pair; it is obvious at once, by finding D, so that BD and CA meet IJ in points harmonic in regard to I, J, and that CD and AB meet IJ in such points; then I, J are the double points of the involution determined on the line IJ by the joins of the points A, B, C, D, so that AD and BC meet IJ in points harmonic in regard to I and J.

There is then, by the preceding example, one conic having A, B, C for a self-polar triad, and having the point D and the line IJ as pole and polar of one another. For this conic the points A and D are the poles, respectively, of BC and IJ, and the line AD is the polar of the point where BC meets IJ. Thus the conic meets the line IJ in points which are harmonic conjugates in regard to the points where IJ is met by BC and AD. There is a similar statement for CA and BD, and for AB and CD. The conic, therefore, passes through I and J, and is the conic we desired to construct. The tangents at I and J are the lines DI and DJ. The reader will compare this work with that of Ex. 17.

Ex. 24. *The polar reciprocal of one conic in regard to another. A particular case.* It follows from what has been said above in regard to duality with respect to a conic, and the dual definition of a conic (p. 22), that if we have two conics, S and Ω, and take the polars, in regard to Ω, of all the points of S, these polars touch another conic, say S'. Any two points of S thus give two tangents of S', and the polars, in regard to Ω, of the points of their join, become the lines passing through the point of intersection of these two corresponding tangents of S'. In particular the pole, in regard to Ω, of any tangent of S, is a point of S'. Thus S can be obtained from S' by the same rule as was S' from S. Two such conics S, S' are called polar reciprocals in regard to Ω. Conversely, it will be seen below, in Chap. III, that, given two arbitrary conics S, S', there are, in general, four conics Ω, in regard to which S and S' are polar reciprocals.

Now consider two conics, S and S', which are such that there is a triad of points of S, say A, B and C, for which the joins BC, CA, AB are tangents of S'. If then we draw from any point, P, of S, the two tangents to S', and these meet S again in Q and R, the six lines BC, CA, AB, PQ, PR and QR all touch a conic (above, Ex. 7); and this is the conic S', because S' touches five of these lines. In other words, an infinite number of triads of points P, Q, R can be found on S, for which the joins QR, RP, PQ touch S'. Take now one of these triads, P, Q, R. By what is proved above, in Ex. 21, there is a conic, say Ω, for which both A, B, C and P, Q, R are self-polar triads. Consider the conic which is the polar reciprocal of S

in regard to Ω. This touches the six lines BC, CA, AB, QR, RP, PQ (which are the polars in regard to Ω, of the points of the triads); it is therefore the conic S'. It follows from this that, if P', Q', R' be another triad of points of S for which $Q'R'$, $R'P'$, $P'Q'$ touch S', then P', Q', R' must be a self-polar triad in regard to Ω.

To prove this, let the polar of P', in regard to Ω, meet S in Q_1 and R_1; then, if the polar of Q_1, in regard to Ω (which polar passes through P'), meet Q_1R_1 in R_2, a conic contains A, B, C, P', Q_1, R_2 (Ex. 12, above), which is therefore S; thus R_2 coincides with R_1, and P', Q_1, R_1 is a self-polar triad in regard to Ω. Therefore, as P', Q_1, R_1 are points of S, the lines $P'Q_1$, $P'R_1$, Q_1R_1 are tangents of the polar reciprocal, S'. Hence Q_1, R_1 are the same as Q', R'. We have therefore an infinite number of triads of points of S, of which the joins touch S', which are self-polar in regard to Ω.

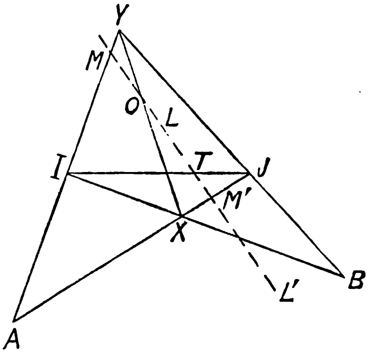
There are, in fact, as has been remarked, three other conics beside Ω for which S and S' are polar reciprocals. These have not the same property in regard to the triads P, Q, R. And, in general, when S and S' are polar reciprocals in regard to a conic Ω, it is not the case that triads of points of S are self-polar in regard to Ω. The matter arises again for consideration in Chap. III (Ex. 24).

A conic S which contains triads of points A, B, C for which the joins BC, CA, AB all touch another conic Σ, may be spoken of as triangularly circumscribed to Σ. Let S_1, S_2, ... be a set of conics all triangularly circumscribed to Σ. Let Ω_1 be the conic by which S_1 is the polar reciprocal of Σ, the triads spoken of being self-polar in regard to Ω_1; let Ω_2 similarly arise from S_2 and Σ, and so on. The set of conics Ω_1, Ω_2, ... are then all outpolar to Σ.

Similarly, if we have a set of conics Σ_1, Σ_2, ..., all triangularly inscribed to a conic S, we can find a set of conics Ω_1, Ω_2, ... all inpolar to S.

Ex. 25. *Gaskin's theorem for conics through two fixed points outpolar to a given conic.* Let Ω be a given conic, and I, J be two given points. We have defined (Ex. 3, above) the director conic of Ω in regard to I and J. We have considered two conics such that the tangents at their four common points intersect, in fours, in two points (Ex. 19, above). We have defined a conic as outpolar to Ω when there are triads of points of the conic which are self-polar in regard to Ω (Ex. 11). We now prove that all conics outpolar to Ω passing through the points I, J, are in the special relation with the director conic which was considered in Ex. 19. Such an outpolar conic is obtained by taking any triad which is self-polar in regard to Ω, and drawing the conic through I, J which contains the points of this triad. We have proved (Ex. 11) that any conic through the common points of two conics which are both outpolar to Ω is also outpolar to Ω; in particular any one of the line pairs through these common points consists of two lines conjugate to one another in regard to Ω. Thus, if any two conics outpolar to Ω be described through the points I, J, and X, Y be the remaining intersections of these, then the line XY passes through the pole of IJ in regard to Ω; this pole we denote by O. We denote, also, by A, the intersection of the lines IY and JX; as these lines are conjugate in regard to Ω, the point A is on the director conic of Ω in regard to I and J. Similarly, the point B, in which the lines IX, JY intersect, is on the director conic. This director conic passes through I and J, and, as we have remarked (Ex. 3), the tangents at I, J, of this conic, meet in the pole, O, of the line IJ taken in regard to the conic Ω.

Now draw any line through O, and consider the involution of pairs of points determined thereon by all conics through the four points I, J, X, Y. We prove that the double points of this involution are the points, say L and L', in which this line meets the director conic. One pair of this involution consists of the point O and the point, T, in which the line meets IJ. As O is the pole of IJ in regard to the director conic, the points O and T are harmonic conjugates in regard to L and L'. Let the line, drawn through O, meet the lines IY and JX, respectively, in M and M'. As these lines join the point A, of the director conic, to the points I, J of this, and O is the pole of IJ in regard to this conic, it follows (Ex. 5, above) that M, M' are conjugate points in regard to the director conic, and are, therefore, harmonic conjugates in regard to L and L'. The points M, M' are, however, another pair of points determined, on the line OLL', by a conic through the four points I, J, X, Y; this pair, together with O and T, determine the involution. It is, therefore, shewn that L and L' are the double points of this involution.



Hence, the conic drawn through I, J, X, Y to contain L has, for its tangent at L, the line OLL'; as this passes through the pole, O, of IJ in regard to the director conic, the conic $IJXYL$ is related to the director conic as were the conics in Ex. 19. The conic $IJXYL$ was obtained as passing through the intersections of any two conics through I, J which are outpolar to Ω, and through any point L of the director conic. It is, therefore, in effect, any conic through I, J outpolar to Ω. And thus we have proved the result. [Add.]

The result was given by Gaskin (*Geometrical construction of a conic section subject to five conditions*, Cambridge, 1852, p. 33, Prop. 15).

Ex. 26. *Two related ranges upon a conic obtained by composition of two involutions.* Referring for a moment to the symbolic representation, in order to explain the meaning of the result to which we now pass, we have remarked (Preliminary, above, Ex. 2), that two related ranges of points, $O+xU$, $O+yU$, are given by one of the two relations

$$\frac{y-\beta}{y-\alpha}=\sigma\frac{x-\beta}{x-\alpha},\qquad \frac{1}{y-\alpha}=\frac{1}{x-\alpha}+\frac{1}{\lambda}.$$

Considering the former, if we take μ arbitrary, and take z so that

$$\frac{x-\beta}{x-\alpha}\cdot\frac{z-\beta}{z-\alpha}=\mu,$$

then we have

$$\frac{y-\beta}{y-\alpha}\cdot\frac{z-\beta}{z-\alpha}=\sigma\mu.$$

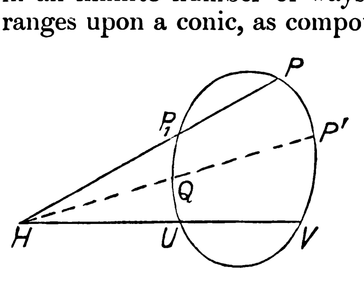
The first of these expresses that the points, $O+xU$, $O+zU$, are a pair of a certain involution; the second expresses that $O+zU$, $O+yU$ are a pair of another involution.

Similarly the latter equation is obtained by the composition of the two,

$$\frac{1}{x-\alpha}+\frac{1}{z-\alpha}=\frac{1}{\nu},\qquad \frac{1}{y-\alpha}+\frac{1}{z-\alpha}=\frac{1}{\nu}+\frac{1}{\lambda},$$

wherein ν is arbitrary; and each of these connects a pair of points belonging to an involution.

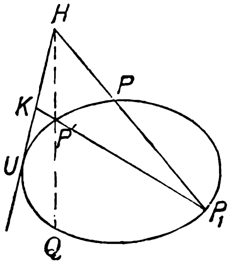
Now consider two related ranges upon a conic, the general point of one range being P, the point of the other range corresponding to this being P'. Suppose that the pairs, P, P', do not belong to an involution; and, in the first instance, that the common corresponding points of the two ranges are not coincident. It is to be shewn that we can, in an infinite number of ways, find a range of points P_1, upon the conic, such that the pairs, P, P_1, form an involution, and the pairs P_1, P' also form an involution. Thus we may, in an infinite number of ways, regard a correspondence of related ranges upon a conic, as compounded of two involutions.



Let U, V be the common corresponding points of the two related ranges, and, on the line UV, take an arbitrary point, H. Let the lines joining H to any two corresponding points, P and P', of the two ranges, meet the conic again, respectively, in P_1 and Q. Then, as P varies, since the lines P_1P, QP', UV meet in a point, the pairs P_1, P; Q, P'; U, V belong to an involution. Therefore, the range P, Q, U, V is related to the range P_1, P', V, U, and is, hence, related to the range P', P_1, U, V. Wherefore, in the two original related ranges, to the position Q of the point P, corresponds the position P_1 of the point P'.

The ranges (P_1), (P') are related; for, as P_1, P are pairs of an involution, the range (P_1) is related to (P); and this, by the original hypothesis, is related to (P'). And, when P_1 is at P', the point P, by the construction, is at Q. To this position of P, however, we have shewn, corresponds the position P_1 for P'. Namely, in the two related ranges (P_1), (P'), when P_1 is at P', the point P' is at P_1. Thus (Preliminary, above, p. 2), the two related ranges (P_1), (P'), together, form an involution. Therefore, the line P_1P' passes through a fixed point, say, K. This point is on UV; for when P is at V, then P_1 is at U and P' is at V.

We can, therefore, pass from the range (P) to the range (P'), by first passing to (P_1), where P, P_1 are pairs of an involution, the lines P_1P meeting UV in a fixed point, H, which is arbitrary on this line, and, then, passing from (P_1) to (P'), the pairs P_1P' forming another involution, the lines P_1P' meeting UV in another fixed point, K, whose position depends upon that taken for H. Clearly K may be taken arbitrarily on UV, if preferred, and H determined accordingly.



The theorem, and the proof, are, essentially, unchanged when the points U, V coincide. The line UV is then replaced by the tangent at U.

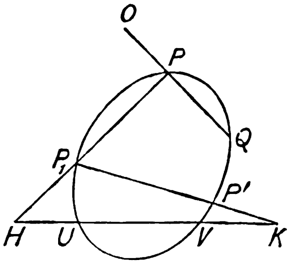
The proposition will be of use, later, in discussing the composition of two general screw motions. For this, and other purposes, it is desirable to state it in terms only of points which lie on the conic, without H and K. If L and L' be the points of contact with the conic of the two tangents which can be drawn from H, then, thinking first of U and V as distinct, the points L and L', as points of the conic, are harmonic conjugates in regard to U and V; for the lines joining L to H and L' are harmonic conjugates in regard to the lines LU, LV Similarly, L and L', are harmonic conjugates in regard to P_1 and P. Thus, in place of taking H arbitrarily on UV, we may take any two points L, L', of the conic, which are harmonically conjugate, on the conic, in regard to U and V. Then P_1 will be obtained from P by the fact that, as points of the conic, P_1 and P are harmonic conjugates in regard to L and L'. There will then be two other fixed points of the conic, say, M and M', also harmonic conjugates in regard to U and V, in regard to which P_1 and P' are harmonic conjugates. The points M, M' are the points of contact of the tangents from K. When U and V coincide, in U, we are, similarly, to take a quite arbitrary point, L, of the conic, and determine P_1 as the harmonic conjugate of P in regard to U and L; there will then be another fixed point, M, of the conic, such that P_1 and P' are harmonic conjugates in regard to U and M.

Ex. 27. *Three triads of points of a conic are all in perspective with another triad of points of the conic.* If A, B, C; A', B', C', and A'', B'', C'', be any three triads of points of a conic, we can find another triad of points of the conic, A_1, B_1, C_1, which is in perspective with each of the three given triads, from three properly chosen centres. For if we regard A, B, C and A', B', C' as determining two related ranges upon the conic, wherein the points A, B, C, of the first range, correspond, respectively, to the points A', B', C', of the second, these ranges will have two common corresponding points on the conic, whose joining line is the axis of relation of the two ranges, containing all such cross intersections as that of AB' and $A'B$, of BC' and $B'C$, etc. By taking the three triads in pairs, we have, thus, three such axes of relation, whose intersections are three in number. These intersections are the centres of perspective spoken of. This follows from Ex. 26; the range (A, B, C) is related to the range (A', B', C') by means of two involutions, one of these, with the pairs A, A_1; B, B_1; C, C_1, with the use of one of the centres of perspective, the other, with the pairs A_1, A'; B_1, B'; C_1, C', with the use of another of the centres of perspective. And so on.

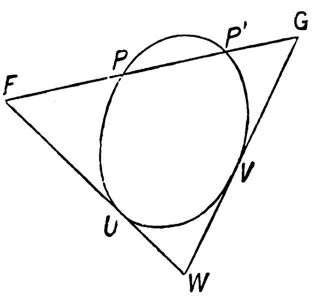
Though out of logical order, it may be as well to remark that this result is connected with one that, in the language of metrical geometry, which we consider later (Chap. v, below), is expressed by saying that three congruent figures in a plane may be regarded as images, by reflexion, of the same figure, in three suitably taken mirrors.

Ex. 28. *The envelope of the line joining two corresponding points of two related ranges on a conic.* We have seen that the joins of pairs of points of an involution on a conic, all pass through a point. Also, that the joins of corresponding points, of two related ranges of points, on two lines, are tangents of a conic. In generalisation of these we now prove that the joins of two corresponding points, of two related ranges of points on a conic, are tangents of another conic, which touches the original, at the two (generally distinct) points which are the common corresponding points of the two ranges.

Let the corresponding points be P, P'; let O be an arbitrary point of the plane. There are two of the lines PP' which pass through O. For, let the relation between the ranges (P), (P') be given, as in Ex. 26, by means of lines, PP_1, passing through a fixed point, H, and lines, P_1P', passing through a fixed point, K. Let OP meet the conic again in Q. The pairs (Q, P), (P, P_1), (P_1, P') are then pairs of three involutions; from this it can be shewn that the ranges (Q), (P') are related. These two related ranges will have two common corresponding points, in general distinct; but when Q coincides with P, there is a line PP' passing through O.

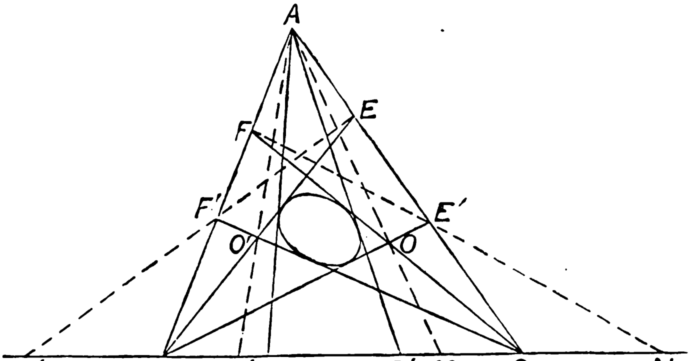


Assuming this result, let U, V be the common corresponding points of the ranges (P), (P'); and let F be any point of the tangent at U. There will then pass through F two lines PP', of which, however, FU is one. The other will meet the tangent at V in a definite point, G. Similarly, to an arbitrary position of G, on the tangent at V, there corresponds a single definite position of F, on the tangent at U. Hence, by an abbreviated argument, spoken of in the Preliminary remarks (above, p. 8), we infer that the ranges (F), (G) are related; and hence, that the lines PP' touch a conic; this conic touches the tangent at U at the point where F lies when G is at the intersection, W, of the tangents at U, V, that is at U. It similarly touches VW at V.



It is easy to state the corresponding result when U and V coincide.

Ex. 29. *The tangents from three points, A, B, C, to a conic, meet BC, CA, AB in six points of a conic.* Let the tangents from three points, A, B, C to an arbitrary conic Ω, meet BC, CA, AB, respec-



tively in D, D', in E, E', and in F, F'. These six points then lie on another conic.

Let BE', CF meet in O, and AO meet BC in M; let BE, CF' meet in O', and AO' meet BC in M'. Let $E'F$ meet BC in N, and EF' meet BC in N'.

We have shewn that the tangents from an arbitrary point to the conics touching four lines are a pencil in involution; and of such conics there are three point pairs, through each of which the four lines pass. For the four lines constituted by the tangents to the conic Ω from B and C, two of these point pairs are B, C and O, O'. Hence we see that the pairs D, D'; M, M'; B, C, lying on tangents from A, are in involution.

The point N is, however, the harmonic conjugate of M in regard to B and C, and the point N' is the harmonic conjugate of M', in respect of the same. Thus (see Preliminary, above, Ex. 3), the points N, N' also belong to the involution spoken of.

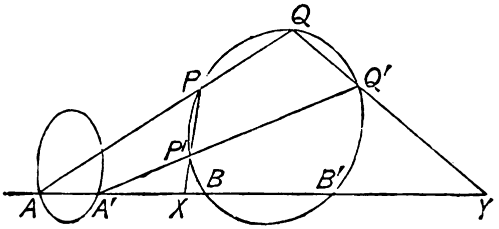
Now the conics drawn through the four points E, E', F, F' meet the line BC in pairs of points belonging to an involution; of this involution, however, B, C and N, N' are two pairs; so that this is the same involution as the former.

Wherefore, the conic through E, E', F, F' which is drawn through D, equally contains D'. And this is what we desired to prove.

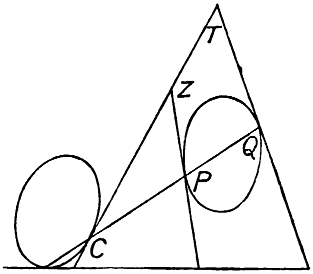
If this conic be called Σ, the reader may proceed to prove that there is a conic, through the intersections of Ω and Σ, which touches BC, CA, AB.

A particular case of the result proved, arises when the conic Ω is a point pair. Of this again a particular case is that, if AD, BE, CF meet in a point, then a conic exists touching BC, CA, AB, respectively, at D, E, F. And the theorems lead, by duality, to other theorems worth notice. (See also Chap. III, Ex. 14.)

Ex. 30. *Two tangents of one conic meet two tangents of another in four points lying on a conic through the common points of the two conics, if the four points of contact are in line.* Let a line joining two points, A and A', of one conic, Φ, meet another conic, Ω, in B and B'. Let any line through A meet Ω in P, Q, and any line through A' meet Ω in P', Q'. Let PP', QQ' meet the line AA', respectively, in X and Y.



The pairs of points A, A'; B, B'; X, Y then belong to the involution determined on the line AA' by all conics through the four points P, Q, P', Q'. And as this involution is determined by the first two pairs, it is the same as the involution determined on the line AA' by all conics through the common points of the conics Φ, Ω. To this latter involution, therefore, the pair X, Y belong.



By supposing A' to coincide with A, and P' with P, and Q' with Q, we reach the result that, if any line, drawn through a point, A, of the conic Φ, meet the conic Ω in P and Q, and the tangents at P, Q meet the tangent at A in X and Y, then these last are a pair of the involution determined, on the tangent at A, by the conics which pass through the four common points of Φ and Ω.

Now let the line APQ meet the conic Φ again in C, and let the tangents at P and Q meet the tangent at C in Z and T. Then, by parity of reasoning, taking P on Ω instead of A on Φ, and A, C on Φ instead of P, Q on Ω, we infer that the conic drawn through the intersections of Φ and Ω to contain X also contains Z; from what was said, this conic contains Y. By a similar argument, the same conic contains T.

This is the result enunciated, the points A, C, P, Q being in line.

Dually, if from an arbitrary point the two pairs of tangents be drawn to two conics, the four joins, of the points of contact on one conic, to the points of contact on the other conic, are all tangents of a conic which touches the four common tangents of the two conics.

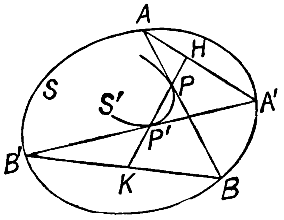
From the combination of these theorems many interesting results can be deduced. It will be proved in Chap. III, that, if the tangents of the conics Φ, Ω, respectively at A and P, meet in X, and the conic, through the common points of Φ, Ω, which contains X, be S, while that, touching the common tangents of Φ and Ω, which touches AP, be Σ, then, to another position of X on S, there corresponds, by the same construction, another tangent, such as AP, of Σ.

Ex. 31. *Poncelet's theorem, for sets of points A, B, C, ... of one conic, of which the joins AB, BC, ... separately touch other fixed conics, all the conics having four points in common.*

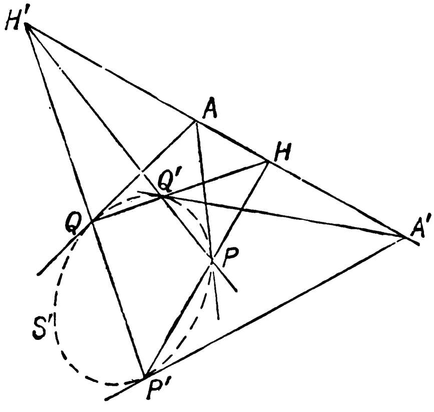
The reference is to Poncelet's *Propriétés Projectives*, Paris, 1865, t. I, p. 316.

Let A, A', B, B' be four points of a conic, S, such that the lines AB, $A'B'$ touch another conic, S', respectively, in P and P'. Let PP' meet AA', BB', respectively, in H and K. We shew, first, that a conic can be drawn, through the common points of S and S', to touch AA' in H, and BB' in K.

For consider the four points consisting of P repeated and P' repeated; and the involution determined on AA' by the conics through these four points. Three pairs of this are (1), the points A, A', on the conic S, (2), the points in which the conic S' meets AA', (3), the point H repeated. The pairs (1), (2) shew that this is the same involution as that determined on AA' by the conics through the common points of S and S'. Of this involution, then, H is a double point, and is, thus, a point of contact with AA' of a conic, say, T, through the common points of S and S'. We prove, now, that this conic, T, touches BB' at K. This we do by shewing, from the preceding example, that, if T meet HK again in K_1, the tangent of it, at K_1, passes through B and B'. By that example, the conics S', T being met by a line in P, P' and H, K_1, the tangents at P, P' must meet the tangents at H, K_1, in four points lying on a conic passing through the common points of S' and T, which are the common points of S and S'. These four points are A, A', and the points where the tangent at K_1 meets AB and $A'B'$; the two former points identify the conic through them as S. This proves that the tangent at K_1 passes through B, B'; so that K_1 is K.



It follows, of course, in the same way, that if PP' meet AB' and $A'B$ in L and M, a conic through the common points of S and S' touches these lines in L and M.



If the other tangent from A to S' touch this in Q, and the other tangent from A' to S' touch this in Q', it follows, similarly, that PQ' meets AA' in a point, H', which is also a point of contact with AA' of a conic through the common points of S and S'. Thus, H' is the other double point of the involution spoken of, and is the harmonic conjugate of H in regard to A and A'. But, then, by the same argument, $Q'Q$ meets AA' in the harmonic conjugate of H', that is, in H; and, so, $P'Q$ passes through H'.

Now take a further conic, S'', passing through the common points of S and S', or, say, for brevity, a conic of the system (S, S'). Let either one of the tangents drawn from B to S'' meet S again in C, touching S'' in R. It follows, then, from what has just been said, that it is possible to pair with this such an one of the two tangents from B' to S'', touching S'' in R', that the line RR' shall pass through K; this point, K, is one of the two points of possible contact, with BB', of conics of the system (S, S'). If the appropriate tangent meet S again in C', the line RR' will meet CC' in a point, J, in which the conic (S, S'), which touches BB' at K, touches CC'.

There are now two pairs of points of S, the pairs A, A' and C, C', such that the lines AA' and CC' both touch the same conic of the system (S, S'), respectively at H and J. Call this conic Σ. By applying to the lines AA', CC', and the conics S and Σ, the argument we originally applied to the lines AB, $A'B'$, and the conics S and S', we can then infer that the lines AC, $A'C'$ are both touched by a further conic, S''', of the system (S, S'), the points of contact being on the line JH.

It is thus possible to have an infinite number of triads A, B, C; A', B', C'; ... of points of S, for which the joins $BC, B'C', \ldots$; $CA, C'A', \ldots$; $AB, A'B', \ldots$; touch, respectively, three fixed conics S', S'', S''', of the system (S, S'). The argument can clearly be extended to the case when the points of a set are not triads, but of any greater number.

A particular case of the foregoing arises when, with four points, A, A', B, B', of the conic S, of which AB, $A'B'$ meet in a point O, we have, instead of the proper conic S', a pair of lines which intersect in O. There are then no distinct points of contact, P, P', with which to determine the points H, K of AA' and BB'. But, a point H can be determined on AA', as one of the two which are harmonic, both in regard to A, A', and in regard to the two points in which AA' is met by the lines of the line pair, S'. The line HO then meets BB' in a point, K, which is one of the two, which are harmonic conjugates, both in regard to BB', and in regard to the two points in which BB' is met by the lines of the line pair, S'. A conic of the system (S, S') can then be drawn to touch AA' at H, and to touch BB' at K.

If, in the preceding theory, we start with two proper conics, S and S', a line pair, of the system (S, S'), can intersect only in a point, O, which is one of the three points of the common self-polar triad of the system. Such a line pair can then replace S'' or S'''.

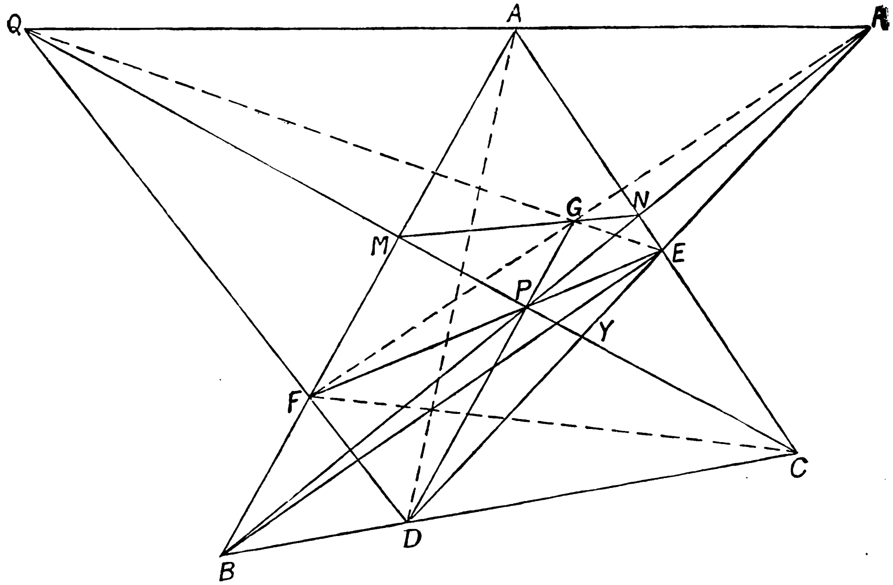
In particular, we may have triads, A, B, C, of points of S, such that AB, BC touch two proper conics, S', S'' of the system (S, S'), respectively in P and Q, say, while CA passes through one of the three points, O, of the self-polar triad of (S, S'). It will appear (in Chap. III) that, in this case, the line PQ is a tangent of a fixed conic, which touches the common tangent lines of S' and S''; and that the line pair of (S, S') which meet in O are harmonic conjugates in regard to the lines OP, OQ.

A very particular case is the intersecting simple result that triads A, B, C exist on a conic, of which the joins BC, CA, AB pass, respectively, each through one of the points of any given triad which is self-polar in regard to the conic. (Cf. Ex. 5, p. 28.)

Ex. 32. *Construction of a fourth tangent, with its point of contact, of a conic touching three lines at given points.*

The present example, which, in effect, collects various results already given, is introduced in view of the following example.

Respectively on the joins BC, CA, AB, of three points A, B, C, let D, E, F be points such that the lines AD, BE, CF meet in a point. Let the lines DE, DF, respectively, meet any line, drawn through A, in R and Q. Let CQ, BR meet AB, AC, respectively, in M and N, the lines CQ, BR intersecting in P. We prove that the lines QR, MN, BC meet in a point; and that the lines MN, DP, EQ, FR, meet in a point; while P is on EF. Also that MN is a tangent of the conic which can be drawn to touch BC, CA, AB, respectively, at D, E, F, whose point of contact is on DP, as well as on EQ and FR.



In fact FE meets BC in the point harmonic to D with respect to B and C; thus F, E are harmonic with respect to the points in which FE is met by DA and BC. Therefore, Q, R are harmonic with respect to A, and the point where QR is met by BC. This shews that QB, RC meet on AP, so that the triads QMB, RNC are in perspective, the lines QR, MN, BC meeting in a point. That P lies on FE is a direct application of Pappus' theorem for the two triads Q, A, R and B, D, C. That QE, FR meet on MN is, then, also a consequence of Pappus' theorem, applied to the two triads Q, A, R and F, P, E. To shew that DP passes through the point, say, G, where MN is met by EQ or FR, it is necessary to prove that E, F are harmonic in regard to P, and the point where QR meets EF; or, that AP, AR are harmonic in regard to AE, AF, which follows because QB, RC meet on AP. Finally, that the conic which (Ex. 29) can be drawn to touch BC, CA, AB at D, E, F, respectively, touches MN, on DP, follows from Ex. 14, above; or can be deduced, independently, by considering, with fixed points D, E, F, various lines QAR, giving rise to related ranges (M), (N), on AB, AC. In regard to this conic, P, Q, R will be a self-polar triad.

It is clear that B and N are harmonic conjugates in regard to P and R; and, further, if QP meet DE in Y, that D and E are harmonic conjugates in regard to Y and R.

Ex. 33. *Hamilton's extension of Feuerbach's theorem.* (See, also, p. 117.)

With the results of the previous example in mind, consider two conics. One of these, Ω, is to be any conic having P, Q, R for a self-polar triad. The other, Σ, is to be the conic through D, E, F which meets the line MN in the points in which it is met by Ω. We prove that these conics meet further on the line, which we call l, which contains the three intersections (BC, EF), (CA, FD), (AB, DE). This is the line which, for the conic touching BC, CA, AB at D, E, F, we have called the Hessian line of D, E, F (Ex. 9). In fact, the conic Ω meets the line DR in points harmonic in regard to Y and R; the conic Σ also meets this line in points, D and E, harmonic in regard to Y and R. Wherefore the common chord MN, of Ω and Σ, and the line joining the other two common points of these conics, meet DR in points harmonic in regard to Y and R. The point harmonic to the intersection of MN with DR, with respect to Y and R, is, however, we have seen, the point where MB meets DR. This is a point of the line, l, spoken of. Another point of the common chord of Ω and Σ, other than MN, is proved, similarly, to be on l, by considering the line DQ, in place of DR.

This is a slight extension of a result ascribed to Sir W. R. Hamilton (Salmon's *Conics*, 1879, p. 313). For what follows we take for Ω any conic touching BC, CA, AB and touching MN, say, at K; this will have P, Q, R for a self-polar triad; then Σ will be the conic through D, E, F which touches MN at K. The result is that the join, of the two remaining intersections of these two conics, is the line l.

Conversely, let A, B, C be any points, and l any line. Let D, on BC, be the harmonic, in regard to B and C, of the point where l meets BC; and make a similar construction for E and F. Let Ω be any conic touching BC, CA, AB. Let its intersections with l be I and J. Then the conic Σ, defined as that containing D, E, F, I, J, touches Ω. This is the theorem desired.

The point of contact of Σ with Ω may be defined as the point of contact with Ω of the fourth common tangent of Ω with the conic which touches BC, CA, AB at D, E, F, respectively. But it may be defined without this conic. For, if Ω touch BC, CA, AB respectively at D', E', F', the point P (one of the three points of the common self-polar triad of Ω and the conic which touches BC, CA, AB at D, E, F) is the intersection of $E'F'$ and EF, by what we have seen. Hence the points M, N can be constructed, as in the preceding example, by the lines CP, BP (as also the points Q and R). The point, K, required, is then the intersection of $D'P$ with MN (which also lies on $E'Q$ and $F'R$). Incidentally, given the points of contact of two conics with BC, CA, AB, we thus construct their remaining common tangent, and its points of contact with the two conics.

When the points A, B, C, I, J are given, we can (Ex. 10, above) find a point, T, such that each of the line pairs, AT, BC; BT, CA; CT, AB, meet the line IJ in points which are harmonic in regard to I and J. The conic Σ will then be (Ex. 18) the locus of the poles of the line IJ in regard to all conics through A, B, C, T. Let then A, B, C, T be four arbitrary points, and let I, J be any two points conjugate to one another in regard to all conics through A, B, C, T. Four conics, such as Ω, can (Ex. 17) be constructed, to pass through I, J, and touch the joins of three of the four points A, B, C, T. All the sixteen conics so obtained are touched by the conic which is the locus of the poles of IJ in regard to the conics through A, B, C, T.

Feuerbach's theorem in its original form will be referred to in the next Chapters. (See pp. 75, 89, 113, 122.)

Ex. 34. *Theorem in regard to the remaining common tangents of three conics which touch three given lines.*

If three conics, S_1, S_2, S_3, all touch the lines BC, CA, AB, the points of contact being, in the case of S_1, respectively D_1, E_1, F_1, with a similar notation for the others, and, if the remaining common tangent of S_2 and S_3 meet BC in T_1, with a similar notation for the other pairs, then the pairs T_1, D_1; T_2, D_2; T_3, D_3 are in involution.

For, by what has been said, in the two preceding examples, the remaining common tangent of S_2 and S_3 meets AB, AC, respectively, in M_1 and N_1, where N_1 and M_1 are, respectively, on the lines joining B and C to the point, P_1, in which F_2E_2 intersects F_3E_3.

Thus the point T_1, where M_1N_1 meets BC, is the harmonic conjugate, in regard to B and C, of the point in which AP_1 meets BC. The point D_1 is also the harmonic conjugate in regard to B and C, of the point where F_1E_1 meets BC. This line F_1E_1 is the same as the line P_2P_3. It is a fundamental property of the joins of four points, that the joins of A, P_1, P_2, P_3 meet BC in pairs of points in involution. As these points are the harmonic conjugates, in regard to B and C, of the pairs T_1, D_1; T_2, D_2; T_3, D_3, it follows that these are also in involution. (Preliminary, above, p. 7, Ex. 4.)

The dual theorem is that, if three conics be drawn through three points A, B, C, and we draw the three lines from A to the remaining intersections of the pairs of these conics, and pair these, properly, with the tangents at A of the three conics, there is obtained a pencil in involution.

A more general proposition is that, if any four conics be drawn through the three points A, B, C, say S_1, S_2, S_3, S_4, then the three pairs of lines through A constituted by (1) the common chords through A of S_2, S_3 and of S_1, S_4; (2) the common chords through A of S_3, S_1 and of S_2, S_4; (3) the common chords through A of S_1, S_2 and of S_3, S_4, are in involution. This may be proved as above, or also, by the transformation of Ex. 16 preceding. It depends on the fact that the pairs of lines from an arbitrary point, to the opposite pairs of intersections of four lines in a plane, belong to an involution. (Preliminary, p. 4, above.) When the fourth conic breaks up into two lines, of which one passes through A, we obtain the theorem referred to in regard to three conics through A, B, C.

Ex. 35. *The four conics, through two points, of which each contains three of four other given points.*

If the two given points be O, Q, and the four given points be denoted by 1, 2, 3, 4, and the conic which contains O, Q, 2, 3, 4 be called S_1, its tangent at O being called t_1, with a similar notation for three other conics, then the four pairs of lines $(O1, t_1)$, $(O2, t_2)$, $(O3, t_3)$, $(O4, t_4)$ are in involution. For the conics S_1, S_2, S_3 have in common the three points O, Q, 4, while S_2, S_3 have also the point 1 in common, S_3 and S_1 the point 2, and S_1, S_2 the point 3. The pairs of lines $(O1, t_1)$, $(O2, t_2)$, $(O3, t_3)$ are, thence, in involution, by the preceding example. Again the conics S_2, S_3, S_4 have in common the points O, Q, 1; the residual intersections of the pairs of these are, for S_3, S_4, the point 2; for S_4, S_2, the point 3; and, for S_2, S_3, the point 4. Wherefore $(O2, t_2)$, $(O3, t_3)$, $(O4, t_4)$ are in involution; this is, therefore, the same involution as before.

Ex. 36. *The polars of any point in regard to the six conics which can be drawn through the sets of five, from six arbitrary points of a plane, touch a conic.*

This result was suggested by Mr H. M. Taylor, Trinity College, Cambridge. (See W. W. Taylor, *Messenger of Mathematics*, XXXVI, 1907, p. 113; and particularly, E. J. Nanson, *Mess. of Math.* XLVI, 1917, p. 183.)

Let the six given points be denoted by 1, 2, ..., 5, 6, the other point being O, the conic containing the five points 2, 3, 4, 5, 6 be S_1, etc., the polar of O in regard to S_1 being p_1, etc. Any two of the conics have four points in common, and a further, auxiliary, conic can be drawn through O and these four points. As the polars of O, in regard to conics through four points, meet in a point (Ex. 15, above), the polars of O in regard to the two conics meet on the tangent, at O, of the auxiliary conic spoken of.

Consider then the intersections of p_5 with p_1, p_2, p_3, p_4. These lie, by what has been said, on the tangents, at O, respectively of the four conics containing the points

$$O, 2, 3, 4, 6; \quad O, 1, 3, 4, 6; \quad O, 1, 2, 4, 6; \quad O, 1, 2, 3, 6.$$

These however are conics, through the points O and 6, containing, respectively, the various sets of three points which can be chosen from the four 1, 2, 3, 4. By the result of the preceding example, it follows that the range of four points in which p_5 is met by p_1, p_2, p_3, p_4 (these points lying on the tangents at O of the four conics just named) is related to the pencil of lines joining O to the points 1, 2, 3, 4. The range in which p_6 is met by p_1, p_2, p_3, p_4 is, similarly, related to this pencil. Whence the six polar lines touch a conic.

The above interesting proof of Taylor's theorem is derived from that given by Mr S. G. Soal, of the East London College, in J. L. S. Hatton's *Principles of Projective Geometry*, Cambridge, 1913, p. 299.

From this demonstration we can state the theorem: *Given six arbitrary points of a plane, there exists, corresponding to any further arbitrary point, O, of the plane, a conic, and, upon this conic, six points subtending, at any point of the conic, a pencil which is related to that of the lines which join O to the six given arbitrary points.* For the points of contact of p_1, p_2, p_3 p_4 with the conic are, as a range on the conic, related to the range determined by these tangents on the tangent p_5.

If seven points be taken, and each in turn omitted, the seven conics obtained have three common tangents.

Ex. 37. Let A, B, C, A', B', C', D be seven given points, and p a given line through D. Any conic through A, B, C, D meets p in a further point, M, and there is a second conic through A', B', C', D, M. The chord of these two conics, other than p, passes through a fixed point.

CHAPTER II

PROPERTIES RELATIVE TO TWO POINTS OF REFERENCE

Introductory. The multiplicity of metrical results often found in books on Geometry are in fact relations of the figure which is considered, either to two points of reference, regarded as given, or, more generally, to a conic of reference, regarded as given. This is one of the striking discoveries, essentially due to Poncelet, of the early part of the nineteenth century. The fact is generally proved by beginning, on the basis of Euclid's geometry, with the metrical relations, and gradually shewing how these can be summarised under more general points of view. We adopt the converse procedure. Without assuming the metrical basis, and as consequences of the foundations which have been explained, we obtain a number of results which are easily recognised as containing those usually developed as consequences of the notion of distance. Finally, from this and Chapter v, it should emerge quite clearly that distance is only a special way of using the algebraic symbols; and that it is an arbitrary limitation to regard it as fundamental.

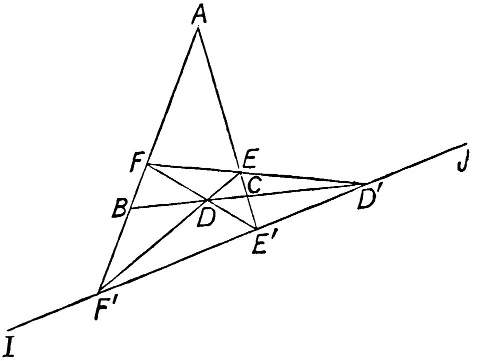
For the sake of clearness we shall employ the terms, such as *perpendicular*, *circle*, *rectangular hyperbola*, and so on, which are current in metrical geometry. They have here, however, a more general meaning, depending on the choice of the absolute points of reference.

Middle points, Perpendicular lines, Centroid, Orthocentre. Suppose that two points of reference are given; they may be called the Absolute points, and will often be denoted by I and J.

If any line, AB, meet the line IJ in K, the point, say C, which is the harmonic conjugate of K in regard to A and B, may be called the *middle point* of AB, in regard to I and J. Any two lines whose intersection is on the line IJ may be said to be *parallel* to one another, in regard to I and J. Any two lines which meet the line IJ in two points which are harmonic conjugates of one another, in regard to I and J, may be said to be *at right angles*, or, to be *perpendicular*, in regard to I and J. With these definitions, all the theorems of incidence involving these terms remain true. Thus, for instance, two lines, which are both parallel to a third line, are parallel to one another; all lines which are at right angles to a line are parallel to one another; if a series of parallel lines be drawn, meeting any two lines, say a and b, the former in A and the latter in B, the middle point of AB lies on a definite line through the intersection of a and b.

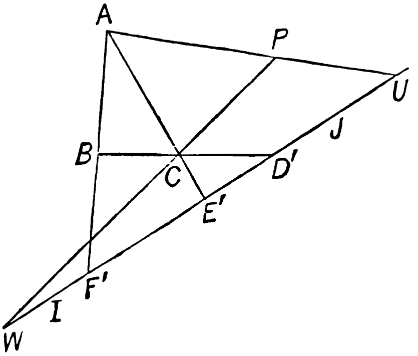
Or, passing to somewhat less obvious results, the ordinary constructions for the centroid, and for the orthocentre, of three given points A, B, C, can be carried out, these words being defined as we now explain.

For the *centroid*, this is defined as the intersection of the lines joining A, B, C, respectively, to the middle points of BC, CA, AB. If the line IJ be met by BC, CA, AB, respectively, in D', E', F', and D be the harmonic conjugate of D' in regard to B, C, while E is similarly defined from E' by means of C and A, and F from F' by A and B, we have only to shew that AD, BE, CF meet in a point. In fact, it is clear, from the construction, that E, F, D' are in line; as, likewise, are F, D, E' and D, E, F'; and, hence, that the line, joining A to the intersection of BE and CF, passes through D.



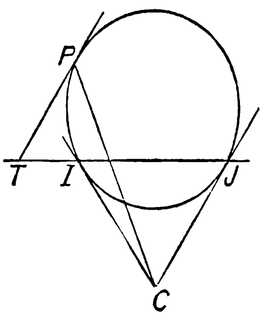
For the *orthocentre*, we are to shew that the lines, drawn through A, B, C, respectively perpendicular to BC, CA, AB, meet in a point. As before, let D', E', F' be the points in which IJ is met by BC, CA, AB, respectively; and let U, V, W be the harmonic conjugates, in regard to I and J, respectively of D', E', F'. We are to shew that AU, BV, CW meet in a point.

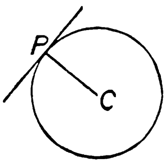
If AU, CW meet in P, the involution determined on the line IJ by the joins of A, B, C, P (Preliminary, above, p. 4), is determined by the two pairs D', U and F', W. In regard to these pairs, I, J are harmonic conjugates; these are then the double points of the involution, and thus AC, BP meet the line IJ in points harmonic in regard thereto. Thus BV passes through P. Each of A, B, C, P is then the orthocentre of the other three in regard to I and J.



Considering I, J as a degenerate conic, regarded as an envelope (above, p. 24), the theorem of the orthocentre is evidently a particular case of the theorem (above, p. 31) that, if, of the line pairs containing four points, two pairs consist of lines which are conjugate in regard to a conic, so does the third pair. [Add.]

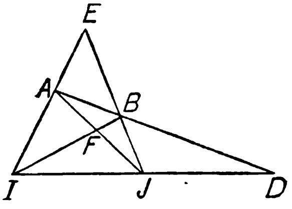
Circles. Any conic which passes through the two absolute points of reference, I, J, will be spoken of as a circle in regard to I and J, or, simply, as a circle. By the *centre* of the circle will be meant the pole, in regard thereto, of the line IJ. As a conic is determined by five points, a circle is determined by three points, beside I and J.

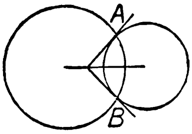




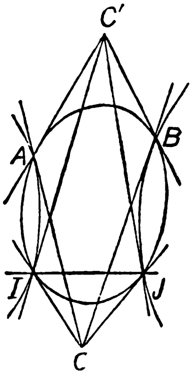
If the tangent, at a point P, of a circle, whose centre is C, meet the line IJ in T, this point, T, lying on the polar of C, and on the tangent at P, is the pole of the line PC. Thus PT, PC meet IJ in points which are harmonic in regard to I and J. In other words, the tangent at any point of a circle is at right angles to the line joining this point to the centre of the circle.

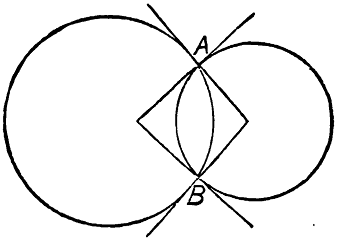
Two circles will have two points of intersection, beside I and J. The tangents of one of the circles, at these two points, meet on the line joining the centres of the circles. For, if these two points be denoted by A and B, and D, E, F be the intersections, respectively, of IJ, AB, of IA, JB, and of IB, JA, the polar, in regard to either circle, of the point D, is the line EF (above, p. 23), and this line contains the intersection, both of the tangents at A, B, and of the tangents at I, J. This line, EF, is thus the line joining the centres of the two circles.





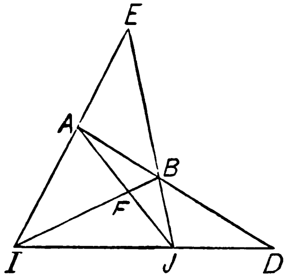
Suppose that, at one of the common points, A, of two circles, the tangents of the circles are at right angles to one another, in regard to I and J. As the tangent of a circle at any point is at right angles to the line joining this point to the centre, it follows that the tangent, at A, of either circle, passes through the centre of the other. Thence, as the tangents of either circle, at the points A, B, meet on the line joining the centres of the circles, it follows that the tangent at B, of either circle, passes through the centre of the other circle. Wherefore the tangents at *B*, of the two circles, are at right angles. Two such circles are said to *cut at right angles*, or *orthogonally*; they are such a pair of associated conics as were considered before (Chap. I, Ex. 19, above, p. 42). It follows from what was then said that, any line, drawn through the centre of either of two circles which cut at right angles, meets the two circles, respectively, in two pairs of points which are harmonic in regard to one another; and, conversely, that if, on any line drawn through the centre of a circle, there be taken two points which are harmonic conjugates of one another in regard to the points in which the line cuts the circle, then any circle drawn through these two points cuts the given circle at right angles. It follows, also, that two points, which are such that every circle, passing through them, cuts a given circle at right angles, must be conjugate points in regard to this circle, and lie on a line through its centre.

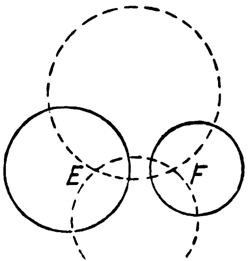




Coaxial circles. The common chord, *AB*, of two circles, other than *IJ*, may be called the *radical axis* of the two circles. Circles passing through *A* and *B*, that is, conics through *A*, *B*, *I*, *J*, may be said to be *coaxial.*

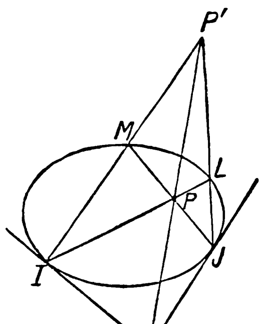
Of such conics, beside the lines *IJ*, *BA*, there are two which are line pairs, the pair *IA*, *JB*, meeting, say, in *E*, and the pair *IB*, *JA*, meeting, say, in *F*. The points *E*, *F* may be called the *limiting points* of the coaxial circles through *A* and *B*. The line *EF* contains the centres of all the coaxial circles; and *E*, *F* are conjugate points in regard to any one of these circles. The triad of points self-conjugate in regard to all the coaxial circles consists of the limiting points *E*, *F*, together with the point, *D*, where the radical axis intersects the line *IJ*. Thus, by what we have said above, all circles through the limiting points, *E*, *F*, cut every circle of the coaxial system at right angles; these circles form a second coaxial system, of which *A* and *B* are the limiting points, their centres being on the radical axis, AB, of the first system. Conversely, a circle whose centre is at any point on the radical axis, AB, of the first system of circles, can be drawn, to cut all the circles of this system at right angles; it passes through the limiting points.





Inversion in regard to a circle. There is a process, which has been much employed, by which, when we are given a circle in a plane, we can pass from any point of the plane to another point. It will subsequently be found, in dealing with the relations of geometry in two and in three dimensions, that the process can be regarded from a very simple point of view. It can also be shewn that, in its effect, the process does not differ from the transformation explained in Chap. I, Ex. 16 (above, p. 38). We give, however, some provisional results obtained from the present point of view.

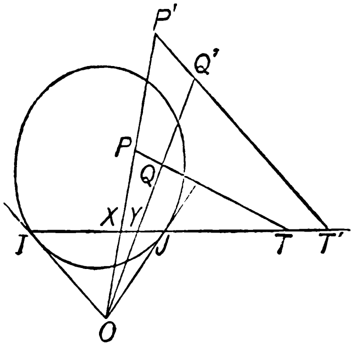
Two points, P and P', are said to be inverses of one another in regard to a given circle when, (*a*) they are conjugate in regard to the circle, and, (*b*) their join passes through the centre of the circle. When this is so, it follows, by what has already been remarked in the preceding chapter (Ex. 5, above, p. 28), that the lines IP, JP' meet in a point, say, L, lying on the circle, and the lines IP', JP also meet in a point of the circle, say, M. The relation of the points P, P' is evidently reciprocal. The definition can be held to include the inverse of a point in regard to a line, which, taken with the line IJ, may be regarded as a degenerate circle; the inverse of P in regard to the line, say, P', will be such that PP' is at right angles to the line, and the mid-point of PP' is on the line.



When P describes any locus, P' will describe a corresponding locus. The most important case is that, when P describes any circle, then P' also describes a circle. Of this a particular case arises when P describes a circle which passes through the centre, O, of the given circle by which we define the inversion; then the circle described by P' degenerates into a line, taken with the line IJ. Conversely, we may say that the inverse of a line is a circle passing through the centre of the fixed circle of inversion. These statements are easy to prove: When P describes any conic passing through I and J, the pencils $I(P)$, $J(P)$ are related; these, however, give, respectively, the ranges (L), (M), of the given fundamental conic, which are, therefore, also related. Thus P', the intersection of JL and IM, describes another conic passing through I and J, that is, a circle. While, if P, on its locus, can take the position O, the points L, M, take, then, respectively, the positions I and J; then the ray IJ, of the pencil $I(P')$, corresponds to the ray JI, of the pencil $J(P')$, and the locus of P' degenerates (above, p. 10) into the line IJ and another line. Conversely, if P' describes a line, the pencils, $I(P')$, $J(P')$, are related; so then, also, are the pencils $I(P)$, $J(P)$; thus P describes a conic passing through I and J. But, in particular, to the point P' which is on IJ, correspond positions, of L and M, respectively at I and J; the corresponding position of P is then at O. The locus of P is, thus, a circle passing through the centre, O, of the circle of inversion.

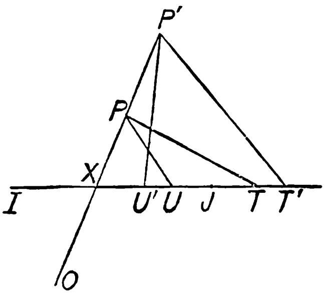
Examples of inversion in regard to a given circle.

Ex. 1. Let the point P, of one circle, invert into the point P', of the inverse circle; and let the tangents of these circles, respectively at P and P', meet the line IJ, in T and T'; let the line OPP' meet IJ in X. Then X is a double point of the involution determined by the pairs I, J and T, T'. Consider, first, two points, P and Q; and their inverses, respectively P' and Q'. Let PP' and QQ' meet IJ, respectively, in X and Y, while PQ and $P'Q'$ meet IJ in T and T'. Then, by what has been said above, the conic drawn through I, J, P, Q, P' also contains Q'; or, the points P, Q, P', Q' lie on a circle, cutting the circle of inversion at right angles. Thus, the pairs X, Y; T, T'; I, J belong to the involution, of points on IJ, determined by conics through P, Q, P', Q'. When Q coincides with P, and Q' with P', this leads to the result stated.



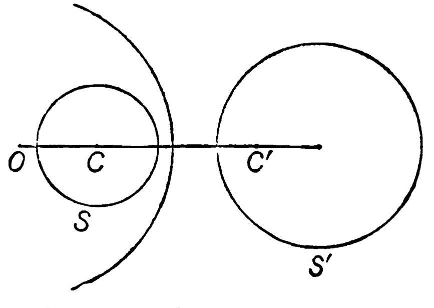
Ex. 2. The circles which are the inverses of two circles which cut at right angles are two circles which also cut at right angles.

Let the tangents of two circles, which meet at P, and cut at right angles, intersect the line IJ, respectively, in T and U; let the tangents of the, respectively, inverse circles, at the inverse point P', meet IJ in T' and U'. Consider the involution determined by the two pairs of points, I, J and T, T'. The point, U, which is the harmonic conjugate of T in regard to I and J, and the point, V, which we may take, which is the harmonic conjugate of T' in regard to I and J, form a pair also belonging to this involution (above, p. 7). If the line $P'PO$ meet IJ in X, we have seen in the preceding example that the involution determined by I, J with X as a double point, contains the pairs T, T' and U, U'. By what we have now said, this same involution, as determined by I, J and T, T', contains the pair U, V. Thus V coincides with U', and the points T', U' are harmonic conjugates in regard to I, J. Therefore, the inverse circles cut at right angles at P'. It is easy to prove, also, that the inverses of two circles which touch one another are two circles, which also touch one another.



Ex. 3. Suppose we are given a circle and two points which are inverse to one another in regard to this circle, that is, these points are conjugate to one another in regard to this circle, and lie on a line passing through the centre of this circle. If we invert this system in regard to a circle, we obtain another circle and two points. The new points are inverse to one another in regard to the new circle.

For, every circle through the two original inverse points cuts at right angles the circle in regard to which they are inverse, as we have proved. Wherefore, by Ex. 2 above, every circle through the two new points cuts the new circle at right angles. This shews, by what we have said, that the two new points are inverse points in regard to the new circle.

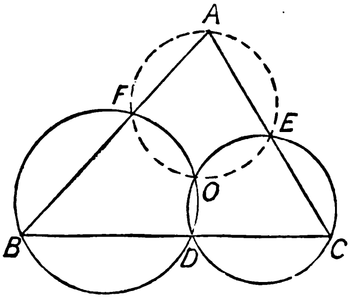


Ex. 4. It follows, from Ex. 3, that, when a circle, say S, of centre C, inverts into a circle, S', the point C' which is the inverse of C (in regard to the circle of inversion), is the inverse of the centre, O, of the circle of inversion, taken in regard to S'. For, if K be any point of the line IJ, the points C

and K are inverses of one another in regard to S. The inverse of K (in regard to the circle of inversion) is, however, O. Thus O, C' are inverse points in regard to S'. In particular, when S' is a line, it passes through the middle point of OC' and is at right angles to this.

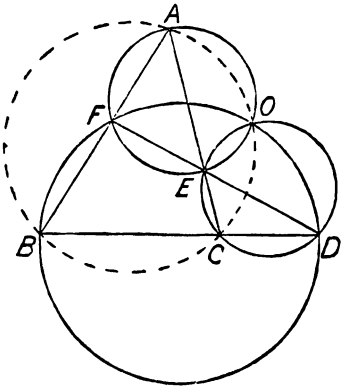
Examples in regard to circles. *Ex.* 1. If any three circles be taken, and the common chord, other than IJ, of each of the three pairs of these circles, these three common chords meet in a point. Let the circles be S, U, V; consider the common points of S with U, and, also, of S with V; let the former be P, P', and the latter be Q, Q', and let the chords PP', QQ' meet in T. By what has been seen, a circle can be described with T as centre, to cut the circle S at right angles; it is the conic touching TI and TJ, respectively at I and J, which passes through one (and therefore the other) of the points of contact of the tangents drawn from T to S. The points P, P', being on S, are inverse in regard to this circle, as also are Q, Q'. This circle, therefore, cuts U and V, also, at right angles. Its centre T is, therefore, on the common chord of U and V. For, if H be one of the common points of U and V, and TH meet U and V again, respectively, in L and M, it follows that L and M are both inverses of H, in regard to the circle of centre T, and, therefore, coincide.

Ex. 2. Let A, B, C be any three points; and D, E, F be any three points lying respectively on BC, on CA and on AB. Through the three points F, B, D (beside I, J) a circle can be drawn; through the three points E, C, D another circle. These, beside D (and I, J), will have common a further point O. This point lies on the circle passing through A, E, F. (Miquel, *Liouv. J.*, iii, 1838.)



Denote the points in which the line IJ is met by BC, CA, AB, respectively, by A', B', C'; and the points in which the lines OD, OE, OF, respectively, meet IJ, by D', E', F'. Then the pairs of points I, J; A', F'; C', D' are in involution, that determined on IJ by the conics through the four points F, B, D, O; so that the range C', F', J, I is related to the range D', A', I, J. Again, by conics through E, C, D, O, the pairs I, J; A', E'; B', D' are in involution, and, therefore, the range D', A', I, J is related to the range B', E', J, I, and is, hence, related to the range E', B', I, J. But, then, the ranges, C', F', J, I and E', B', I, J, being related, the pairs I, J; B', F'; C', E' are in involution. This shews that a circle passes through the points A, E, F, O. [Add.]

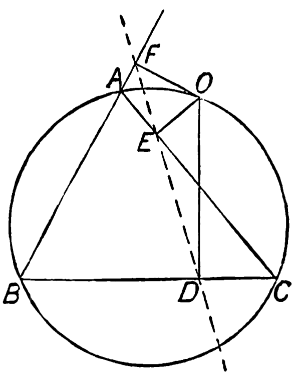
Ex. 3. Wherefore, by supposing the points D, E, F to be in line, we infer that, if four lines be taken in a plane, and, from every three of these lines, by intersections of pairs, a triad of points be defined, then the three circles determined by three of these triads have a common point. Therefore, by symmetry of reasoning, the four circles determined by the four triads have a point in common.



Conversely, let points, D, E, F, lying, respectively, on the three joins BC, CA, AB, be such that the circle ABC contains the point of concurrence, O, of the three circles AEF, BFD, CDE, whose existence is proved in Ex. 2. Then the points D, E, F are in line. For, let the line EF meet BC in P. The point O is determined by the intersection of the circles AEF and ABC, and the circle EPC passes through O. But this circle has, with the circle CDE, the points E, O, C, I, J in common, and thus coincides with it. Hence P coincides with D.

The circle ABC may thus be regarded as the locus of the point of intersection of the circles AEF, BFD, CDE, when D, E, F are determined by a varying line.

Ex. 4. A particular corollary from this is the fact (already proved, Chap. I, Ex. 8) that, if, from a point, O, of the circle ABC, lines be drawn at right angles to BC, CA, AB, meeting these, respectively, in D, E and F, then these points D, E, F are in line. For, the pencil of lines $E(I, J, O, A)$, being harmonic, is related to the pencil $F(I, J, O, A)$. Thus a circle can be drawn through O, E, F, A, I, J. Similarly, circles can be drawn through O, F, B, D and O, D, C, E.



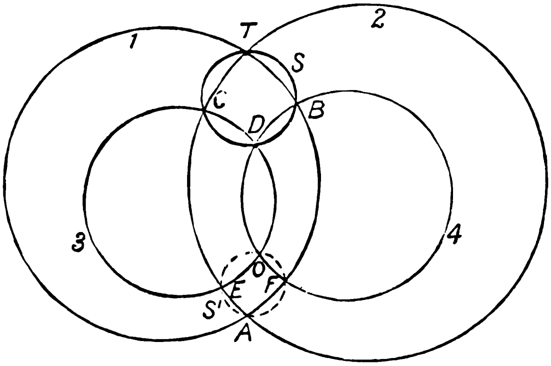
If the lines OD, OE, OF meet the line IJ, respectively, in D', E', F'; and L be the harmonic conjugate, on the line OD, of the point O in regard to D and D', so that D is the middle point of OL; and, similarly, M, N be taken, on OE, OF, respectively, such that E is the middle point of OM, and F the middle point of ON, it follows that L, M, N are in line.

Ex. 5. If, in the preceding example, there be four lines, instead of only the three BC, CA, AB, and O be the point of concurrence of the four circles, determined, as in Ex. 3, by every three of these, there will be obtained four points, L, M, N, P, lying in line. The rule for these is, that the middle point of OL lies on one of the four given lines, the line OL being at right angles to this line.

Now apply the theory of inversion, explained above, to this figure, the circle of inversion being any circle of which O is the centre. The four given lines then invert into circles passing through O; the centres of these circles will (Ex. 4, above, in regard to inversion) be the points obtained by inversion of L, M, N, P. These centres will then lie on a circle passing through O. In the original figure, the point O was obtained as a point of concurrence of four circles; of these, every two have an intersection beside O, which is also an intersection of two of the given lines. These circles will invert into four lines, say a', b', c', d'. The original figure can, then, be obtained, by inversion, from four arbitrary lines a', b', c', d', and an arbitrary point, O.

We thus infer a further property of the four circles considered in Ex. 3 above (given, with other properties, by Steiner, *Ges. Werke*, i, p. 223)—namely, that their centres lie on a circle which passes through O. This result may, also, be regarded as built up from the theorem, similarly obtainable from Ex. 4, by inversion, that, if three circles with a common point be such that the other three points of intersection of the pairs of these lie in line, then the centres of the three circles lie on a circle passing through their point of concurrence.

Ex. 6. In the figure of Ex. 3, we have four lines, and four circles which meet in O. This arose from the figure of Ex. 2, in which there



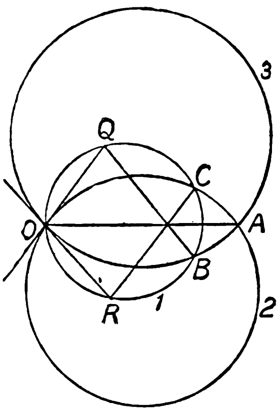
were three lines, and three concurrent circles. There is a generalisation, of importance, in which the lines are replaced by circles.

Let any four circles, which we suppose considered in a definite order, be denoted, respectively, by 1, 2, 3, 4. Let the pairs of intersections (other than I, J), of the successive pairs of these circles, be denoted, respectively, as follows:

for 1, 2 by T, A; for 2, 3 by C, E; for 3, 4 by D, O; for 4, 1 by B, F.

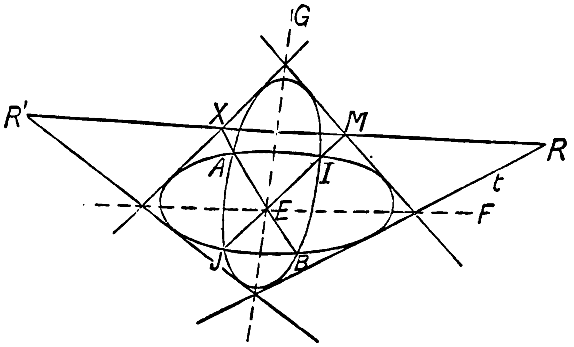
Then it can be shewn that, if the four points T, C, D, B lie on a circle, say S, the four remaining points, A, E, O, F, also lie on a circle, say S'.

This result is obtainable by inversion in regard to any circle of centre T, whereby we are led back to Miquel's theorem of Ex. 2. But, as will be seen below, in dealing with the relations of geometry in two and three dimensions, this is not the simplest way of regarding the theorem.



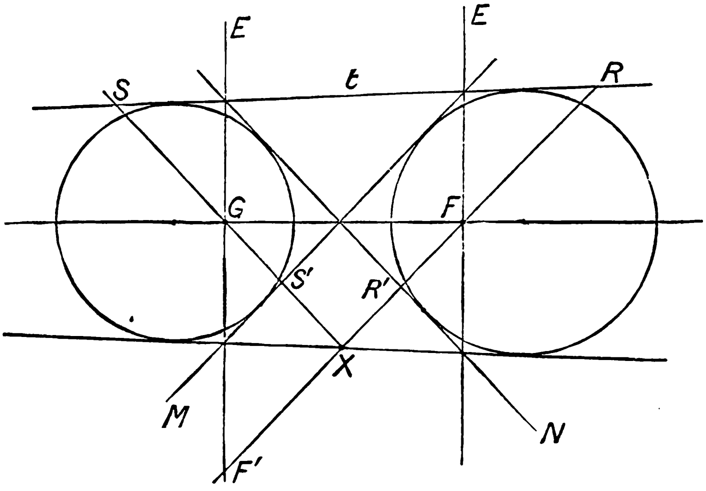
Ex. 7. Three circles, 1, 2, 3, meet in a point O. The tangents at O, respectively of the circles 2 and 3, meet the circle 1 in Q and R; the points of intersection of the circles are A or (2, 3), B or (3, 1), C or (1, 2). Prove that OA, QB, RC meet in a point. (Chap. I, Ex. 34.)

Ex. 8. If a pair of common chords of two conics, which meet in E, meet two common tangents of the conics, respectively, in X and M, the line XM passes through one of the two points, F, which, with E and another point G, form the common self-polar triad of the two conics. The points X, M belong to the involution determined on the line XM by the two conics, the double points of this involution being the point F, and the point of XM which lies on EG; the points, R, R', in which the line XM meets the other two tangents common to the conics, also belong to this involution.



The result follows at once from the fact remarked (Chap. I, Ex. 14), that the figure consists of pairs of points which are harmonic images of one another in regard to the line EG and the point F.

Ex. 9. In the case of two circles, the common self-polar triad consists of the two limiting points, F, G, and the point, E, in which the common chord, AB, meets the line IJ. Let one of the four common tangents of the circles be called t, and consider, with this, the common tangent which intersects t on the line of centres, FG, of the two circles; let X be the intersection of this latter tangent with the common chord AB. Another common tangent passes through the intersection of the line EF with t; let this meet the line IJ in M, the line XM passing through the limiting point F. The remaining common tangent passes through the intersection of the line EG with t, and meets the line IJ, say, in N. Let XM meet the tangent t, and the tangent through N, respectively in R and R'. (Cf. the diagram of Ex. 8.)



If then RX meet EG in F', there is on the line RX an involution of which F and F' are the double points, of which X, M and R, R' are two pairs, and two other pairs are the intersections of the line XR with the two circles. (Ex. 8.)

Now a circle, Ω, can be drawn with centre X, to contain the limiting points F, G, by what we have shewn above; this will cut both the given circles at right angles, and will, therefore, pass through the points of contact with the circles of the common tangent containing X. As M is on the polar, IJ, of X, in regard to this circle, and X, M are harmonic in regard to F and F', this circle passes also through F'. Thus the points R, R' are inverses of one another in regard to this circle, Ω. In the same way, if XG meet the tangent t, and the tangent through M, respectively in S and S', it follows that S, S' are inverses of one another in regard to this same circle, Ω. The lines XF, XG, likewise, meet the given circles in points which are inverses of one another in regard to this circle, Ω.

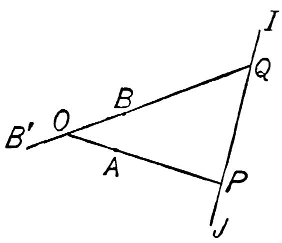
If we now invert the figure in regard to this circle, Ω, both the given circles invert into themselves, but the common tangent t inverts into a circle, passing through the points X, R', S', which touches both the given circles.

Conversely, suppose that the three lines, which appear here as the common tangents containing respectively X, M, N, are given. It can be shewn that there are four circles touching these three lines; the proof has in fact been given (Chap. I, Ex. 17). Also, as F, F' are harmonic in regard to X, M, and XF meets the tangent MS' on IJ, while XG meets the tangent NR' also on IJ, it follows easily that X, R', S' are the mid points, in regard to I, J, of the pairs of points formed by the intersections of the three given lines. Thus the circle $XR'S'$ is determined by these lines. Hence these four circles are all touched by another circle. (See also Chap. I, Ex. 33.)

The present proof (usually based on metrical considerations here not utilised) was given by J. P. Taylor, *Quarterly Journal of Mathematics*, XIII (1875), p. 197; and by Fontené, *Nouvelles Annales*, VII (1907), p. 161. The theorem is considered again, in Chap. III, below.

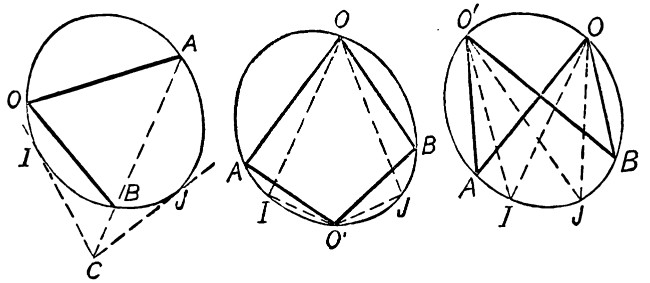
Angle properties for a circle. In the present volume, no use is made of any measure of an angle until Chap. V is reached. In Euclid's treatment of circles great use is made of the fact that the angles in any segment of a circle are equal to one another; it will be convenient to give here a condition under which we may, if we desire, speak of two angles as being equal. In Euclid's theory, if two angles α, β be respectively equal to two angles α', β', then an angle, formed by the *sum* of the two former, is regarded as being equal to the angle formed by the *sum* of the two latter. This is a step which we do not take before Chap. V is reached, the *sum* of two angles being at present undefined.

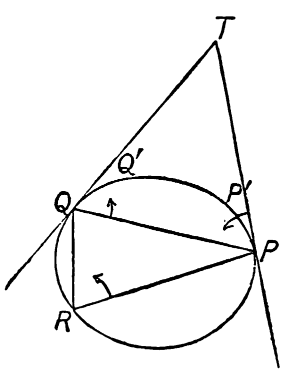
Let OA, OB be two lines, meeting in a point, O; and let $O'A'$, $O'B'$ be two other lines, meeting in O'. We may say that, in regard to the fundamental points I, J, the line OB makes with OA an angle equal to that made with $O'A'$ by $O'B'$, if the pencil $O(B, A, I, J)$ be related to the pencil $O'(B', A', I, J)$. If OB, OA meet the line IJ, respectively, in Q and P, while $O'B'$, $O'A'$ meet IJ in Q' and P', respectively, the ranges (Q, P, I, J), (Q', P', I, J) are then related.



It may be remarked that, when we are dealing with real points, if B' be a point of the line OQ separated from B by O and Q, this provisional definition would render the angle made by OB' with OA equal to the angle made with OA by OB. These, however, are, in general, different from the angle made by OA with OB, or with OB'. The angle made by OB with OA is equal to the angle made by OA with OB, when OA, OB are harmonic conjugates in regard to I, J; that is, in the phraseology here used, when OA, OB are at right angles, in regard to I, J.

From this definition, if the points O, O', A, B lie on a circle, the angle which $O'B$ makes with $O'A$ is equal to the angle which OB makes with OA. And, when the line AB passes through the centre, C, of the circle, in which case the lines joining O to A and B are harmonic conjugates in regard to OI, OJ, the lines OB, OA are at right angles.



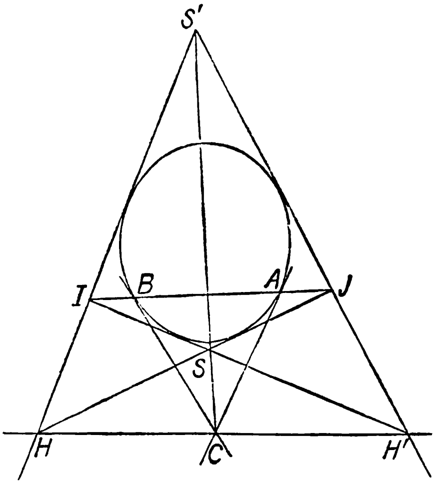


Ex. If the tangents at two points, P, Q, of a circle meet in T, the angle made by QT with QP is equal to the angle made by PQ with PT; and, if R be any other point of the circle, this is equal to the angle made by RQ with RP.

For, if P', Q' be any other two points of the circle, the range P', Q', I, J, of four points of the circle, gives related pencils when joined to any two points of the circle. Thus, the pencils $Q(Q', P', I, J)$, $P(Q', P', I, J)$ are related. Recalling, what has appeared in the definition of a conic, that the tangent contains two coincident points of the conic, we see, by supposing P' to be at P, and Q' to be at Q, that the pencils $Q(T, P, I, J)$, $P(Q, T, I, J)$ are related. Further, the pencils

$Q(Q', P', I, J)$, $R(Q', P', I, J)$ are related, so that each of the preceding pencils is related to $R(Q, P, I, J)$.

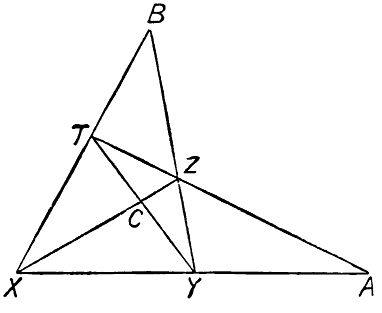
Foci and axes of a conic. The *foci* of a conic, in regard to I and J, are defined as the two pairs of intersections, S, S' and H, H', of the two pairs of tangents to the conic from I, J. By the properties of conics touching four lines (above, p. 24), the three lines SS', HH', IJ form a self-polar triad, the intersection, C, of SS' and HH', being the pole of the line IJ in regard to the conic, that is, the *centre*, in regard to I and J. The lines CSS', CHH' meet IJ in two points which are harmonic conjugates in regard to I and J; these lines are called the *axes* of the conic; they are at right angles. The tangents CA, CB, to the conic, from the centre C, are called the *asymptotes*. Any two lines, drawn through the centre C, conjugate to one another in regard to the conic, that is, harmonic conjugates in regard to the asymptotes CA, CB, are called *conjugate diameters* of the conic. The axes are that single pair of conjugate diameters which are at right angles.



Ex. As a conic can be drawn to have five given tangents, a conic can be drawn to touch the joins BC, CA, AB, of three points A, B, C, and have a given point S for a focus. The tangents, other than IS, JS, drawn to this conic from I and J will intersect one another in a further focus, S', which we may call the focus conjugate to S.

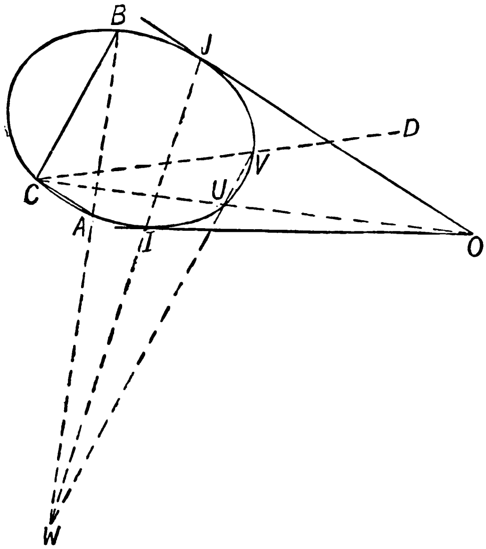
We consider the correspondence of S and S', either of which determines the other. In particular we shew that if S be at the orthocentre of A, B, C, in regard to I, J, then S' is at the centre of the circle A, B, C.

The line pairs AI, AJ; AS, AS'; AB, AC are pairs of tangents drawn from A to particular conics touching the four lines IS, JS, IS', JS'; they are therefore in involution (Chap. I, Ex. 1). Similarly for the pairs of lines joining B to I, J; S, S'; C, A; and for the pairs of lines joining C to I, J; S, S'; A, B. Let the double rays of the first involution, of lines through A, intersect the double rays of the second involution, through B, in the four



points X, Y, Z, T; then it follows, because the lines AYX, AZT, say, are harmonic in regard to AC, AB, and the lines BZY, BTX, say, are harmonic in regard to BA, BC, that the lines XZ, TY, say, meet in C. And, then, as the line IJ is divided harmonically by the line pair AYX, AZT, and also by the line pair BZY, BTX, it is divided harmonically by the line pair XCZ, YCT (above, p. 4); so, likewise, is the line SS'. Wherefore the lines XCZ, YCT are the double lines of the involution containing the line pairs CI, CJ; CS, CS'; CA, CB. Thus the relation of S, S' is that of two points which are conjugate to one another in regard to all conics passing through X, Y, Z, T (as in Ex. 16 of Chap. I), and I, J are conjugate to one another, also, in regard to such conics. Thus when S describes a line, S' describes a conic passing through A, B, C; and conversely.

A point D can be taken so that IJ is divided harmonically by AC, BD and by BC, AD; it will then (above, p. 4) also be divided harmonically by AB, CD. The point D will be the orthocentre of A, B, C, in regard to I, J, and I, J will be conjugate to one another in regard to all conics through A, B, C, D. Let the conic (circle) be drawn through A, B, C, I, J, and let O be the intersection of the tangents of this conic at I, J, the centre of this conic. We shew then that when S is at D, then S' is at O. For let CO, CD meet the conic A, B, C, I, J, again, respectively in U, V, and let BA, JI meet in W. As the points I, J are separated harmonically by the lines AB and CD, while, as points on the conic, U and C are harmonic conjugates in regard to I and J, it follows that VU passes through W. Thus the line

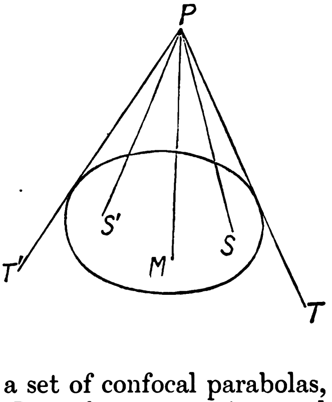


pairs *CI*, *CJ*; *CA*, *CB*; *CO*, *CD* are in involution. By a similar argument the line pairs joining *A* to the point pairs *I*, *J*; *B*, *C*; *O*, *D* are in involution; as, also, those joining *B* to *I*, *J*; *C*, *A*; *O*, *D*. This shews that *O*, *D* are related as was stated.

Confocal Conics. A system of conics having the same foci is called a confocal system. It is the same as a system of conics with four common tangents, of which two meet in the point *I*, and the other two in the point *J*. The four foci are in pairs, on two lines which meet in the centre of the conic; if one of these pairs be common to two conics, so are the other pair.

If the two pairs of foci be denoted, as above, by *S*, *S′* and *H*, *H′*; and if *PT*, *PT′* be the tangents, to one conic of the confocal system, drawn from an arbitrary point *P*, it is known (above, p. 25) that the four pairs of lines *PT*, *PT′*; *PI*, *PJ*; *PS*, *PS′*; *PH*, *PH′*, are in involution. The double lines of this involution are harmonic in regard to the lines of each pair; in particular in regard to *PI*, *PJ*, and so are at right angles. These double lines are, however, the tangents at *P*, of the two conics which can be drawn, through *P*, to touch the four lines *IS*, *IS′*, *JS*, *JS′* (above, p. 25). Thus, through any point *P* can be drawn two conics of the confocal system, and these cut one another at right angles, at the point *P*. It can be shewn, similarly, that they also cut at right angles at each of their other three points of intersection.

Further, if we refer again to the provisional phraseology in regard to equal angles, the angle made by *PS′* with *PT′* is equal to the angle which *PT* makes with *PS*. For, as the pairs of lines, *PS*, *PS′*; *PT*, *PT′*; *PI*, *PJ* are in involution, the pencils *P* (*S′*, *T′*, *I*, *J*), *P* (*S*, *T*, *J*, *I*) are related, and the second is related to *P* (*T*, *S*, *I*, *J*). Also, if one of the double rays of this involution be *PM*, the pencils *P* (*M*, *S′*, *I*, *J*), *P* (*S*, *M*, *I*, *J*) are related; so that the angle made by *PM* with *PS′* is equal to the angle made with *PM* by *PS*. In particular, when *P* is on the conic, the angles *S′PM*, *MPS* are equal.



A particular set of confocal conics is a set of confocal parabolas, that is, conics all touching the line *IJ* at the same point, and having two other common tangents, *IS*, *JS*. [Add.]

Common chords of a circle with a conic. If the axes of a conic meet the line *IJ* in *X* and *Y*, as the axes meet in the pole of the line *IJ* and are at right angles and conjugate in regard to the conic, the points *X*, *Y* are the double points of the involution determined by the two pairs consisting of *I*, *J* and the points, *F*, *F'*, in which the conic meets *I*, *J*; this is the involution determined on the line *IJ* by the conic and any circle. If, then, a pair of chords, *LM*, *L'M'*, of the conic and the circle, meet *IJ* in *P* and *P'*, the pair *P*, *P'* will belong to this involution. Wherefore, the ranges, *P*, *X*, *I*, *J* and *X*, *P'*, *I*, *J*, are related. Thus, the angle made by the common chord, *LM*, of a conic, and any circle, with either axis, *CX*, is equal to the angle made by this axis with the complementary common chord, *L'M'*, of the conic and circle.

The director circle of a conic. The director circles of conics touching four lines are coaxial. This is the conic which has been defined (Chap. I, Ex. 3) as the director conic in regard to *I* and *J*. From its definition it is the locus of points from which the two tangents to the conic are at right angles; it passes through the points in which the conic is met by the polar of any one of the four foci; and its centre coincides with the centre of the conic. As has been shewn (Chap. I, Ex. 25), if, through three points which form a self-polar triad in regard to the conic, the circle be drawn, this cuts the director circle at right angles.

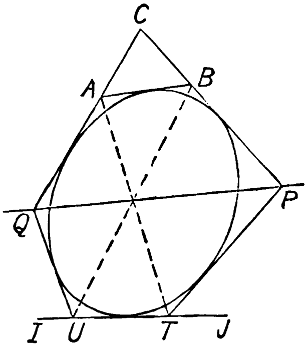
Further, the director circles of all conics which touch four given lines have two points in common, beside *I* and *J*, that is, are a coaxial system of circles. For, let *P* be one of the intersections of the director circles of two particular conics touching these four lines. The pairs of tangents from *P*, to all these conics, are a pencil in involution (above, p. 25); and these meet the line *IJ*, which is supposed to have no special relation to the four lines, in pairs of points in involution. Of this involution, however, the points *I*, *J* are the double points; for, the tangents from *P* to each of the two selected conics are at right angles, and an involution is determined by two of its pairs. Wherefore, the tangents from *P* to any other of the conics are at right angles, and *P* lies on the director circle of this conic. From this, the statement made follows.

Particular, degenerate, conics touching the four lines, are the point pairs, three in number, through which the lines pass in pairs. If *A*, *C* be such a pair, the lines *PA*, *PC* are the tangents to this from a point *P*, and the corresponding director circle is the locus of *P* when the lines *PA*, *PC* are at right angles. The centre of this circle is the middle point of *AC* in regard to *I* and *J*. If *B*, *D*, and *E*, *F*, be the other point pairs, the middle points of *AC*, *BD*, *EF* lie on a line, the line containing the centres of all the coaxial director circles, which are the centres, the poles of *IJ*, in regard to the conics touching the four lines.

One conic can be drawn touching the four lines which also touches *IJ*. The director circle of this, as was remarked, degenerates into the line *IJ* and another line; if the other two tangents which can be drawn, from I, J, to this conic, meet in S, the line in question is the polar of S in regard to this conic. (Cf. Ex. 5 of Chap. I.) This line is then the radical axis of all the director circles, and is at right angles to the line of centres of the conics. [Add.]

Parabolas. The directrix of a parabola. A conic touching the line IJ is called a *parabola*. If this touch IJ in S', and if S be the intersection of the tangents from I, J, the four foci of the parabola are S, S' and I, J. The polar of S in regard to the parabola is called the *directrix*; it was this name which suggested, to Gaskin, to call the director circle of any conic, the *director*. As a particular case of what has been said, since a line cuts a circle at right angles only if it passes through the centre of the circle, it follows that the circle drawn through three points, which are a self-polar triad in regard to a parabola, has its centre on the directrix of the parabola.

The directrix of a parabola also passes through the orthocentre of any three points whose joins are three tangents of the parabola. This follows at once from Brianchon's theorem. For let A, B, C be three points such that BC, CA, AB touch the parabola; let CB, CA, respectively, meet the directrix in P and Q; let the other tangents from P, Q, to the parabola, meet IJ, respectively, in T and U. As then, by definition, the tangents, PT, CB, are at right angles, the line AT is the line drawn from A at right angles to BC. So BU is the line drawn from B at right angles to AC. The point of intersection of these lines is the orthocentre of A, B, C (above, p. 64); by Brianchon's theorem (Chap. I, Ex. 2) this point of intersection is on the directrix PQ. Conversely, it may be seen that a parabola can be drawn, to touch three given lines, to have for directrix any line through the orthocentre of the points formed by the intersections of the lines.



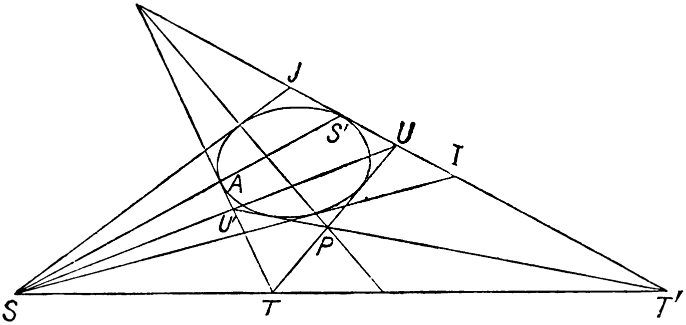
Examples in regard to a parabola. *Ex.* 1. If A, B, C be three points such that the lines BC, CA, AB touch the parabola, the circle through A, B, C contains also the focus, S, of the parabola. For, as the lines joining S to the points I, J are, by the definition of S, tangents of the parabola, and the line IJ is also a tangent, this is the theorem (Chap. I, Ex. 7) that, when the six lines, which are the joins of the pairs of each of two triads of points, touch a conic, the two triads are points of another conic.

Ex. 2. If we have four lines, any three of these, by their intersections in pairs, give rise to a triad of points, through which there passes a circle. The four circles so obtained meet in a point, namely the focus of the parabola which can be drawn to touch the four lines (beside the line IJ). This is a result obtained above (p. 71). It will be seen later that the theorem is equivalent to Moebius' theorem of the in- and circumscribed tetrads (Vol. I, p. 62).

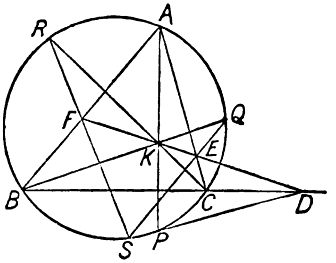
A conic, S, containing a triad of points whose joins touch another conic, Σ (and, therefore, an infinite number of such triads), may be spoken of as triangularly circumscribed to Σ. Thus, any circle passing through the focus of a parabola is triangularly circumscribed to the parabola.

Ex. 3. Let the line joining the focus, S, of a parabola, to the point, S', in which the curve touches IJ, meet the curve again in A. Then A may be called the vertex. It can be shewn that the line drawn through the focus, S, at right angles to any tangent of the curve, meets this tangent at a point lying on the tangent at the vertex, A.

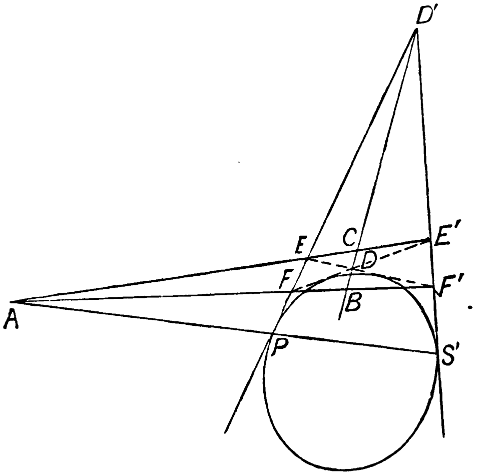
For, take any point P of the directrix, the polar of S. Draw from P the two tangents of the curve; let these meet the line IJ, respectively, in U and T', and meet the tangent at A, respectively, in T and U', so that, as is easy to prove, S, U, U' are in line, and S, T, T' are in line. Then U, T' are harmonic conjugates in regard to I and J. That is, the line ST, drawn from S at right angles to the tangent TU, meets this tangent in a point, T, of the tangent at the vertex. (Cf. Ex. 5, p. 28.)



Combining with the last example, we infer that the lines drawn from the Miquel point of four lines, there obtained, each at right angles to one of the lines, meet these lines respectively in points which are in line. Also, the orthocentres of the four triads, determined by each set of three of the lines, are in line. Further, the pedal line of a point, O, of a circle, in regard to three points of the circle, of which the orthocentre is K, passes through the middle point of OK. (Cf. Ex. 8, of Chap. I; and Ex. 4, above, regarding the circle.)



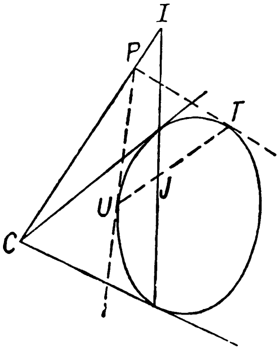
Ex. 4. Any line drawn through the orthocentre, K, of three points, A, B, C, of a circle, meets BC, CA, AB, respectively, in the points D, E, F, and the lines AK, BK, CK meet the circle again, respectively, in P, Q, R. Prove that the lines DP, EQ, FR meet in a point, S, of the circle. And that this is the focus of the parabola which, as has been remarked, can be described to touch BC, CA, AB and have the given line as directrix.



Ex. 5. The joins of three points, A, B, C, which form a self-polar triad in regard to a conic, meet any tangent of the conic, respectively, in D', E', F'; on these joins are taken D, E, F such that D, D' are harmonic conjugates in regard to B, C, and so on. Prove that the lines EF, FD, DE are tangents of the conic. Thus, if A, B, C be a self-conjugate triad in regard to a parabola, the circle through the middle points of BC, CA, AB, contains the focus of the parabola.

Definition of a rectangular hyperbola. A conic which meets the line IJ in two points which are harmonic conjugates of one another in regard to I and J, is called a *rectangular hyperbola* in regard to I and J. In particular, a degenerate rectangular hyperbola consists of two lines which are at right angles. A conic which touches the line IJ, at one of the two points I, J, is both a parabola and a rectangular hyperbola. [Add.]

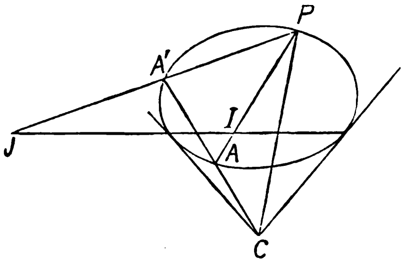
By considering the involution of pairs of points determined on the line IJ by the conics which pass through the four intersections of two rectangular hyperbolas, we see that all these conics are rectangular hyperbolas. In particular, if K be the orthocentre of a triad of points, A, B, C, all conics through A, B, C, K are rectangular hyperbolas; conversely, if any rectangular hyperbola drawn through A, B, C meet the line, which is drawn through A at right angles to BC, in the further point K, the line BK is at right angles to CA; thus any rectangular hyperbola drawn through A, B, C also contains the orthocentre.



The centre, C, of a rectangular hyperbola, and the points I, J, form, by definition, a self-polar triad in regard to the rectangular hyperbola. The lines PI, PJ, when P is any point of the line CI, are thus conjugate in regard to the curve, and are, therefore, harmonic in regard to the tangents, PT, PU, which can be drawn from P to the curve. These tangents are therefore at right angles. Thus we see that the director circle of a rectangular hyperbola degenerates into the line pair joining its centre to the points I and J.

Examples in regard to a rectangular hyperbola. *Ex.* 1. Any pair of lines CP, CD, drawn through the centre, C, of a rectangular hyperbola, which are conjugate lines in regard to the curve, are harmonic in regard to the tangents of the curve, CL and CM, at the points, L and M, where the curve meets the line IJ. In particular CI, CJ are such a pair of conjugate lines. If CP meet the curve in P, and the tangent at P meet CL, CM in L' and M', then P is the middle point of $L'M'$ in regard to IJ. Also the angle which CD makes with CL is equal to the angle which CL makes with CP.

Ex. 2. If the lines joining any point, P, of a rectangular hyperbola, to the points I and J, meet the curve again, respectively, in A and A', the line AA' passes through the centre, C, of the curve, and is at right angles to the line CP. In fact the lines $A'I$, AJ, PC meet in a point of the curve.



Further, if the tangents of the conic at A and A' intersect in T (which will lie on IJ), and any line through T meet the rectangular hyperbola in M and N, prove that PM, PN are at right angles, in regard to I and J.

In general, for any conic, not necessarily a rectangular hyperbola, if through a fixed point, P, of the conic, lines be drawn at right angles, meeting the conic again, respectively, in M and N, the pairs M, N, so arising, belong to an involution of points of the conic.

Thus MN passes through a fixed point. It can be seen that this is on the line, drawn through P, at right angles to the tangent at P.

Ex. 3. The centre of a given rectangular hyperbola is R, and O is any point of the curve. A circle is drawn, with centre O, to cut the curve in A, B, C, D. If we form the polar reciprocal of the rectangular hyperbola in regard to the circle, that is, the envelope of the polars, in regard to the circle, of the points of the rectangular hyperbola, prove that the curve obtained is a parabola touching the tangents of the circle at A, B, C, D, whose focus is the pole, in regard to the circle, of the line through R at right angles to OR, that is, the inverse of R in regard to the circle, the directrix of the parabola passing through O.

Auxiliary circle of a conic. We now explain a method by which we may pass from any conic to another, or conversely from this other to the original, the deduction in the latter case being exactly the dual of the former.

Let Ω be any conic, and C, S any two fixed points, the line CS meeting Ω in A and A'. Let R be any variable point of Ω. Let CR and SR, respectively, meet Ω again in Z and Z'.

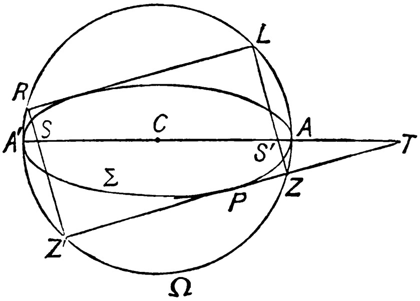
As then Z, R are pairs of an involution of points on the conic Ω, the range (Z) on the conic is related to the range (R). This in turn is, similarly, related to the range (Z'). Wherefore, by a result proved above (Ex. 28, Chap. I), the line ZZ' touches another conic, say, Σ. The points A, A' are the common corresponding points of the two ranges (Z), (Z'), and the conic Σ touches Ω at A and A'. If P be the point of contact of ZZ' with Σ, and T the intersection of ZZ' with AA', it is easy to see that P, T are harmonic, both with respect to Z, Z', and also with respect to the points in which ZZ' meets the tangents at A and A'.

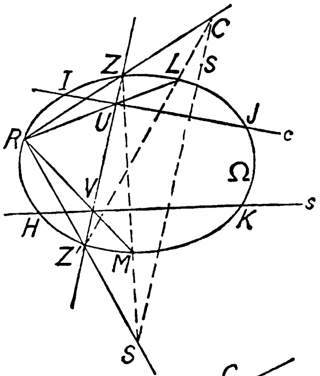
Now denote by c the polar of C in regard to Ω, and by s the polar of S; let c meet Ω in I, J, and s meet Ω in H, K. It can at once be proved that the conic Σ touches CH, CK at the points where these meet the line c, and touches SI, SJ at the points where these meet the line s.

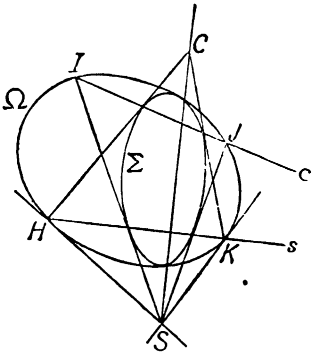
If the line ZZ', which we denote by r, meet the lines c, s, respectively, in U and V, and RU, RV meet the conic Ω in L and M, respectively, C, L, Z' will be in line, as, also, S, M, Z will be in line. The line RL is then a tangent of the conic Σ, being the position of ZZ' when R is at Z'; for a like reason, RM is a tangent of Σ.

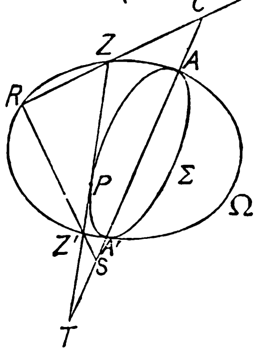
Hence, if the conic Σ be given, and the lines c and s, the conic Ω can be defined by finding the points, U, V, where a variable tangent, r, of Σ, meets c and s, drawing from U, V the tangents (other than r) UL, VM, and finding the locus of their intersection, R. This is exactly the dual of the construction by which Σ is found when Ω

and the points C, S, are given. Further, as the line $Z'L$ passes through the pole of IJ in regard to Ω, the points Z', L are, as points of the conic Ω, harmonic conjugates in regard to the points I, J of Ω; thus, the line SR is the line through S drawn to the point, of the line IJ, which is the harmonic conjugate, in regard to I and J, of the point, U, where the tangent RL, of Σ, meets the line IJ. If we suppose the points S, I, J to be given, this shews how we may determine either of the conics Σ, Ω when the other is given. The lines $Z'R$, LZ meet on IJ; thus, if we take the point, S', of SC, which is the harmonic conjugate of S, in regard to C and the point where SC meets IJ, then the lines $S'LZ$ and RL will meet the line IJ in points which are harmonic conjugates of one another in regard to I and J.





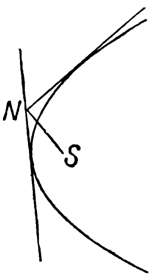




If we now regard I, J as the absolute points, the conic Ω will be a circle, of which C is the centre. The conic Σ will then also have C for centre, and S, S' for foci. The circle, Ω, is, then, the locus of the foot of the perpendicular drawn from S (or from S'), to a tangent of Σ; or, conversely, Σ is thereby determined from Ω.

The circle Ω is that generally called the auxiliary circle of the conic Σ.

Ex. 1. If we be given a fixed point, S, and a line; and from a variable point, N, of this line, the line be drawn at right angles to SN, then this line envelopes a parabola, which touches the given line.



Ex. 2. If two conics, Σ, Ω, have two points of contact, and a variable tangent of one of these, Σ, meet Ω in P and Q, the further tangents from P and Q, drawn to Σ, meet in a point T, whose locus is a further conic touching both of Σ and Ω at their two points of contact.

Ex. 3. If A, B, C, O be four points, and the lines drawn from O at right angles to BC, CA, AB meet these, respectively, in D, E, F, the circle D, E, F is generally called *the pedal circle* of O in regard to A, B, C. It is obviously the auxiliary circle of the conic which can be drawn with O as focus to touch BC, CA, AB.

Let any conic be given, on which O is a fixed point, and A, B, C are variable points. We prove that the pedal circle passes through a fixed point.

For this, let the lines OI, OJ meet the given conic again, respectively, in M and N. There is then, as the six points O, M, N and A, B, C are on the conic, another conic touching BC, CA, AB, OM, ON and MN (Chap. I, Ex. 7). This will be the same as the conic having O for focus which touches BC, CA, AB; which conic, therefore, touches MN. Therefore, the pedal circle of O in regard to A, B, C passes through the fixed point in which MN is met by the line drawn from O at right angles to MN.

In particular, we have remarked (Ex. 2, p. 84) that, when the conic is a rectangular hyperbola, this point is the centre. Thus the pedal circle of any point of a rectangular hyperbola, in regard to any other three points of the curve,. passes through the centre of the rectangular hyperbola.

When the conic is a circle, or passes through the points I, J, it has been seen (Chap. I, Ex. 8, and p. 71, above) that the pedal circle, defined with reference to I and J, degenerates into a line, together with the line IJ. It can, however, be shewn, in this case (see below, Chap. III, p. 139), that, if we consider three variable points, A, B, C, of the circle, which are a self-polar triad in regard to another fixed conic (which must then be taken so that this is possible, being inpolar to the circle; see Chap. I, Ex. 11), then the pedal line of a fixed point of the circle in regard to these points A, B, C, passes through a fixed point.

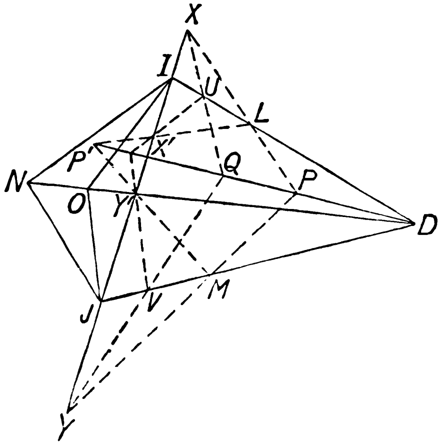
The pedal circle of a point O, in regard to three points A, B, C, being the auxiliary circle of a conic, having O for focus, which touches *BC*, *CA*, *AB*, also arises as the pedal circle, in regard to *A*, *B*, *C*, of another point, *O'*, the conjugate focus of this conic. We have shewn above (p. 78), that when *O* describes any conic passing through *A*, *B*, *C*, then *O'* describes a line; and that, when *O* is at the orthocentre of *A*, *B*, *C*, then *O'* is at the centre of the circle through *A*, *B*, *C*. Thus, when the point *O* is on a rectangular hyperbola through *A*, *B*, *C*, which then also passes through the orthocentre (above, p. 84), the point *O'* lies on a line through the centre of the circle *A*, *B*, *C*. We thus have the result, that, if a point *O'* describe any definite line passing through the centre of the circle *A*, *B*, *C*, the pedal circle of *O'* in regard to *A*, *B*, *C*, passes through a fixed point, which is in fact the centre of the rectangular hyperbola through *A*, *B*, *C* which corresponds to the line. It will be seen, below, that this point is the middle point, in regard to *I*, *J*, of the two points consisting of the orthocentre, and the fourth intersection of the hyperbola with the circle *A*, *B*, *C*.

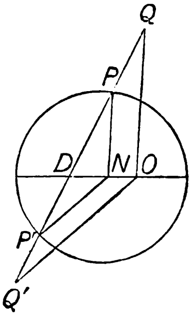
Ex. 4. If a circle be described, with centre at a point, *O*, of a rectangular hyperbola, to meet this in *A*, *B*, *C*, *D*, and *DO* meet the circle again in *S*, prove that the pedal line of *S*, in regard to *A*, *B*, *C*, passes through the middle point of *O*, *S*. Hence shew that the directrix of the parabola, described with focus at *S* to touch *BC*, *CA*, *AB*, passes through *O*; and obtain this result by the transformation referred to above.

Ex. 5. Prove that to any point *P* of the conic Σ there corresponds a point *P'* of the conic Ω such that *CP'*, *SP* meet on the line *IJ*, while *CP*, *SP'* meet on the line *HK*. Taking two arbitrary points, *C*, *S*, and two arbitrary lines *IJ*, *HK*, investigate the correspondence between an arbitrary point *P* and the point *P'* determined by these two conditions. (Boscovich, *Sectionum Conicarum Elementa nova quadam methodo concinnata*, *Elementa Universae Matheseos*, Venetiis, 1757. See Taylor's *Ancient and Modern Geometry of Conics*, p. 3, etc.)

The locus of the centres of rectangular hyperbolas passing through three given points. We considered in Chap. I (Ex. 18) the locus of poles of a given line in regard to all conics passing through four given points, say, *A*, *B*, *C*, *D*. We found that it is a conic whose intersections with the line are two points, *I* and *J*, which are conjugate in regard to all conics passing through *A*, *B*, *C*, *D*; and we specified nine other points thereon, namely, first, the three intersections of the line pairs passing through these four points, and, second, the points, each on the join of two of these points, harmonic, in regard to these two points, to the point in which the join meets the given line *IJ*. When we take the points *I*, *J* as absolute points, each of the points *A*, *B*, *C*, *D* is the orthocentre of the other three, and the conics passing through these four points are rectangular hyperbolas. We thus have the result that, the locus of the centres of the rectangular hyberbolas which pass through three given points A, B, C (and therefore, also, through the orthocentre, D, of these) is a circle, which contains the three points where AD, BD, CD, respectively, meet BC, CA, AB, and contains, also, the middle points, in regard to the line IJ, of each of the six pairs of points, BC, CA, AB, AD, BD, CD. This is usually called the nine points circle of A, B, C. It is, however, equally the nine points circle of any three of the four points A, B, C, D.

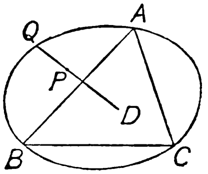
The nine points circle can however be defined as the locus of the middle point, in regard to the line IJ, of the points D, Q, where Q is any point of the circle A, B, C, the point D being the orthocentre of A, B, C.



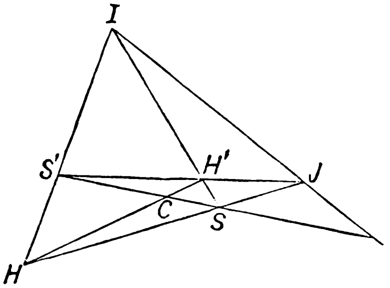


It is, in fact, the case that if a variable line be drawn, through a fixed point, D, to meet a given circle in P and P', and a point Q be taken on DP, such that P is the middle point of DQ in regard to the line IJ, then the locus of Q is another circle. This contains the point, Q', of the line DPP', which is such that P' is the middle point of DQ'; and, if the centre of the given circle be N, the tangents of the new circle at I, J meet in a point O on ND such that IO, ID are harmonic in regard to IJ and IN, namely N is the middle point of DO. To prove the result, let DI, DJ meet the original circle in L, M, respectively; take U harmonically conjugate to D, in regard to I and L, and take V similarly on DJ, in regard to J and M. Then QU, PL meet, in X, on IJ; and QV, PM meet, in Y, on IJ. Thus the pencil, of fixed centre U, denoted by $U(Q)$, is related to the range (X), and, thence, to the pencil $L(P)$; this last is related to $M(P)$, and, thence, to $V(Q)$. Thus the locus of Q is a conic containing U and V. In the same way UQ', LP' meet, in X', on IJ, and VQ', MP' in Y'; and the proof of the statement may be completed.

If the given circle be the nine points circle of A, B, C, of which D is the orthocentre, the point Q, as P describes the nine points circle, describes a circle. We have seen however that the nine points circle passes through the middle points of DA, DB, DC; the new circle thus passes through A, B, C. Thus the suggested definition of the nine points circle is justified; and we see that its centre, N, is the middle point of D and O, where O is the centre of the circle A, B, C. The point P, of the nine points circle, is the centre of a rectangular hyperbola containing A, B, C; this hyperbola, as it contains D, also passes through Q. Three other points of this hyperbola can be assigned, lying respectively on AP, BP, CP. [Add.]



The polars of a point in regard to a system of confocal conics touch a parabola. The theorem obtained from the dually corresponding figure deserves mention also. The polars of a point in regard to a system of conics touching four given lines envelope a conic, of which eleven tangents can be easily assigned.



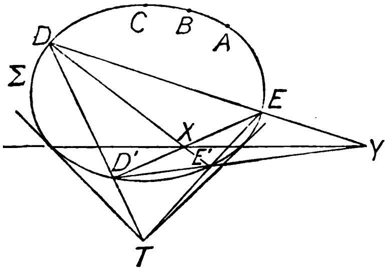
Let the point pairs through which the four lines pass be the absolute points I, J, together with the pairs S, S' and H, H'. The conics are then a system of confocals in regard to I and J, whose pairs of conjugate foci are S, S' and H, H', their common centre being the intersection, C, of the axes SS' and HH'. Denote the fixed point by O.

Then three of the eleven tangents of the envelope are, by what has been seen above, the line IJ, and the lines CS, CH; thus the envelope is a parabola touching the axes of the confocal conics. And, as these axes are at right angles, in regard to I and J, the directrix of the parabola passes through the common centre, C. Two other tangents of the envelope are the tangents at O of the two confocals which pass through O; these tangents, being harmonic in regard to OI, OJ, are at right angles, and the directrix of the parabola passes through O. Other six tangents of the envelope are lines passing through the points I, J, S, S', H, H', in which the four given lines intersect; to find the line in question through one of these six points, we are to join this point to O, and take the harmonic conjugate of this join, in regard to the two of the four given lines which meet in the point selected. In particular, let Q be the point such that, IQ, IO are harmonic in regard to IS, IH, while JQ, JO are harmonic in regard to JS', JH; then QI, QJ are tangents of the envelope, and Q is the focus of this. This point Q is the conjugate of O in regard to all conics through the four foci S, S', H, H'.

Now let Σ denote a particular one of the confocal conics, and let one of the four common tangents, of Σ and the parabola, touch Σ in P; as this tangent touches the parabola there is one of the confocal conics in regard to which O is the pole of this line. On the other hand, the poles of this tangent in regard to all the confocals lie on a line, through P, at right angles to the tangent, this new line being conjugate to the tangent in regard to all the confocals, and, in particular, in regard to the point pair I, J. (Chap. I, Ex. 15.) Wherefore, OP is the line drawn through P at right angles to the tangent at P, which is called the normal at P. It is thus possible to draw through O four normals to the particular conic Σ.

If we take the poles, in regard to this particular conic Σ, of all the tangents of the parabola, these lie upon another conic, the polar reciprocal (p. 47) of the parabola in regard to Σ. This new conic evidently contains the four points of Σ, such as P, whereat the normal of Σ passes through O. It is easy to see that it is a rectangular hyperbola, containing the centre C of all the confocal conics, and the point O, and the points of the line IJ which lie upon the axes, CS, CH, of the confocal conics. The lines CS, CH, IJ are in fact tangents of the parabola, and their poles, in regard to Σ, are three of the points spoken of; the point O is, by definition of the parabola, the pole of a tangent of this, in regard to Σ. That the four normals to a conic, Σ, from a point O, can be constructed by this rectangular hyperbola, was known to Apollonius (Conics, Lib. V, Props. 58–63); we consider the theorem again below (p. 93).

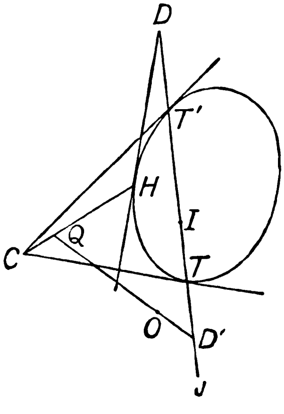
A particular theorem. For clearness we give here a particular result which will be found to be of interest.



Let A, B, C, D be any four points, and X, Y two points which are conjugate to one another in regard to all conics passing through these. Take one such conic, say, Σ, and let T be the pole of XY in regard to Σ. Let TD meet Σ again in D'. We prove that the six points A, B, C, X, Y, D' lie on a conic. The theorem has already been proved (above, p. 89). For, with X, Y as the absolute points, the point D is the orthocentre of A, B, C, and the conic Σ is a rectangular hyperbola of which T is the centre; the conic A, B, C, X, Y is the circle A, B, C, which was proved to meet the rectangular hyperbola in a point D', such that T is the middle point of DD' in regard to the line XY, as here. But we give another proof. If the lines YD, YD' meet the conic Σ, again, in E and E', respectively, the lines DD', EE' meet on the polar of Y, in T; and the lines DE', $D'E$ meet on the polars of Y and T, at X. As DB, AC meet the line XY in two points harmonic in regard to X and Y, as, likewise, do DC, AB, the pencils, $D\,(X, Y, B, C)$ and $A\,(X, Y, C, B)$, are related; the former is the pencil $D\,(E', E, B, C)$, which is related to the pencil $D'\,(E', E, B, C)$ or $D'\,(Y, X, B, C)$, which is related to $D'\,(X, Y, C, B)$. This is therefore related to $A\,(X, Y, C, B)$. Thus the points D', A, X, Y, C, B lie on a conic.

Conversely, let two conics, Σ, Σ', intersect in A, B, C, D', and a chord, XY, of one conic, Σ', meet the other conic, Σ, in two points harmonic in regard to X and Y; and let the line joining the pole, T, of this chord, in regard to Σ, to one of the four common points, D', of the two conics, meet Σ again in D. Then the points X, Y are conjugate to one another in regard to all conics through A, B, C, D. Wherefore, if I, J be any two points of the line XY which are harmonic in regard to X and Y, a conic can be drawn through A, B, C, D, I, J.

From this, if we regard I, J as the absolute points, in which case the conic Σ' is a rectangular hyperbola, we have the theorem that, if a conic, Σ, of centre T, be met by a rectangular hyperbola, Σ', which meets the line IJ in two points which are conjugate in regard to the conic Σ, and meets Σ in A, B, C, D'; and D be the other point in which TD' meets Σ; then the four points A, B, C, D lie on a circle. We shall see that this result is, particularly, of interest when A, B, C are such that the rectangular hyperbola contains the centre, T.

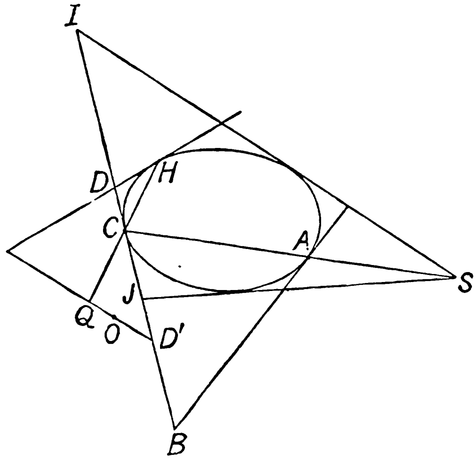


The normals of a conic which pass through a given point. Let O be any fixed point, and H a variable point of a given conic. Let C be the centre of the conic. If through O the line be drawn which is at right angles to the tangent at H, and this line meet CH in Q, we consider the locus of Q as H varies. If QO meet the line IJ in D', and the tangent at H meet the line IJ in D, so that D, D' are harmonic conjugates in regard to I and J, the pencil $O(Q)$, or $O(D')$, is related to the range (D), on the line IJ. The line CH, through the pole, C, of the line IJ, and through the point of contact, H, of a tangent from D, is, however, the polar of D; the line CH meets IJ in a point which is the harmonic of D in regard to the points, T and T', in which the line IJ meets the conic. Thus the pencil $C(Q)$, or $C(H)$, is related to the range (D), and, hence, to the pencil $O(Q)$. Therefore the locus of Q is a conic, passing through the centre C, and the point O. If the axes of the conic meet IJ in A and B, these points are harmonic conjugates both in regard to T, T' and in regard to I, J. Thus, when D is at A, the line CH goes through B, and the point D' is at B; in this case Q is at B. The conic locus of Q is thus a rectangular hyperbola, containing both A and B.

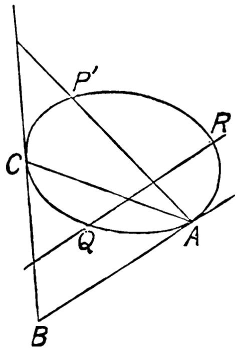
At a position of Q which is on the given conic, and coincides with H, the line OQ is at right angles to the tangent of the given conic at Q, namely is the normal at Q. The four normals of the given conic, which pass through O, are then the normals of this conic at the points where it is met by the rectangular hyperbola.

It follows from what is said in the last section that, if these be the points Q_1, Q_2, Q_3, Q_4, and CQ_4 meet the given conic again in Q', then the points Q_1, Q_2, Q_3, Q' lie on a circle.

The preceding construction remains valid when the given conic is a parabola, touching the line IJ at C. If then, S being the focus, the line SC meet the parabola again in A, and the tangent at A meet the line IJ in B, the locus of Q passes through O and C, but also passes through B, and has the line CS for its tangent at C. As B is the pole of CS, and C, B are harmonic conjugates in regard to I, J, the locus of Q is still a rectangular hyperbola. To see that it passes through B, we have only to take H at C; then D is at C, and D' is at B; to see that it has CS for its tangent at C, we have only to take H at A, in which case D is at B and D' is at C; so that the ray CA of the pencil $C(Q)$ corresponds to the ray OC of the pencil $O(Q)$. In this case, one of the four normals through O is always OC; and there are three others.



If OP, OQ, OR be these normals, it can be shewn, in this case, that the circle P, Q, R passes through A. For take P' on the given parabola, such that AP' and QR meet the line IJ in points which are harmonic in regard to C and B. As the parabola itself, touching IJ at C, has also this property, it follows, by considering the involution on the line CB by all conics through A, Q, R, P', that each of the line pairs AQ, RP' and AR, $P'Q$ also meets IJ in a pair of points which are harmonic in regard to C and B. Thus the pencil $P'(C, B, R, Q)$ is related to the pencil $A(C, B, Q, R)$; this, however, is related to the pencil of lines joining any point of the parabola to C, A, Q, R, and, thus, to the pencil $C(B, A, Q, R)$, and, therefore, to the pencil $C(A, B, R, Q)$. As, then, the pencils $P'(C, B, R, Q)$, $C(A, B, R, Q)$ are related, we can infer that a conic constructed to pass through P', Q, R, C, B has CA for its tangent at C. Conversely, then, the rectangular hyperbola containing the points of the parabola whereat the normals pass through O, having been shewn to contain Q, R, C, B and to have CA for its tangent at C, must pass through P'; so that P' is the same as P. We thus infer that any conic through P, Q, R, A meets the line IJ in points which are harmonic in regard to C and B; or, conversely, that the conic through P, Q, R, and any two points of the line CB which are harmonic in regard to C and B, passes through A. Of such conics, the circle P, Q, R is one; for we have seen that C, B are harmonic in regard to I and J.

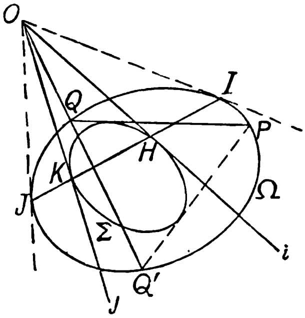


Ex. 1. Prove that an arbitrary given line is a normal of one conic of a given confocal system. If this line be normal at P, and meet this conic again in P', and if it meet another conic of the confocal system at Q and Q', prove that the normals of this other conic, at Q and Q', meet on the tangent at P', of the former conic.

Ex. 2. The dual conception to a normal of a conic is found by taking two arbitrary fixed lines i and j, and finding, upon a variable tangent of the conic, which touches this at P, the point P', which is the harmonic conjugate of P in regard to i and j. Prove that the locus of P' is a curve which meets an arbitrary line in four points, passes twice through the point of intersection of the lines i, j, and twice through each of two other points, which lie on the polar, in regard to the conic, of this point of intersection.

The dual of the director circle of a conic. The tangents to a conic from a point of the director circle are such as meet the fixed line IJ in a pair of an involution of points on this line, of which I, J

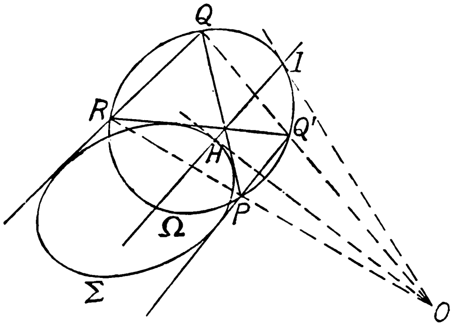
are the double points. Dually, if, from a given point, be drawn a pair of lines, belonging to an involution, and one of the points of intersection, with a given conic, of one of these lines, be joined to one of the intersections of the other line, then the join of the two intersections envelopes another conic. If i, j be the double rays of the involution, the tangent of this second conic is any line whose intersections with the given conic are harmonic in regard to its intersections with the given lines i and j.



In fact both the constructions, that for the director circle, and that we have now described, arise in the same figure. Let the points P, Q, of a given conic, Ω, be such that the lines, OP, OQ, joining them to a fixed point O, are a pair of a pencil in involution, of which the lines, i, j, passing through O, are the double rays. Thus the line PQ is met by the lines i, j, respectively, in two points which are conjugate in regard to the conic Ω. Now, to any point on the line i, there may be made to correspond the point of the line j, which is conjugate to the former in regard to the conic Ω; as these two points describe, on i and j, respectively, two ranges which are related, the line which joins them envelops a conic, say, Σ. This conic touches the lines i and j, say, in H and K, respectively. It also touches the four tangents, of the conic Ω, at the points where Ω is met by the lines i and j. If OQ meet Ω again in Q', the conic Σ also touches PQ'. And, if the other possible tangents be drawn to Σ from Q and Q', these meet, on the conic Ω, in the further point of the line OP which lies on Ω. Further, by considering the position of the line PQ in which it coincides with the line i, we see that O, H are conjugate to one another in regard to Ω; as, likewise, are O and K; thus HK is the polar of O in regard to Ω, meeting Ω at points, I, J, whereat the tangents pass through O. From this we infer that the lines PI, PJ are conjugate to one another in regard to the conic Σ. The conic Ω is thus the director circle of Σ in regard to the points I, J; while Σ is the dual of the director circle of Ω in regard to the lines i and j. (Cf. the diagram on p. 96.)

It is seen that P, Q, Q' is a triad of points, on the conic Ω, of which two joins PQ, PQ' touch the conic Σ, while the third join QQ' passes through one point, O, of the self-polar triad common to Ω and Σ. Similarly, if the second tangent to Σ from Q' meet Ω in R, then RQ, QP, PQ' is a triad of three tangents of Σ, two of whose intersections, Q and P, lie on Ω, while the third, the intersection of RQ and PQ', lies on the line, HI, of the common self-polar triad of

Ω and Σ. The tetrad P, Q, R, Q', of points of Ω, is such that the joins of these, taken in order, are tangent lines of Σ.



Conversely, if any two conics, Ω, Σ, be such that there exists a tetrad of points, P, Q, R, Q', of Ω, whose four joins, when the points are taken in order, are tangents of Σ, the diagonals PR and QQ' meet in one point of the common self-polar triad of the two conics; and there is then an infinite number of such tetrads of points. A sufficient condition for this relation is that the tangents of Σ, at two of its intersections with Ω, should meet in a point lying on Ω. Particular examples of two conics in this relation are, (1) Two circles Ω, Σ, of which Ω contains the centre of Σ, (2) A circle Ω passing through a pair of conjugate foci of Σ, (3) A circle Ω and a conic Σ which is the envelope of a chord PQ of Ω which subtends a right angle at a fixed point (not lying on Ω). In this last case, the middle point of P, Q describes a circle whose centre is the middle point of the two points constituted by O and the centre of Ω. [Cf. p. 56; also Add.]

CHAPTER III

THE EQUATION OF A LINE, AND OF A CONIC

Preliminary remarks. All the properties of conics so far given have been proved without the use of the symbols; and the same plan could be carried out to the end. For many of these properties it is, at present, more usual to give either a proof with the help of the symbols, or a proof by the metrical methods of what is called elementary geometry, in which, to a segment, or to an angle, a number is associated called its measure. We shall see that this metrical geometry can be regarded as an abbreviated use of algebraical symbols; it is, moreover, in the writer's view, in virtue of the system of symbols which it presupposes, founded on an arbitrary limitation of the geometrical possibilities. There is, therefore, logical interest in shewing that the symbols are not necessary; and in finding, if, for practical reasons, it is desired to reach the usual geometrical results, what are the narrowest geometrical limitations sufficient for this. But the employment of the symbols, by fixing the ideas, often renders a demonstration easier to follow. Moreover it happens that, since the time of Descartes, much labour has been devoted to elaborating systematic methods of algebraic computation; and, thus, methods which, from a geometrical point of view, are highly artificial, have come to be regarded as more natural.

The coordinates of a point in a plane. If A, B, C be three given points in a plane, not lying in line, of which the symbols are also represented by A, B, C, any other point in the plane has a symbol of the form $xA+yB+zC$, wherein x, y, z are algebraic symbols, of the system employed. These we suppose commutative in multiplication. If the point be specified for which the symbol is $A+B+C$, the symbols x, y, z are definite when P is assigned, save that they may be replaced, respectively, by mx, my, mz, where m is any symbol other than 0; thereby the symbols $x^{-1}y$, $x^{-1}z$ would be unaffected. The aggregate (x, y, z) are then called the coordinates of P, in regard to A, B, C, it being understood that (mx, my, mz) are equally the coordinates of P.

If we take any other three points, A', B', C', not lying in line, we may, similarly, specify the coordinates of all points of the plane in terms of these. In particular, if A, B, C be given in terms of these by the respective syzygies

$$A=a_1A'+a_2B'+a_3C',\quad B=b_1A'+b_2B'+b_3C',$$
$$C=c_1A'+c_2B'+c_3C',$$

the symbol for P, with respect to A', B', C', becomes

$$x(a_1A' + a_2B' + a_3C') + y(b_1A' + b_2B' + b_3C') + z(c_1A' + c_2B' + c_3C'),$$

or, say, $$x'A' + y'B' + z'C',$$

where

$$x' = a_1x + b_1y + c_1z, \quad y' = a_2x + b_2y + c_2z, \quad z' = a_3x + b_3y + c_3z.$$

The coordinates are thus changed by a linear transformation.

There can be no objection to agreeing to represent the relations involved, respectively, by

$$(A, B, C) = \left|\begin{matrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{matrix}\right| (A', B', C'), \quad (x', y', z') = \left|\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{matrix}\right| (x, y, z);$$

herein the matrix of transformation by which we pass from the coordinates, x, y, z, to the new coordinates, x', y', z', is obtained from that by which we pass from the *new* points of reference, A', B', C', to the *original* points of reference, A, B, C, by the change, often called transposition, of rows into columns. This is often expressed by saying that (x, y, z) and (A, B, C) are transformed *contragrediently*.

The equation of a line. Let two points which determine a line be of respective symbols

$$P_1 = x_1A + y_1B + z_1C, \quad P_2 = x_2A + y_2B + z_2C;$$

any other point of the line is then of symbol $P_1 + \lambda P_2$, and has coordinates $x_1 + \lambda x_2$, $y_1 + \lambda y_2$, $z_1 + \lambda z_2$. Now consider the three symbols

$$u = y_1z_2 - y_2z_1, \quad v = z_1x_2 - z_2x_1, \quad w = x_1y_2 - x_2y_1;$$

we notice, first, that these are unaffected, save by multiplication with the same factor, if (x_1, y_1, z_1), (x_2, y_2, z_2) be, respectively, replaced by the coordinates, (x_1', y_1', z_1'), (x_2', y_2', z_2'), of any other two points of the line; thus, when the line is given, the symbols u, v, w are determined, save for a common multiplier; then we notice that, if (x, y, z) be the coordinates of any point whatever, of the line, we have

$$ux + vy + wz = 0;$$

finally, if u, v, w be given, any point whose coordinates (x, y, z) satisfy this equation, is a point of the line. For, (x_1, y_1, z_1), (x_2, y_2, z_2) being two points of the line, this equation, coupled with

$$ux_1 + vy_1 + wz_1 = 0, \quad ux_2 + vy_2 + wz_2 = 0,$$

enables us to infer that there are two symbols λ_1, λ_2 such that

$$x = \lambda_1x_1 + \lambda_2x_2, \quad y = \lambda_1y_1 + \lambda_2y_2, \quad z = \lambda_1z_1 + \lambda_2z_2.$$

The equation $$ux + vy + wz = 0,$$
wherein u, v, w are given (save for a common multiplier), and x, y, z have any values for which the equation is verified, being the relation satisfied by the coordinates of all points of the line, and by no other points, is called the *equation of the line.*

The coordinates of a line. The three symbols u, v, w, which, as we have seen, determine a line, are called the *coordinates of the line.* If m be any algebraic symbol, other than 0, the symbols mu, mv, mw are equally the coordinates of the line.

In particular, the coordinates of the fundamental points B, C, being, respectively, $(0, 1, 0)$ and $(0, 0, 1)$, the coordinates of the line BC, joining these, are $(1, 0, 0)$. If, then, we represent the *lines* BC, CA, AB, respectively, by the symbols $\mathrm{a}, \mathrm{b}, \mathrm{c}$, we may regard the line whose coordinates are u, v, w as represented by the symbol

$$l = u\mathrm{a} + v\mathrm{b} + w\mathrm{c}.$$

This is analogous to the representation of any point of the plane by means of the three fundamental points A, B, C. The analogy extends further. For we can shew that if l_1, l_2 be the symbols of any two lines, the symbol, formed in the same way, for any line through the intersection of these two, is of the form $pl_1 + ql_2$. To this end, let (x_0, y_0, z_0) be the coordinates of the point of intersection, O, of the lines, (x_1, y_1, z_1) of any other point of the first line, and (x_2, y_2, z_2) of any other point of the second line. Any line through O is determined by O and a point, say P, not lying on either of the two given lines; if we put

$$O = x_0A + y_0B + z_0C, \quad P_1 = x_1A + y_1B + z_1C, \quad P_2 = x_2A + y_2B + z_2C,$$

we can suppose the symbol of P to be

$$P = \xi O + \eta P_1 + \zeta P_2;$$

thus, if P be $xA + yB + zC$, we have

$$x = \xi x_0 + \eta x_1 + \zeta x_2, \quad y = \xi y_0 + \eta y_1 + \zeta y_2, \quad z = \xi z_0 + \eta z_1 + \zeta z_2,$$

and these give

$$yz_0 - y_0z = \eta(y_1z_0 - y_0z_1) + \zeta(y_2z_0 - y_0z_2),$$

with similar expressions for $zx_0 - z_0x$ and $xy_0 - x_0y$. Therefore, in terms of the coordinates of the two given lines, the coordinates of the line OP are

$$\eta u_1 + \zeta u_2, \quad \eta v_1 + \zeta v_2, \quad \eta w_1 + \zeta w_2,$$

and the symbol for this line is, therefore,

$$(\eta u_1 + \zeta u_2)\,\mathrm{a} + (\eta v_1 + \zeta v_2)\,\mathrm{b} + (\eta w_1 + \zeta w_2)\,\mathrm{c},$$

or $$\eta l_1 + \zeta l_2.$$

The analogy between the expression of any point in terms of three points, and of any line in terms of three lines, is, thus, complete. And, as in the former case, so in the latter, when the symbols, a, b, c, of three fundamental lines are given, in order to specify without ambiguity the symbol of any other line, it is necessary to specify the symbol of some one fourth definite line beside a, b, c; we may, in particular, specify the line of which the symbol is a + b + c.

The equation of a point. We introduced the equation of a line as the equation satisfied by the coordinates of all points of the line. From what has just been said, we may, similarly, speak of the equation of a point, this being the equation satisfied by the coordinates of all the lines which contain this point. Clearly, if the point be of coordinates (x_0, y_0, z_0), the equation of the point will be

$$ux_0 + vy_0 + wz_0 = 0,$$

where u, v, w are variable line coordinates. For the sake of distinctness, the coordinates of a line are often called the *tangential* coordinates, and the equation of a point, its *tangential* equation.

And, just as the coordinates of the line joining the two points, whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2), are

$$u = y_1z_2 - y_2z_1, \quad v = z_1x_2 - z_2x_1, \quad w = x_1y_2 - x_2y_1,$$

so the coordinates of the point, which is the intersection of the lines of coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2), are

$$x_0 = v_1w_2 - v_2w_1, \quad y_0 = w_1u_2 - w_2u_1, \quad z_0 = u_1v_2 - u_2v_1.$$

For, if (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of two points, one on each line, the coordinates u_1, v_1, w_1 are the cofactors of x_2, y_2, z_2, respectively, in the determinant

$$\begin{vmatrix} x_0, & y_0, & z_0 \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

and u_2, v_2, w_2 are, similarly, the cofactors of x_1, y_1, z_1, respectively. If we form the determinant whose elements are the minors of this determinant, the minors of the new determinant are known to be proportional to the elements of this one; this gives the statement made.

Thus, instead of regarding the symbols of the *points* of a plane as being fundamental, we might equally well have regarded the symbols of the *lines* as fundamental. Or, if we regard the *coordinates* as fundamental, we may regard that which we have from the first spoken of as the symbol of a point, as being, in fact, that expression, linear in undetermined line coordinates, which occurs above as the first member, or left side, of the equation of a point.

The equation of a conic. That the coordinates of any point of a line are connected by an equation may be regarded as a consequence of the fact that these coordinates are functions of a varying parameter, that is, of the forms

$$x = \theta x_1 + x_2, \quad y = \theta y_1 + y_2, \quad z = \theta z_1 + z_2,$$

wherein θ varies from point to point of the line. Now, we have shewn (Chap. I, above) that the symbol of any point of a non-degenerate conic is of the form

$$P = \theta^2 A + \theta B + C,$$

A, C being any two points of the conic, and B the pole of AC in regard to the conic; with reference to these points, the coordinates of any point of the conic are, then, $x = \theta^2$, $y = \theta$, $z = 1$. Thus every point of the conic is such that its coordinates satisfy the equation

$$xz = y^2.$$

With reference, however, to any three general points of the plane, A', B', C', which are not in line, the coordinates of a point of the conic, by what we have seen above, will be of the forms

$$x' = a_1\theta^2 + b_1\theta + c_1, \quad y' = a_2\theta^2 + b_2\theta + c_2, \quad z' = a_3\theta^2 + b_3\theta + c_3,$$

wherein the coefficients $a_1, b_1, \ldots, c_3$ remain the same, but θ varies from point to point of the conic. As we suppose that the conic does not degenerate into two straight lines, so that there is no relation of the form $Lx' + My' + Nz' = 0$, holding for all values of θ, these equations lead to equations

$$\theta^2 = A_1x' + A_2y' + A_3z', \quad \theta = B_1x' + B_2y' + B_3z', \quad 1 = C_1x' + C_2y' + C_3z',$$

and, hence, to

$$(A_1x' + A_2y' + A_3z')(C_1x' + C_2y' + C_3z') = (B_1x' + B_2y' + B_3z')^2.$$

Thus, referred to any three general points of the plane, the coordinates of all the points of a conic satisfy a homogeneous equation of the second order.

Conversely, any such equation, say

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

connecting the coordinates x, y, z of a point, involves that the point lies on a definite conic. This we now prove:

(*a*) The quadric expression on the left side of this equation may be the product of two factors, both linear in x, y, z, being identically equal, say, to

$$(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z),$$

for all values of x, y, z. When this is so, by forming the partial differential coefficients in regard to x, y, z, we see that all the three linear functions of x, y, z given by

$$ax + hy + gz, \quad hx + by + fz, \quad gx + fy + cz$$

vanish for the same set of values of x, y, z, namely for that set of values which satisfies both the equations

$$l_1x + m_1y + n_1z = 0, \quad l_2x + m_2y + n_2z = 0;$$

for the moment we suppose that the lines of which these last are the equations are not the same. The simultaneous vanishing of the three linear functions requires, however, that the determinant

$$\Delta, = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

should vanish. And, conversely, it will appear that it is easy to prove that when this determinant vanishes, the quadric expression, on the left side of the given equation, necessarily breaks up into the product of two linear factors. These two factors may be the same; but, then, there are other conditions connecting the coefficients $a, b, c, \ldots$; in fact the minors, of two rows and columns, in Δ, also vanish. Also, it is true that if the given quadric equation is satisfied by the coordinates of every point of a line, say, the line of which the equation is $L = 0$, where L is linear in x, y, z, then the quadric expression on the left of the equation breaks up into the product of L and another linear factor. For, by ordinary division, we can, if L contain x, write this quadric expression in such a form as

$$LM + A(y - mz)(y - nz),$$

where M is linear in x, y, z, and A, m, n are independent of x, y, z. This cannot be satisfied by every point for which $L = 0$ unless the coefficient A be zero.

(*b*) Now, suppose the determinant Δ is not zero. Let P_1, P_2 be any two points whose coordinates satisfy the given equation. From the fact that the given equation is of the second order, it can be shewn that every line drawn through P_1 contains another point, say P, whose coordinates also satisfy the given equation, so that there is associated to every line drawn through P_1, a definite line through P_2, joining this to P. If the coordinates of P_1 and P_2 be, respectively, (x_1, y_1, z_1) and (x_2, y_2, z_2), the equations of lines through P_1 and P_2, respectively, are of the forms

$$yz_1 - y_1z = \theta(xz_1 - x_1z), \quad yz_2 - y_2z = \phi(xz_2 - x_2z),$$

for proper values of θ and ϕ. By what we have seen, to every θ we can associate a definite ϕ, and conversely. Thus (cf. the Preliminary remarks to this volume, p. 8, above), the pencils $P_1(P)$ and $P_2(P)$ are related. Therefore P describes a conic in accordance with our original definition.

(*c*) We may reach this conclusion more directly, as follows: Let the quadric expression on the left side of the given equation be denoted by F, and the results of substituting therein, for x, y, z, respectively x_1, y_1, z_1 and x_2, y_2, z_2, by F_1 and F_2; let p, q, r denote, respectively, $ax+hy+gz$, $hx+by+fz$, $gx+fy+cz$, and p_1, q_1, r_1 what these become when x_1, y_1, z_1 are put for x, y, z, with, similarly, p_2, q_2, r_2, when x_2, y_2, z_2 are put for x, y, z; we then have

$$x_1p_2+y_1q_2+z_1r_2=x_2p_1+y_2q_1+z_2r_1,$$

either of these being the same as

$$ax_1x_2+by_1y_2+cz_1z_2+f(y_1z_2+y_2z_1)+g(z_1x_2+z_2x_1)+h(x_1y_2+x_2y_1),$$

which we denote by F_{12}. If we now put

$$X=xp_1+yq_1+zr_1, \quad Z=xp_2+yq_2+zr_2,$$
$$Y=x(y_1z_2-y_2z_1)+y(z_1x_2-z_2x_1)+z(x_1y_2-x_2y_1),$$

and suppose the points (x_1, y_1, z_1), (x_2, y_2, z_2) to satisfy the given equation $F=0$, so that both F_1 and F_2 vanish, we can shew that, for all values of x, y, z, the equation $F=0$ is the same as

$$XZ-\mu Y^2=0,$$

where $\mu=\frac{1}{2}\Delta/F_{12}$. For, whatever x_1, y_1, z_1 and x_2, y_2, z_2 may be, we have, by multiplication of determinants, the product

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

equal to

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} \begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{vmatrix},$$

and, hence, equal to

$$\begin{vmatrix} F, & X, & Z \\ X, & F_1, & F_{12} \\ Z, & F_{12}, & F_2 \end{vmatrix};$$

namely, we have the identity,

$$Y^2\Delta=F(F_1F_2-F_{12}^2)+2XZF_{12}-F_1Z^2-F_2X^2.$$

When we substitute for x, y, z, in F, respectively, $x_1+\lambda x_2, y_1+\lambda y_2, z_1+\lambda z_2$, the result is $F_1+2\lambda F_{12}+\lambda^2F_2$; we are supposing $F_1=0$, $F_2=0$, and that the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) does not consist of points all satisfying $F=0$; thus F_{12} does not vanish. The identity obtained thus leads to

$$F_{12}^2F=2F_{12}(XZ-\mu Y^2),$$

where $\mu=\frac{1}{2}\Delta/F_{12}$. Wherefore, the equation $F=0$ is satisfied by all points (x, y, z) which, for any value of θ, satisfy the equations

$$X=\theta^2, \quad \mu Y=\theta, \quad \mu Z=1;$$

thus the equation $F=0$ belongs to a conic, provided the linear functions of x, y, z which we have denoted by X, Y, Z, are independent, that is, provided the lines $X=0$, $Y=0$, $Z=0$ do not meet in a point; and, in fact, $X=0$, $Y=0$ intersect in the point (x_1, y_1, z_1), while $Z=0$, $Y=0$ meet in the distinct point (x_2, y_2, z_2).

A particular form of the equation of a conic. We have seen that, with reference to three points, consisting of any two points of a conic and the pole of the line joining them, the equation of the conic can be supposed to have the form $xz-y^2=0$, any point of the conic having coordinates of the form $(\theta^2, \theta, 1)$. Many results which hold for the general form of the equation can be well illustrated by this case.

(1) Evidently the line represented by the equation

$$x-y(\theta+\phi)+z\theta\phi=0$$

contains both the points $(\theta^2, \theta, 1)$ and $(\phi^2, \phi, 1)$; this is, then, the equation of the chord of the conic. Or, a line represented by the equation $ax+by+cz=0$ meets the conic in two points for which the values of the parameter θ are those given by the equation $a\theta^2+b\theta+c=0$. In particular the tangent of the conic at the point $x'=\theta^2$, $y'=\theta$, $z'=1$ has the equation $x-2y\theta+z\theta^2=0$; it is to be remarked that this is the same as $xz'+x'z-2yy'=0$, and, again, that this is the same as

$$\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\right)(xz-y^2)=0.$$

From this remark we can at once find the equation of the tangent at any point of the conic represented by the general equation

$$aX^2+bY^2+cZ^2+2fYZ+2gZX+2hXY=0;$$

for, if X, Y, Z be any linear functions of x, y, z, say

$$X=a_1x+b_1y+c_1z,\quad Y=a_2x+b_2y+c_2z,\quad Z=a_3x+b_3y+c_3z,$$

and X', Y', Z' be precisely the same linear functions of x', y', z', it is easy to see that

$$X'\frac{\partial}{\partial X}+Y'\frac{\partial}{\partial Y}+Z'\frac{\partial}{\partial Z}=x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z};$$

now we have shewn that the general equation of the conic, say $F=0$, in variable coordinates X, Y, Z, can, by choosing x, y, z to be suitable linear functions of these, be reduced to the form $xz-y^2=0$. Wherefore, the equation of the tangent, at a point of coordinates X', Y', Z', is

$$\left(X'\frac{\partial}{\partial X}+Y'\frac{\partial}{\partial Y}+Z'\frac{\partial}{\partial Z}\right)F=0.$$

This, however, can be obtained, also, as follows: The point

(X', Y', Z') being upon the conic, and (X, Y, Z) being any other point, the point $(X' + \lambda X, Y' + \lambda Y, Z' + \lambda Z)$, on the line joining these, lies upon the conic, $F = 0$, if

$$\lambda^2 F + \lambda \left(X' \frac{\partial F}{\partial X} + Y' \frac{\partial F}{\partial Y} + Z' \frac{\partial F}{\partial Z}\right) + F' = 0,$$

where F' is what F becomes when X', Y', Z' are written for X, Y, Z, and is therefore zero. The equation will be satisfied by $\lambda = 0$ only, and the line joining (X', Y', Z') to (X, Y, Z) will meet the conic only at (X', Y', Z'), if, and only if, (X, Y, Z) satisfy the equation which we have given as the equation of the tangent at (X', Y', Z').

We can hence infer the equation of the polar of a point in regard to a conic. For the equation just obtained for the tangent is symmetrical in regard to (X, Y, Z) and (X', Y', Z'). Therefore, if the tangents of the conic at the points (x_1, y_1, z_1) and (x_2, y_2, z_2) intersect in the point (ξ, η, ζ), so that we have two equations, arising as expressing this fact, say

$$(x_1, y_1, z_1;\ \xi, \eta, \zeta) = 0, \quad (x_2, y_2, z_2;\ \xi, \eta, \zeta) = 0,$$

each linear in both the two sets of coordinates which it contains, and symmetrical in regard to these two sets, we can infer that the line expressed by the equation

$$(x, y, z;\ \xi, \eta, \zeta) = 0,$$

wherein x, y, z are the coordinates of a varying point, passes through (x_1, y_1, z_1) and (x_2, y_2, z_2). It is thus the polar line of (ξ, η, ζ). Its equation is therefore

$$\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}\right) F = 0,$$

where now, in F, the coordinates x, y, z are written for X, Y, Z. The same result is obtained by regarding the polar as the locus of a point, (x, y, z), which is the harmonic conjugate of (ξ, η, ζ), in regard to the points in which the conic is met by a line through (ξ, η, ζ). For these intersections must then have coordinates of the respective forms $(x + \lambda\xi, y + \lambda\eta, z + \lambda\zeta)$ and $(x - \lambda\xi, y - \lambda\eta, z - \lambda\zeta)$; the quadratic equation for λ which expresses that one of these points satisfies the equation $F = 0$, must then be without the term in λ.

In particular, the polar of the point (ξ, η, ζ), in regard to the conic $xz - y^2 = 0$, is given by the equation $x\zeta + z\xi - 2y\eta = 0$.

With this form for the equation of the conic, the tangent line, whose equation, we have seen, is $x - 2y\theta + z\theta^2 = 0$, has coordinates $u = 1$, $v = -2\theta$, $w = \theta^2$. Wherefore the coordinates of all tangent lines of the conic are such as to satisfy the equation

$$4wu - v^2 = 0.$$

This equation is often called the tangential equation of the conic. In view of what has been said in regard to the point equation, and as to the dual nature of a conic, and from the similarity of the relations for the coordinates of a line and of a point, it is at once clear, (1) that for a conic whose equation is given in general form, the tangential equation must be homogeneously of the second order in the coordinates, u, v, w, of any tangent of the conic, (2) that any such quadric equation connecting u, v, w must be the tangential equation of a conic, becoming a point pair when the equation breaks into two linear factors. But we may prove directly that the condition for the line, whose equation is $ux + vy + wz = 0$, to touch the conic given by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

is $$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0,$$

where A, B, C, F, G, H are respectively the cofactors of a, b, c, f, g, h in the determinant Δ, so that $A = bc - f^2$, $F = gh - af$, etc. And then, from the well-known determinant identities such as

$$BC - F^2 = a\Delta, \quad GH - AF = f\Delta,$$

we see that we can pass from the tangential equation to the point equation by the same rule as that by which the former equation is deduced from the latter. To prove the statement made, we express that the line $ux + vy + wz = 0$ is the tangent at some point, (x', y', z'), of the conic, so that this equation is of the form

$$x(ax' + hy' + gz') + y(hx' + by' + fz') + z(gx' + fy' + cz') = 0.$$

Whence we have

$$\frac{ax' + hy' + gz'}{u} = \frac{hx' + by' + fz'}{v} = \frac{gx' + fy' + cz'}{w},$$

which, however, lead to

$$\frac{x'}{Au + Hv + Gw} = \frac{y'}{Hu + Bv + Fw} = \frac{z'}{Gu + Fv + Cw};$$

these, when we express that $ux' + vy' + wz' = 0$, give the condition specified.

As examples, the reader may prove the simple facts that the conic whose equation is obtained by rationalising $(fx)^{\frac{1}{2}} + (gy)^{\frac{1}{2}} + (hz)^{\frac{1}{2}} = 0$ has for its tangential equation $fvw + gwu + huv = 0$, the tangent at any point (x', y', z') of the conic having the equation

$$x(f/x')^{\frac{1}{2}} + y(g/y')^{\frac{1}{2}} + z(h/z')^{\frac{1}{2}};$$

this conic touches the joins, BC, CA, AB, of the three points of reference for the coordinates. Dually, the conic, through A, B, C, whose equation is $fyz + gzx + hxy = 0$, has a tangential equation obtainable from $(fu)^{\frac{1}{2}} + (gv)^{\frac{1}{2}} + (hw)^{\frac{1}{2}} = 0$, the tangent at any point (ξ, η, ζ) having the equation $\xi^{-2}fx + \eta^{-2}gy + \zeta^{-2}hz = 0$. [Add.]

We return now again to the equation $xz - y^2 = 0$. Let X, Z be any points of the line AC which are harmonic conjugates in regard to A and C, say, of symbols $mA - C$, $mA + C$, respectively, in terms of the symbols of the points A and C; their coordinates will then be, referred to A, B, C, respectively $(m, 0, -1)$ and $(m, 0, 1)$. The equations of the lines BX, BZ will then be, respectively, $x + mz = 0$ and $x - mz = 0$. Now we have agreed that our system of symbols shall be such that there is a symbol i for which $i^2 = -1$ (Vol. I, p. 165); for this we use always the same sign; the other symbol satisfying the same equation will then be $-i$. This being understood, let

$$\xi = i(x - mz), \quad \zeta = (x + mz), \quad \eta = 2m^{\frac{1}{2}}y,$$

where $m^{\frac{1}{2}}$, as usual, is one of the symbols, t, for which $t^2 = m$. We then have, as another form of the equation $xz - y^2 = 0$, the equation

$$\xi^2 + \zeta^2 = \eta^2;$$

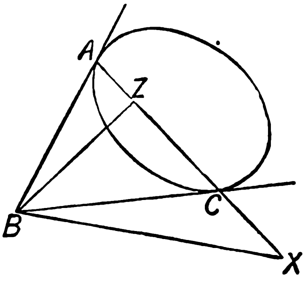
and we have, writing $m^{-\frac{1}{2}}$ for $(m^{\frac{1}{2}})^{-1}$,

$$\begin{aligned} xA + yB + zC &= \tfrac{1}{2}(\zeta - i\xi)A + \tfrac{1}{2}m^{-\frac{1}{2}}\eta B + \frac{1}{2m}(\zeta + i\xi)C \\ &= \frac{\xi}{2im}(mA - C) + \eta\frac{1}{2m^{\frac{1}{2}}}B + \frac{\zeta}{2m}(mA + C). \end{aligned}$$

The triad, B, X, Z, is self-polar in regard to the conic. Conversely, if, in the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the coordinates are referred to a triad which is self-polar in regard to the conic, we must have $f = 0$, $g = 0$, $h = 0$. For the polar of the point $(1, 0, 0)$, which in general is of equation $ax + hy + gz = 0$, must then be expressed by $x = 0$, with a similar remark for the points $(0, 1, 0)$, $(0, 0, 1)$. If, then, we put $\xi = xa^{\frac{1}{2}}$, $\eta = iyb^{\frac{1}{2}}$, $\zeta = zc^{\frac{1}{2}}$, the equation reduces to $\xi^2 + \zeta^2 = \eta^2$. The algebraic problem of reducing the general equation to this form, equivalent to the problem of finding a self-polar triad of the conic given by the general equation, is solved below (p. 141).



Equation of a circle. In the phraseology of the preceding chapter, with reference to the points A, C, taken as absolute points, the conic is a circle, whose centre is B; the lines BX, BZ are any two lines through its centre which are at right angles to one another in regard to A and C.

If, retaining X, Z as two of the points of reference, we take for the third point, instead of B, another point, B', the equation of the circle takes the form

$$(\xi' - a\eta)^2 + (\zeta' - c\eta)^2 = \eta^2.$$

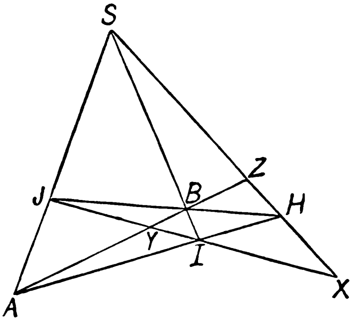
In fact, writing (see p. 107)

$$X_1 = \frac{1}{2im}(mA - C), \quad Z_1 = \frac{1}{2m}(mA + C), \quad B_1 = \frac{1}{2m^{\frac{1}{2}}}B,$$

and taking B' so that

$$B' = B_1 - aX_1 - cZ_1,$$

the symbol of a general point found above, $\xi X_1 + \eta B_1 + \zeta Z_1$, is $(\xi + a\eta) X_1 + \eta B' + (\zeta + c\eta) Z_1$, and this we write in the form $\xi' X_1 + \eta B' + \zeta' Z_1$. The points of reference are now, any point B' of the plane, together with any two points X, Z, on the absolute line AC, which are such that $B'X$, $B'Z$ are at right angles in regard to A and C. The centre of the circle is now the point for which the coordinates ξ', ζ', η are, respectively, a, c, 1.



In further illustration of the point of view adopted in the last chapter, let us now find the equation of a general circle of a coaxial system, that is, of a conic through four given points. Denote these points by I, J, A, B, using I, J as the notation for the two absolute points. The points of intersection, S, H, respectively of AJ, BI and of AI, BJ, are (Chap. II, above, p. 66) the limiting points of the coaxial system of circles. Let AB, IJ meet in Y, also IJ, SH in X, and AB, SH in Z; the triad S, H, Y is self-polar in regard to all conics through I, J, A, B. We refer the circles to the points Z, X, Y, and suppose the symbols so chosen that $I = \frac{1}{2}(X + iY)$, $J = \frac{1}{2}(X - iY)$, as is possible since I, J are harmonic in regard to X and Y.

For any particular one of the circles, let the pole of the line IJ, which lies on XZ, be of symbol $Z + aX$; then, by what we have seen above, the symbol of any point of this circle is expressible, with a parameter θ, in the form

$$\theta^2 I + \theta m (Z + aX) + J,$$

where m is a fixed appropriate symbol. Denoting this by $xX + yY + zZ$, we have

$$x = \tfrac{1}{2}(\theta^2 + 1) + ma\theta, \quad y = \tfrac{1}{2}i(\theta^2 - 1), \quad z = m\theta,$$

which satisfy the equation

$$(x - az)^2 + y^2 = m^{-2}z^2,$$

other circles with the same centre having the same form of equation with a different value for m. But, if the points A, B be $Z \pm hY$, that is, be of coordinates $(0, \pm h, 1)$ with reference to X, Y, Z, this circle passes, by hypothesis, through A, B, and we have $m^{-2} = a^2 + h^2$. The circle may therefore be represented by

$$x^2 + y^2 - 2axz - h^2z^2 = 0,$$

and the other circles of the coaxial system, through A and B, have the same form of equation with a different value for a. The triad of reference for the coordinates, Z, X, Y, is determined by, (1) the line of centres of all the circles, which is the line XZ, (2) the common chord, YZ, of all the circles, (3) the absolute line, XY. The symbol h is such that the common points of the circles are of coordinates $(0, \pm h, 1)$.

The relation, to the points I, J, and the triad X, Y, Z, of the points S, H, is similar to that of A, B. But while A, B have symbols $Z \pm hY$, the symbols of S, H are $Z \pm kX$. Thus the general circle passing through the limiting points S, H has an equation

$$x^2 + y^2 - 2byz - k^2z^2 = 0,$$

where k is the same for all of these, but b varies from circle to circle. Here, as the points J, B, H, which may be supposed to have respective symbols $X - iY$, $Z + hY$, $Z + kX$, are in line, we may suppose $h = ik$; so that the general circle through the limiting points has an equation of the form

$$x^2 + y^2 - 2byz + h^2z^2 = 0.$$

We may now verify, what was proved in Chap. II (p. 66, above), that any one of these circles cuts any one of the coaxial circles at right angles. For this purpose we first obtain the condition that, at a point (x_0, y_0, z_0) where they intersect, two conics should cut at right angles, in regard to I and J, according to the definition we have given for this (Chap. II, p. 63). With regard to the points X, Y, Z, by which the coordinates are defined, we assume only that X, Y are on the absolute line, IJ, and are harmonic in regard to I and J; for these two absolute points, then, we take, as symbols, $X \pm iY$. If the equations of the two conics be, respectively, $\phi = 0$ and $\psi = 0$, and the values, at their common point (x_0, y_0, z_0), of the partial derivatives $\partial\phi/\partial x$, $\partial\phi/\partial y, \ldots, \partial\psi/\partial z$, be denoted, for brevity, by u, v, w, p, q, r, the tangents of these conics, at this point, are, we have seen, of equations $ux + vy + wz = 0$, and $px + qy + rz = 0$. These meet the absolute line, for which $z = 0$, respectively where $ux + vy = 0$ and $px + qy = 0$, that is, at the points whose symbols are $vX - uY$ and $qX - pY$. These symbols, however, are, respectively,

$$(v + iu)(X + iY) + (v - iu)(X - iY),$$
$$(q + ip)(X + iY) + (q - ip)(X - iY),$$

or $$(v + iu) I + (v - iu) J, \quad (q + ip) I + (q - ip) J;$$

the condition that the two tangents should be at right angles, in regard to I and J, is that these points should be harmonic conjugates in regard to I, J, namely, that

$$(v + iu)^{-1}(v - iu) + (q + ip)^{-1}(q - ip) = 0,$$

or $$up + vq = 0;$$

more fully this is, that, at the common point of the conics $\phi = 0$, $\psi = 0$, we should have

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} = 0.$$

Applying this to the case of the circles whose equations are

$$x^2 + y^2 - 2axz - h^2z^2 = 0, \quad x^2 + y^2 - 2byz + h^2z^2 = 0,$$

it becomes $$(x - az)x + y(y - bz) = 0;$$

this is satisfied at each common point of the two circles, as we see by adding their equations.

Incidentally, we have seen that the condition that any two lines

$$ux + vy + wz = 0, \quad px + qy + rz = 0,$$

should be at right angles, in regard to I and J, is $up + vq = 0$.

Examples of the algebraic treatment of the theory of conics. The preceding explanations are probably sufficient to exhibit the application of the algebraic symbols to the theory of conics. We collect together now a succession of illustrations, with, in many cases, omission of detailed discussion. Most of the results obtained will be known to the reader of the preceding chapters; but there are some, of which the symbolic treatment is briefer than a purely geometrical treatment would be, which have not been referred to previously. The reader will be able to convince himself, by devising a geometrical method, that the symbols are not indispensable even for these. But there is one assumption which is made in some of the concluding examples to which reference should be made. It is assumed in these examples that the system of symbols employed is such that a rational algebraic equation of order n has n roots, and can be written as the product of n corresponding linear factors. Logically, the examples in which this assumption is made should have been deferred till after the considerations given in Chap. IV; it seemed more convenient to include them here.

Ex. 1. *Condition that a pair of points should be harmonic conjugates in regard to another pair.* On a line, determined by two points

O, U, we consider two points whose symbols are $O+\alpha U$, $O+\beta U$, where α, β are the values of x for which $ax^2+2hx+b=0$, and also two points $O+\alpha' U$, $O+\beta' U$, where α', β' are the values of x for which $a'x^2+2h'x+b'=0$. Prove that the condition, that the two latter points should be harmonic conjugates in regard to the two former, is that $(\alpha'-\alpha)(\alpha'-\beta)^{-1}+(\beta'-\alpha)(\beta'-\beta)^{-1}=0$, and that this is the same as $ab'+a'b-2hh'=0$.

Ex. 2. *A result in regard to involutions.* If A, B and A', B' be two pairs of points, on a line, or on a conic, and X, Z be the double points of the involution which is determined by these two pairs, prove that the three pairs of points, A, A'; B, B'; X, Z, are in involution, as also are the three pairs, A, B'; A', B; X, Z.

Ex. 3. *Representation of all the pairs of an involution.* If, as in Ex. 1, two points, $A, = O+\alpha U$, and $B, = O+\beta U$, be determined by the roots α, β of the equation $ax^2+2hx+b=0$, and, also, two points $A', = O+\alpha' U$, and $B', = O+\beta' U$, by the equation $a'x^2+2h'x+b'=0$, then the two points, $O+\xi U$, $O+\eta U$, where ξ, η are the roots of the equation $\theta(ax^2+2hx+b)+a'x^2+2h'x+b'=0$, for any symbol θ, are a pair of the involution determined by the two pairs A, B and A', B'. Further, every pair of this involution is so determined, with an appropriate value for θ.

Ex. 4. *A general result for involutions.* In the previous example, the points $O+\xi U$, $O+\eta U$ can be expressed by the points A and A'; let the expressions, for the symbols of the points, be

$$O+\xi U = m(A+\lambda A'), \quad O+\eta U = n(A+\mu A').$$

Prove, then, that $\lambda\mu^{-1}=(\xi-\alpha)(\xi-\alpha')^{-1}/(\eta-\alpha)(\eta-\alpha')^{-1}$; and, denoting this by ϵ, prove that

$$\frac{4\epsilon}{(1+\epsilon)^2}=\frac{\theta\,[2(\Delta\Delta')^{\frac{1}{2}}-H]}{[\theta(\Delta)^{\frac{1}{2}}+(\Delta')^{\frac{1}{2}}]^2},$$

where $\Delta=h^2-ab$, $\Delta'=h'^2-a'b'$, $H=2hh'-ab'-a'b$.

In particular, (Ex. 1), the points $O+\xi U$, $O+\eta U$ are harmonic conjugates of one another, in regard to A and A', when θ is such that

$$\theta^2(h^2-ab)=h'^2-a'b'.$$

Ex. 5. *Condition that two circles, in general form, should cut at right angles.* Shewing, from the text, that the equations of any two circles can be taken in the forms

$$x^2+y^2+2gxz+2fyz+cz^2=0, \quad x^2+y^2+2g'xz+2f'yz+c'z^2=0,$$

prove that the condition that they should cut at right angles is

$$2gg'+2ff'=c+c'.$$

Ex. 6. *The radical axis of two circles; the radical centre of three circles.* If the equations of any two conics, referred to any triad of points, be $F=0$, $F'=0$, prove that the equation of any conic which passes through their four intersections is $\theta F + F' = 0$, for an appropriate symbol θ. In particular, shew that the radical axis of the two circles in the preceding example is (by $\theta = -1$)

$$2(g-g')x + 2(f-f')y + (c-c')z = 0.$$

Also that the three radical axes, for the pairs of any three circles, meet in a point. Shew, further (see Ex. 3), that the conics which pass through the common points of two given conics cut an arbitrary line in pairs of points belonging to the same involution.

Ex. 7. *The common tangents and centres of similitude of two circles.* Prove that the condition that the line whose equation is

$$ux + vy + wz = 0$$

should touch the circle whose equation is

$$(x-az)^2 + (y-bz)^2 = h^2z^2$$

is that $$(au + bv + w)^2 = h^2(u^2 + v^2).$$

Hence, find the equations of the four tangents, common to this circle and the circle given by $(x-a'z)^2 + (y-b'z)^2 = h'^2z^2$. There are three point pairs such that every one of the four tangents passes through one or other of the two points of a pair; and the three lines, each containing the points of a pair, form a triad self-polar in regard to both the circles. Shew that these lines have for intersections the two limiting points of the two circles and a point on the absolute line, IJ. The centres of the two circles, in terms of the points of reference, X, Y, Z, have, for respective symbols, $C = aX + bY + Z$, and $C' = a'X + b'Y + Z$; shew that there are two points on the line joining the centres, with symbols $hC' \pm h'C$, through each of which pass two of the common tangents of the circles. These two points are called the centres of similitude of the two circles. It may be proved that the six centres of similitude of three circles lie, in threes, upon four lines.

The circle passing through the centres of similitude of two circles, with its centre on the line joining the centres of these, is of importance. Prove that the equation of this circle is $h^2F' - h'^2F = 0$, where $F=0$, $F'=0$ are the equations of the two circles, written as in this example.

Ex. 8. *Feuerbach's theorem. Apolar triads.* Let A, B, C be three points such that the joins BC, CA, AB touch a circle, say, Ω; and let D, E, F be the middle points, respectively, of B, C; of C, A; and of A, B. Then the circle D, E, F touches Ω. The definitions of a circle, and of middle points, are with reference to two absolute points I, J.

Let K be the centre of Ω, and take coordinates in regard to I, J, K, any point of the plane being $xI + zJ + yK$. The equation of Ω may then be supposed to be $xz - y^2 = 0$. Any point of this has, then, coordinates of the form $(\theta^2, \theta, 1)$; in particular, let the points of contact of BC, CA, AB be given, respectively, by the parameters α, β, γ, so that, for instance, the equation of BC is $x - 2y\alpha + z\alpha^2 = 0$. The coordinates of A are then easily found to be $\beta\gamma$, $\frac{1}{2}(\beta + \gamma)$, 1, and the coordinates of D to be

$$\alpha(\alpha\beta + \alpha\gamma + 2\beta\gamma), \quad (\alpha + \beta)(\alpha + \gamma), \quad \beta + \gamma + 2\alpha.$$

Now write, for brevity,

$$p = \alpha + \beta + \gamma, \quad q = \beta\gamma + \gamma\alpha + \alpha\beta, \quad r = \alpha\beta\gamma, \quad \mu = q/p$$

and consider the conic whose equation is

$$(xz - y^2)(rp^{-2} - \mu) + y(x - 2y\mu + z\mu^2) = 0.$$

It can be verified that this equation is satisfied by the coordinates of D, E and F. The points common to this conic and $xz - y^2 = 0$ are the points of intersection of $xz - y^2 = 0$ with the two lines whose equations are $y = 0$, $x - 2y\mu + z\mu^2 = 0$. The former of these meets $xz - y^2 = 0$ in the points I and J; the latter meets it in two coincident points of coordinates $(\mu^2, \mu, 1)$. Thus the equation written down represents the circle D, E, F; and this touches Ω at the point $(\mu^2, \mu, 1)$.

As has appeared (Chap. I, Ex. 33; Chap. II, p. 89) the circle D, E, F is the locus of the centres of rectangular hyperbolas passing through A, B, C. This may be verified directly. The equation

$$\Sigma P(x - 2y\alpha + z\alpha^2)^{-1} = 0,$$

containing three terms with undetermined coefficients P, Q, R, evidently represents a conic through the three points A, B, C; by suitable choice of P, Q, R this can be taken to pass through two other arbitrary points; it is therefore the general conic through the points A, B, C. It is a rectangular hyperbola if its intersections with the line $y = 0$ are harmonic in regard to I and J; for this the equation, involving terms in x^2, xz, z^2, which gives its intersections with $y = 0$, must contain no term in xz; it is at once seen that the condition for this is $\Sigma P(\beta^2 + \gamma^2) = 0$. If we now form, by the rule given above, the equation for the polar of a point (x', y', z'), in regard to this conic, and express that this reduces to $y = 0$, (in which case (x', y', z') will be the centre of the conic), we obtain two further conditions for P, Q, R; in virtue of the first condition these become

$$\Sigma P[x' - y'(\beta + \gamma)] = 0 \quad \text{and} \quad \Sigma P\beta\gamma[y'(\beta + \gamma) - z'\beta\gamma] = 0.$$

Eliminating P, Q, R from the three conditions, we find that (x', y', z') must satisfy the equation

$$(xz - y^2)(r - pq) + y(xp^2 - 2ypq + zq^2) = 0,$$

which is that of the conic D, E, F.

The values of the parameters α, β, γ, for the points of contact of BC, CA, AB with the circle Ω, are the values of θ for which the polynomial $u = \theta^3 - p\theta^2 + q\theta - r$, vanishes. If we similarly take together the values of the parameter for the points I, J, of the circle Ω, and the point where it is touched by the circle D, E, F, these being, respectively, ∞, 0, μ, these may be said to be the roots of the polynomial $v = \theta^2 - \mu\theta$, regarded as a cubic polynomial. Now, for any two quadratic polynomials, say $a\theta^2 + 2h\theta + b$ and $a'\theta^2 + 2h'\theta + b'$, we have remarked (above, p. 111) an expression involving their coefficients, $ab' - 2hh' + ba'$, which is of geometrical interest. For any two cubic polynomials $a\theta^3 + 3b\theta^2 + 3c\theta + d$ and $a'\theta^3 + 3b'\theta^2 + 3c'\theta + d'$, there is a similar expression, which is also of geometrical interest, namely $ad' - 3bc' + 3cb' - da'$. Two cubics are said to be *apolar* to one another when the expression vanishes; this is the case for the two cubics, u, v, considered here. We may, therefore, say, that the circle D, E, F touches the circle Ω at the point of Ω which is the apolar complement of I and J, with reference to the points where Ω is touched by BC, CA, AB. (See Ex. 26, below.) [Add.]

Ex. 9. *Continuation of the preceding example.* The point where the nine points circle, of A, B, C, that is, the circle D, E, F, touches the inscribed circle Ω, may also be described as the fourth intersection of Ω with a particular conic, which we now explain, passing through the points where Ω touches BC, CA, AB. Let these points of contact be called P, Q, R ; the equation of the line QR is

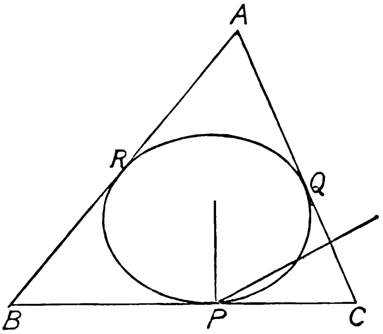
$$x - y(\beta + \gamma) + z\beta\gamma = 0.$$

Denote this by $U = 0$. Similarly, let RP and PQ be $V = 0$ and $W = 0$. The conic represented by the equation

$$\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right) UVW = 0,$$

may be called the polar conic of the point (x', y', z') in regard to the degenerate cubic locus represented by $UVW = 0$. When (x', y', z') is the centre, $(0, 1, 0)$, of the circle Ω, this polar conic becomes $\Sigma(\beta + \gamma)VW = 0$. It may easily be verified that this conic passes through the point $(\mu^2, \mu, 1)$ of Ω, and that it passes through P, Q and R. Its tangent at P is the harmonic conjugate, in regard to PQ, PR, of the line joining P to the centre of the inscribed circle Ω, as is easy to verify, with a similar statement for Q and R; this suffices to identify the conic. Thus, the nine points circle of A, B, C touches the inscribed circle on the polar conic of the centre of the inscribed circle, taken in regard to the degenerate cubic formed of the three joins of the points of contact of the inscribed circle with BC, CA, AB.

Another conic which touches the inscribed circle Ω at the point



$(\mu^2, \mu, 1)$ is obtained as follows: Consider again the line drawn through P harmonic, in regard to PQ, PR, to the line joining P to the centre of the inscribed circle; and the similar lines through Q and R. A conic can be drawn touching these lines to have a focus at the centre of the inscribed circle. We may prove that this conic has the equation

$$[x - 2y(\mu - rp^{-2}) + z\mu^2]^2$$
$$= 4xzr^2p^{-4};$$

in fact, it is clear that this conic meets each of the lines $x = 0$, $z = 0$ only in one point, or touches these, and so has the centre, $(0, 1, 0)$, of the inscribed circle, as a focus. It is also clear that it meets the inscribed circle $xz - y^2 = 0$ where this is met by the two lines

$$x - 2y\mu + z\mu^2 = 0, \quad x - 2y(\mu - 2rp^{-2}) + z\mu^2 = 0,$$

of which the former is the tangent at $(\mu^2, \mu, 1)$, and so touches the inscribed circle at this point. That it touches the line, spoken of, through P, whose equation is $W(\alpha + \gamma) + V(\alpha + \beta) = 0$, or $x(\alpha + p) - 2y(\alpha^2 + q) + z(\alpha q + r) = 0$, may be verified by shewing that the result of eliminating y between this equation and the equation of the conic, is a perfect square in x and z; or by first finding the tangential equation of the conic, which is

$$[2u(pq - r) + vp^2][2w(pq - r) + vq^2] = v^2r^2,$$

and then verifying that this is satisfied by $u = \alpha + p$, $v = -2(\alpha^2 + q)$, $w = \alpha q + r$. The algebra is the same in both cases.

We may also give an algebraic proof of Feuerbach's theorem with coordinates which are referred to the points A, B, C. Without any loss of generality the equation of the circle touching BC, CA, AB may be taken to be

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0,$$

the absolute line, IJ, having the equation $lx + my + nz = 0$. Now consider the conic, say, N, expressed by the equation

$$2lmnC + fgh(lx + my + nz)(f^{-1}x + g^{-1}y + h^{-1}z) = 0,$$

where C is the expression $x^2 - 2yz + \dots$ which occurs on the left side of the equation of the inscribed circle, and $f = m - n$, $g = n - l$, $h = l - m$. Its intersections with the circle expressed by $C = 0$ are the intersections of this with the line $lx + my + nz = 0$, and with the line $f^{-1}x + g^{-1}y + h^{-1}z = 0$. The former is the absolute line, so that the conic considered, N, is also a circle; the latter is in fact a tangent of the circle $C=0$; thus the conic considered, N, is a circle touching the circle $C=0$. To prove that the line $f^{-1}x+\ldots=0$ touches the circle $C=0$, we remark, more generally, that a conic whose equation is that obtained by rationalising the equation

$$(ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=0,$$

which is the most general conic touching BC, CA, AB, has for its tangent at any point, (x', y', z'), the line given by

$$x\,(a/x')^{\frac{1}{2}}+y\,(b/y')^{\frac{1}{2}}+z\,(c/z')^{\frac{1}{2}}=0,$$

and has the tangential equation $avw+bwu+cuv=0$. Consider now the intersections of the conic N with BC, CA, AB; it is easily verified that three of these are the middle points, respectively, of B, C, of C, A, and of A, B, in regard to the absolute line. This suffices to identify the conic N with the nine points circle; and so we have proved Feuerbach's theorem. If the other intersections of N, respectively with BC, CA, AB, be D, E, F, and D' be the harmonic conjugate of D in regard to B and C, with a similar definition for E' and F', on CA and AB, it may be proved that D', E', F' lie on the line whose equation is

$$(x+y+z)\,(mn+nl+lm)-(l^2x+m^2y+n^2z)=0.$$

Ex. 10. *Hamilton's modification of Feuerbach's theorem. Introductory results.*

The following results can be verified (cf. Chap. I, Ex. 33). Let A, B, C be any triad of points, not in line, and O, O' be two further points. Let AO, BO, CO meet BC, CA, AB, respectively, in D, E, F, and AO', BO', CO' meet BC, CA, AB, respectively, in D', E', F'; let EF, $E'F'$ meet in P, while FD, $F'D'$ meet in Q, and DE, $D'E'$ meet in R. Then the lines QR, RP, PQ, respectively, contain A, B, C, while the lines AP, BQ, CR meet in a point. Also, the lines DP, EQ, FR meet in a point, say, K, and the lines $D'P$, $E'Q$, $F'R$ meet in a point, say, K'. Further, the line KK' passes through the intersection of QR with BC, of RP with CA, and of PQ with AB.

A conic, say, S, can be drawn to touch BC, CA, AB, respectively, in D, E, F, and a conic, say, Ω, can be drawn to touch BC, CA, AB, respectively, in D', E', F'. And a conic, say, T, can be drawn to contain the six points D, D', E, E', F, F'. The points K, K' are the points of contact, respectively, with S and Ω, of their fourth common tangent, and these points lie on T. The triad P, Q, R is self-polar in regard to each of the three conics S, Ω, T. Further, if EF, FD, DE, respectively, meet BC, CA, AB in U, V, W, these points are in line, and the two intersections of this line with the conic Ω form, with K', a triad of points of Ω which is apolar (see below) with the triad D', E', F', of points thereon. Incidentally we have a construction, when three common tangents of two conics, and their points of contact, are given, for finding the points of contact of the remaining common tangent.

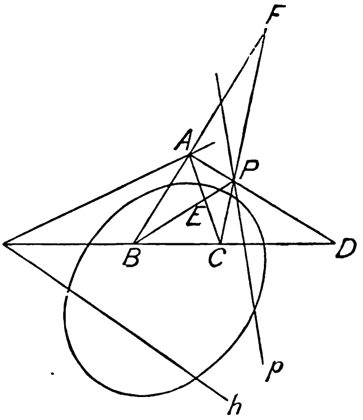
If the conics S and Ω be expressed, respectively, by the equations

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0, \quad (l'x)^{\frac{1}{2}} + (m'y)^{\frac{1}{2}} + (n'z)^{\frac{1}{2}} = 0,$$

and we denote $mn' - m'n$, $nl' - n'l$, $lm' - l'm$, respectively, by p, q, r, it will be found that the point P is $(-p, q, r)$, with similar forms for Q and R, and that K and K' are, respectively, $(p^2 l, q^2 m, r^2 n)$ and $(p^2 l', q^2 m', r^2 n')$. We can express the points of the conic Ω in terms of a parameter, θ, by the equations $l'x = \theta^2$, $m'y = (\theta - 1)^2$, $n'z = 1$, the points D', E', F' being then given, respectively, by $\theta = 0$, $\theta = 1$, $\theta = \infty$, and the point K' by $rn'\theta + pl' = 0$. The line UVW, whose equation is $lx + my + nz = 0$, meets Ω in points, I, J, given by the equation $\theta^2 (l/l') + (1-\theta)^2 (m/m') + n/n' = 0$. For the three points I, J, K' the parameter values are then given by an equation of the form $\mathrm{a}\theta^3 + \theta^2 - \theta + \mathrm{b} = 0$, while the cubic equation for D', E', F' reduces to $\theta^2 - \theta$. This is what is meant by saying that the triad I, J, K' is apolar with the triad D', E', F' (see, above, Ex. 8). The equation of the conic T is

$$\Sigma ll'x^2 - \Sigma (mn' + m'n) yz = 0.$$

Ex. 11. *Hamilton's modification of Feuerbach's theorem.* Let the lines joining a point, P, of a conic, to three other points of the conic, A, B, C, meet BC, CA, AB, respectively, in D, E, F, so that D, E, F is a self-polar triad for the conic. We have remarked that the tangents at A, B, C meet BC, CA, AB, respectively, in the points of a line (Chap. I, Ex. 9), which we denote by h; we denote the tangent at P by p. We have already proved (Chap. I, Ex. 33) that if any conic having D, E, F for a self-polar triad be drawn, its four intersections with the lines p and h lie on a conic passing through A, B, C. We can take coordinates referred to D, E, F, in such a way that the tangents of the original conic at P, A, B, C have, respectively, the equations



$$x + y + z = 0, \quad -x + y + z = 0, \quad x - y + z = 0, \quad x + y - z = 0;$$

the original conic will then have an equation of the form

$$a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = 0,$$

wherein $a + b + c = 0$; any conic having D, E, F for self-polar triad will have an equation of the form $Ax^2 + By^2 + Cz^2 = 0$. The conic containing the four intersections of this with the lines p, h, and also the points A, B, C, will then have the equation

$$(Aa^2 + Bb^2 + Cc^2)(ax + by + cz)(x + y + z) = 4abc\,(Ax^2 + By^2 + Cz^2).$$

Conversely, if any conic be taken through A, B, C, meeting h and p, then a conic can be drawn through the four intersections, having D, E, F for a self-polar triad. If we take coordinates referred to A, B, C, and the equation of the original conic be put in the form $yz + zx + xy = 0$, while the conic taken through A, B, C has the equation $fyz + gzx + hxy = 0$, then the conic, through the four intersections of this with p and h, which has D, E, F for a self-polar triad, is

$$(f\eta\zeta + g\zeta\xi + h\xi\eta)(\xi^{-2}x + \eta^{-2}y + \zeta^{-2}z)(x + y + z) \\ + fyz + gzx + hxy = 0,$$

where (ξ, η, ζ) are the coordinates of P.

Ex. 12. *Conic referred to its centre and axes.* The two lines drawn through the centre, C, of a conic, (the pole of the absolute line, IJ, in regard thereto), which are harmonic conjugates of one another both in regard to the two tangents which can be drawn to the conic from C, and in regard to CI, CJ, are, as we have previously said, called the axes of the conic. If these meet the line IJ in A and B, the triad A, B, C is self-polar, and, in reference to these, the equation of the conic is of the form $px^2 + qy^2 = z^2$. This fails if the conic is a parabola, that is, touches the line IJ, a possibility we exclude. We may suppose the coordinates x, y so chosen that the points I, J are on the lines $x \pm iy = 0$, so that, in any circle the terms of the second degree in x and y are $x^2 + y^2$. We replace p, q, respectively, by a^{-2} and b^{-2}. The expression in terms of a parameter, t, of the points of the conic, may then be taken to be

$$x = a(1 - t^2), \quad y = 2bt, \quad z = 1 + t^2,$$

so that $t = b^{-1}y/(z + a^{-1}x)$. It is usual to develop many results in this notation.

The equation

$$a^{-1}x(1 - t_1t_2) + b^{-1}y(t_1 + t_2) = z(1 + t_1t_2)$$

is at once verified to be that of the line joining the points t_1 and t_2; it gives the equation of the tangent by putting $t_2 = t_1$. The tangents to the curve from the points I, J pass through one of the two foci, S, H, given by $x = \pm (a^2 - b^2)^{\frac{1}{2}}$, $y = 0$, $z = 1$, and also through one of the other two foci, S', H', given by $x = 0$, $y = \pm (b^2 - a^2)^{\frac{1}{2}}$, $z = 1$. The normal of the conic at the point (ξ, η, ζ) has the equation

$$a^2\xi^{-1}x - b^2\eta^{-1}y = (a^2 - b^2)\,\zeta^{-1}z,$$

and this same equation, if we regard (x, y, z) as given, and ξ, η, ζ as variable, is that of the rectangular hyperbola of Apollonius, meeting the given conic at the feet of the four normals from (x, y, z), (p. 93).

If we consider the intersections of any circle with the given conic, we immediately verify that any one of the three pairs of chords common to the conic and the circle meets the line IJ in two points which are harmonic in regard to A and B (a proposition often expressed by saying that these chords are equally inclined to one of the axes, cf. Chap. II, p. 79); and further, substituting the above parametric expressions for the points of the conic in the equation of the circle, if t_1, t_2, t_3, t_4 be the parameters of the four intersections, and we assume (in illogical anticipation of the limitation of the symbols which is explained in the following Chapter) that the equation of the fourth order which these satisfy is capable of being written in the form $(t-t_1)(t-t_2)(t-t_3)(t-t_4)=0$, we infer that $p_1=p_3$, where p_r is the sum of the products of sets of r of t_1, t_2, t_3, t_4. In a similar way, from the equation for the Apollonius rectangular hyperbola, we infer that the parameters of the feet of the four normals which can be drawn to the conic from any point satisfy $1-p_2+p_4=0$. This last changes into the former if we replace t_4 by $-t_4^{-1}$. Thus if P_1, P_2, P_3, P_4 be the feet of the four normals, and Q_4 be the other point of the conic lying on the line P_4C, it follows that P_1, P_2, P_3, Q_4 lie on a circle, as has already been proved (Chap. II, p, 93). If three points of the conic, with parameters t_1, t_2, t_3, be such that the normals thereat meet in a point, it can be verified that

$$\Sigma t_2 t_3 = \Sigma (t_2 t_3)^{-1}.$$

Lastly, if a line be drawn through one of the foci $(c, 0, 1)$, where $c^2=a^2-b^2$, at right angles to the tangent of the conic at any point, of equation $a^{-1}x(1-t^2)+2b^{-1}yt-z(1+t^2)=0$, this line will have the equation $2b^{-1}t(x-cz)-a^{-1}(1-t^2)y=0$, and will meet the tangent in the point whose coordinates are $x=a(1-\lambda^2)$, $y=2a\lambda$, $z=1+\lambda^2$, where $\lambda=(a-c)b^{-1}t$, this being a point of the circle $x^2+y^2=a^2z^2$. This has been called the auxiliary circle (Chap. II, p. 85).

Ex. 13. *Confocal conics.* The points I, J, S, H, S', H', in the preceding example, having, respectively, the coordinates

$$(1, i, 0),\ (1, -i, 0),\ (c, 0, 1),\ (-c, 0, 1),\ (0, ic, 1),\ (0, -ic, 1),$$

where $c^2=a^2-b^2$, any two conics touching the four lines $IH'S$, $JS'S$, $IS'H$, $JH'H$, may be taken to have equations

$$a^{-2}x^2+b^{-2}y^2=z^2,\quad (a^2+\lambda)^{-1}x^2+(b^2+\lambda)^{-1}y^2=z^2;$$

these are *confocals*, in regard to I and J. The condition that any line, of equation $lx+my+nz=0$, should touch the second of these conics is $a^2l^2+b^2m^2-n^2+\lambda(m^2+l^2)=0$; the tangential equation of the first conic is $a^2l^2 + b^2m^2 - n^2 = 0$, and the tangential equation of the point pair I, J is $l^2 + m^2 = 0$. We have previously remarked (p. 112, above) that if $f = 0, f' = 0$ be the point equations of any two conics, the equation of the general conic through their four intersections is $\kappa f + f' = 0$, for a proper parameter κ; thus the tangential equation of a confocal conic appears, as it should, as that of the general conic touching the tangents from the points I, J to any one conic of the system. It is easy to verify the properties of a system of confocal conics, many of which have already been referred to (pp. 79, 90, 93), by means of these equations. In particular, the polar of a point (ξ, η, ζ), in regard to the conic

$$(a^2 + \lambda)^{-1}x^2 + \dots - z^2 = 0,$$

has the equation

$$(a^2 + \lambda)^{-1}\xi x + (b^2 + \lambda)^{-1}\eta y - \zeta z = 0,$$

which we may write in the form $\lambda^2 Z + \lambda Y + X = 0$, where

$$Z = \zeta z, \quad Y = (a^2 + b^2)\,\zeta z - \xi x - \eta y, \quad X = a^2b^2\zeta z - a^2\eta y - b^2\xi x.$$

By what we have previously seen, this is the equation of the tangent of the conic whose equation is $4XZ - Y^2 = 0$. It can be verified that this is the same as is obtained by rationalising the equation

$$(\xi x)^{\frac{1}{2}} + (-\eta y)^{\frac{1}{2}} + c\,(\zeta z)^{\frac{1}{2}} = 0.$$

This conic touches the line $x = 0$ (for $\lambda = -a^2$), the line $y = 0$ (for $\lambda = -b^2$), and the line $z = 0$ (for $\lambda = \infty$); it is thus a parabola, in regard to I and J, the point C, at which two perpendicular tangents meet, being on the directrix. The point (ξ, η, ζ) is equally on the directrix; for there are two of the confocals passing through this point, those namely for which λ has either of the values for which $(a^2 + \lambda)^{-1}\xi^2 + \dots - \zeta^2 = 0$, and these cut at right angles; for the tangents are $(a^2 + \lambda_1)^{-1}\xi x + \dots - \zeta z = 0$ and $(a^2 + \lambda_2)^{-1}\xi x + \dots - \zeta z = 0$, which are at right angles (p. 110, above), if

$$(a^2 + \lambda_1)^{-1}(a^2 + \lambda_2)^{-1}\xi^2 + (b^2 + \lambda_1)^{-1}(b^2 + \lambda_2)^{-1}\eta^2 = 0,$$

an equation obtainable from

$$(a^2 + \lambda_1)^{-1}\xi^2 + \dots - \zeta^2 = 0 \quad \text{and} \quad (a^2 + \lambda_2)^{-1}\xi^2 + \dots - \zeta^2 = 0;$$

and these tangents are the polars of (ξ, η, ζ) in regard to these two confocals. The focus of the parabola is the intersection of two of its tangents passing respectively through I and J; the polar $(a^2 + \lambda)^{-1}\xi x + \dots - \zeta z = 0$ passes through $(1, i, 0)$ for the value of λ given by $(a^2 + \lambda)^{-1}\xi + (b^2 + \lambda)^{-1}i\eta = 0$; and we similarly find the polar passing through $(1, -i, 0)$. The intersection of these polars gives the focus, which is found to be $(c^2\xi\zeta, \ -c^2\eta\zeta, \ \xi^2 + \eta^2)$. It is easy to verify that the general conic passing through the four foci has the equation $x^2 - y^2 - c^2z^2 + 2hxy = 0$, where h is variable, and hence that the focus of the parabola is conjugate to (ξ, η, ζ) in regard to all these conics. Further we may verify that the polar reciprocal of the parabola in regard to $a^{-2}x^2 + b^{-2}y^2 - z^2 = 0$, is the Apollonius rectangular hyperbola containing the feet of the normals of this conic which pass through (ξ, η, ζ); for, in order that the polar of a point (x', y', z'), in regard to the conic $a^{-2}x^2 + b^{-2}y^2 - z^2 = 0$, may coincide with a tangent of the parabola whose equation is

$$(a^2 + \lambda)^{-1}\xi x + (b^2 + \lambda)^{-1}\eta y - \zeta z = 0,$$

it is necessary and sufficient that

$$a^{-2}x'/(a^2 + \lambda)^{-1}\xi = b^{-2}y'/(b^2 + \lambda)^{-1}\eta = z'/\zeta;$$

this can be satisfied by a proper value of λ if

$$a^2\xi(x')^{-1} - b^2\eta(y')^{-1} = (a^2 - b^2)\zeta(z')^{-1}.$$

Ex. 14. *Conics derived from a conic intersecting the joins of three points.* If a conic, S, which meets the lines BC, CA, AB, respectively, in D, D', E, E'; F, F', have the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the lines AD, AD' are given by $by^2 + cz^2 + 2fyz = 0$. The coordinates of either of these lines, whose equation may be written $vy + wz = 0$, are then such that, $b^{-1}v^2 + c^{-1}w^2 - 2b^{-1}c^{-1}fvw = 0$. The six lines AD, AD', BE, BE', CF, CF' are thus tangents of the conic, Σ', whose tangential equation is

$$bcu^2 + cav^2 + abw^2 - 2afvw - 2bgwu - 2chuv = 0,$$

or, say, $\Sigma' = 0$. If $\Sigma = 0$ be the tangential equation of the conic S, so that Σ is the sum of three pairs of terms such as

$$(bc - f^2)u^2 + 2(gh - af)vw,$$

we see that $\Sigma' - \Sigma$ is the sum of three pairs of terms such as $f^2u^2 - 2ghvw$. We have remarked that a conic whose equation is obtained by rationalising the equation $(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0$ touches BC, CA, AB; similarly a conic whose tangential equation, $\Omega = 0$, is obtained by rationalising $(fu)^{\frac{1}{2}} + (gv)^{\frac{1}{2}} + (hw)^{\frac{1}{2}} = 0$, is one which passes through A, B, C; its point equation is, in fact, $fyz + gzx + hxy = 0$. In the case in hand we have $\Sigma' - \Sigma = \Omega$. The four common tangents of the conics S and Σ' are thus touched by the conic Ω, which passes through A, B, C. Similarly, by taking the point equation of the conic Σ', we see that there is a conic passing through the four common points of S and Σ' which touches BC, CA, AB, its point equation being $(afx)^{\frac{1}{2}} + (bgy)^{\frac{1}{2}} + (chz)^{\frac{1}{2}} = 0$.

A particular case of the preceding is when S is such that Σ' degenerates into a point pair, so that the lines AD, BE, CF meet in a point, say, O, and AD', BE', CF' also meet in a point, say, O'.

Then the points, on BC, CA, AB, which are the respective harmonic conjugates of D, E, F with regard to B, C; C, A; A, B, lie on a line, say, l; and the points, on BC, CA, AB, similarly derived from D', E', F', also lie on a line, say, l'. In general, these derived six points lie on the conic whose equation is

$$ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0,$$

as is easy to see; in the case now considered, this conic breaks into the two lines, l and l'. When this is so, it follows, from what has been said, that there is a conic through A, B, C which touches the tangents drawn to S from O and O'. From the equations it also follows easily that this conic passes through the intersections of S with the lines l and l'; and also, through these intersections there passes a conic in regard to which A, B, C are a self-polar triad, the lines l and l' being the polars, respectively, of O' and O, in regard to this. The point equation of a conic, given in tangential form, which breaks into a point pair, represents the line joining these points, taken twice over, as is easy to see. Thus, in the case now considered, there is a conic touching BC, CA, AB which touches S at its two intersections with the line OO'; and, further, the tangents of the conics, at their two points of contact, meet in the intersection of the lines l, l'. In the present case, we may regard S as the nine points circle of the triad, A, B, C; we have only to take for absolute points the intersections of S with the line l; it can be proved that these are conjugate points in regard to all conics through the four points A, B, C, O', so that, for instance, AO' is at right angles to BC in reference to these points. Thus the results here stated are equivalent to saying that the two points O, O', respectively the intersection of the lines from A, B, C to the middle points of B, C, of C, A, and of A, B, and the orthocentre of A, B, C, are the centres of similitude of the nine points circle and the circle A, B, C, the radical axis of these circles being the line l', derived from the orthocentre, O', by the rule given; further that there is a circle, coaxial with these two circles, with centre at the orthocentre, for which A, B, C is a self-polar triad, and for which O is the pole of the radical axis spoken of; and, also, that there is a conic touching BC, CA, AB, which touches the nine points circle at two points of the line OO' whereat the tangents are parallel (with regard to the absolute line l) to this radical axis. (See also Chap. I, Ex. 29.)

It would seem to be worth while to speak of the conic whose equation, referred to three points A, B, C, is

$$ax^2 - 2fyz + by^2 - 2gzx + cz^2 - 2hxy = 0,$$

as the harmonically conjugate of the conic whose equation is

$$ax^2 + 2fyz + by^2 + 2gzx + cz^2 + 2hxy = 0.$$

If the latter meet BC in P_1 and P_2, the former meets BC in two points, P_1', P_2', which are the harmonic conjugates, in regard to B and C, respectively, of P_1 and P_2, with a similar statement for CA and AB. And, further, to speak of the latter conic as being bipunctual, in regard to A, B, C, when the six lines such as AP_1, AP_2, meet in two points. The condition for this is that the harmonically conjugate conic break up into two lines.

If X, Y, Z be a self-polar triad in regard to a conic, and O be any point, and XO, YO, ZO meet YZ, ZX, XY, respectively, in A, B, C, then the conic is bipunctual in regard to A, B, C. For if we suppose O to be $(1, 1, 1)$, relatively to X, Y, Z, and the equation of the conic relatively to these be $ax^2 + by^2 + cz^2 = 0$, by taking

$$\xi = \tfrac{1}{2}(-x+y+z), \quad \eta = \tfrac{1}{2}(x-y+z), \quad \zeta = \tfrac{1}{2}(x+y-z),$$

we have, for the equation in regard to A, B, C, the form

$$a(\eta+\zeta)^2 + b(\zeta+\xi)^2 + c(\xi+\eta)^2 = 0.$$

The harmonically conjugate conic, in regard to A, B, C, is then

$$a(\eta-\zeta)^2 + b(\zeta-\xi)^2 + c(\xi-\eta)^2 = 0,$$

which is evidently a line pair.

Conversely, if a conic be bipunctual in regard to A, B, C, it is possible, and in only one way, to draw lines, YZ, ZX, XY, passing, respectively, through A, B, C, such that the triad X, Y, Z, formed by their intersections, is self-polar in regard to the conic; and the lines AX, BY, CZ meet in a point. For, if, referred to A, B, C, the conic be

$$ll'\xi^2 + mm'\eta^2 + nn'\zeta^2 - (mn'+m'n)\eta\zeta - (nl'+n'l)\zeta\xi - (lm'+l'm)\xi\eta = 0,$$

and we attempt to write this equation in the form

$$P(\eta+p\zeta)^2 + Q(\zeta+q\xi)^2 + R(\xi+r\eta)^2 = 0,$$

we find

$$p = (nl'-n'l)(lm'-l'm)^{-1}, \quad -2Pp = mn'+m'n,$$

with similar values for q, Q, r, R, (so that $pqr = 1$).

We may similarly consider a conic such that the tangents to it from A, B, C meet BC, CA, AB in points lying, in threes, upon two lines.

Ex. 15. *Two triads reciprocal in regard to a conic are in perspective.* If P, Q, R be the poles, with regard to any conic whose tangential equation referred to A, B, C is

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0,$$

respectively of BC, CA, AB, then the lines AP, BQ, CR meet in a point (Ex. 10, Chap. I). For P, Q, R have the respective coordinates (A, H, G), (H, B, F), (G, F, C), and the lines AP, BQ, CR meet in the point given by

$$Fx = Gy = Hz.$$

Dually, the polars of A, B, C meet BC, CA, AB, respectively, in three points lying in line. If the equation of the conic be

$$ax^2+\ldots+2fyz+\ldots=0,$$

the equation of this line is $f^{-1}x+g^{-1}y+h^{-1}z=0$.

Ex. 16. *The Hessian line of three points of a conic.* If the tangents at three points, A, B, C, of a conic, meet BC, CA, AB, respectively in D, E, F, the points D, E, F are in line (Chap. I, Ex. 9). If the points of the conic be expressed by a parameter, and the values of this for the points A, B, C be the values of x/y for which $ax^3+3bx^2y+3cxy^2+dy^3=0$, the values of the parameter at the two points, H, K, at which the line D, E, F meets the conic, are the two values of x/y for which

$$\frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2}-\left(\frac{\partial^2 F}{\partial x\partial y}\right)^2=0,$$

where $F=0$ is the equation above (cf. Vol. I, p. 174). Further, if the left side of the last quadratic be multiplied by $x+my$, *where m is arbitrary*, and the result be written $a'x^3+3b'x^2y+3c'xy^2+d'y^3=0$, it may be proved that $ad'-3bc'+3cb'-da'=0$. In other words, any three points of a conic are apolar with the Hessian points of these taken with a third arbitrary point. If we take the equation of the conic in the form $fyz+gzx+hxy=0$, the points (x, y, z) can be expressed in terms of a parameter by $x=-f\theta^{-1}$, $y=g(\theta-1)^{-1}$, $z=h$, so that A, B, C correspond, respectively, to $\theta=0$, 1, ∞; the points H, K are then given by $\theta^2-\theta+1=0$, and it is easy to verify the relation in question for the two cubics

$$x^2y-xy^2=0,\quad (x+my)(x^2-xy+y^2)=0.$$

The relation is unaffected if, instead of θ, we use a parameter ϕ connected with θ by an equation $A\theta\phi+B\theta+C\phi+D=0$, wherein A, B, C, D are constants. Similar remarks hold for the dual case of three given tangents of a conic.

Ex. 17. *The director circle of a conic, and generalisations.* It can be proved that the equation of the two tangents drawn from a point (x', y', z') to touch the conic $U=0$ is $UU'-T^2=0$, where U' is what U becomes when x', y', z' are written in place of x, y, z; and T is one half of what is obtained by operating upon U by

$$x'\partial/\partial x+y'\partial/\partial y+z'\partial/\partial z,$$

and is symmetrical in regard to x, y, z and x', y', z'. This result follows by considering that the points, with coordinates of the forms $(x+\theta x', y+\theta y', z+\theta z')$, on the line joining (x, y, z) to (x', y', z'), which lie on the conic, are given by $U+2\theta T+\theta^2 U'=0$.

Further, if these tangents meet the line joining the points, I, J, which have coordinates $(1, \pm i, 0)$, in two points harmonic in regard to these, that is, if the tangents be at right angles, then (x', y', z') lies on the locus whose equation is

$$(a+b)\,U = (ax + hy + gz)^2 + (hx + by + fz)^2,$$

or

$$C\,(x^2 + y^2) - 2Gxz - 2Fyz + (A + B)\,z^2 = 0\,;$$

here U is $ax^2 + \ldots + 2fyz + \ldots$, and $A = bc - f^2$, $F = gh - af$, etc. This gives the equation of the director circle in regard to I and J.

More generally, the locus of the point of intersection of two lines, conjugate to one another in regard to the conic, drawn respectively through the points (x_1, y_1, z_1), (x_2, y_2, z_2), is $UT_{12} = T_1T_2$, where T_1 and T_2 are the forms of T when (x', y', z') are replaced, respectively, by (x_1, y_1, z_1) and (x_2, y_2, z_2), while T_{12} is obtained from T by replacing (x, y, z) and (x', y', z'), respectively, by (x_1, y_1, z_1) and (x_2, y_2, z_2). This locus becomes the director circle when the points (x_1, y_1, z_1), (x_2, y_2, z_2) are the points $(1, \pm i, 0)$.

This however may be generalised; we may seek the locus of a point from which the tangents to a conic are conjugate in regard, not to a point pair but, to another general conic, that is, the locus of a point from which the pair of tangents to one conic are harmonic conjugates of one another in regard to the tangents from that point to another conic. The locus is still a conic; and it is obvious by the definition that this contains the eight points of contact with the two conics of their four common tangents. If the tangential equation of one of the conics be $Au^2 + \ldots + 2Fvw + \ldots = 0$, that of the other being of the same form with $A', \ldots, F', \ldots$ in place of $A, \ldots, F, \ldots$, the pair of tangents to the former from the point (ξ, η, ζ) is given by

$$A\,(y\zeta - \eta z)^2 + \ldots + 2F\,(z\xi - \zeta x)\,(x\eta - \xi y) + \ldots = 0,$$

and these meet $z = 0$ in two points given, say, by $\mathrm{a}x^2 + 2\mathrm{h}xy + \mathrm{b}y^2 = 0$, where

$$\mathrm{a} = B\zeta^2 - 2F\zeta\eta + C\eta^2, \quad \mathrm{h} = -H\zeta^2 + F\xi\zeta + G\eta\zeta - C\xi\eta,$$
$$\mathrm{b} = A\zeta^2 - 2G\zeta\xi + C\xi^2\,;$$

the corresponding values for the second conic being denoted by a', h', b', we require to express that $\mathrm{ab}' - 2\mathrm{hh}' + \mathrm{ba}' = 0$. On reduction this shews that (ξ, η, ζ) describes the conic whose equation is

$$(BC' + B'C - 2FF')\,x^2 + \ldots + 2\,(GH' + G'H - AF' - A'F)\,yz + \ldots = 0.$$

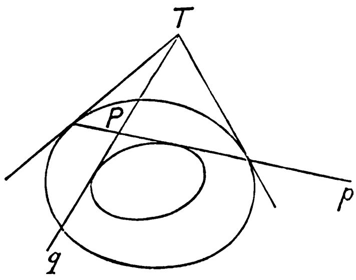
Dually, a line whose intersections with one conic,

$$ax^2 + \ldots + 2fyz + \ldots = 0,$$

are harmonic in regard to its intersections with another such conic, envelopes the conic whose tangential equation is

$$(bc' + b'c - 2ff')\,u^2 + \ldots + 2\,(gh' + g'h - af' - a'f)\,vw + \ldots = 0\,;$$

this conic evidently touches the eight tangents of the two conics at their common points.



For the former case, of two conics given tangentially, we may, by taking a pair of the intersections of their common tangents as the absolute points, I, J, regard the two conics as belonging to a system of confocals, their equations being

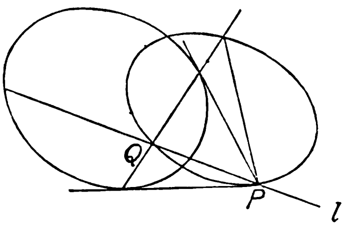
$$a^{-2}x^2+b^{-2}y^2-z^2=0,$$

$$(a^2+\lambda)^{-1}x^2+(b^2+\lambda)^{-1}y^2-z^2=0.$$

The tangents from a point, P, to the former conic, say, p and q, will be conjugate in regard to the second conic, if q passes through the pole, T, of the line p in regard to the second conic. The locus of P, with these coordinates, is found to be

$$2a^2b^2\left[a^{-2}x^2+b^{-2}y^2-z^2\right]+\lambda\left[x^2+y^2-(a^2+b^2)z^2\right]=0;$$

it meets either of the two conics in the four points where this is met by its director circle.



For the latter case, of two conics given in point coordinates, we may regard both the conics as circles. A line, l, will meet the circles harmonically if it join a point P, of one circle, to one of the points, Q, where this circle is met by the polar of P taken in regard to the other circle. The circles being supposed to have equations

$$x^2+y^2+2gxz+cz^2=0,\quad x^2+y^2+2g'xz+cz^2=0,$$

as we have shewn to be possible (above, p. 109), the envelope is found to be given by

$$(u^2+v^2)(c-gg')+(gu-w)(g'u-w)=0.$$

The equation $u^2+v^2=0$ represents the absolute points I, J, and the equation $(gu-w)(g'u-w)=0$ represents the centres of the two circles. The envelope has the centres of the two circles for foci, and touches the tangents of the circles drawn at their intersections. A particular case is that in which the two circles cut at right angles; then $c=gg'$, and the envelope reduces to the point pair given by the centres of the circles. We have previously seen (Chap. II, p. 66) that when two circles cut at right angles, any line through the centre of either is divided harmonically by the circles. [Add.]

A particular case is when one of the two conics is a line pair. For example, if the conic be the circle $x^2+y^2-(a^2+b^2)z^2=0$, and the line pair be $a^{-2}x^2 + b^{-2}y^2 = 0$, the envelope is found to be $a^2u^2 + b^2v^2 - w^2 = 0$, which is the tangential equation of the conic $a^{-2}x^2 + b^{-2}y^2 - z^2 = 0$. Thus any tangent of a conic meets the director circle in two points which are harmonic in regard to the points in which the tangent meets the asymptotes of the conic.

The director circle, S, of a conic Σ, has been shewn (Chap. II, p. 96) to be such that sets of four points can be taken thereon whose joins, in order, are four tangents of the conic. It may be proved that for two conics $S = 0$, $\Sigma = 0$, where

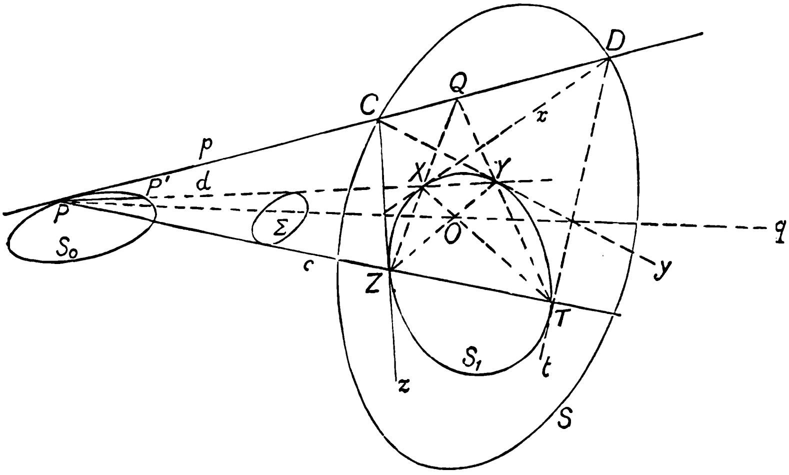
$$S = ax^2 + by^2 + cz^2, \quad \Sigma = Au^2 + Bv^2 + Cw^2,$$

the condition for this possibility is that the sum of two of the quantities aA, bB, cC should be equal to the third.

Ex. 18. *Envelope of a line joining points of two conics which are conjugate in regard to a third.* We now prove the following theorem, which will be found to include many particular results: let

$$S_0 = x^2 + y^2 + z^2, \qquad \Sigma_0 = u^2 + v^2 + w^2,$$
$$S_1 = ax^2 + by^2 + cz^2, \quad \Sigma_1 = a^{-1}u^2 + b^{-1}v^2 + c^{-1}w^2,$$

so that $\Sigma_0 = 0$ is the tangential equation of the conic $S_0 = 0$, and $\Sigma_1 = 0$ is the tangential equation of the conic $S_1 = 0$. In general, any two conics have four distinct intersections, and their equations can be reduced to these forms. Any conic touching the four common tangents of Σ_0 and Σ_1 has a tangential equation $\Sigma_0 - \theta^2\Sigma_1 = 0$. We prove that this conic is the envelope of a line joining two points, lying respectively on the conics S_0 and S_1, say P and Z, which are conjugate to one another in regard to the conic $(abc)^{\frac{1}{2}}S_0 + \theta S_1 = 0$, which passes through the common points of S_0 and S_1; and



conversely. Further, any conic through the intersections of S_0 and S_1 has an equation of the form $\phi^2 S_0 - S_1 = 0$. We prove that this conic is the locus of the intersection of two tangents, respectively of the conics S_0 and S_1, say p and z, which are conjugate to one another in regard to the conic $\phi\Sigma_0 + (abc)^{\frac{1}{2}}\Sigma_1 = 0$, which touches the common tangents of S_0 and S_1. But further, when $\phi = \theta$, the two dually corresponding facts arise in the same figure, the lines p and z being the tangents at the points P, Z, of the conics S_0 and S_1.

Let S_0, S_1 and S be any three conics which have four points in common. Let the tangent line, p, at any point, P, of the conic S_0, meet S in two points of which C is one. Let the two tangents be drawn from C to the conic S_1, the point of contact of one of these being Z. Denote the line PZ by c. From the second intersection, D, with the conic S, of the line p, draw a tangent to S_1, touching this in X. Let ZX meet p in Q. Let the line joining P to the intersection of the tangents of S_1 at X and Z be q.

Let the equations of S_0, S_1 be, respectively $S_0 = 0$, $S_1 = 0$, where $S_0 = x^2 + y^2 + z^2$, $S_1 = ax^2 + by^2 + cz^2$, and the equation of S be $\theta^2 S_0 - S_1 = 0$; let the coordinates of the point P be (ξ, η, ζ), and of the line p be (λ, μ, ν), so that we have $\xi^2 + \eta^2 + \zeta^2 = 0$, $\lambda^2 + \mu^2 + \nu^2 = 0$, $\lambda\xi + \mu\eta + \nu\zeta = 0$. Let (x, y, z) be any other point, and (l, m, n) any other line; let $P_0 = x\xi + y\eta + z\zeta$, and $P_1 = ax\xi + by\eta + cz\zeta$. Also let $\Sigma_0 = l^2 + m^2 + n^2$, $\Sigma_1 = a^{-1}l^2 + b^{-1}m^2 + c^{-1}n^2$, $\Pi_0 = l\lambda + m\mu + n\nu$, $\Pi_1 = a^{-1}l\lambda + b^{-1}m\mu + c^{-1}n\nu$. We can at once verify the following equations, identically true in regard to x, y, z and l, m, n:

$$(\theta^2 - a)(by\zeta - cz\eta)^2 + (\theta^2 - b)(cz\xi - ax\zeta)^2 + (\theta^2 - c)(ax\eta - by\xi)^2$$
$$= -(bc\xi^2 + ca\eta^2 + ab\zeta^2) S_1 + abcP_0^2 - \theta^2 P_1^2,$$

$$abc\{(a^{-1}\theta^2 - 1)(y\zeta - z\eta)^2 + (b^{-1}\theta^2 - 1)(z\xi - x\zeta)^2$$
$$+ (c^{-1}\theta^2 - 1)(x\eta - y\xi)^2\}$$
$$= \theta^2(a\xi^2 + b\eta^2 + c\zeta^2) S_1 + abcP_0^2 - \theta^2 P_1^2,$$

$$(\theta^{-2} - a^{-1})(b^{-1}m\nu - c^{-1}n\mu)^2 + (\theta^{-2} - b^{-1})(c^{-1}n\lambda - a^{-1}l\nu)^2$$
$$+ (\theta^{-2} - c^{-1})(a^{-1}l\mu - b^{-1}m\lambda)^2$$
$$= -[(bc)^{-1}\lambda^2 + (ca)^{-1}\mu^2 + (ab)^{-1}\nu^2]\Sigma_1 + (abc)^{-1}\Pi_0^2 - \theta^{-2}\Pi_1^2,$$

$$(abc)^{-1}\{(a\theta^{-2} - 1)(m\nu - n\mu)^2 + (b\theta^{-2} - 1)(n\lambda - l\nu)^2$$
$$+ (c\theta^{-2} - 1)(l\mu - m\lambda)^2\}$$
$$= \theta^{-2}(a^{-1}\lambda^2 + b^{-1}\mu^2 + c^{-1}\nu^2)\Sigma_1 + (abc)^{-1}\Pi_0^2 - \theta^{-2}\Pi_1^2,$$

which we shall call, respectively, (1), (2), (3) and (4). It will be seen that (3) and (4) are at once deducible from (1) and (2), and it is easy to deduce (2) from (1).

Now, regarding x, y, z as the coordinates of a variable point, the left side of the equation (1) vanishes when the point, which is the intersection of the line p with the polar line of (x, y, z) in regard to S_1, lies on the conic $\theta^2 S_0 - S_1 = 0$, or S; that is, when (x, y, z) is the pole, in regard to S_1, of any line through C or D. In other words, the left side of equation (1), put equal to zero, is the equation of the line pair consisting of the polars, in regard to S_1, of the points C and D. These lines pass, respectively, through Z and X, and meet S_1 in the points of contact of the other tangents to S_1 drawn from C and D, say Y and T. On the right side of equation (1), the terms $abcP_0^2 - \theta^2 P_1^2$, put equal to zero, would represent the line pair consisting of the polars of (ξ, η, ζ) in regard to the two conics $(abc)^{\frac{1}{2}} S_0 \pm \theta S_1 = 0$. Thus, the right side of equation (1), put equal to zero, would represent a conic passing through the points where these two polar lines meet the conic S_1. Therefore, these two polar lines are either the pair XZ, YT, or else the pair XY, ZT. We prove that they are the former pair. For we have shewn (Chap. I, Ex. 30) that the tangents of the conic S_1, at the points where it is met by the line PZ, are such as to meet the tangent, p, of S_0, in points lying on a conic through the common points of S and S_1; and such conic is identified by the single point C in addition to these four common points. Thus P, Z, T are in line, and, similarly, P, X, Y are in line. Wherefore, the three points consisting of the point P, the point of intersection, say, O, of XT and YZ, and the point of intersection of ZX and TY, form a self-polar triad in regard to S_1. Then, as O, the intersection of the polars of C and D in regard to S_1, is the pole of CD, it follows that ZX and TY meet on CD, so that Q, Y, T are in line. The lines whose equations are $(abc)^{\frac{1}{2}} P_0 \pm \theta P_1 = 0$, not containing the point P, save for one particular value of θ, are thus the lines ZX, TY. Therefore, the point Q is the point of intersection of the polars of P taken in regard to all the conics passing through the common points of S_0 and S_1. It follows, thence, that the points P, X are harmonic conjugates of one another in regard to the two points in which the line PX meets one of the two conics $(abc)^{\frac{1}{2}} S_0 \pm \theta S_1 = 0$, while the points P, Y are harmonic conjugates of one another in regard to the two points in which this line meets the other of these two conics. If P' be the other point in which the line PX meets the conic S_0, it will appear that P' and Y are conjugate in regard to the same conic as that for which P, X are conjugate, while P' and X and P, Y are, likewise, two pairs of conjugate points with respect to the same one of the two conics $(abc)^{\frac{1}{2}} S_0 \pm \theta S_1 = 0$.

Next, consider the equation (2). The left side vanishes when the line joining the points (x, y, z) and (ξ, η, ζ) touches the conic whose tangential equation is $\theta^2 \Sigma_1 - \Sigma_0 = 0$. This is a particular conic touching the common tangent lines of S_0 and S_1. The left side of equation (2), put equal to zero, thus represents the two tangents drawn from P to the conic $\theta^2\Sigma_1 - \Sigma_0 = 0$. The right side of equation (2), put equal to zero, represents, however, a conic passing through the points X, Y, Z, T; this consists, then, of the lines PXY, PZT. These lines, therefore, touch the conic $\theta^2\Sigma_1 - \Sigma_0 = 0$, which we may denote by Σ; we may denote the lines PXY, PZT, respectively, by d and c.

Now consider the equation (3). Regarding l, m, n as coordinates of a variable line, the left side vanishes when the pole $(a^{-1}l, b^{-1}m, c^{-1}n)$, in regard to Σ_1, of this line, and the pole, P, or (λ, μ, ν), of the line p, in regard to Σ, determine a line touching $\theta^{-2}\Sigma_0 - \Sigma_1 = 0$, which is the conic Σ. In other words, the left side of equation (3) vanishes for all lines (l, m, n) which are polars, in regard to Σ_1, of points of either of the two lines PXY, PZT; thus, put equal to zero, it represents, tangentially, the point pair constituted by the poles, in regard to Σ_1, of the lines d and c; if the tangents of S_1, at the points X, Y, Z, T, be called, respectively, x, y, z, t, these poles are the points (x, y) and (z, t). On the right side of the equation (3), the terms $(abc)^{-1}\Pi_0^2 - \theta^{-2}\Pi_1^2$ vanish when the line (l, m, n) passes through the pole, in regard to one of the conics $(abc)^{-\frac{1}{2}}\Sigma_0 \pm \theta^{-1}\Sigma_1 = 0$, of the line (λ, μ, ν); that is, these terms, put equal to zero, are the tangential equation of the point pair consisting of the poles of the line p in regard to these two conics. The right side of equation (3), put equal to zero, is, tangentially, the equation of a conic touching the tangents drawn from these poles to the conic S_1. We thus infer that the poles of the line p, in regard to the conics $\theta\Sigma_0 \pm (abc)^{\frac{1}{2}}\Sigma_1 = 0$, are a complementary pair of intersections of the tangents drawn from C and D to the conic S_1, other than the pair (x, y) and (z, t); they are also other than C and D. These poles are then the points (x, z) and (y, t). It is known that these points lie on a line, q, passing through P. Thus, the tangent, z, drawn from C to the conic S_1, is the conjugate of the tangent, p, drawn from C to the conic S_0, in regard to one of the conics $\theta\Sigma_0 \pm (abc)^{\frac{1}{2}}\Sigma_1 = 0$, and the other tangent, y, is the conjugate of p in regard to the other of these conics. It will appear that the other tangent drawn from C to the conic S_0 may be similarly coupled, respectively, with y and z.

Lastly, consider the equation (4). The left side, equated to zero, is the tangential equation of lines meeting the line p upon the conic $S_1 - \theta^2 S_0 = 0$, or the conic S. Thus it represents the point pair C, D. The equation thus expresses the fact we have already found, that the tangents from the points (x, z), (y, t), to the conic S_1, have the points, C, D, as a complementary pair of intersections.

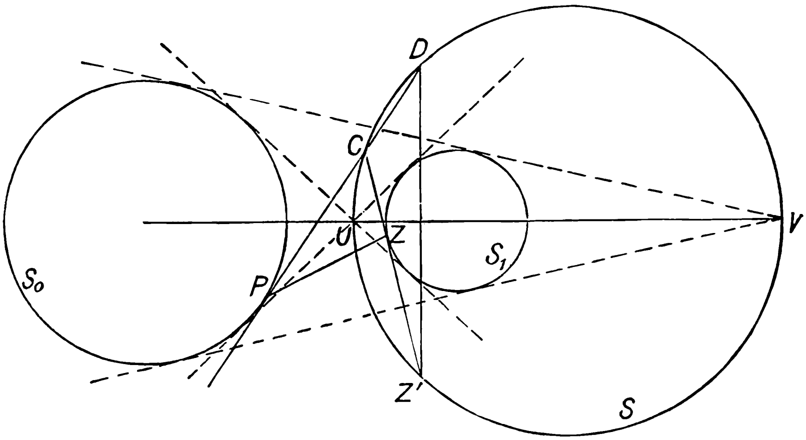
We have thus proved the results stated at the beginning of the example. We consider some particular cases.

A particular case arises when the conic $(abc)^{\frac{1}{2}}S_0 - \theta S_1 = 0$ is a pair of common chords of the conics S_0 and S_1, so that, in brief, the line PZ is divided harmonically by these two lines. The common chords will intersect in one of the three points which form the common self-polar triad for these two conics S_0, S_1. The line joining the other two points of this triad will contain two complementary intersections each of two of the common tangents of S_0 and S_1. It can be shewn that, in this case, the envelope $\Sigma_0 - \theta^2\Sigma_1 = 0$ touches both the common chords considered, and that the conic $\theta^2 S_0 - S_1 = 0$ passes through the two intersections of common tangents spoken of. For, when $(abc)^{\frac{1}{2}}S_0 - \theta S_1 = 0$ is a line pair, say, for $\theta = (bc/a)^{\frac{1}{2}}$, the envelope $\theta\Sigma_0 - (abc)^{\frac{1}{2}}\Sigma_1 = 0$ is the point pair consisting of the two intersections of common tangents spoken of. The coordinates (t, m, n) of the line pair $(abc)^{\frac{1}{2}}S_0 - (bc/a)^{\frac{1}{2}}S_1 = 0$, are, respectively, $l = 0$, $m = (a-b)^{\frac{1}{2}}$, $n = \pm(c-a)^{\frac{1}{2}}$; these touch $\Sigma_0 - (bc/a)\Sigma_1 = 0$, which is $a^{-1}(a^2 - bc)l^2 - (c-a)m^2 + (a-b)n^2 = 0$. The common tangents of S_0 and S_1 have coordinates $[a(b-c)]^{\frac{1}{2}}$, $\pm[b(c-a)]^{\frac{1}{2}}$, $[c(a-b)]^{\frac{1}{2}}$; when $\theta = (bc/a)^{\frac{1}{2}}$, the envelope $\theta\Sigma_0 - (abc)^{\frac{1}{2}}\Sigma_1 = 0$ reduces to $[b(c-a)]^{-1}m^2 - [c(a-b)]^{-1}n^2 = 0$, which represents the two points of intersection of common tangents spoken of. The coordinates of these two points are $x = 0$, $y = [b(c-a)]^{-\frac{1}{2}}$, $z = \pm[c(a-b)]^{-\frac{1}{2}}$; these points lie on the conic $\theta^2 S_0 - S_1 = 0$, which is

$$(bc - a^2)x^2 + b(c-a)y^2 - c(a-b)z^2 = 0.$$

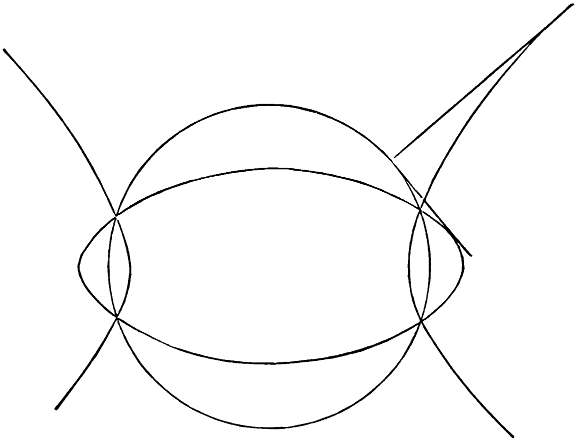
If the two intersections of common tangents spoken of, be called U and V, it appears that the tangents CP, CZ, respectively to S_0 and S_1, drawn from C, are harmonic conjugates in regard to CU and CV; if these tangents meet the conic $\theta^2 S_0 - S_1 = 0$ in further points D and Z', these points will, then, be harmonic conjugates in regard to the points U, V, of this conic. The line DZ' will, thus, pass through the pole of the chord UV of this conic; this is that one originally considered of the points, which are the common self-polar triad of S_0 and S_1, through which the common chords of this conic pass. Thus we have triads of points, C, D, Z', lying on the conic S, of which two joins touch, respectively, the conics S_0 and S_1, while the remaining join, DZ', passes through one of the triad of common conjugate points for S_0 and S_1. (In the general case, Chap. I, Ex. 31, the line DZ' touches a certain conic through the common points of S_0 and S_1.) As, however, the particular case in which the conic $(abc)^{\frac{1}{2}}S_0 - \theta S_1 = 0$ is a pair of common chords of S_0 and S_1, is, as has been remarked, also the particular case in which the envelope $\theta\Sigma_0 - (abc)^{\frac{1}{2}}\Sigma_1 = 0$ is a pair of intersections of common tangents of S_0 and S_1, the fact that $\Sigma_0 - \theta^2\Sigma_1 = 0$ touches the common chords corresponds dually to the fact that the conic

$\theta^2 S_0 - S_1 = 0$ passes through these intersections of common tangents. We have remarked that the tangents CP, CZ are harmonic in regard to CU and CV; these last are, in fact, the double rays of a pencil in involution, of which CP, CZ are a pair, and two other pairs are the lines drawn from C to the pairs of intersections of S_0 and S_1, two intersections being paired when they lie on one of the common chords which we considered. This is obvious because all these four intersections lie on the conic $\theta^2 S_0 - S_1 = 0$ which contains C, and the chords which join these pairs pass through the pole, in regard to this conic, of the line UV. Dually, if we denote the common chords considered by u and v, and consider the envelope $\Sigma_0 - \theta^2\Sigma_1 = 0$ for the particular value of θ necessary that the line c may be a tangent, the tangents, u, v, of this envelope, meet the tangent c in the double points of an involution, of which one pair are the points where c meets S_0 and S_1, and another pair are the points where c meets either pair of common tangents of S_0 and S_1 which meet on the line UV.



We may suppose the conics S_0, S_1 to be two circles, the common chords considered being the radical axis, and the line IJ. Then, if U, V be the intersections of the common tangents of the two circles which lie on the line of centres, the circle, S, coaxial with S_0 and S_1, which passes through U and V, is such that tangents CP, CZ, drawn from a point, C, of S, respectively to S_0 and S_1, meet S again in points, D, Z', lying on a line at right angles to the line of centres; further, the lines joining C to the absolute points I, J, common to the circles, as, also, the lines joining C to the other intersections of the circles, are harmonic in regard to CU and CV. Also, the line PZ is bisected by the radical axis; and this line envelopes a parabola which touches the radical axis, and touches the common tangents of the circles, and has for focus the middle point of the centres of the two circles. The two common tangents which meet in U are met by PZ in two points whose middle point is on the radical axis, as are the two common tangents which meet in V.

We may also regard the conics Σ_0, Σ_1 as two conics of a confocal system, the points of intersection, U, V, of pairs of common tangents, being taken for the absolute points, I, J. There is, then, a circle, passing through the common points of the two confocals, at every point of which intersect a tangent of one conic, and a tangent of the other at right angles to this. The two common chords of the confocals which meet in their centre subtend right angles at any point of this circle. The line joining the points of contact of the two perpendicular tangents envelopes a conic confocal with the given ones, which touches the common central chords of these; this line is divided harmonically by these chords, and divided harmonically by the lines joining I (or J) to the two foci, S, H, of the two conics.



A still more particular case of the general theorem is the fact that if a line be drawn, from a focus, S, of a conic, at right angles to the tangent, at a point, P, of the conic, meeting this in N, then N describes a circle, whose centre is the centre of the conic, having two points of contact with the conic. If the point equation of the conic be $a^{-1}x^2 + b^{-1}y^2 - z^2 = 0$, the lines NP, NS are tangents of the two envelopes expressed by $al^2 + bm^2 - n^2 = 0$ and $(a-b)l^2 - n^2 = 0$, and are harmonic conjugates in regard to the two lines drawn from N which belong to the envelope expressed by

$$al^2 + bm^2 - n^2 - [(a-b)l^2 - n^2] = 0, \text{ or by } l^2 + m^2 = 0;$$

this envelope is the point pair I, J. The point N thus lies on the locus whose point equation is $a^{-1}x^2 + b^{-1}y^2 - z^2 - (b^{-1} - a^{-1})y^2 = 0$, or $x^2 + y^2 - az^2 = 0$; this is the auxiliary circle of the conic. (See Chap. II, p. 86, above.)

Ex. 19. *Indication of a generalisation of the preceding results.* Referring back to the result given above in Ex. 4 (p. 111), we see that, if from the four elements defined by the two roots of the quadratic $f = 0$, and the two roots of the quadratic $f' = 0$, we form two pairs, by taking each root of $f = 0$ with one of the roots of $f' = 0$, which is possible in two ways, then the elements which are harmonic in regard to the constituents of both the pairs are given by one of the equations $(\Delta')^{\frac{1}{2}}f \pm (\Delta)^{\frac{1}{2}}f' = 0$. But the result of Ex. 4 leads to a result for two conics of which the corresponding corollary is in connexion with the result of the preceding Example 18. Let $S = 0$, $\Sigma = 0$ be the point equation, and the tangential equation, of the same conic, where

$$S = ax^2 + \dots + 2fyz + \dots,$$
$$\Sigma = -(bc - f^2)u^2 - \dots - 2(gh - af)vw - \dots;$$

let S', Σ' be what these become when a', b', ... are written, respectively, for a, b, ...; also let $T = 0$ be the condition for a line, of coordinates u, v, w, whose intersections with $S = 0$ are harmonic in regard to its intersections with $S' = 0$ (above, Ex. 17), where

$$T = -(bc' + b'c - 2ff')u^2 - 2(gh' + g'h - af' - a'f)vw - \dots.$$

Then, let the line (u, v, w) meet $S = 0$ in the points A, B, meet $S' = 0$ in the points A', B', and meet $\theta S + \theta' S' = 0$ in the points X, Y. In terms of the symbols of A, A', let the symbols of X and Y be $X = A + \xi A'$ and $Y = A + \eta A'$; and let $\epsilon = \xi^{-1}\eta$. It follows from Ex. 4 that

$$\frac{4\epsilon}{(1+\epsilon)^2} = \frac{\theta\theta'[2(\Sigma\Sigma')^{\frac{1}{2}} - T]}{[\theta(\Sigma)^{\frac{1}{2}} + \theta'(\Sigma')^{\frac{1}{2}}]^2}.$$

Thus, in particular, when X, Y are harmonic in regard to A, A', and so $\epsilon = -1$, the line (u, v, w) envelopes the conic whose tangential equation is $\theta^2\Sigma - \theta'^2\Sigma' = 0$.

Ex. 20. *Of correspondences, in particular of general involutions, upon a conic.* Taking the conic of which any point has coordinates of the form θ^2, θ, 1, we may consider two points, P and Q, of the conic, with parameters, respectively, θ and ϕ, which are so related that there exists an equation of the form

$$\theta^m(a\phi^n + b\phi^{n-1} + \dots) + \theta^{m-1}(a_1\phi^n + b_1\phi^{n-1} + \dots) + \dots$$
$$+ a_m\phi^n + b_m\phi^{n-1} + \dots = 0,$$

wherein the polynomial on the left is supposed to be general, containing every term $\theta^r\phi^s$ with $0 \leqq r \leqq m$ and $0 \leqq s \leqq n$. A particular case is that in which m and n are both 1, so that P and Q describe related ranges upon the conic (Vol. I, p. 140). In general, we say that there is a (m, n) correspondence between P and Q, there being m positions of P corresponding to any position of Q, and n positions of Q corresponding to any position of P. In saying this we are again assuming that a polynomial expression in a parameter vanishes for a number of values of this parameter equal to the degree of the polynomial. See Chap. IV, below.

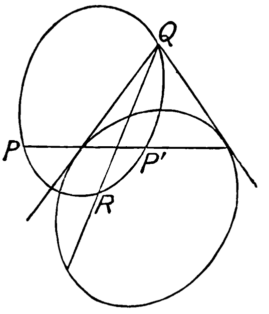
As a rule, it is not the case, in such a correspondence, that when Q takes the position P, then P takes one of the corresponding positions Q. This will be so if $m = n$, and the equation connecting θ, ϕ be symmetrical in regard to these. When $m = n = 1$ this arises if P, Q are pairs of an involution upon the conic. When $m = n = 2$ it arises if the relation be of the form

$$\theta^2 (a\phi^2 + h\phi + g) + \theta (h\phi^2 + b\phi + f) + g\phi^2 + f\phi + c = 0\,;$$

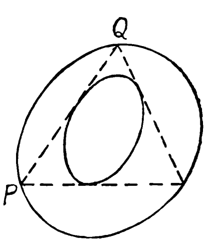
then the two positions of P which correspond to any position of Q are the intersections of the fundamental conic, $xz - y^2 = 0$, with the polar of Q in regard to the conic $ax^2 + 2fyz + \ldots = 0$. The polar of P in regard to this last conic then passes through Q, but meets the fundamental conic, in general, in a new point, R; thus, when Q takes the position P, one of the corresponding positions of P coincides with Q but the other is generally a fresh point. Another interpretation of the relation is obtained by remarking that, if the line joining P and Q, be written $ux + vy + wz = 0$, so that $u = 1$, $v = -(\theta + \phi)$, $w = \theta\phi$, we have

$$aw^2 - hvw + g(v^2 - 2wu) + bwu - fuv + cu^2 = 0,$$

and the line PQ envelopes a conic, of which this is the tangential equation. (Cf. pp. 125, 126.)



A particular case of the general symmetrical correspondence just described, is that in which any point Q, and the m points P which correspond to it, form a set of $(m + 1)$ points which remains the same set when Q takes the position of any one of the $(m + 1)$ points which form the set. Such a series of sets of $(m + 1)$ points is called an *involution* of sets of $(m + 1)$ points, and is of great importance. In the case $m = 2$, $n = 2$, this, clearly, requires that it be possible to take triads of points of $xz - y^2 = 0$ which are self-polar in regard to $ax^2 + 2fyz + \ldots = 0$, or that it be possible to take triads of points of $xz - y^2 = 0$ whose joins touch



the conic whose tangential equation is that written above. We have previously seen (Chap. I, Exx. 12 and 7) that, in either case, if one such triad be possible, an infinite number of such triads is possible. For general values of m, it is convenient to consider the equation satisfied by all the $(m+1)$ points forming the symmetrical set consisting of Q and the m corresponding points P. It can be shewn that this equation must necessarily be of the form

$$\theta^{m+1}(a+\lambda b)+\theta^m(a_1+\lambda b_1)+\ldots+\theta(a_m+\lambda b_m)+a_{m+1}+\lambda b_{m+1}=0,$$

where a, b, a_1, b_1, ... are constants, but λ varies from set to set. Thus the points of a set are given by values of θ satisfying an equation $u+\lambda v=0$, but the points of another set by values of θ satisfying an equation $u+\lambda' v=0$; here u, v are two definite polynomials in θ of degree $m+1$, which, corresponding to $\lambda=0$, $\lambda^{-1}=0$, respectively, give the points of two particular sets of the series. The whole involution is thus determined by two of its sets.

We may give the proof of this result: the m points P corresponding to Q will be given by an equation

$$\theta^m u_0+\theta^{m-1}u_1+\ldots+u_m=0,$$

where $u_0, u_1, \ldots, u_m$ are polynomials in the parameter, ϕ, by which Q is determined. By the definition of the involution, if we replace θ, in this equation, by one of its roots, say θ_1, and ϕ by any other of the roots, say θ_2, the equation remains true. Thus, forming the equation

$$(\sigma-\phi)(\sigma^m u_0+\sigma^{m-1}u_1+\ldots+u_m)=0,$$

or, say,

$$\sigma^{m+1}u_0+\sigma^m v_1+\sigma^{m-1}v_2+\ldots+v_{m+1}=0,$$

each of $u_0, v_1, v_2, \ldots, v_{m+1}$ is a polynomial in ϕ, and this equation is satisfied not only by $\sigma=\phi$, but, also, by each of the values, $\sigma=\theta_1, \sigma=\theta_2, \ldots, \sigma=\theta_m$, which, with ϕ, make up the set of $m+1$ values belonging to a set. These $(m+1)$ roots are, however, also obtained from the equation

$$\sigma^{m+1}u_0'+\sigma^m v_1'+\sigma^{m-1}v_2'+\ldots+v'_{m+1}=0$$

which differs from the preceding in having any one of $\theta_1, \theta_2, \ldots, \theta_m$ written in place of ϕ, in the coefficients; for instance, θ_2. If then we take a value of r for which v_r/u_0 does actually contain ϕ, and denote this by $F(\phi)$, we have $F(\phi)=F(\theta_1)=F(\theta_2)=\ldots=F(\theta_m)$. Thus the $m+1$ points of any set are given by $F(\phi)=\lambda$, for a proper value of λ; it follows, since u_0 is at most of order m in ϕ, that v_r, in $F(\phi)=v_r/u_0$, must be of order $m+1$, at least, if the values $\phi, \theta_1, \theta_2, \ldots, \theta_m$ are all different; and it cannot be of higher order, since none of $u_1, u_2, \ldots, u_m$ is of higher than order m in ϕ. For it is assumed that $\phi, \theta_1, \ldots, \theta_m$ are all different, save for particular values of ϕ.

The equation $\sigma^{m+1}u_0 + \sigma^m v_1 + \ldots = 0$ can thus be arranged in the form $P + \lambda Q = 0$, where P, Q are polynomials in σ, not containing ϕ, being the same as $F(\sigma) - \lambda = 0$. If s be another suffix such that v_s/u_0 does actually contain ϕ, so that, by the same argument, v_s is of order $m+1$ in ϕ, and the values $\phi, \theta_1, \ldots, \theta_m$ are given by the roots of $v_s/u_0 = \mu$, for a proper value of μ, it follows that $v_s/u_0 = AF(\phi) + B$, where A, B do not depend upon ϕ.

We may apply the preceding theory to find when two points $(\theta^2, \theta, 1)$, $(\phi^2, \phi, 1)$, of the conic $xz - y^2 = 0$, which are in $(2, 2)$ correspondence expressed by the symmetrical equation before written,

$$\theta^2(a\phi^2 + h\phi + g) + \ldots = 0,$$

or, say $F(\theta, \phi) = 0$, are two points of a set of three points belonging to a series in involution. For this it is necessary, and sufficient, that there should be two polynomials of the third degree, in the parameter x, say $u(x)$, $w(x)$, such that the $(2, 2)$ relation above written, $F(\theta, \phi) = 0$, is obtainable by elimination of λ from the two equations $u(\theta) + \lambda w(\theta) = 0$, $u(\phi) + \lambda w(\phi) = 0$, the equation $F(\theta, \phi) = 0$ being then of the form $(\theta - \phi)^{-1}[u(\theta)w(\phi) - u(\phi)w(\theta)] = 0$. But, if the two values of θ, corresponding to ϕ in virtue of $F(\theta, \phi) = 0$, taken with ϕ itself, constitute such a set, the equation in σ, $(\sigma - \phi)F(\sigma, \phi) = 0$, will give the set; therefore, by the theory given above, the various sets can be found from the equation $v_3/u_0 = \lambda$, namely

$$-\phi(g\phi^2 + f\phi + c)/(a\phi^2 + h\phi + g) = \lambda,$$

for varying values of λ. In particular, for $\lambda^{-1} = 0$, one set will be that given by $\phi^{-1} = 0$, and $a\phi^2 + h\phi + g = 0$, and another set will be that given by $\phi(g\phi^2 + f\phi + c) = 0$. If we write

$$u(\phi) = a\phi^2 + h\phi + g, \quad w(\phi) = \phi(g\phi^2 + f\phi + c),$$

it is easy to verify the identity

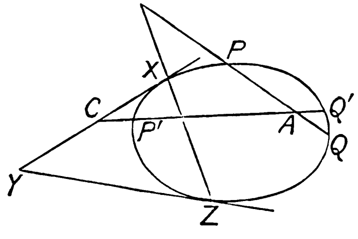
$$(\phi - \theta)^{-1}[u(\theta)w(\phi) - u(\phi)w(\theta)]$$
$$= gF(\theta, \phi) + [hf - bg - (ca - g^2)]\phi\theta.$$

Therefore, we see, supposing g not to be zero, that the condition that the $(2, 2)$ relation in question should connect two points of a set of three points of such a series in involution, is

$$hf - bg = ca - g^2.$$

This condition can also be shewn, if we write $(\sigma - \phi)F(\sigma, \phi)$ in the form $u(\phi).(\sigma^3 + H\sigma^2 + K\sigma + \lambda)$, to secure that $H = g^{-1}(a\lambda + f)$, $K = g^{-1}(h\lambda + c)$, in accordance with the theory given above. It will be obtained below (Ex. 24) as the condition that it be possible to find triads of points of the conic $xz - y^2 = 0$ which are self-polar in regard to the conic $ax^2 + 2fyz + \ldots = 0$.

Taking a fixed point, O, of the conic $xz - y^2 = 0$, and three other fixed points, A, B, C, not lying on this conic, the general conic through these four points will have an equation of the form $U+\lambda V=0$. The three intersections of this conic with $xz-y^2=0$, other than O, will then be given by an equation $u+\lambda v=0$, where u, v are cubic polynomials in the parameter, ϕ, of a point of $xz-y^2=0$; these three intersections will then be a set of an involution lying on this conic, of which the different sets are given by different values of λ. Conversely, let P, Q, R and P', Q', R' be any two triads of points of this conic; take, upon the conic, an arbitrary fixed point, O; then draw any definite conic through P, Q, R, O, and any other definite conic through P', Q', R', O; these two conics will have in common three further points, A, B, C, not generally lying upon the given conic $xz-y^2=0$. The general conic through A, B, C, O will then meet this given conic in three further points P'', Q'', R'', other than O, and these will belong to the involution of sets of three points determined by the two sets P, Q, R and P', Q', R'. Evidently, in this construction, the point O, and one of the points A, B, C may be taken arbitrarily; in particular, denoting by X, Z the points of contact of the given conic, $xz-y^2=0$, with $z=0$, $x=0$, respectively, there will be a set of the involution containing X; say this consists of X, P, Q; and there will be a set, say Z, P', Q', containing Z; assuming that Z is not one of P, Q and X is not one of P', Q', we may write the equations of the lines PQ, $P'Q'$ in the respective forms $ax+hy+gz=0$, $gx+fy+cz=0$. Then the sets of the involution will be given by



$$\lambda\,(a\phi^2+h\phi+g)+\phi\,(g\phi^2+f\phi+c)=0,$$

for varying values of λ, the points X, Z being given by $\phi^{-1}=0$, $\phi=0$, respectively; this is the same as given above. But this involution is obtained by the intersections of the conic $xz-y^2=0$ with the conics of equation $\lambda z\,(ax+hy+gz)+y\,(gx+fy+cz)=0$, for varying λ. These conics have for four common points; first, the point X, on $xz-y^2=0$; then the point, A, where the lines, PQ, $P'Q'$ meet; then B, where XZ, PQ meet; and, finally, C, where XY and $P'Q'$ meet.

A particular result, referred to in Chap. II (above, p. 87), furnishes an example of the theorem obtained here, that the sets of an involution, of three points, are given by an equation $\lambda u+v=0$, for varying values of λ, where u, v are cubic polynomials in the parameter which determines a point. For, if we assume, with continued anticipation of a limitation introduced in Chap. IV, that a cubic equation, in σ, whose roots are a, b, c, is capable of the form $(\sigma - a)(\sigma - b)(\sigma - c) = 0$, this shews that if a, b, c be the values belonging to any set of the involution, we have equations $a + b + c = P + Qabc$, $bc + ca + ab = R + Sabc$, where P, Q, R, S are the same for all the sets.

Now consider the pedal lines of a fixed point of the circle $x^2 + y^2 = z^2$ (the absolute points being $z = 0$, $x \pm iy = 0$), taken in regard to the joins of three points of the circle which are a set of an involution, of sets of three points, lying on the circle. We shew that, for all these sets, the pedal lines have a point in common. Putting $x + iy = X$, $x - iy = Y$, so that the circle is $XY = z^2$, the parameter for a point of the circle may be taken to be $\theta = X/z$. If for the fixed point this be λ, and for the points of a set of the involution the values be $\theta = a$, $\theta = b$, $\theta = c$, it is easy to prove that the equation of the pedal line is

$$2\lambda(Yabc - X\lambda) = [abc + \lambda(bc + ca + ab) - \lambda^2(a + b + c) - \lambda^3]z;$$

with the conditions given above, the right side is, save for the factor z,

$$abc(1 + \lambda S - \lambda^2 Q) + \lambda R - \lambda^2 P - \lambda^3;$$

thus the pedal line passes through the fixed point given by

$$Xz^{-1} = \tfrac{1}{2}(\lambda + P - \lambda^{-1}R),\quad Yz^{-1} = \tfrac{1}{2}(-\lambda Q + S + \lambda^{-1}).$$

(Third year problems, St John's College, Cambridge, Dec. 1889. Cf. Weill, *Liouville's J.* IV, 1878, 270.)

In general, in an involution of sets of n points, there will be $(n - 1)^2$ sets whereof two of the n points coincide, as is easy to see. For $n = 3$, taking two such sets as the fundamental sets, the parameter being chosen so that the points of coincidence are given by $\theta = 0$, $\theta = \infty$, the involution may be represented by

$$\theta^2(b\theta + a) + \lambda(c\theta + b) = 0.$$

Now suppose the points of the involution to be points, $(\theta^2, \theta, 1)$, of the conic $xz - y^2 = 0$. Define the pole of a line in regard to points A, B, C as the point O such that AO and the line meet BC in points harmonic in regard to B and C, with a similar condition for C, A and for A, B. Then we may prove that, for all sets A, B, C of the involution $\theta^2(b\theta + a) + \lambda(c\theta + b) = 0$, the pole of the line joining the two double points, $\theta = 0$, $\theta = \infty$, is the same, being the point $(a, -3b, c)$.

The condition for an involution of sets of three points upon a conic may be approached differently. The following examples lead up to this.

Ex. 21. *Reciprocation of one conic into another when they have a common self-polar triad.* As has already been remarked (above,

p. 107), the equations of two conics which have a common self-polar triad, when the coordinates are taken in regard thereto, are of the forms

$$ax^2 + by^2 + cz^2 = 0, \quad a'x^2 + b'y^2 + c'z^2 = 0.$$

Now let $Ax^2 + By^2 + Cz^2 = 0$ be a further conic for which the same triad is self-polar. If we consider the locus of the poles, taken in regard to this last conic, of the tangents of one of the former conics, it will be another conic. In fact the pole of the line $ax\xi + by\eta + cz\zeta = 0$, in regard to $Ax^2 + \ldots = 0$, is $(A^{-1}a\xi, B^{-1}b\eta, C^{-1}c\zeta)$, and when (ξ, η, ζ) is such that $a\xi^2 + b\eta^2 + c\zeta^2 = 0$, this pole describes the conic

$$a^{-1}A^2x^2 + b^{-1}B^2y^2 + c^{-1}C^2z^2 = 0.$$

If we consider the envelope of the polars, also taken in regard to $Ax^2 + \ldots = 0$, of the points of the same conic, $ax^2 + \ldots = 0$, it is similarly seen to consist of the tangents of the conic $a^{-1}A^2x^2 + \ldots = 0$. This last is then called the polar reciprocal of $ax^2 + \ldots = 0$ in regard to $Ax^2 + \ldots = 0$. There are thus four conics $Ax^2 + \ldots = 0$ in regard to which the polar reciprocal of $ax^2 + \ldots = 0$ is the other given conic $a'x^2 + \ldots = 0$, namely those for which $A^2 = aa'$, $B^2 = bb'$, $C^2 = cc'$.

Conversely, if a conic S be reciprocated to a conic S', a self-polar triad of points for S becomes a self-polar triad of lines for S', as we easily see; thus, when S and S' have a common self-polar triad, this is also self-polar for the reciprocating conic.

It may be remarked that if we take the polar line of a point, P, or (ξ, η, ζ), in regard to the conic $Ax^2 + \ldots = 0$, so obtaining $Ax\xi + By\eta + Cz\zeta = 0$, and then take the pole of this line in regard to one of the three associated conics, say $-Ax^2 + By^2 + Cz^2 = 0$, we obtain the point $(-\xi, \eta, \zeta)$ or P', which can be obtained from P by joining the point $(1, 0, 0)$, or A, to P, by a line, AP, meeting BC in D, and taking P' on AD harmonic to P in regard to A and D.

Ex. 22. *The determination of the common self-polar triad of two conics in general.* The construction for the common self-polar triad of two conics has been given (above, p. 23) when their four common points are given, and these are distinct. When the equations of the conics are given, in general form, say $ax^2 + 2fyz + \ldots = 0$, $a'x^2 + 2f'yz + \ldots = 0$, the algebraic determination will begin by seeking a point, (x, y, z), whose polar line is the same for both conics. For this, we must have

$$\frac{ax + hy + gz}{a'x + h'y + g'z} = \frac{hx + by + fz}{h'x + b'y + f'z} = \frac{gx + fy + cz}{g'x + f'y + c'z};$$

these lead to three equations of the form

$$(a - \lambda a')x + (h - \lambda h')y + (g - \lambda g')z = 0,$$

which, if the two given conics be $S = 0$, $S' = 0$, express that the polar lines of the three points $(1, 0, 0)$, etc., taken in regard to the conic $S - \lambda S' = 0$, meet in a point. Thus the conic $S - \lambda S' = 0$, for the appropriate value of λ, is a pair of lines, through the common points of the given conics, which intersect in (x, y, z). By elimination of x, y, z we find that λ is a root of a cubic equation, say $\Delta(\lambda) = 0$, which arises naturally in determinant form. When the given conics have four distinct intersections, there are three line pairs through these, and, therefore, this cubic equation has three roots which are different. Denote them by $\lambda_1, \lambda_2, \lambda_3$, in this case. Then let l_1, m_1, n_1 be determined from two of the equations of the form

$$(a - \lambda_1 a')\, l_1 + (h - \lambda_1 h')\, m_1 + (g - \lambda_1 g')\, n_1 = 0,$$

the third of these equations being a consequence of these two, and, similarly, let l_2, m_2, n_2 be found from the equations of the same form in which λ_1 is replaced by λ_2 and l_1, m_1, n_1 by l_2, m_2, n_2, respectively. If we denote by S_{12} the expression $al_1l_2 + f(m_1n_2 + m_2n_1) + \dots$, a sum of six terms, which is symmetrical in regard to (l_1, m_1, n_1) and (l_2, m_2, n_2), and denote by S'_{12} the similar expression in which $a, f, \dots$ are replaced by $a', f', \dots$, the first set of three equations enables us to deduce $S_{12} - \lambda_1 S'_{12} = 0$, and the second set of three equations similarly enables us to deduce $S_{12} - \lambda_2 S'_{12} = 0$. As we are supposing λ_1 and λ_2 to be different we can infer that $S_{12} = 0$, $S'_{12} = 0$. In the same way we infer that $S_{23} = S'_{23} = S_{31} = S'_{31} = 0$. From these it follows that if, in S and S', we replace x, y, z, respectively, by $l_1\xi + l_2\eta + l_3\zeta$, $m_1\xi + m_2\eta + m_3\zeta$, $n_1\xi + n_2\eta + n_3\zeta$, then S has a form $P\xi^2 + Q\eta^2 + R\zeta^2$ and S' a form $P'\xi^2 + Q'\eta^2 + R'\zeta^2$; here, for instance, P is what S becomes when l_1, m_1, n_1 are written for x, y, z, with similar values for Q, R, P', Q', R', and, as is easy to see, $P = \lambda_1 P'$, $Q = \lambda_2 Q'$, $R = \lambda_3 R'$. As the equations from which l_1, m_1, n_1 are found are homogeneous in these, we may suppose them taken so that $P' = 1$, and, similarly, may suppose l_2, m_2, n_2 and l_3, m_3, n_3 so taken that $Q' = 1$ and $R' = 1$.

When the roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic equation above are not all different, the previous theory, and, in particular, the equations such as $S_{12} = 0$, may cease to hold, and it may not be possible to choose ξ, η, ζ so that S and S' involve only squares of these. Without entering here into a question of the general theory of quadratic forms, we give below the forms, for all cases, to which two conics can be reduced (Ex. 25).

Ex. 23. *Fundamental invariants for two conics.* As the cubic equation for λ, which arises above, has the geometrical interpretation in regard to the line pairs which pass through the common points of the conics $S = 0$, $S' = 0$, the four coefficients in this equation have values which, save for a common factor, are independent of the base points in terms of which the coordinates are taken. If we denote the cubic, written at length, by $\Delta - \lambda\Theta + \lambda^2\Theta' - \lambda^3\Delta' = 0$, the expression Δ is that polynomial, homogeneously of the third degree in the coefficients of the conic S, whose vanishing is the condition for $S = 0$ to be a line pair (above, p. 102), while Δ' is the similar polynomial for S', the expression Θ is only of the second degree in the coefficients of S, but is of the first degree in the coefficients of S', being, explicitly, the sum of three pairs of terms such as $Aa' + 2Ff'$, where $A = bc - f^2$, $F = gh - af$, etc., which are the coefficients in the tangential equation of S, while Θ' is similarly related to S'. We proceed to find the geometrical significance of the equation $\Theta = 0$, or of $\Theta' = 0$. [Add.]

Ex. 24. *The algebraical condition for one conic to be outpolar to another, or to be triangularly circumscribed to another.* It was proved in Chap. I (Ex. 11), and will appear independently here, that if two conics S and Σ' be such that a triad of points of S be self-polar in regard to Σ', then there is an infinite number of other triads of points of S with the same property; the polar, in regard to Σ', of any point, P, of S, meets S in points, Q, R, such that P, Q, R is self-polar in regard to Σ'. And, when this is so, there are triads of tangents of Σ' which are self-polar in regard to S, also in infinite number. It was suggested that S should be spoken of as outpolar to Σ', and Σ' as inpolar to S. The property is one relating to the points of the outpolar conic, S, and to the tangents of the inpolar conic, Σ'; and its algebraic condition involves, most conveniently, the coefficients in the point equation of the outpolar conic, and the coefficients in the tangential equation of the inpolar conic. The necessary and sufficient condition is linear in both these sets of coefficients, being, in fact, the vanishing of the sum of the three pairs of terms such as $aA' + 2fF'$, which, above, we have called Θ'. As the expression is of the first degree in the coefficients of S, it follows that, if S_1, S_2, ... be conics all outpolar to Σ, so also is any conic $A_1S_1 + A_2S_2 + \ldots = 0$.

It was also proved in Chap. I (Ex. 7), and will appear here, that if two conics S and Σ' be such that a triad of points A, B, C exists on the former, whose joins, BC, CA, AB, touch the latter, then there is an infinite number of other such triads, the tangents drawn to Σ', from any point of S, meeting S again in two points whose join is a tangent of Σ'. The condition for two conics to be in this relation is again primarily one for the points of S, and the tangents of Σ'. This condition is in fact

$$\tfrac{1}{4}(aA' + 2fF' + \ldots)^2$$
$$= (bc - f^2)(B'C' - F'^2) + 2(gh - af)(G'H' - A'F') + \ldots,$$

where, on both sides of the equation, there is a sum of three such pairs of terms as are written. If we suppose that the point equation, and not the tangential equation, of Σ', is given, being $a'x^2 + 2f'yz + \ldots = 0$, we have $B'C' - F'^2 = a'\Delta'$, $G'H' - A'F' = f'\Delta'$, etc.; then writing, as usual, $A = bc - f^2$, $F = gh - af$, etc., the condition can also be written $\frac{1}{4}(\Theta')^2 = \Delta'\Theta$, the conic which is triangularly inscribed to the other being Σ', or S'.

It is to be noticed that if we take a conic by which S is reciprocated into Σ', a triad of points of S which is self-polar in regard to Σ' becomes a triad of tangents of Σ' which is self-polar in regard to S, so that the condition $\Theta' = 0$ involves two geometrical facts; but a triad of points of S whose joins touch Σ', becomes, when the reciprocating conic is properly chosen, the triad of these three joins. (Cf. Chap. I, Ex. 24.) This is brought out by writing the condition $\Theta'^2 = 4\Delta'\Theta$ symmetrically in regard to the point equation of S and the tangential equation of Σ', as we have done. It may also be remarked, in regard to the two theorems, that the points in which a tangent line of Σ' meets S are conjugate points in regard to the conic whose equation is $2F - S\Theta' = 0$, where $F = 0$ is the conic locus of points from which the tangents to S and Σ' form a harmonic pencil (above, Ex. 17).

To prove that the algebraic conditions above given are necessary and sufficient, is easy, if we remark that they are both expressible by the invariants referred to in Ex. 23; it is necessary then only to verify them for any conveniently chosen triad of points of reference for the coordinates, that is, for any two conveniently chosen forms to which the equations of the conics can be simultaneously reduced.

Thus, for the first condition, suppose that a triad of points, A, B, C, lying on S, is self-polar in regard to Σ'. We can then, referring coordinates to A, B, C, suppose $S = 2fyz + 2gzx + 2hxy$, and $\Sigma' = b'c'u^2 + c'a'v^2 + a'b'w^2$. For these two we evidently have $\Theta' = 0$, which is thus *necessary*. Conversely, suppose $\Theta' = 0$; take the polar, in regard to Σ', of any point, A, of S, and let this meet S in B and C. In regard to A, B, C we can then suppose, as before, that $S = 2fyz + 2gzx + 2hxy$. The tangential equation $\Sigma' = 0$ must be such that the pole of the line BC, in regard to Σ', which in general is the point whose coordinates are A', H', G' (being the point whose equation is $u'(A'u + H'v + G'w) + v'(H'u + \ldots) + w'(G'u + \ldots) = 0$, when (u', v', w') are the coordinates, $(1, 0, 0)$, of the line BC), must, in this case, be the point A, or $(1, 0, 0)$; thus $G' = 0$ and $H' = 0$. The general form of Θ' is however, here, $2fF' + 2gG' + 2hH'$; thus, as we are assuming $\Theta' = 0$, we must have $fF' = 0$. We can only have $f = 0$ if the conic S reduces to the line pair $2x(gz + hy) = 0$, which we exclude. Wherefore $F' = 0$, and Σ' has the form

$$A'u^2 + B'v^2 + C'w^2 = 0.$$

Therefore $\Theta' = 0$ is *sufficient* for the triad A, B, C to be self-polar in regard to Σ'.

In regard to this relation of two conics, we may remark, further, that we have shewn (Chap. I, Ex. 11) that when S is outpolar to Σ', there can be found, on S, sets of four points, A, B, C, D, such that the pair of lines AD, BC are conjugate in regard to Σ', as also are BD, CA and CD, AB. In general, if, from three points A, B, C, in regard to which the equation of Σ' is $A'u^2 + 2F'vw + \ldots = 0$, there be drawn lines AD, BD, CD respectively conjugate to BC, CA, AB, with respect to Σ', they will meet in a point, D, of coordinates x, y, z, given by $xF' = yG' = zH'$; if the points A, B, C be on S, which then has an equation of the form $2fyz + 2gzx + 2hxy = 0$, the condition that this point D should also be on S is $2fF' + 2gG' + 2hH' = 0$, which is exactly $\Theta' = 0$. The equation $\Sigma' = 0$ can be written in the form

$$\frac{1}{F'}(G'H' - A'F')u^2 + \frac{1}{G'}(H'F' - B'G')v^2 + \frac{1}{H'}(F'G' - C'H')w^2$$
$$- F'G'H'\left(\frac{u}{F'} + \frac{v}{G'} + \frac{w}{H'}\right)^2 = 0,$$

and $u = 0$, $v = 0$, $w = 0$, $(F')^{-1}u + (G')^{-1}v + (H')^{-1}w = 0$ are the respective equations of the points A, B, C, D. In general, if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be any four points, and A_1, A_2, A_3, A_4 be any constants, the equation

$$\Sigma A_r(ux_r + vy_r + wz_r)^2 = 0,$$

containing four terms, for $r = 1, 2, 3, 4$, is easily seen to be the tangential equation of a conic in regard to which the line joining any two of the four points is conjugate to the line joining the other two. Dually, sets of four tangents of Σ' exist, whose point pair intersections are conjugate in regard to S. (Cf. p. 218.)

The particular cases of the relation referred to in Ex. 13 of Chap. I are easy to verify. In particular, a single point, (ξ, η, ζ), regarded tangentially as a coincident point pair, of equation $(u\xi + v\eta + w\zeta)^2 = 0$, is inpolar to S, if (ξ, η, ζ) lie thereon; this suggests immediately the form above given for Σ' in terms of the equations of four points of S, if we notice that, when two or more conics, $\Sigma_1' = 0$, $\Sigma_2' = 0, \ldots$ are all inpolar to S, so also is any conic $A_1\Sigma_1' + A_2\Sigma_2' + \ldots = 0$. We easily verify that the equation

$$(u\xi + v\eta + w\zeta)^2 = 0,$$

if $\xi = m_1n_2 - m_2n_1$, $\eta = n_1l_2 - n_2l_1$, $\zeta = l_1m_2 - l_2m_1$, is that formed, by the rule, as the tangential equation of the line pair

$$(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) = 0,$$

consisting of any two lines meeting in (ξ, η, ζ).

If $\Sigma_1 = 0$, $\Sigma_2 = 0$, ..., $\Sigma_5 = 0$ be any five conics, a conic $S = 0$ can be found which is outpolar to all of them, this requiring five linear equations for the six coefficients of S. Dually, a conic can be found inpolar to any five given conics. As an application of this, we can prove that, if $S = 0$, $T = 0$, $U = 0$ be the point equations of any three conics, all the conics given by the equation $\theta^2 S + 2\theta T + U = 0$, for varying values of θ, are inpolar to another conic; for the tangential equation corresponding to this is of the fourth degree in θ, and we can find a conic outpolar to the five conics whose tangential equations are those obtained by equating to zero the coefficients of all the powers (θ^r, for $r = 4, 3, 2, 1, 0$) of θ occurring in this tangential equation. This outpolar conic will, we have seen, contain the point of intersection of every line pair contained, for a proper value of θ, in the system $\theta^2 S + 2\theta T + U = 0$. The condition for a conic of this form to be a line pair, being of the third degree in the coefficients, is of the sixth degree in θ. If we assume, once more illogically anticipating the point of view explained in the following Chapter IV, that an equation of the sixth degree has six roots among the system of symbols which we are employing, we can then infer that the six line pairs of the system of conics $\theta^2 S + 2\theta T + U = 0$ intersect in six points lying on a conic.

Coming now to the relation between conics of which one is triangularly circumscribed to the other, suppose that three points, A, B, C, of a conic, S, are such that the joins, BC, CA, AB, touch another conic Σ'. Taking coordinates relatively to these points, we may then suppose that

$$S = 2fyz + 2gzx + 2hxy, \quad \Sigma' = 2F'vw + 2G'wu + 2H'uv.$$

Then, the expression $aA' + 2fF' + \dots$ reduces to $2fF' + 2gG' + 2hH'$; and the expression

$$(bc - f^2)(B'C' - F'^2) + 2(gh - af)(G'H' - A'F') + \dots$$

reduces to $(fF' + gG' + hH')^2$. The condition $\Theta'^2 = 4\Delta'\Theta$ is thus *necessary*. Conversely, if this condition be assumed, and we draw, from an arbitrary point, A, of S, the two tangents to Σ', meeting S again in B and C, and refer the coordinates to A, B, C, we have $S = 2fyz + 2gzx + 2hxy$, $\Sigma' = A'u^2 + 2F'vw + 2G'wu + 2H'uv$; thus the condition leads to $ghA'F' = 0$. We suppose S not to reduce to two lines, and Σ' not to reduce to two points; wherefore none of g, h, F' is zero, and we infer $A' = 0$, which shews that BC touches Σ'; or the condition is *sufficient*. [Add.]

We have already seen (Chap. I, Ex. 24) that the triads of points of S whose joins touch Σ' are all self-polar in regard to another conic, Ω. When we take

$$S = 2fyz + 2gzx + 2hxy, \quad \text{and} \quad \Sigma' = 2F'vw + 2G'wu + 2H'uv,$$

and take two points of S of coordinates x_1, y_1, z_1 and x_2, y_2, z_2, the equation of the line joining these is

$$(x_1x_2)^{-1}fx + (y_1y_2)^{-1}gy + (z_1z_2)^{-1}hz = 0,$$

as we verify at once; the condition that this line should touch Σ', which is $f^{-1}F'x_1x_2 + g^{-1}G'y_1y_2 + h^{-1}H'z_1z_2 = 0$, is however also the condition that the points (x_1, y_1, z_1), (x_2, y_2, z_2) should be conjugate in regard to $f^{-1}F'x^2 + g^{-1}G'y^2 + h^{-1}H'z^2 = 0$. It is easily seen that, in regard to this conic, S and Σ' are polar reciprocals; this is then the conic Ω. If we take another of the four conics in regard to which S and Σ' are polar reciprocals, say Ω', a triad of points, A, B, C, of S, whose joins touch Σ', reciprocates, in regard to Ω', into a triad of tangents of Σ' whose intersections, say A', B', C', lie on S. The point A' is thus obtained from A by the sequence of two reciprocations, in regard to Ω' and Ω, and, as has been remarked, the line AA' passes through one of the triad of points self-polar in regard to both S and Σ', as also do BB' and CC'. The aggregate of the triads A, B, C may thus be arranged in sets of four, of which any two are in perspective.

If the conics S, Σ' be referred to their common self-polar triad, it is easy to choose a parameter for the points of S, such that the condition, that a chord of S should be a tangent of Σ', takes the form $(\theta^2\phi^2 + 1)a + \theta\phi b - (\theta^2 + \phi^2) = 0$, and to see (see Ex. 20, above) that the condition that two points, θ, ϕ, so connected should be two of a set of an involution of sets of three points is $a^2 = b + 1$. Introducing this value of b, and adjoining the factor $\theta - \phi$, we obtain, for the cubic equation satisfied by ϕ and the two values, θ_1, θ_2, associated therewith,

$$\sigma^3 - \sigma^2 a\phi(\phi^2 - a)(a\phi^2 - 1)^{-1} - \sigma a + \phi(\phi^2 - a)(a\phi^2 - 1)^{-1} = 0.$$

Thus, in accordance with the theory, the involution of sets of three points is given by $\lambda(a\phi^2 - 1) - \phi(\phi^2 - a) = 0$, for varying values of λ. Four associated sets of this involution are given by λ, $-\lambda$, λ^{-1}, $-\lambda^{-1}$, corresponding to the values ϕ, $-\phi$, ϕ^{-1}, $-\phi^{-1}$, as we easily see. The expression $\Theta'^2 - 4\Delta'\Theta$ can be written as the product of four (irrational) factors; it vanishes when the roots α, β, γ, of the cubic equation $\Delta - \lambda\Theta + \lambda^2\Theta' - \lambda^3\Delta' = 0$, are subject to

$$\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} + \gamma^{\frac{1}{2}} = 0.$$

Ex. 25. *Enumeration of reduced forms for the equations of any two conics.*

When two conics intersect in four points which are not distinct, the preceding determination of the equations $ax^2 + by^2 + cz^2 = 0$, $a'x^2 + b'y^2 + c'z^2 = 0$, fails. When the conics touch in two points, they have an infinite number of common self-polar triads, and their equations can, in an infinite number of ways, be reduced to

$x^2+y^2+z^2=0$, $x^2+y^2+cz^2=0$. In this case they can be reciprocated into one another in an infinite number of ways.

Prove that when the conics have one point of contact, their equations can be taken to be $2xy+z^2=0$, $2xy+cz^2+y^2=0$. And further that they are, then, polar reciprocals of one another in regard to either of the two conics $2xy \pm c^{\frac{1}{2}}z^2+\frac{1}{2}y^2=0$.

Also, that when three of the points of intersection coincide at one point, their equations can be taken to be

$$2xy+z^2=0, \quad 2xy+z^2+2yz=0.$$

And that, then, they are polar reciprocals of one another in regard to $2xy+z^2+yz-\frac{1}{8}y^2=0$.

Finally, when the four intersections coincide, that their equations can be taken to be $2xy+z^2=0$, $2xy+z^2+y^2=0$. And that, then, they are polar reciprocals of one another in regard to

$$2xy+z^2+\tfrac{1}{2}y^2-\tfrac{1}{2}\sigma(my+2z)^2=0,$$

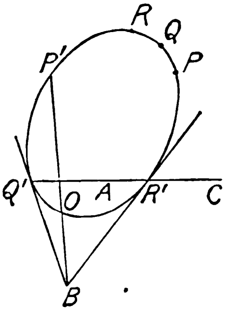
for any value of m, σ being 0 or 1. [Add.]

Ex. 26. *A particular involution of sets of three points. Apolar triads.*

It has been seen that an involution of sets of three points on a conic is obtained by the intersections of this conic with a set of conics all passing through a point, O, of the conic and through three other points, A, B, C, not lying on the conic. A special case which is of interest is when A, B, C are a self-polar triad in regard to the given conic. It can then be seen that if the parameters, θ, of two sets of the involution be such that, respectively,

$$a\theta^3+3b\theta^2+3c\theta+d=0 \quad \text{and} \quad a'\theta^3+3b'\theta^2+3c'\theta+d'=0,$$

then $ad'-3bc'+3cb'-da'=0$. This relation is clearly unaffected if we replace a, b, c, d, respectively, by $a+\lambda a'$, $b+\lambda b'$, $c+\lambda c'$, $d+\lambda d'$, whatever λ may be. It is also unaffected if we use, in place of θ, a parameter, ϕ, given by $(m\theta+n)/(r\theta+s)$; namely, if the cubic equations then become $A\phi^3+3B\phi^2+\ldots=0$, and $A'\phi^3+\ldots=0$, we have $AD'-3BC'+\ldots=0$; as may readily be verified. Every two sets of the involution are then apolar triads. Conversely, let P, Q, R and P', Q', R' be any two triads upon a conic; upon the line $Q'R'$ two points, A, C, can be taken which are conjugate to one another in regard to all conics through the four points P', P, Q, R; in particular A, C will then be conjugate in regard to the given conic. If B be the pole of $Q'R'$, in regard to the given conic, and BP' meet this conic again in O, it can be shewn that the six points P, Q, R, A, C, O lie



on a conic (above, p. 92). But, when P', Q', R' and P, Q, R are apolar, this conic also passes through B, as may be proved. The points A, B, C are a self-polar triad in regard to the given conic; and there are two conics through these, and through the point O, whose further intersections with the given conics are, respectively, the triad P', Q', R' and the triad P, Q, R; namely, the first of these is the degenerate conic consisting of the lines AC and BO; the second of these is the conic above remarked. Thus every conic through A, B, C, O meets the given conic in a triad of points apolar with P, Q, R, and with P', Q', R'.

If the given conic, referred to Q', R', B, have the equation $xz - y^2 = 0$, any conic through A, B, C has an equation of the form $x^2 - \lambda^2 z^2 = 2y(fz + hx)$; if this pass through O, or $(m^2, m, 1)$, we can replace $2f$ by $m^3 - \lambda^2 m^{-1} - 2hm^2$. The three other intersections of this conic with the given one are then determined by

$$\theta^3 + \theta^2 m + \theta m^2 + \lambda^2 m^{-1} - 2h\theta(\theta + m) = 0,$$

which, for every value of h, is apolar with the cubic $\theta^2 + \theta m = 0$, belonging to Q', R', P'. It is easy to see that A, C are conjugate in regard to all conics through P', P, Q, R.

If, for $f = ax^3 + 3bx^2y + 3cxy^2 + dy^3$, the Hessian form

$$H = f_{11}f_{22} - f_{12}^2, \quad \text{where } f_{11} = \partial^2 f/\partial x^2, \text{ etc.},$$

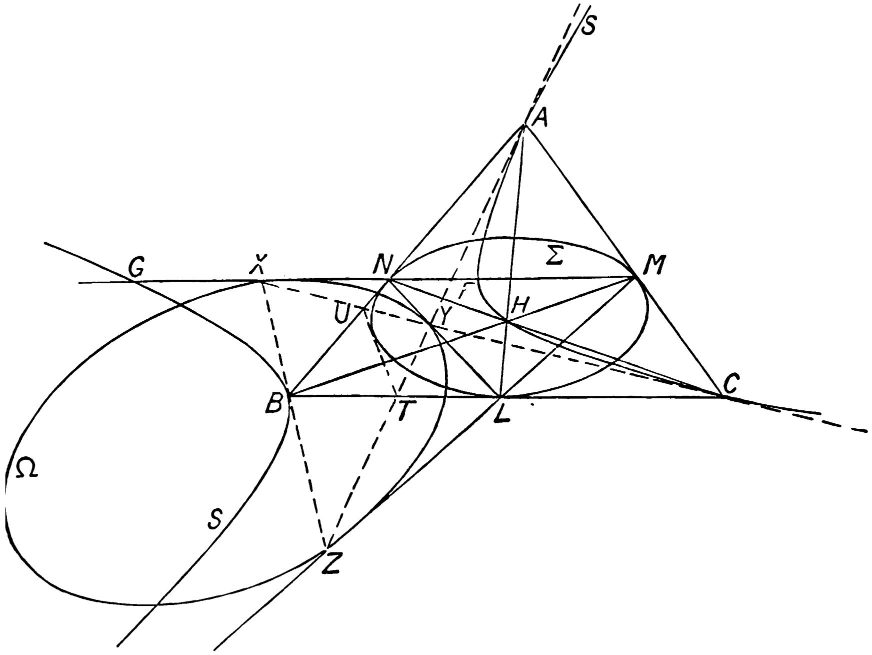
be denoted by $px^2 + qxy + ry^2$, we at once verify that $ar - bq + cp = 0$ and $br - cq + dp = 0$. This gives the result, previously mentioned (above, p. 124), that the cubic form $H(Ax + By)$, wherein A, B are arbitrary, is apolar with f. We have also mentioned (see above, p. 30) that if the tangents at three points, P, Q, R, of a conic, meet QR, RP, PQ in D, E, F, the line DEF meets the conic in the Hessian points, upon the conic, of the points P, Q, R. Wherefore, if P, Q, R and Q', R' be any points of a conic, the point, P', such that P', Q', R' are apolar with P, Q, R, may be constructed by drawing the conic touching QR, RP, PQ, $Q'R'$ and the line DEF; the tangents to this conic, other than $Q'R'$, drawn from Q' and R', meet in the point, P', of the given conic, which is required. A further construction for P' arises from what is said above; we have only to draw the conic through P, Q, R and B (the pole of $Q'R'$), which meets $Q'R'$ in points, A, C, conjugate in regard to the given conic; if this meet the given conic again in O, then P' is on BO.

The line DEF, spoken of above, may, as before (Chap. I, Ex. 9), be called the Hessian line of P, Q, R. The reader may verify, for example, that for the three points of a circle, of centre O, consisting of the absolute points I, J and any point, P, of the circle, the Hessian line is that through the middle point of O, P, at right angles to OP. Given three tangents of a conic, we may, dually, define the Hessian point of these. For example, in the case of a parabola, of focus S, the Hessian point of the three tangents consisting of SI, SJ and the absolute line IJ, is the point H, such that the directrix is at right angles to SH and passes through the middle point of S, H.

It may also be proved that if two conics, S and S', meet in A, B, C, D, the polar line, in regard to the conic S, of the Hessian point of the tangents of S' drawn at A, B, C, meets S in two points, which, with D, form, on S, a triad apolar with the triad A, B, C. Dually, if the common tangents of two conics, Σ and Σ', be a, b, c, d, and we take the Hessian line of the three points where Σ' touches a, b, c, this Hessian line meets Σ in two points which, taken with the point of contact of d with Σ, form a triad apolar with the points of contact of a, b, c with Σ. (Cf. Chap. I, Ex. 32.)

Ex 27. Three conics related in a particular way.

Suppose that a conic, Σ, touches three lines, BC, CA, AB, respectively at L, M, N; and that H is the Hessian point of the three tangents, being the point common to AL, BM, CN. Further, that



S is any conic, through A, B, C, which passes also through H. Then, for S, the triad L, M, N is self-polar. The tangents of S, at A and C, thus meet on NL, say, in Y, and if these tangents meet BC, BA, respectively, in T and U, the line TU touches Σ. For, taking various conics S, through A, B, C, H, the lines TU envelope a conic, and this is identified with Σ by remarking that particular conics S are the line pairs AB, NC; AL, CB; AC, BH, for which the line TU is, respectively, AB; BC; CA. The line TU is the Hessian line of A, B, C, for the conic S; the relation of the conics S, Σ is thus reciprocal.

The conic S meets MN in two points, F, G, which are harmonic in regard to M and N; thus the point triad A, F, G, of points of S, is self-polar in regard to Σ, and S is outpolar to Σ. If the tangents of S, at A, B, C, meet MN, NL, LM, (in pairs), respectively in X, Y, Z, the line triad YZ, ZX, XY is self-polar for all conics which touch BC, CA, AB and TU, in particular for Σ. Thus S is also inpolar to Σ.

Further, if we consider the locus of the poles, in regard to Σ, of the tangents of S, that is, the polar reciprocal of S in regard to Σ, whose tangents are the polars, in regard to Σ, of the corresponding points of S, we obtain a third conic, say, Ω. As the polar of XZ, in regard to Σ, is the point Y, and the polar of B, in regard to Σ, is NL, with a similar remark for ZY and XY, it appears that Ω touches MN, NL, LM, respectively at X, Y and Z. It will appear that Ω is equally the polar reciprocal of Σ in regard to S; and that any two of the three conics, S, Σ, Ω are related to one another in the same way as were S and Σ.

We see that the condition that a conic, S, through A, B, C, should contain the Hessian point of the lines BC, CA, AB, in regard to a conic Σ touching BC, CA, AB, is that S should be outpolar to Σ. With the symbols, if S be $2fyz + 2gzx + 2hxy = 0$, and Σ be, tangentially, $vw + wu + uv = 0$, which is not a restriction, the Hessian point referred to is $(1, 1, 1)$, and the condition that S should contain this is $f + g + h = 0$, which is the condition that S should be outpolar to Σ. Thus it follows that, if A', B', C' be any other triad of points of the conic S, for which $B'C'$, $C'A'$, $A'B'$ touch Σ, the Hessian point of these lines, in regard to Σ, also lies on S; and the Hessian line of A', B', C', in regard to S, also touches Σ; and the figure may be developed from A', B', C' as it was from A, B, C. In the figure we have considered, there are three triads of points; A, B, C; L, M, N, and X, Y, Z; the points of one triad lie on the joins of the points of another triad, while the joins of the points of the first triad contain the points of the third triad. Keeping the conics, S, Σ, Ω, fixed, such a set of three point triads can be initiated from any one of the nine points; and the triad, A', B', C', of points of S, belonging to such a set, will be apolar with A, B, C. This follows from what has preceded.

The conics S, Σ, being such that S is both outpolar to Σ, and triangularly circumscribed to Σ, are such that the two invariants

Θ, Θ' for these two conics (above described, Ex. 23) are both zero. For we have both $\Theta' = 0$ and $\Theta'^2 = 4\Delta'\Theta$. Referring S and Σ to their common self-polar triad, we find that these, and the conic Ω, may be taken to have equations $x^3 + y^2 + z^2 = 0$, $x^2 + \omega y^2 + \omega^2 z^2 = 0$, $x^2 + \omega^2 y^2 + \omega z^3 = 0$, where ω is such that $(2\omega + 1)^2 + 3 = 0$. Another form to which three such conics can be reduced is $x^2 + 2yz = 0$, $y^2 + 2zx = 0$, $z^2 + 2xy = 0$ (see the concluding problems on Analytical Conics in Wolstenholme's problems).

Another deduction of the three conics may be made by taking three points, A, B, C, of the conic S, then drawing, through the pole, Y, of AC, any line to meet BA, BC, respectively, in N and L, and then describing the unique proper conic, Σ, which touches BA, BC, respectively, at N and L, and also touches AC. If this conic touch AC at M, the conic Ω then arises as touching NL at Y, and touching MN and ML where these are met by the tangent at B of the conic S.

Particular figures in which three such conics arise may be referred to: (1) If O, P, Q, R be the feet of the four normals drawn to a conic, S, from any point, T, and, the point O being kept fixed, the point T be allowed to vary on the normal at O, the triads of lines, QR, RP, PQ, so obtained, are all tangents to a parabola, Σ, and the poles of these lines, in regard to S, describe a rectangular hyperbola, Ω. These three conics are related as above. Their respective equations, that of Σ being tangential, may be taken to be

$$a^{-1}x^2 + b^{-1}y^2 - z^2 = 0, \quad z_0 uv - b^{-1}y_0 uw - a^{-1}x_0 vw = 0,$$
$$xyz_0 + xzy_0 + yzx_0 = 0,$$

where $a^{-1}x_0{}^2 + b^{-1}y_0{}^2 - z_0{}^2 = 0$. (2) If a circle, S, be described to pass through the focus of a parabola, Σ, with its centre on the directrix of the parabola, there is associated with these a rectangular hyperbola, Ω. The equations may be taken to be

$$(x + az)^2 + y^2 - 4a^2z^2 = 0, \quad y^2 - 4axz = 0, \quad x^2 - y^2 - 3a^2z^2 - 2axz = 0,$$

the tangential equation of Ω being $-3l^2 + 4m^2 + a^{-2}n^2 + 2a^{-1}nl = 0$. In connexion with this set we may deduce the result that when a parabola touches the lines BC, CA, AB, in such a way that its directrix contains the centre of the circle A, B, C, the nine points circle of the triad which is formed by the intersections of the tangents of the circle drawn at A, B, C, touches this circle at the focus of the parabola.

Ex. 28. *The director circle of a conic touching three lines is cut at right angles by the circle for which the three lines form a self-polar triad.* Let a, b, c be the lines, Σ the conic touching them, and C the circle. As Σ is inpolar to C, therefore C is outpolar to Σ; thus, by Gaskin's theorem, C cuts the director circle of Σ at right angles.

Ex. 29. The circle in regard to which a triad of points is self-polar has its centre at the orthocentre of the triad. If two triads of points on a conic have the same orthocentre, there is a circle in regard to which both triads are self-polar. (Cf. Ex. 21, p. 45.)

Ex. 30. Prove that conics touching three lines whose director circles have a common point have a fourth common tangent.

Ex. 31. *A transformation of conics through four points into conics having double contact.* The equations

$$\xi^{-1}(xy - z^2) = \eta^{-1}y(x - y) = -\zeta^{-1}z(x - y)$$

lead to $$x^{-1}(\xi\eta - \zeta^2) = y^{-1}\eta(\xi - \eta) = -z^{-1}\zeta(\xi - \eta)$$

and transform conics, of equation $y^2 - z^2 + \lambda(z^2 - x^2) = 0$, passing through the four points $(1, \pm 1, \pm 1)$, into conics, of equation $(x - y)^2 + \lambda(z^2 - x^2) = 0$, all having double contact on $x - y = 0$.

Other Examples are found in the Additions at the end of the Volume.

CHAPTER IV

RESTRICTION OF THE ALGEBRAIC SYMBOLS. THE DISTINCTION OF REAL AND IMAGINARY ELEMENTS

Review of the development of the argument. In the first three chapters of this volume we have followed the plan, adopted in Chapter I of Volume I, of avoiding reference to the relative order of the points of a line, or to characteristics of the algebraic symbols which would follow from such order, save for the occasional anticipation, in some of the latter examples in Chapter III, involved in referring to the roots of an equation. Our discussion of the Real Geometry, in Chapter II of Volume I, led us to regard the figures of such a real geometry, when amplified by means of the postulated elements, as existing among those of the Abstract Geometry. Thus we came to recognise, upon a line of which three points, O, E, U, are given, the existence of an aggregate of points satisfying the conditions for an abstract order (Vol. I, pp. 121, 128); while, however, in the real geometry, two points determined a segment, *two* segments were determined in the abstract geometry by the assignment of two points; and, instead of the possibility of speaking of a point as lying between two others, we were led to speak of the separation of two points of a line by two others (*ibid.* pp. 118, 119). At the same time, the points of a line obtainable, from three given points of the line, by a finite number of repetitions of the process of forming the fourth harmonic point from three given points, were associated, relatively to the three fundamental points, with certain iterative symbols, analogous to the integers of ordinary arithmetic (*ibid.* pp. 86, 138); and it was found, on the basis of the conception of the separation of two points by two others, that the points obtainable by this harmonic process arise in every segment of the line (*ibid.* p. 132). It was suggested (*ibid.* p. 146) that a consequence of the assumption of the existence of an aggregate of points upon the line satisfying the condition (3) for an abstract order (*ibid.* p. 122), would be the introduction of algebraic symbols forming a natural extension of the iterative symbols, as the irrational numbers of arithmetic form an extension of the rational numbers; it would follow from the equable distribution of the points of the harmonic net, that points for which such extended symbols are appropriate would be equably distributed. The deduction of Pappus' theorem (*ibid.* p. 129) was for points satisfying the conditions for an abstract order, it being assumed that the lines involved contained all the points necessary to the satisfaction of these conditions. More fully, the geometrical scheme was built up by supposing certain points to be given, and other points, in infinite number, to be deduced from these by prescribed rules of construction; it was assumed that if the fundamental points were among those for which the conditions for an abstract order, on any lines containing them, were satisfied, the points derived therefrom by the prescribed conditions might equally be regarded as satisfying such conditions. This underlying assumption of the existence of what, relatively to the rules of construction, may be called a *closed* set of points, which is to be made more precise with the use of the algebraic symbols, was avoided, in the Abstract Geometry of Chapter I of Vol. I, by the frank assumption of Pappus' theorem.

We were not content, however, with only points satisfying the conditions for an abstract order. We recognised, also, the existence of points such as would arise if the algebraic symbols appropriate to the geometry were such that the equation $z^2 = c$ could always be satisfied by a symbol, z, of the system employed, whatever symbol of this system c might be. When, in particular, c is -1, this condition requires the existence of points not previously recognised (see Vol. I, p. 162). We shall, as is customary, use i to denote a symbol for which $i^2 = -1$. By the adjunction of this symbol, i, to the symbols previously introduced on the basis of the iterative symbols, it is then possible to obtain a solution of the equation $z^2 = c$ for every symbol c which is itself formed from i and the previous symbols. If we call the symbols consisting of the iterative symbols, and those derived from them as the irrational numbers of arithmetic are derived from the rational numbers, the *real* symbols, all the symbols arising in consequence of the fundamental geometrical constructions, and the possibility of Steiner's problem (Vol. I, p. 155), in virtue of the laws of combination of the symbols (Vol. I, p. 64), will evidently be of the form $x + iy$, where x and y are real symbols.

It is the case now that, if we introduce certain laws of order of succession, or, if the phraseology is allowed, certain conceptions of *magnitude*, the real symbols are, in manipulation, indistinguishable from the real numbers of arithmetic, and the symbols $x + iy$ are indistinguishable from the numbers of ordinary Analysis. To those who have considered the theoretical basis of Analysis, this conclusion will be evident. But this basis is often discussed with the help of conceptions derived from the consideration of the measurement of quantity, and with a metrical Euclidean plane[1]. Some brief further indication of the ideas involved may therefore be allowed.

---

[1] Cf. Gauss, *Werke*, III, 1866, p. 79. Ich werde die Beweisführung in einer der Geometrie der Lage entnommenen Einkleidung darstellen, weil jene dadurch die grösste Anschaulichkeit und Einfachheit gewinnt. Im Grunde gehört aber der

To some it will appear that it would have been better, as it would have been simpler, frankly to assume from the first that all points to be considered in the geometry are such as those for which the numbers of Analysis, $x + iy$, are appropriate. To the writer this attitude appears to beg one of the main, and most interesting, questions arising in discussing the foundations of geometry. It is admitted that, by help of the symbols, we can define a closed system of points and figures to which this closed system of symbols is appropriate. But this does not appear to preclude the existence of other points and figures susceptible of geometrical theory. It is true of course that in the present volumes we do not obtain any geometrical results for which corresponding results, identical in statement, do not hold in the realm of points representable by symbols $x + iy$. But this appears to be a consequence of the arbitrary limitation, adopted for practical reasons, involved in the assumption, first of Pappus' theorem, and then of the possibility of Steiner's construction, or, more generally, as in some examples in Chap. III, of the possibility of the solution of problems depending, when expressed algebraically, on equations of higher than the second order. Nor, even, is it the case that the rules for the order of succession of the real symbols, to which we presently proceed, are the only ones we might adopt. For instance, an interesting theory arises by considering a system of symbols representable in the form $a + bx + cx^2 + \ldots$, wherein $a, b, c, \ldots$ may be supposed to be real symbols, but x *not* one of these, and agreeing that such a symbol shall be held to follow another symbol of the same form, when, in the difference of these, $a - a' + (b - b')x + (c - c')x^2 + \ldots$, the first of the coefficients $a - a', b - b', c - c', \ldots$, which is not zero, is *positive*. [Add.]

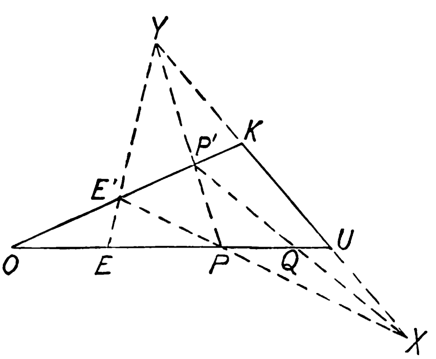
The distinction of positive and negative for the real symbols. The real symbol, a, of a point, $O + aU$, belonging to the ordered aggregate which we may suppose to exist upon a line, upon which three points of the aggregate, O, U, and E, of symbol $E = O + U$, are given, is said to be *positive* when the point lies in the segment OU which contains E. When the point belongs to the other segment OU, the symbol a is said to be negative. We saw that the points $O + aU$, $O - aU$ are harmonic conjugates in regard to O and U, and, therefore, separated by O and U (Vol. I, pp. 74, 119); thus, when a is a positive symbol, the symbol $-a$ is negative. In particular, as $0 = -0$, it is immaterial whether we regard the symbol of the point O as positive or negative; the point U is exceptional, as will appear, and our definition does not apply to it.

eigentliche Inhalt der ganzen Argumentation einem höhern von räumlichen unabhängigen Gebiete der allgemeinen abstracten Grössenlehre an, dessen Gegenstand die nach der Stetigkeit zusammenhängenden Grössencombinationen sind, einem Gebiete, welches zur Zeit noch wenig angebauet ist, und in welchem man sich auch nicht bewegen kann ohne eine von räumlichen Bildern entlehnte Sprache.

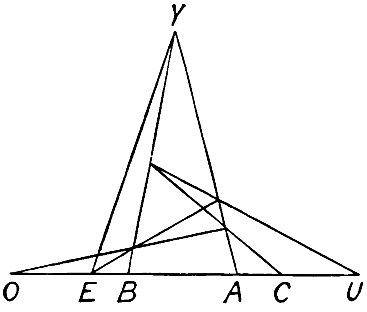
Whether c be a positive or negative real symbol, the symbol c^2 is positive; in virtue of $(-c)(-c) = c^2$ (Vol. I, p. 85), it is sufficient to prove this when c itself is positive. To pass from $P, = O + cU$, to $Q, = O + c^2U$, the construction given (Vol. I, p. 84) was as follows: draw, from O and U, respectively, two arbitrary lines OK, UK, meeting in K; take Y arbitrarily on UK; let EY meet OK in E', and $E'P$ meet KU in X; then, if PY meet OK in P', the point Q is the intersection of $P'X$ with OU. Conversely, if

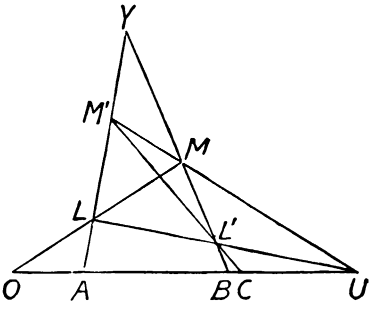
$$Q = O + aU,$$

and a be a positive real symbol, there is one positive real z for which $z^2 = a$, as well as a negative real $-z$. To obtain P from Q is, in fact, the problem considered in Vol. I, p. 157 (iii): we are to draw through the given points E', Q, Y, respectively in the segments $OE'K$, OQU, $UXYK$, three lines, $E'PX$, XQP', $YP'P$, whose intersections in pairs, respectively P', P and X, shall lie respectively in the specified segments.



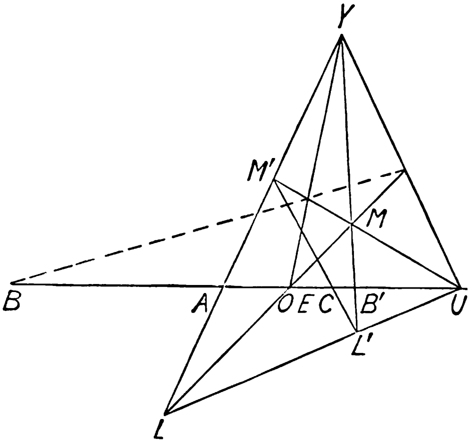
If a, b be both positive real symbols, the symbol $c, = a + b$, is also positive; if a, b be both negative, then c is also negative. In each case, the equation $a + b = 0$ requires that both $a = 0$ and $b = 0$. The product ab is, likewise, positive when a, b are both positive, or both negative.





Order of magnitude for the real symbols. The real symbols may then be put in order, which we may call the order of magnitude, by the rule that $a > b$ when $a - b$ is a positive symbol. If $A = O + aU$, $B = O + bU$, it can be proved that the geometrical condition for this is that the order U, B, A shall be the same as that of O, E, U. In this definition it is understood that neither A nor B is at U. It follows that if x be any real symbol, and $a > b$, then $a + x > b + x$; also if a, b, x, y be positive real symbols, and $a > b, x > y$, then $ax > by$. And so on.

With these definitions it is at once possible to pass to the notion of the upper (or lower) *bound*, of any indefinitely continued sequence of real symbols, when such exists; and to the notion of a continuous function, and the theory of such functions.



The roots of a polynomial. It requires then only an elaboration of these conceptions to shew that a polynomial

$$z^n + (a_1 + ib_1) z^{n-1} + \ldots + a_n + ib_n,$$

wherein $a_1, b_1, \ldots, a_n, b_n$ are real symbols, can be written as the product of n factors of the form $z + p + iq$, where p, q are real symbols.

Writing for z the form $x + iy$, and collecting the real symbols together, the polynomial takes the form $u + iv$, when u and v are polynomials in x and y with only real symbols as coefficients. A proof of the theorem may be constructed by proving that $u^2 + v^2$, which is a real continuous function of x and y, which is necessarily positive, has zero for its least value. In particular, when all of $b_1, b_2, \ldots, b_n$ are zero, if there is a root $x + iy$, with y not zero, there is also a root $x - iy$, so that, when n is odd, there is at least one root, x, which is real. [Add.]

The exponential and logarithmic functions. We also assume the existence of the so-called exponential function, exp (z), or e^z, which is unaltered when z is replaced by $z + 2\pi i$, and the functions

$$\cos z, = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z, = \frac{1}{2i}(e^{iz} - e^{-iz}),$$

$$\cosh z = \tfrac{1}{2}(e^{z} + e^{-z}), \quad \sinh z = \tfrac{1}{2}(e^{z} - e^{-z}).$$

When $e^z = \zeta$, we write $z = \log \zeta$, the function $\log \zeta$ being indeterminate by additive integer multiples of $2\pi i$. The fundamental properties of this function are also assumed.

A simple application to the double points of an involution. If O, U and A, B be two pairs of points, in line, there is a pair of points, P, Q, of the line, which are harmonic both in regard to O, U and in regard to A, B. If, beside O, U, there be given another point, E, of the line, which, in terms of the symbols of O, U, has the symbol $E = O + U$, then the symbols of A, B, and of P, Q, have definite expressions in terms of O and U. It is of importance to know when the expressions of P and Q, in terms of O and U, contain only real symbols.

Let the expressions of A and B be $O + aU$, $O + bU$, respectively; let p, q be two symbols such that $p^2 = a$ and $q^2 = b$; we clearly have

$$q(O + p^2U) + p(O + q^2U) = (p + q)(O + pqU),$$
$$-q(O + p^2U) + p(O + q^2U) = (p - q)(O - pqU);$$

comparing these, remembering that points of symbols $A \pm mB$ are harmonic in regard to A and B, we see that the points $O + pqU$, $O - pqU$ are the points desired, harmonic both in regard to O, U, and in regard to A, B.

Now, when a, b are real, and, therefore, ab is real, the symbol pq, which is such that $(pq)^2 = ab$, is real only when ab is positive, that is, only when a, b are both positive, or both negative. Thus, when O, U, E and A, B are regarded as real, *the points P, Q,* which are harmonic both in regard to O, U and A, B, *are real provided A and B are not separated by O and U, but not otherwise.*

If, however, relatively to O, U, and E, the points A, B are such that $a = x + iy$, and $b = x - iy$, where x, y are real, then ab, which is $x^2 + y^2$, is real and positive; in this case, then, P, Q are real. Two such points, A, B, are commonly said to be *conjugate imaginaries* in regard to O, U and E.

As the results are of frequent application, we may regard the matter from another point of view. Let us refer both O, U and A, B to two other points of the line, say, X and Y, which we suppose to be given, together with another point of the line whose symbol is $X + Y$; suppose then that $O = X + \xi Y$, $U = X + \eta Y$, where ξ, η are the roots of the equation $ax^2 + 2hx + b = 0$, and also that $A = X + \alpha Y$, $B = X + \beta Y$, where α, β are the roots of the equation $a'x^2 + 2h'x + b' = 0$. The general pair of points, of the involution determined by the two pairs O, U and X, Y, is then $X + mY$, $X + nY$, where m, n are the roots of the equation

$$ax^2 + 2hx + b + \lambda(a'x^2 + 2h'x + b') = 0,$$

for a proper value of λ. If the symbols ξ, η be either real, or conjugate imaginaries, so that $\xi + \eta$ and $\xi\eta$ are real, we may suppose a, h, b to be real, the case when ξ, η are real being that in which $h^2 - ab$ is positive. Conversely, when a, h, b are real, ξ and η are either real or conjugate imaginaries. Similarly for α and β. We limit ourselves now to the consideration of the case when a, h, b, a', h', b' are all real. The points P, Q, which are harmonic both in regard to

O, U and to A, B, are the double points of the involution determined by these two pairs. These arise, then, for the values of λ which make the last written quadratic a perfect square, namely when $(a+\lambda a')(b+\lambda b')=(h+\lambda h')^2$; the values of λ satisfying this will either be real, or be conjugate imaginaries. If these values be λ_1 and λ_2, the last written quadratic will arise by squaring

$$(a+\lambda_1 a')x+h+\lambda_1 h', \text{ or } (a+\lambda_2 a')x+h+\lambda_2 h',$$

and the symbols of P, Q will be

$$(a+\lambda_1 a')X-(h+\lambda_1 h')Y, \quad (a+\lambda_2 a')X-(h+\lambda_2 h')Y;$$

these points P, Q will then be real in regard to X, Y and $X+Y$, if, and only if, λ_1 and λ_2 be real, for which the condition is that

$$(2hh'-ab'-a'b)^2-4(h^2-ab)(h'^2-a'b')$$

should be positive. If we put

$$M=-a^{-2}(h^2-ab), \quad M'=-(a')^{-2}(h'^2-a'b'),$$

this expression is equal to $a^2a'^2$ multiplied into

$$[(a^{-1}h-(a')^{-1}h')^2+M-M']^2+4M'(a^{-1}h-(a')^{-1}h')^2.$$

We see thus that the expression is positive if either M or M' be positive; that is, if either h^2-ab or $h'^2-a'b'$ be negative; that is, if one, or both, of the two pairs O, U and A, B, be conjugate imaginaries in regard to X, Y and $X+Y$. But, when M, M' are both negative, so that ξ, η, α, β are all real, or O, U and A, B are all real in regard to X, Y and $X+Y$, the expression is equal to $a^2a'^2$ multiplied into $(\xi-\alpha)(\xi-\beta)(\eta-\alpha)(\eta-\beta)$; it is, therefore, only positive when O, U are not separated by A and B.

We may interpret ξ, η and α, β as values of the parameter, θ, on the conic expressed by $x=\theta^2$, $y=\theta$, $z=1$, speaking of points given by real values of θ as real points. Then the double points of the involution given by the two pairs of points of the conic, O, U and A, B, are the points of contact of the tangents to the conic which can be drawn from the point in which the chords OU and AB intersect. The previous remarks, therefore, give the result that, from any point of a line which does not cut the conic in two real points, there can be drawn two real tangents of the conic. Two real tangents can also be drawn from a point which is such that, from it, two lines can be drawn whose respective pairs of intersections with the conic are both real and do not separate one another, regarded as points of the aggregate of real points of the conic which is given by the real values of θ. But from a point from which two lines can be drawn intersecting the conic each in a pair of real points, those of one pair separating those of the other pair, no real tangents can be drawn to the conic.

We may interpret the result somewhat differently, regarding O, U

as the double points of one involution, and A, B as the double points of another involution; then P, Q are the pair of points common to both involutions. These are then real when at least one of the involutions has for its double points a pair of conjugate imaginary points. Regarding the points, as before, as lying on the conic $xz - y^2 = 0$, the result shews that, if T and T' be two points, not lying on the conic, one of which, at least, is such that no real tangents can be drawn from it to the conic, then the line TT' meets the conic in two real points. For, the involution of which the double points are the points of contact of tangents from T to the conic, consists of pairs of points of the conic lying on lines through T.

Another application is as follows: With reference to two absolute points, I and J, we have defined the axes of a conic, whose centre is Y (the pole of the line IJ), as the lines joining Y to the points, P, Q, of the line IJ, which are harmonic, both in regard to I, J, and in regard to the two points, X and Z, in which the conic meets the line IJ. Then, when, with respect to points regarded as real on the line IJ, the points I and J are conjugate imaginaries, the axes of the conic are real, provided the centre, Y, is real, and the points X, Z, where the conic meets the line IJ, are either real or are conjugate imaginaries.

Remark. In what precedes we have dealt with the reality of P, Q in two ways, first considering A, B relatively to O, U, and afterwards considering both O, U and A, B relatively to two other points X, Y. The distinction is not immaterial; if O, U be imaginary in regard to X and Y, a point which is real in regard to X and Y may be imaginary in regard to O and U. For instance, if $\lambda^2 + \mu^2 = 1$, and $O = X + (p + iq)Y$, $U = X + (p - iq)Y$, then the point given by $O + (\lambda + i\mu)U$ is the same as that given by

$$X + [p + \mu^{-1}(1 - \lambda)q]\,Y.$$

General statement of the meaning of real and imaginary points in a plane. As a basis for geometry in a plane, we suppose that four points are given, say A, B, C and D. These are regarded as real. By the intersections of the joining lines of pairs of these points, three other points are obtained. By the intersections of the joining lines of the pairs of all the points now obtained, other points are found, and the method may be indefinitely repeated. The derivation may be arranged as a series of applications of the process of finding the fourth harmonic point of three given points of a line, or of the process of finding the fourth harmonic line of three lines which meet in a point. The points and lines so obtained are unlimited in number. Further, with a proper definition of the meaning of the statement, it appears that the points so derived occur everywhere in the plane. This generalises the theorem for the points of a harmonic net upon a line (Vol. I, p. 131). The symbols of the points A, B, C being fixed relatively to one another by the condition that the symbol of the point D is $A + B + C$, the points so derived are in fact all those expressed by symbols of the form $lA \pm mB \pm nC$, where l, m, n are iterative symbols. This has not been formally proved, but seems sufficiently clear from what was said, Vol. I, p. 86 (see also Moebius, *Ges. Werke*, Leipzig, 1885, Vol. I, p. 247). Thus, as in the case of points of a line, in virtue of the notion of an abstract order (Vol. I, p. 128), these points form a net sufficient to identify a multitude of points of the plane, all expressed by symbols $pA + qB + rC$, where p, q, r are real symbols in the sense which has been explained. It is these points, and no others, which we speak of as the points of the plane which are real with respect to A, B, C and D. All other points of the plane, of symbols $xA + yB + zC$, for which one at least of $x^{-1}y$, $x^{-1}z$, $y^{-1}z$, $y^{-1}x$, $z^{-1}x$, $z^{-1}y$ is a complex symbol $(p + iq)$, the others of these being real, we speak of as imaginary or complex points, relatively to A, B, C and D.

This statement is capable of much more minute analysis. For the purposes we have in view here, it is necessary to trace the connexion between the algebraic symbols employed and the numbers of Analysis; it is to this end that the Moebius net is introduced.

Some applications of the definition. Imaginary points and lines. A line which has two real points, say $pA + qB + rC$ and $lA + mB + nC$, has an infinite number of other points which are real, with symbols $(p + \sigma l) A + (q + \sigma m) B + (r + \sigma n) C$, where σ is any real symbol. Similarly, if two lines which meet in a real point, P, have each a further real point, respectively Q and R, then an infinite number of lines can be drawn through P having each an infinite number of real points; they are those, namely, which contain real points of the line QR. A line which has two real points has an equation of the form $ax + by + cz = 0$, where a, b, c are real symbols. Conversely, a line with such an equation contains an infinite number of real points, and may be called a *real line*. Through the common point of two real lines, which is necessarily real, may be drawn an infinite number of real lines. A real line contains, however, an infinite number of imaginary points; and, through a real point can be drawn an infinite number of imaginary lines.

Conversely, an imaginary line contains one, but only one, real point; and, through an imaginary point there passes one, and only one, real line. For, to an imaginary point,

$$(l + ip) A + (m + iq) B + (n + ir) C,$$

where l, m, n, p, q, r are real but $l^{-1}p$, $m^{-1}q$, $n^{-1}r$ not all equal, there corresponds a *conjugate* imaginary point,

$$(l - ip)A + (m - iq)B + (n - ir)C;$$

and, similarly, to an imaginary line there corresponds another, the conjugate imaginary line. The join of an imaginary point to its conjugate is a real line through the first point; the intersection of an imaginary line with its conjugate is a real point of the first line. Conversely, a line given by the equation

$$(l + ip)x + (m + iq)y + (n + ir)z = 0,$$

where l, m, n, p, q, r are all real, but $l^{-1}p$, $m^{-1}q$, $n^{-1}r$ are not all equal (since otherwise this would be a real line), contains only the real point for which both the equations $lx + my + nz = 0$, $px + qy + rz = 0$ are satisfied; through this point the conjugate imaginary line passes. Similarly, for an imaginary point.

The reader will compare this theory with that given in Vol. I (p. 166), of aggregates of real elements having the mutual relations of imaginary elements. Consider any imaginary point. We can, conveniently, suppose its symbol to be $3^{\frac{1}{2}}L + iP$, where

$$3^{\frac{1}{2}}L = lA + mB + nC, \quad P = pA + qB + Cr,$$

its conjugate being $3^{\frac{1}{2}}L - iP$. The values $i/3^{\frac{1}{2}}$ and $-i/3^{\frac{1}{2}}$ are the roots of the equation $3x^2 + 1 = 0$. The quadratic $3x^2 + 1$ is the Hessian form of the cubic $(x - s)[(x - s)^2 - (1 + 3sx)^2]$, as is easy to verify, the symbol s being arbitrary. The roots of this cubic are s, $(s - 1)/(1 + 3s)$ and $(s + 1)/(1 - 3s)$; if we use $\vartheta(s)$ to denote $(s - 1)/(1 + 3s)$, and $\vartheta^2(s)$ to denote $\vartheta[\vartheta(s)]$, these roots are s, $\vartheta(s)$, $\vartheta^2(s)$; we easily see that $\vartheta^3(s)$, or $\vartheta[\vartheta^2(s)]$, is equal to s. Thus (see Vol. I, p. 173), the theory referred to would represent the given imaginary point, whose symbol is $3^{\frac{1}{2}}L + iP$, by a set of three points taken, in a definite order, on the real line containing the given point, namely the three points whose symbols are $L + sP$, $L + \vartheta(s)P$, $L + \vartheta^2(s)P$. The conjugate imaginary point, whose symbol is $3^{\frac{1}{2}}L - iP$, would then be represented by the triad, $L + sP$, $L + \vartheta^2(s)P$, $L + \vartheta(s)P$, consisting of the same points in a different order. And, s being arbitrary, the same imaginary point would be represented by an infinite number of equivalent triads.

Real and imaginary conics. More care is necessary in defining a real conic than in defining a real line, since a conic can be expressed by an equation in which all the coefficients are real and still not have any real point. For instance, the conic whose equation, relatively to the given real points of the plane, is $x^2 + y^2 + z^2 = 0$, contains no point whose coordinates are all real. A conic can also have two real points, and no other real points; or, can have four real points, and no other real points. Consider, for example, the conic expressed by the equation

$$z^2 + zx(l+i) + zy(m+i) - xy[h^2 - lm + k + i(2h - l - m)] = 0,$$

where l, m, h, k are real symbols, i having the usual significance ($i^2 = -1$). This conic contains the two points (1, 0, 0) and (0, 1, 0), which we regard as real. Any other point of real coordinates lying on this conic must be such as to satisfy both the equations

$$z^2 + lzx + mzy - xy(h^2 - lm + k) = 0,\quad zx + zy - xy(2h - l - m) = 0\,;$$

these, however, by elimination of z, require the equation

$$[(h-l)x - (h-m)y]^2 + k(x+y)^2 = 0,$$

and this can be satisfied by no real values of x and y if k be positive (other than $x = 0$, $y = 0$, which, by recurring to the original equations, does not lead to a point of the conic). When k is negative, there are two values of $x^{-1}y$ which satisfy this equation, each of which leads to a value of $x^{-1}z$. Thus the conic has two, or four, real points, according as k is positive or negative.

In general, the real points of a conic expressed by an equation $(a + ia')x^2 + \ldots = 0$, whose coefficients are not real, are the points common to two conics, each with real coefficients, $ax^2 + \ldots = 0$, $a'x^2 + \ldots = 0$. These common points have coordinates which depend upon the roots of an equation of the fourth degree, wherein the coefficients are real; of such an equation, as we have remarked, the real roots are in number, none, two, or four. Thus, a conic whose coefficients are not real possesses none, or two, or four real points, unless it possesses an infinite number (as when it degenerates into two lines of which one is real and the other is not real).

A conic whose equation has real coefficients may have no real point, as we have seen. If, however, such a conic have one real point, and be not a line pair meeting at this point, it has an infinite number of real points; for any real line drawn through the real point meets the conic in another real point. A conic which is a line pair, consisting of two conjugate imaginary lines, contains one real point, the intersection of these, and no other.

Thus we see that we may conveniently speak of a conic which is not a line pair, as real, regarded as a locus of points, when it is expressed by an equation with real coefficients, and possesses one real point. Such a conic has a real tangent at every real point, and is thus real, also, regarded as an envelope. When we speak of a conic as imaginary, it will be desirable to specify whether its equation has real coefficients or not.

It is obvious that a conic which has five real points is a real conic; it may be defined, in the original way, by two related pencils of lines whose centres are at two of these points. But, a real conic possesses imaginary points, and such points may be among five points which are utilised to define such related pencils of lines. It is unnecessary to enter into an examination of the possibilities.

The interior and exterior points of a real conic. In regard to a real conic, supposed not to be a line pair, we may speak of any real point of the plane which does not lie on the conic, as being either *interior*, or *exterior*, to the conic. For the polar line of a real point, say, P, in regard to a conic expressed by an equation with real coefficients, is a real line, and will meet the conic either in two real points, or in two conjugate imaginary points; in the latter case the two tangents that can be drawn to the conic from the point P are conjugate imaginary lines, and we speak of the point P as *interior* to the conic. It was shewn above (p. 160) that every real line drawn through such an interior point meets the conic in two real points. A point, from which two real tangents can be drawn to the real conic, is spoken of as an *exterior* point; from such a point can be drawn some lines which meet the conic in two conjugate imaginary points, and others which meet it in two real points. It can be proved that, if we take two interior points, and consider the two segments of the line which joins these, which are determined by the two real points in which this line meets the conic, then all the real points of one of these segments are interior points. The real points of a plane are thus divided into two categories by means of the real points of a real conic which is not a line pair.

The behaviour for an imaginary conic which is determined by a real equation is different. The tangents drawn to such a conic from any real point are conjugate imaginary, so that, in one sense, any point might be called an interior point; but every real line meets the conic in two conjugate imaginary points.

The reality of the common points, and of the common self-polar triad, of two conics. We consider two conics which are both expressed by equations with real coefficients; one, or both, of the conics may be real. As has been remarked, the determination of the common points of two such conics may be reduced to the solution of an equation of the fourth degree, with real coefficients. To any root of such an equation which is not real there corresponds another root which is the conjugate imaginary of the former; therefore, it may be shewn, to any one of the four common points of the two conics which is not real, there corresponds another common point, conjugate imaginary to the former; and, the line joining these two common points is real. There is, therefore, always, at least one line pair, of two real lines, passing through the four common points of two such conics. If the four common points of the two conics are all real, and different, there are six lines, each joining two of these points, which are all real; in this case the common self-polar triad, of points and lines, constructed from these joins, consists of three real points and three real lines. If the common points of the two conics consist of the two real, and different, points A, B, with the two conjugate imaginary points C and C', the lines AB, CC' are both real, the lines AC, AC' are conjugate imaginaries, and the lines BC, BC' are conjugate imaginaries. The point, P, of intersection of AB and CC' is, therefore, real; the point, Q, of intersection of AC and BC' is conjugate imaginary to the intersection, R, of the lines AC', BC, which are, respectively, conjugate imaginary to AC and BC'; thus the line QR is real, and the common self-polar triad, P, Q, R, consists of one real point and one real line, together with two conjugate imaginary points, and two conjugate imaginary lines. Lastly, if the intersections of the two conics consist of two conjugate imaginary points A, A' and two conjugate imaginary points C, C', the lines AA', CC' are real, and meet in a real point, P; the lines AC, $A'C'$ are conjugate imaginary and meet in a real point, Q, and the lines AC', $A'C$ are conjugate imaginary and meet in a real point, R. The common self-polar triad is thus wholly real. This last case arises, for example, when one of the conics is real, and the other, though having a real equation, is imaginary. Incidentally, we thus give a proof that the cubic equation expressed by

$$\begin{vmatrix} a-\lambda a', & h-\lambda h', & g-\lambda g' \\ h-\lambda h', & b-\lambda b', & f-\lambda f' \\ g-\lambda g', & f-\lambda f', & c-\lambda c' \end{vmatrix} = 0,$$

upon which, as we have seen (Chap. III, Ex. 22), the determination of the common self-polar triad of the two conics $ax^2 + \ldots = 0$, $a'x^2 + \ldots = 0$ depends, has real roots when one at least of the two quadric forms occurring on the left in the equations of the conics, is such that it preserves a constant sign for all real values of x, y, z— it being understood that all the coefficients $a, \ldots, a', \ldots,$ are real.

The tangential equation of a conic, whose point equation is real, has also real coefficients. We can therefore state the preceding results also in terms of the reality of the common tangents. But the dependence of the reality of the common tangents upon the reality of the common points is a matter also for enquiry.

We have disregarded the possibilities of coincidence of the four common points of the two conics in what has been said; the omission may easily be supplied. [Add.]

Ex. A conic passes through three points, A, B, C, and has the points D, D' as conjugate points, and the points E, E' as conjugate points. Find two other (real) points of the conic. In particular, when E, E' are on the line DD', and separate these.

CHAPTER V

PROPERTIES RELATIVE TO AN ABSOLUTE CONIC. THE NOTION OF DISTANCE. NON-EUCLIDEAN GEOMETRY

The interval of two points of a line relatively to two given points of the line. If M, N, P, Q, be four points of a line, and the symbols of these points be such that $aP = M + xN$, $bQ = M + yN$, then the symbol yx^{-1} is such that: (1), it is unaltered by replacing any one or more of the symbols P, Q, M, N, respectively, by pP, qQ, mM, nN, where p, q, m, n are any symbols of the system employed, of which the commutativity in multiplication is, of course, assumed; (2), it has the same value as for any other four points, M', N', P', Q', which form a range related to M, N, P, Q; with a slight change of notation this is the same statement as that a range of points, M, N, $M+N$, $M+\lambda N$, is related to a range of points represented by M', N', $M'+N'$, $M'+\lambda N'$. A proof has been given, Vol. I, p. 154; (3), the symbol yx^{-1} is equal to the symbol formed by the analogous rule from the equations which express M, N in terms of P and Q, that is, the equations

$$-a^{-1}(x-y)M = yP - xba^{-1}Q, \quad a^{-1}y(x-y)N = yP - yba^{-1}Q.$$

The symbol can then be represented by $(P, Q;\ M, N)$, or by $(M, N;\ P, Q)$. What is of the greatest importance to us here is the fact that, if P, Q, R be any three points of the line MN, we have

$$(P, Q;\ M, N) \,.\, (Q, R;\ M, N) = (P, R;\ M, N).$$

Now, and in all that follows, we suppose that the algebraic symbols employed are those arrived at in the preceding chapter, which have the same properties in computation as the complex numbers of ordinary Analysis, with which they may thus be identified. If then $\phi(\xi)$ be any function for which $\phi(\xi\eta) = \phi(\xi) + \phi(\eta)$, and ξ denote $(P, Q;\ M, N)$, η denote $(Q, R;\ M, N)$, and ζ denote $(P, R;\ M, N)$, we shall have $\phi(\zeta) = \phi(\xi) + \phi(\eta)$. We apply this to the case when $\phi(\xi) = (2i)^{-1}\log(\xi)$.

Let the expression

$$\frac{1}{2i}\log[(P, Q;\ M, N)]$$

be called the *interval* from P to Q with respect to M and N. The interval from P to R is thus the sum of the interval from P to Q and that from Q to R, subject however to the ambiguity inherent in the definition. As $(Q, P;\ M, N)$ is the inverse of $(P, Q;\ M, N)$, the interval from Q to P is the negative of the interval from P to Q. But further, in virtue of the property of the logarithm, the interval, in default of further specification, is subject to the addition of an arbitrary integral multiple of π. We may agree that when P and Q coincide, in which case $(P, Q;\ M, N) = 1$, the interval vanishes, and, then, that when the interval is π, the two points again coincide. We may, however, apply the definition to cases in which the interval is not real, as will be seen. If the interval be denoted by θ, we have $(P, Q;\ M, N) = e^{2i\theta}$, and, when P, Q, M, N are all real points, it is $i\theta$ which is real. When $\theta = \frac{1}{2}\pi$, we have $(P, Q;\ M, N) = -1$; then P and Q are harmonic in regard to M and N.

Angular interval in regard to two absolute points. We have considered, in Chapter III, some of the consequences of supposing two points, I, J, of the plane, to be given, and taken as absolute points. With two such points, we may employ the considerations just given, to define the interval from one line, OP, to another line, OQ; namely, if these lines, respectively, meet the line IJ in P' and Q', we may take for this interval the value $(2i)^{-1} \log [(P', Q';\ I, J)]$. This depends only on the points, P' and Q', where the lines meet the line IJ, and not on the position of the intersection, O, of the lines. Suppose in particular, when the points of the plane are referred to three points, C, A, B, which we regard as real, that $I = A + iB$, $J = A - iB$, while P', Q' are given by

$$P' = \cos\alpha \,.\, A + \sin\alpha \,.\, B \quad \text{and} \quad Q' = \cos(\alpha+\theta) \,.\, A + \sin(\alpha+\theta) \,.\, B\,;$$

the lines CI, CJ, CP', CQ' can then be expressed by the respective equations $y = ix$, $y = -ix$, $y = \tan\alpha \,.\, x$, $y = \tan(\alpha+\theta) \,.\, x$. These relations give

$$2e^{i\alpha} P' = I + e^{2i\alpha} J, \quad 2e^{i(\alpha+\theta)} Q' = I + e^{2i(\alpha+\theta)} J$$

and hence $$(P', Q';\ I, J) = e^{2i\theta}.$$

This relation, approached from a metrical point of view, was, it seems, first used by Laguerre (*Nouv. Annal.* xii, 1853, p. 57; *Oeuvres* (1905) II, p. 12). As will be seen, it may be regarded as the foundation of all that follows in this Chapter. (Cp. p. 195.)

It is clear that, if O be any point of a conic containing I and J, of which P and Q are two fixed points, the interval from OP to OQ, in regard to I and J, is the same for all positions of O on the conic. In particular, when PQ passes through the pole of IJ, in regard to this conic, this interval is $\frac{1}{2}\pi$.

Interval in regard to an absolute conic. The postulation of two absolute points thus furnishes a ready means of defining the interval from one line to another. It does not however, except by a limiting process, furnish a means of defining the interval between two points, unless they lie on the line determined by the absolute points. The two absolute points may, however, be thought of as a degenerate conic envelope. This may suggest that we take a fixed undegenerate conic, given in the plane, as a basis for reference. If we do this, two points, say M and N, are determined on any line, PQ, which does not touch the conic, namely the points of intersection of PQ with this absolute conic; and then the interval from P to Q may be defined, as above, in respect to M and N. But this conic also determines two lines through an arbitrary point, O, of the plane, not lying on the conic, namely the tangents from O to this absolute conic; and then the angular interval between any two lines which intersect in O may be defined, as above, by means of these tangents. We have thus a method of defining both the interval from one point to another, and the interval from one line to another, in harmony with the duality otherwise existing in the geometry of the plane. When the absolute conic, regarded tangentially, becomes a point pair, we thence obtain what has, on other grounds, been chosen as the measure of an angle; we could, similarly, obtain a measure of the interval between two points, if we supposed the existence of two absolute lines in the plane, as a degenerate form of the absolute conic. From this general point of view, the ordinary metric relations, usually employed at least since the time of the Greeks, appear as an unsymmetric and incomplete scheme. Moreover, *distance*, as usually understood, which we shall shew to be deducible by a limiting process from what has been said, so far from being an obvious notion, characterising the space of experience, appears as an artificial, if useful, application of the algebraic symbols.

The idea of length is, presumably, suggested by our experience of what are called rigid bodies. We may refer, at once, here, to the notion which replaces this in the present point of view. Let P, Q, M, N be four points in line, of respective coordinates (x, y, z), (x', y', z'), (ξ, η, ζ), (ξ', η', ζ'). Let P_1, Q_1, M_1, N_1 be other four points, the coordinates of P_1 being definite linear functions of the coordinates of P, say

$$x_1 = ax + by + cz, \quad y_1 = lx + my + nz, \quad z_1 = px + qy + rz,$$

the coordinates of Q_1 being the *same* linear functions of the coordinates of Q, those of M_1 the *same* linear functions of the coordinates of M, and those of N_1 the same linear functions of the coordinates of N. It may then be shewn without difficulty that P_1, Q_1, M_1, N_1 are in line, and are a range which is related to the range P, Q, M, N. The linear transformation employed is, in fact, the representation of the correspondence between two related planes (here coinciding) which was considered in Vol. I, p. 149.

Of such linear transformations, depending on eight parameters (the ratios of the coefficients a, b, c, l, m, ...), there is a set, wherein the eight parameters are such functions of three parameters that any point of an arbitrarily chosen conic gives rise to another point of the same conic; as we shall see in more detail later. Any two transformations of this set, carried out in succession, lead from a point of this conic to another point of this conic, and so, in combination, constitute a transformation of the same set.

If the conic be the absolute conic by which we measure the interval between two points P and Q, as has been explained, M and N being points of the conic, the interval between the transformed points, P_1 and Q_1, has, clearly, from what has been said, the same measure as the interval PQ.

The change, by a linear transformation leaving the absolute conic unaltered, from the interval PQ to the interval P_1Q_1, is the operation which, from our present point of view, replaces what, in the metrical geometry, is described as the movement of PQ into the position P_1Q_1.

The reader may shew that the general linear transformation changing any point of the conic $x^2+y^2+z^2=0$ into another point of this conic, is given by

$$\begin{aligned}
x_1 &= \tfrac{1}{2}(D^2+A^2-B^2-C^2)x + (AB-CD)y + (CA+BD)z,\\
y_1 &= (AB+CD)x + \tfrac{1}{2}(D^2+B^2-C^2-A^2)y + (BC-AD)z,\\
z_1 &= (CA-BD)x + (BC+AD)y + \tfrac{1}{2}(D^2+C^2-A^2-B^2)z,
\end{aligned}$$

where A, B, C, D are arbitrary.

We now consider the matter more in detail: Let the absolute conic be given, with respect to real points of reference for the coordinates, by an equation with real coefficients; it may, thus, be either a real conic or an imaginary conic. We can suppose the points of reference, still being real, to be so chosen that this equation is of the form $z^2+\kappa(x^2+y^2)=0$, it being supposed that this conic is not degenerate. The coefficient, κ, is then real; it is negative for a real conic, but positive for an imaginary conic.

The proof of this form may be given, as follows; First, a homogeneous quadratic form in *two* variables, x, y, with real coefficients, can be reduced, by a real linear change of coordinates, to a sum of squares, with real coefficients. For, let the form be $ax^2+2hxy+by^2$. If both a and b be zero, this is $\frac{1}{2}h\,[(x+y)^2-(x-y)^2]$, which consists of two squares. If not, suppose a not to be zero; then the form is $a(x+a^{-1}hy)^2+(b-a^{-1}h^2)y^2$, which again consists of two squares. Second, consider the form in *three* variables, consisting of three pairs of terms such as $ax^2 + 2fyz$. If one of the three coefficients a, b, c, say the coefficient a, be not zero, the form is

$$a(x + a^{-1}hy + a^{-1}gz)^2 + (b - a^{-1}h^2)y^2$$
$$+ (c - a^{-1}g^2)z^2 + 2(f - a^{-1}gh)yz;$$

here the three terms after the first are a quadratic form in y and z only, with real coefficients, and can be written as a sum of two (or fewer) squares, with real coefficients. If, however, all of a, b, c be zero, but none of f, g, h be zero, the original form in x, y, z is

$$2fgh(f^{-1}x + \tfrac{1}{2}g^{-1}y + \tfrac{1}{2}h^{-1}z)^2 - 2f^{-1}ghx^2 - \tfrac{1}{2}fgh(g^{-1}y - h^{-1}z)^2,$$

which consists of three squares. If all of a, b, c be zero, and one, or two, of f, g, h be zero, the form reduces to a product of two real linear forms, PQ, which is $\frac{1}{4}(P + Q)^2 - \frac{1}{4}(P - Q)^2$. In any case, therefore, the form is reducible, in many ways, to $pX^2 + qY^2 + rZ^2$, where p, q, r are real, and X, Y, Z are real linear functions of x, y, z. Assuming that the form, when equated to zero, does not represent a line pair, no one of p, q, r is zero. And, in the equation so obtained, we may suppose that all of p, q, r are positive, or only one is positive. When r is positive, and p, q are also positive, by taking $X_1 = X(p/\kappa)^{\frac{1}{2}}$, $Y_1 = Y(q/\kappa)^{\frac{1}{2}}$, $Z_1 = Zr^{\frac{1}{2}}$, where κ is an arbitrary real positive number, we obtain the equation $Z_1^2 + \kappa(X_1^2 + Y_1^2) = 0$. When r is positive, and p, q are both negative, we can put

$$X_1 = X(-p)^{\frac{1}{2}}/(-\kappa)^{\frac{1}{2}}, \quad Y_1 = Y(-q)^{\frac{1}{2}}/(-\kappa)^{\frac{1}{2}}, \quad Z_1 = Zr^{\frac{1}{2}},$$

where κ is an arbitrary real negative number, and so obtain the equation $Z_1^2 + \kappa(X_1^2 + Y_1^2) = 0$.

Case of an imaginary conic as Absolute. Let (x_0, y_0, z_0) and (x, y, z) be any two points not lying on the absolute conic. Let f denote the expression $z^2 + \kappa(x^2 + y^2)$, and f_0 what this becomes when x_0, y_0, z_0 are put for x, y, z, respectively; also let ψ denote the expression $zz_0 + \kappa(xx_0 + yy_0)$. For the case of an imaginary conic, κ is positive, and f, f_0 are positive; and $ff_0 - \psi^2$, which is equal to

$$\kappa\left[\kappa(xy_0 - x_0y)^2 + (zx_0 - z_0x)^2 + (zy_0 - z_0y)^2\right],$$

is also positive. Let M, N denote the points in which the line joining the point (x_0, y_0, z_0) to the point (x, y, z) meets the conic. We can suppose that, for a positive real value of θ, one of these points has the coordinates

$$x_0f_0^{-\frac{1}{2}} - e^{i\theta}xf^{-\frac{1}{2}}, \quad y_0f_0^{-\frac{1}{2}} - e^{i\theta}yf^{-\frac{1}{2}}, \quad z_0f_0^{-\frac{1}{2}} - e^{i\theta}zf^{-\frac{1}{2}},$$

where the square roots, $f_0^{-\frac{1}{2}}, f^{-\frac{1}{2}}$, have their real positive values; for the substitution of these coordinates in the equation of the conic gives the condition

$$1 - 2e^{i\theta}\psi f_0^{-\frac{1}{2}}f^{-\frac{1}{2}} + e^{2i\theta} = 0,$$

namely $$\cos\theta = \psi f_0^{-\frac{1}{2}} f^{-\frac{1}{2}},$$

of which the value is less than unity. If then, relatively to the points of reference for the coordinates, A, B, C, we define the symbols, O and P, of the points (x_0, y_0, z_0) and (x, y, z), respectively, by means of

$$O = f_0^{-\frac{1}{2}}(x_0 A + y_0 B + z_0 C), \quad P = f^{-\frac{1}{2}}(xA + yB + zC),$$

we may write, for the symbols of the points M and N.

$$M = O - e^{-i\theta}P, \quad N = O - e^{i\theta}P,$$

where $\cos\theta = \psi f_0^{-\frac{1}{2}} f^{-\frac{1}{2}}$, $\sin\theta = (ff_0 - \psi^2)^{\frac{1}{2}} f_0^{-\frac{1}{2}} f^{-\frac{1}{2}}$,

and the symbol, $(O, P;\ M, N)$, defined above (p. 166), is, then, given by

$$(O, P;\ M, N) = e^{2i\theta}.$$

Now let the point where the line OP meets the polar line of O in regard to the absolute conic be denoted by U. The coordinates (ξ, η, ζ) of this point are given by $\xi = xf_0 - x_0\psi$, $\eta = yf_0 - y_0\psi$, $\zeta = zf_0 - z_0\psi$; for these belong to a point on the line joining (x_0, y_0, z_0) to (x, y, z), and are such that $\zeta z_0 + \kappa(\xi x_0 + \eta y_0) = 0$. With these values we have $\zeta^2 + \kappa(\xi^2 + \eta^2)$ equal to $f_0(ff_0 - \psi^2)$, which is positive. If we denote the positive square root of this by μ, we have three equations, for x, y, z, of the following form

$$xf^{-\frac{1}{2}} = \psi f_0^{-\frac{1}{2}} f^{-\frac{1}{2}} . x_0 f_0^{-\frac{1}{2}} + (ff_0 - \psi^2)^{\frac{1}{2}} f_0^{-\frac{1}{2}} f^{-\frac{1}{2}} . \xi\mu^{-1}.$$

We may therefore write, using for the point U the symbol $U = \mu^{-1}(\xi A + \eta B + \zeta C)$,

$$P = \cos\theta . O + \sin\theta . U.$$

Conversely, this equation may, if preferred, be used as giving the definition of θ. If we regard O and U as fixed, it gives a real position of P upon the line OU for every real value of θ, all such positions being obtained by restricting θ to the range from 0 to π. In particular, when $\theta = 0$, the point P is at O; and when $\theta = \frac{1}{2}\pi$, the point P is at U, the conjugate point to O in regard to the absolute conic.

By a precisely similar argument any other point Q, of the line OU, can be expressed, in terms of P and the point, P', conjugate to P upon this line, by the equation

$$Q = \cos\phi . P + \sin\phi . P'.$$

As the original formula was equivalent to $(O, P;\ M, N) = e^{2i\theta}$, so this formula is equivalent to $(P, Q;\ M, N) = e^{2i\phi}$. Thus we have $(O, Q;\ M, N) = e^{2i(\theta+\phi)}$, so that, also

$$Q = \cos(\theta + \phi) . O + \sin(\theta + \phi) . U.$$

This, however, is

$$\cos\phi\,(\cos\theta\,.\,O+\sin\theta\,.\,U)+\sin\phi\,(-\sin\theta\,.\,O+\cos\theta\,.\,U)\,;$$

thus we see that

$$P'=\cos(\theta+\tfrac{1}{2}\pi)\,.\,O+\sin(\theta+\tfrac{1}{2}\pi)\,.\,U,$$

and, when P is given, in terms of O and U, by θ, then P', the conjugate of P, in regard to the conic, upon the line OU, is given by $\theta+\frac{1}{2}\pi$. This fact can, of course, be directly verified, the coordinates of P' being, as we easily see,

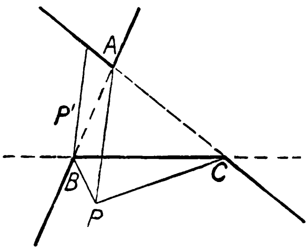
$$-x_0\,(ff_0-\psi^2)+\xi\psi,\quad -y_0\,(ff_0-\psi^2)+\eta\psi,\quad -z_0\,(ff_0-\psi^2)+\zeta\psi.$$

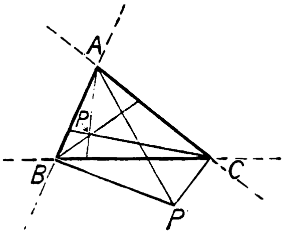
In words, the interval, in regard to the conic, from a point P to any point on its polar line, is $\frac{1}{2}\pi$.

In a precisely similar way, the interval from a line OA to a line OB may be defined with reference to the tangents, OM', ON', drawn from O to the absolute conic. If the points A, B be on the polar of O, the interval is, by definition, equal to the interval from the point A to the point B. In particular when OB is conjugate to OA, in regard to the absolute conic, the interval between these lines is $\frac{1}{2}\pi$.

Application to a triangle. We may illustrate these definitions by applying them to the intervals between any three given points, not lying in line, and the intervals between the lines which join these points. But some preliminary remarks will be useful, in view of the ambiguity we have noticed, in the definition of the interval between two given elements.

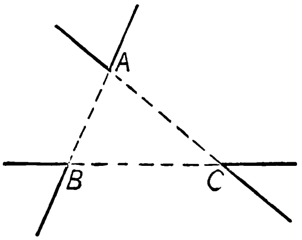
Any two points determine *two* segments, in one or other of which any other real point, of the line determined by the two given points, must lie. If then we have three points, A, B, C, which are not in line, and speak of the segments BC, CA, AB, there are eight ways in which this may be understood. We desire, however, to associate with each of the segments, say, with BC, which may be regarded as an aggregate of points, a certain aggregate of lines, all passing through the point, A, not on the line BC, so that every one of the lines shall contain one of the points of the segment. Limiting ourselves to *real* points and lines, relatively to A, B, C, if we speak of this aggregate of lines as an *angle*, we may say that there are, for each of the points such as A, two angles. We have then in all eight possibilities, of three pairs, each pair being constituted by a segment and its corresponding angle. These eight possibilities fall, however, into two sets, each of four possibilities. In any one of the first set of four possibilities, the three segments, say, BC, CA, AB, and the associated angles, say, A, B, C, have, in regard to real points, the two following properties: (1) Any line of the plane either contains no point of any of the three segments, or contains a point of each of two of the segments; (2) If any point P of the plane be joined to A, B and C, and we consider the intersections of the joining lines PA, PB, PC, respectively, with the lines BC, CA, AB, either, one, of these intersections, or all, lie in the segments of these lines which

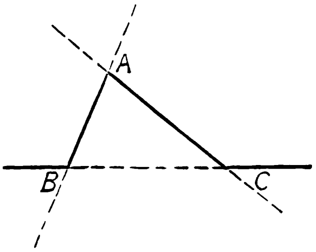




define the possibility considered. If the two segments of the line BC be denoted by a and a', and the two respectively associated angles at A by A and A', with a similar notation for B and C, and one of the four possibilities spoken of consist of A, a; B, b; C, c, then another of these consists of A, a; B', b'; C', c', and the other two are respectively A', a'; B, b; C', c' and A', a'; B', b'; C, c.

The other four possibilities consist of the associations, A', a'; B, b; C, c and A', a'; B', b'; C', c', together with A, a; B', b'; C, c





and A, a; B, b; C', c'. For these also a statement can be made as to the intersections of an arbitrary line with the segments considered, and the lines joining an arbitrary point, to A, B, C, which belong to the angles associated therewith. But this statement will be different from that which holds for the first four possibilities. It appears unnecessary to enter into a detailed analysis of the statement made (cf. Vol. I, p. 100).

In what follows we confine ourselves to one of the first four possibilities. It is immaterial which one; but we could, if desired, describe that one by supposing an additional line given in the plane, whose intersections with the lines BC, CA, AB would enable us to distinguish between the two segments of the line. If we denote the intervals BC, CA, AB chosen, measured relatively to the absolute conic, respectively by a, b, c, the other supplementary segments

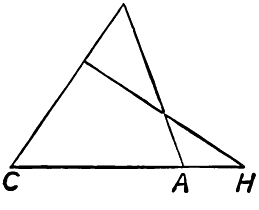
BC, *CA*, *AB* may be taken to be, respectively $\pi - a$, $\pi - b$, $\pi - c$, as we have seen. Similarly the interval between *AB* and *AC*, associated with a, being denoted by A, the supplementary interval between these lines will be $\pi - A$; the intervals adopted at B and C will likewise be denoted by B and C. So far, as we have seen, this statement is ambiguous. But if we agree that a shall be positive and between 0 and $\frac{1}{2}\pi$, when *the point*, B', of the line *BC*, *which is conjugate to B*, in regard to the absolute conic, *does not lie in the segment BC* which is denoted by a, then a will be without ambiguity. In this case, the point, C', of the line *BC*, which is conjugate to C, in regard to the absolute conic, will also not lie in the chosen segment *BC*; for, as B, B' and C, C' are pairs of an involution whose double points are the imaginary intersections of the line *BC* with the imaginary absolute conic, the points B, B' are separated by the pair C, C'. (Above, p. 158.) Similarly we agree that A shall be positive and between 0 and $\frac{1}{2}\pi$, when the line through A which is conjugate to *AB* does not lie in the angle called A, in which case, for a similar reason, the line through A conjugate to *AC* will not lie in the chosen angle A. And we make the like agreement for b, c, B and C.

We can, for the sake of algebraic symmetry, suppose, for the present, without loss of generality, that the equation of the absolute conic is $x^2 + y^2 + z^2 = 0$. If then, for example, (x, y, z) and (x', y', z') be the coordinates of the points A and B, respectively, the interval c, of A, B, will be such (p. 171, above) that

$$\cos c = \pm \frac{xx' + yy' + zz'}{(fx)^{\frac{1}{2}} (fx')^{\frac{1}{2}}},$$

$$\sin c = \frac{[(yz' - y'z)^2 + (zx' - z'x)^2 + (xy' - x'y)^2]^{\frac{1}{2}}}{(fx)^{\frac{1}{2}} (fx')^{\frac{1}{2}}},$$

where fx denotes $x^2 + y^2 + z^2$, and, for each of the square roots, the positive real value is to be taken. In $\cos c$, the numerator vanishes and changes sign when (x', y', z') passes over the point, of the line joining this to (x, y, z), which is conjugate to (x, y, z). We can, therefore, agree to attach to the coordinates of (x', y', z') such a sign that the numerator in $\cos c$ is positive when c, as defined above, is less than $\frac{1}{2}\pi$, and then omit the ambiguous sign $\pm$ in the above formula for $\cos c$.



We now take, first, the particular case in which the lines *AB*, *AC* are conjugate to one another in regard to the absolute conic. We denote the coordinates of the points A, B, C, respectively, by (x, y, z), (x', y', z'),

(x'', y'', z''), and denote by Δ the absolute, positive, value of the determinant

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

the minors in this determinant, each with its proper sign, being denoted, respectively, by

$$\begin{matrix} u, & v, & w, \\ u', & v', & w', \\ u'', & v'', & w''; \end{matrix}$$

any minor of the determinant formed by these last elements is equal to the complementary minor of the original determinant, multiplied by the value of this determinant itself, as is well known. We have the identity

$$(yz' - y'z)^2 + (zx' - z'x)^2 + (xy' - x'y)^2$$
$$= (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) - (xx' + yy' + zz')^2;$$

from this, or independently, we have

$$\Sigma (yz' - y'z)(yz'' - y''z) = \Sigma x^2 . \Sigma x'x'' - \Sigma xx' . \Sigma xx'',$$

which, still using fx for $f(x)$, we denote by

$$-f(u', u'') = fx . f(x', x'') - f(x, x')f(x, x'') \text{ ......(i).}$$

We similarly have, if fu mean $f(u)$,

$$-\Delta^2 f(x', x'') = fu . f(u', u'') - f(u, u')f(u, u'') \text{ ...(ii).}$$

Also, as $ux + vy + wz = 0$ is the equation of the line BC, and so on, we have, as equivalent to the condition that AB and AC are conjugate, the equation

$$f(u', u'') = 0 \text{ ....(iii).}$$

We shall write P, P', P'', U, U', U'', respectively, for the real positive numbers $(fx)^{\frac{1}{2}}, (fx')^{\frac{1}{2}}, (fx'')^{\frac{1}{2}}, (fu)^{\frac{1}{2}}, (fu')^{\frac{1}{2}}, (fu'')^{\frac{1}{2}}$.

For the interval c we have, then,

$$\cos c = f(x, x')/PP', \quad \sin c = U''/PP',$$

the $f(x, x')$ being taken positive, or negative, according as $c < \frac{1}{2}\pi$, or $c > \frac{1}{2}\pi$. For the angle C we have, similarly,

$$\cos C = \pm f(u, u')/UU', \quad \sin C = \Delta P''/UU';$$

but, the pole of the line CA is on the line AB, and is the point thereon which is conjugate to A; the line joining C to this point is the line through C which is conjugate to CA. Thus C is greater or less than $\frac{1}{2}\pi$, according as c is greater or less than $\frac{1}{2}\pi$. Also, from equations (ii), (iii), we have $-\Delta^2 f(x, x') = f(u'')f(u, u')$, which shews that $f(u, u')$ and $f(x, x')$ are of opposite sign. Wherefore we have

$$\cos C = -f(u, u')/UU'.$$

We can now shew that, of the three intervals a, b, c, either all are less than $\frac{1}{2}\pi$, or, just one of them is less than $\frac{1}{2}\pi$. For, let the polar line of the point B meet the line CA in H, so that H will be the pole of the line AB, and will be the point of the line CA which is conjugate to A. By our definition of the triangle considered, if this line, the polar line of B, do not contain a point of either of the segments BA, BC, then H will not be in the segment AC; so that, if each of the intervals a, c is less than $\frac{1}{2}\pi$, then b is, also, less than $\frac{1}{2}\pi$. But, if this line contain a point of each of the segments BA, BC, then, also, H is not in AC; so that, if each of a, c is greater than $\frac{1}{2}\pi$, then, also, b is less than $\frac{1}{2}\pi$. When the polar line of B contains a point of only one of the segments BA, BC, we reach the same conclusion.

But equations (i), (iii) give $f(x', x'')f(x) = f(x, x')f(x, x'')$, which is the same as

$$\frac{f(x', x'')}{P'P''} = \frac{f(x, x')}{PP'} \cdot \frac{f(x, x'')}{PP''};$$

if then we suppose, when x, y, z have been chosen, that the signs of x', y', z' and of x'', y'', z'' are so taken that, not only

$$\cos c = f(x, x')/PP',$$

as has already been agreed, but also $\cos b = f(x, x'')/PP''$, we can infer, remembering that, as has just been seen, one or all of a, b, c are less than $\frac{1}{2}\pi$, that

$$\cos a = \cos b \cos c \quad \text{..................} (1),$$

and also $\cos a = f(x', x'')/P'P''$. Then, from the equation

$$\Delta^2 f(x) = f(u')f(u'') - [f(u', u'')]^2,$$

by (iii), we have

$$\frac{\Delta P''}{UU'} = \frac{U''}{PP'} \bigg/ \frac{U}{P'P''},$$

and hence $$\sin C = \sin c/\sin a \quad \text{..................} (2).$$

Also, from equations (ii), (iii), we have

$$f(x', x'') = \frac{f(u, u')f(u, u'')}{\Delta^2}, \quad f(x, x'') = -\frac{fu' \cdot f(u, u'')}{\Delta^2},$$

so that $$-\frac{f(u, u')}{fu'} = \frac{f(x', x'')}{f(x, x'')},$$

which is $$-\frac{f(u, u')}{UU'} = \frac{U'}{f(x, x'')} \bigg/ \frac{U}{f(x', x'')},$$

namely $$\cos C = \tan b/\tan a \quad \text{..................} (3).$$

From the formulae (1), (2), (3), we deduce at once

$$\sin B = \sin b/\sin a, \quad \cos B = \tan c/\tan a, \quad \cos a = \cot B \cot C$$

together with

$$\cos B/\cos b = \sin C, \quad \cos C/\cos c = \sin B, \quad \tan B = \tan b/\sin c,$$
$$\tan C = \tan c/\sin b.$$

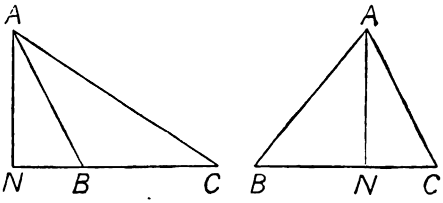
The sum of the angular intervals for a triangle. An immediate consequence of these formulae is that the angular intervals B, C have a sum which is greater than $\frac{1}{2}\pi$. For we have

$$\cos(B + C) = \cos B \cos C - \sin B \sin C = \sin B \sin C (\cos b \cos c - 1),$$

and $\cos b \cos c - 1$ is in all cases negative. Thus $\cos(B + C) < 0$, and therefore $B + C > \frac{1}{2}\pi$. Wherefore $A + B + C > \pi$.

And this last result is true for any triangle A, B, C, whether AB, AC be conjugate lines or not. For, let the lines drawn through A, B, C, each conjugate to the respectively opposite side, meet in the point P (Chap. I, Ex. 10). By our definition of the triangle, one at least of the lines AP, BP, CP contains a point of the opposite segment; for example, suppose that AP contains the point N of the segment BC. We can then apply the above argument to each of the triangles A, N, B and A, N, C. If the angular intervals BAN, NAC be called, respectively, ϕ and ψ, we have $\phi + B > \frac{1}{2}\pi$, $\psi + C > \frac{1}{2}\pi$, $\phi + \psi = A$, and hence $A + B + C > \pi$. This is an important result.

Formulae for a general triangle. Let the line through A which is conjugate to the line BC meet this in N, which is not necessarily a point of the segment BC. Denote by p the interval AN, and by ϕ, ψ, respectively, the angular intervals BAN, NAC, with the conventions introduced before in dealing with the right-angled triangle.



We have

$$\sin p = \sin c \sin B = \sin b \sin C;$$

so that we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the first of these being introduced by a construction similar to that here described.

Further, using NC for $-CN$, and BN for $-NB$, with a similar convention for ϕ and ψ, when necessary, we have

$$\cos a = \cos(BN + NC) = \cos BN \cos NC - \sin BN \sin NC,$$
$$\cos b \cos c = \cos BN \cos NC \cos^2 p;$$

hence

$$\cos a - \cos b \cos c = \cos BN \cos NC \sin^2 p - \sin BN \sin NC;$$

we have, however,

$$\sin b \sin c \cos A = \sin b \sin c \cos(\phi + \psi)$$
$$= \sin b \sin c (\cos\phi \cos\psi - \sin\phi \sin\psi),$$

while $$\frac{\cos\phi}{\cos BN} = \sin B = \frac{\sin p}{\sin c}, \quad \frac{\cos\psi}{\cos NC} = \sin C = \frac{\sin p}{\sin b},$$

and $$\sin\phi = \frac{\sin BN}{\sin c}, \quad \sin\psi = \frac{\sin NC}{\sin b};$$

hence

$$\sin b \sin c \cos A = \cos BN \cos NC \sin^2 p - \sin BN \sin NC.$$

Thus we infer

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \;\ldots\ldots\ldots\ldots(2).$$

By a similar proof we have

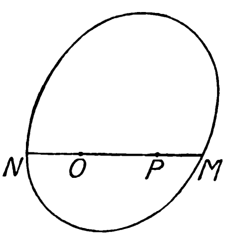
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \;\ldots\ldots\ldots(3).$$

From these formulae it follows, also, that $\cos a - \cos(b + c)$, being equal to $2 \sin b \sin c \cos^2 \frac{1}{2}A$, is necessarily positive. Wherefore, if $\sigma = \frac{1}{2}(a + b + c)$, we see that each of the products $\sin\sigma \sin(\sigma - a)$, $\sin\sigma \sin(\sigma - b)$, $\sin\sigma \sin(\sigma - c)$ is positive; and from these we can deduce that $a + b + c < 2\pi$ and $b + c > a$ (Euclid, Book XI, Props. 21, 22). The latter of these two results is deducible from the former by considering the triangle whose angles are A, $\pi - B$, $\pi - C$; and the former is deducible from the previously obtained result, $A + B + C > \pi$, by considering the triangle whose vertices are the poles of the sides of the given triangle. [Add.]

It should perhaps be remarked that though the formulae here obtained are identical with those usually given for the metrical trigonometry of the sphere, there is an important distinction between that case and this. For on the plane two points coincide if the interval between them is π, while two lines have only one point of intersection. What are two diametrically opposite points of the surface of a sphere are here replaced by a single point. The equivalent, in the present theory, of the geometry of the surface of a sphere, arises below (p. 196).

The case of a real conic as Absolute. When we take as absolute conic of reference a *real*, undegenerate, conic, of which we may suppose the equation to be $fx = 0$, where $fx = -x^2 - y^2 + z^2$, it is necessary, when we are discussing only the real points of the plane, to distinguish between points which are interior points of this conic, and those which are exterior. The former are characterised by the twofold property (above, p. 164) that every real line through an interior point meets the conic in two real points, while the tangents to the conic from the point are imaginary. For an exterior point, O, the tangents to the conic from O are real; some real lines drawn from O meet the conic in two real points, others in two imaginary points, and a line of the former sort is separated by the two real tangents from a line of the second sort.

It seems unnecessary to enter into great detail. We consider only real points which are interior to the conic; and our main object is to shew that the results obtained are those which were found by Lobatschewsky and Bolyai in their (metrical) so-called non-Euclidean geometry. Bibliographical references are given at the end of the volume (Note III).



Let the line joining two interior points, O, P, meet the conic in M and N, so chosen that N, O, P, M are in order; then the expressions of the symbols of O and P in terms of those of M and N will be of the forms $O = M + hN$, $P = M + kN$, where, if the point whose symbol is $M + N$ be an interior point of the conic, the real symbols h, k will be positive and $h > k$. Thus $(O, P;\, M, N)$, or $h^{-1}k$, is real and positive and less than 1, and we may take for $\log [(O, P;\, M, N)]$ a value -2σ, where σ is real and positive. Thus the interval θ as previously defined, by $(2i)^{-1} \log [(O, P;\, M, N)]$, is given by $\theta = i\sigma$. If, for fixed positions of N, O, M, the point P vary, from N, through O, to M, the value of σ varies from $-\infty$, through 0, to $+\infty$; and $\cos\theta$, $\sin\theta$ are, respectively, $\cosh\sigma$ and $i \sinh\sigma$. The conic, $-x^2 - y^2 + z^2 = 0$, meets the lines $x = 0$, $y = 0$ each in real points, but does not meet the line $z = 0$ in real points; thus the points $(1, 0, 0)$ and $(0, 1, 0)$ are exterior points, but the point $(0, 0, 1)$ is an interior point; for this point the expression $-x^2 - y^2 + z^2$ is positive; this expression is therefore constantly positive for all interior points. Similarly, if $u, = yz' - y'z$, $v, = zx' - z'x$, $w, = xy' - x'y$, be the coordinates of a line, the expression $-u^2 - v^2 + w^2$, which vanishes when the line touches the conic, has one sign for all lines which meet the conic in two real points, and the opposite sign for lines not meeting the conic in real points; the expression is negative for the line $x = 0$, and is thus negative for all lines which meet the conic in two real points. Again, if (x, y, z) and (x', y', z') be two interior points, and the signs of x, y, z, as of x', y', z', be properly taken, the expression $-xx' - yy' + zz'$ has a definite sign; for the polar of an interior point contains only exterior points; the expression is thus positive for any two interior points. We see thus that it is appropriate to replace the formulae

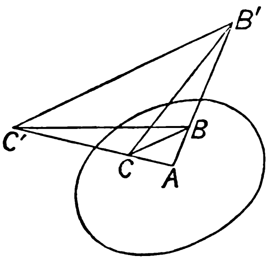
$$\cos\theta = (-xx' - yy' + zz')(fx \, . \, fx')^{-\frac{1}{2}}, \quad \sin\theta = (fu)^{\frac{1}{2}}(fx \, . \, fx')^{-\frac{1}{2}},$$

by the formulae

$$\cosh\sigma = (-xx' - yy' + zz')(fx \, . \, fx')^{-\frac{1}{2}}, \quad \sinh\sigma = (-fu)^{\frac{1}{2}}(fx \, . \, fx')^{-\frac{1}{2}},$$

replacing $(fu)^{\frac{1}{2}}$ by $i(-fu)^{\frac{1}{2}}$. All the square roots, as well as $\cosh\sigma$ and $\sinh\sigma$, are now to be taken positively. These formulae can, if desired, be continued, to give the interval between an interior point, O, and an exterior point, P; the interval σ will then be of the form $\frac{1}{2}i\pi + \lambda$, where λ is real.

The interval between two lines which intersect at an interior point of the conic is to be taken with respect to the two imaginary tangents drawn to the conic from this point. Thus, as in the case of the imaginary conic considered above, the functions that enter are the trigonometrical functions, cos and sin, two lines which are at an interval π being coincident. In particular, as before, two conjugate lines are at an interval $\frac{1}{2}\pi$.



If we consider three points, A, B, C, all interior to the conic, such that AB and AC are conjugate lines, the polar line of A lies entirely exterior to the conic; and, if this meet AB and AC, respectively, in B' and C', these are the poles of AC and AB, respectively. The line, CB', through C, conjugate to CA, thus meets AB, at B', outside the conic and, therefore, the angular interval between CA and CB is less than $\frac{1}{2}\pi$. The same is true of the interval between BA and BC. Recalling, then, what was seen above, for the case when the absolute conic was imaginary, if we use a, b, c for the values of the interval σ belonging, respectively, to BC, CA, AB and B, C for the angle intervals ABC and ACB, the formulae for the triangle ABC are obtainable from those above given for an imaginary conic by merely replacing $\cos a$, $\sin a$, $\cos b$, $\sin b$, $\cos c$, $\sin c$ by $\cosh a$, $\sinh a$, $\cosh b$, $\sinh b$, $\cosh c$, $\sinh c$, respectively. In particular,

$$\cos(B + C), = \sin B \sin C (\cosh b \cosh c - 1),$$

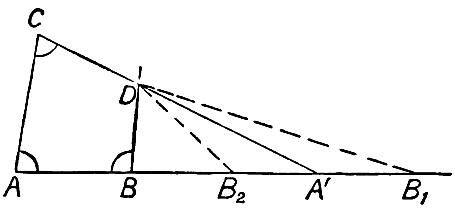
is necessarily positive. Thus we have $B + C < \frac{1}{2}\pi$, the angle between AB and AC being $\frac{1}{2}\pi$.

From this, as before, for any triangle A, B, C for which A, B, C are interior to the conic, we infer that $A + B + C < \pi$.

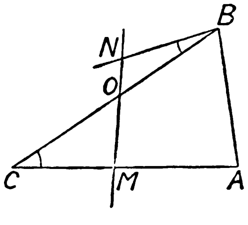
The angle interval, θ, between two lines $ux + vy + wz = 0$, $u'x + v'y + w'z = 0$, which intersect at a point (x, y, z), is such that $\sin\theta$ is of the form $\Delta(fx)^{\frac{1}{2}}(fu \,.\, fu')^{-\frac{1}{2}}$; this interval is then zero for two lines which intersect on the absolute conic. It is thus possible, in this case, to have a triangle for which every one of the three angle intervals is zero, if we allow the vertices to be on the conic.

The sum of the angles of a triangle and a quadrangle of three right angles. If we recall (see above, Chap. IV, p. 158) that two pairs of real conjugate points, on the same line, separate one another, or do not separate one another, according as the line meets the absolute conic in imaginary, or in real, points, we can prove the important theorem for the sum of the angles of a triangle with a single assumption of congruence; we give indications of how this may be done.

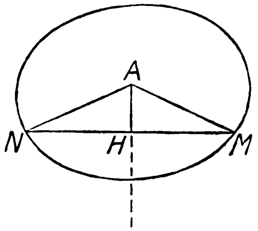
Let A, B, D, C be four points such that AB, CD both pass through the pole, A', of AC, while, also, AC, BD both pass through the pole of AB. The angles at A, B, C are then all equal to $\frac{1}{2}\pi$. Let B_1 be the pole of BD; this point, which is on BA', is separated from B by A and A' if AA' meet the absolute conic in imaginary points. In this case, the angular interval between DB and DB_1 is $\frac{1}{2}\pi$, and, therefore, the angular interval between DB and DC, which is the fourth angle of the quadrangle $ABDC$, is greater than $\frac{1}{2}\pi$. When however the line AA' meets the absolute conic in real points, the pole of BD is not separated from B by A and A'; in this case, let this pole be denoted by B_2, instead of B_1. The angle BDB_2 being $\frac{1}{2}\pi$, the fourth angle, BDC, of the quadrangle $ABDC$, is now *less* than $\frac{1}{2}\pi$.



Bearing in mind the important result thus obtained for a quadrangle of which three angles are right angles, let A, B, C be any triangle with a right angle at A; and, for convenience, suppose the segments AB, AC are both less than $\frac{1}{2}\pi$. Let O be the point of the segment CB such that the intervals CO and OB are equal. From B draw a line BN making the angle CBN equal to the angle ACB. From O draw a line to the pole of AC, meeting this, at right angles, in M, and meeting BN in N. For the triangles CMO and BNO we assume, then, (cf. (3), p. 178), that ON is at right angles to NB. As the angles at A, M, N are right angles, we can by the preceding result, applied to the quadrangle $AMNB$, make an inference as to the angle at B, of this quadrangle; and this angle at B is equal to the sum of the angles ABC, ACB, of the triangle ABC. [Add.]



Comparison with the Lobatschewsky-Bolyai geometry resumed. Now consider a line meeting the real absolute conic in the points M and N, and any point, A, interior to the conic, not lying on this line. We have seen that the σ, (cf. p. 179), between A and any real point of the conic is to be regarded as infinite, and that the interval between A and a point exterior to the conic, if considered at all, is to be regarded as imaginary. In accordance with our usual preconception of the position of a point as being determined by its distance from accessible points, it may seem natural to regard points which are exterior to the conic as not existent, or as at least inaccessible. If we do this, we shall regard all the lines through A which have intersections with the line MN as being limited by the lines AM and AN; and then we may speak of AM, AN as the *two* parallels to MN which can be drawn through A; their intersections with the line MN are at an infinite interval from A, and an infinite number of lines can be drawn through A which do not intersect MN (at any point interior to the conic). Let the line drawn through A conjugate to MN intersect this in H, a point easily seen to be in the segment MN which is interior to the conic; let θ, ϕ denote, respectively, the angular intervals HAM and HAN, and let p be the value of the interval above denoted by σ, for AH. As the angles AMH, ANH are both zero, we infer, from the formulae



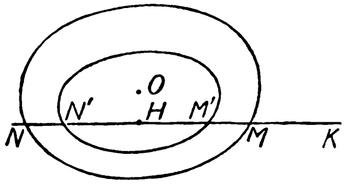
$$\sin\theta = \cos AMH \operatorname{sech} p,$$
$$\sin\phi = \cos ANH \operatorname{sech} p,$$

which we have obtained, that θ and ϕ are equal, both being $\sin^{-1}(\operatorname{sech} p)$. This would then be what, in the Lobatschewsky geometry, is called the *angle of parallelism*.

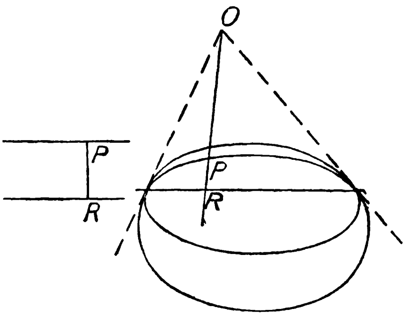
Further, if O be any fixed point interior to the absolute conic, and P a variable interior point such that the interval OP is constant, then P describes a conic having two contacts with the absolute conic, namely at the (imaginary) points where this conic is met by the polar line of O. This conic has the equation

$$(zz' - xx' - yy')^2 = (z^2 - x^2 - y^2)(z'^2 - x'^2 - y'^2)\cosh^2\sigma,$$

where (x', y', z') are the coordinates of O, and σ is the constant interval OP. This conic is that called by Lobatschewsky a *circle*, of which (x', y', z') is the centre. If any chord, $M'N'$, of this circle meet the absolute conic in M and N, and H, K be the two points which are the double points of the involution determined by the pairs M, N and M', N', the one of these points which is interior to the absolute conic being H, it is easy to see that the intervals $N'H$ and HM' are equal, and that the line drawn through H conjugate to $M'N'$ passes through O. In words, the lines, drawn

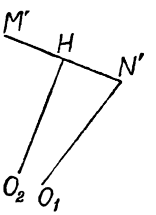


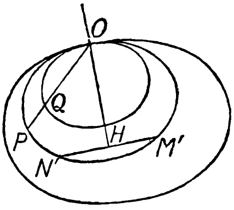
through the middle points of the chords of a circle at right angles to these chords, all pass through the centre of the circle.



If through every point R, interior to the absolute conic, of a line which meets the conic in two real points, a line be drawn at right angles to this line, and thereon a point, P, be taken so that the interval RP is always the same (the point P being, in a sense to be explained, always on the same side of the given line) then the locus of P is that called by Lobatschewsky an *equidistant curve*. In fact, if O, the pole of the given line, be a point exterior to the given conic, the locus of a point P such that the interval OP is constant, is precisely such an equidistant curve.

The corresponding locus, of P, may be considered when O is a point of the absolute conic. This will then be a conic whose four intersections with the absolute conic all coincide at one point. This is the *horocycle* of Lobatschewsky. The lines bisecting at right angles the chords of the horocycle are all parallel, that is, they all pass through a point (O) of the absolute conic. This fact enables us to construct a horocycle passing through a point N'; for draw through this a definite line $N'O_1$; then draw any line $N'HM'$, the point H being such that the angle $HN'O_1$ is the angle of parallelism for a line through N' which is to be parallel to the line, HO_2, drawn through H at right angles to $N'H$, namely $\cosh N'H$ is equal to $\operatorname{cosec}(O_1N'H)$ (see above). The point M' is then such that $M'H = HN'$. If $N'O_1$ remain the same, the angle $O_1N'H$ being allowed to vary, the locus of M' is a horocycle. (This construction is given by Lobatschewsky, *Geometrische Untersuchungen zur Theorie der Parallellinien*, Reprint, Berlin, 1887, § 31, p. 37.) If two horocycles have the same centre, O, and a variable line through O meet these, respectively, in P and Q, the interval PQ is constant; or, two concentric horocycles cut equal segments on two parallel lines passing through their common centre (Lobatschewsky, *loc. cit.* § 33, p. 39).





Case of degenerate conic as Absolute. In the case of a conic represented in point coordinates by $z^2 + \kappa (x^2 + y^2) = 0$, or in line coordinates by $\kappa w^2 + u^2 + v^2 = 0$, say, respectively, $fx = 0$ and $Fu = 0$, the interval, p, between two points (x, y, z), (x', y', z') is, save for sign, such that

$$(fx \, . \, fx')^{\frac{1}{2}} \cos p = zz' + \kappa (xx' + yy'),$$

$$(fx \, . \, fx')^{\frac{1}{2}} \sin p = \{\kappa [\kappa (xy' - x'y)^2 + (xz' - x'z)^2 + (yz' - y'z)^2]\}^{\frac{1}{2}},$$

and the angle, θ, between two lines

$$ux + vy + wz = 0, \quad u'x + v'y + w'z = 0,$$

save for sign, is such that

$$(Fu \, . \, Fu')^{\frac{1}{2}} \cos \theta = \kappa ww' + uu' + vv',$$

$$(Fu \, . \, Fu')^{\frac{1}{2}} \sin \theta = \{\kappa (uw' - u'w)^2 + \kappa (vw' - v'w)^2 + (uv' - u'v)^2\}^{\frac{1}{2}}.$$

Consider what these become, supposing κ to be positive, when, first, κ approaches to zero, and, second, with the usual phraseology, κ becomes indefinitely great. In the former case the absolute conic becomes the point pair given by $z = 0$, $x \pm iy = 0$, or by $u^2 + v^2 = 0$. In this case the interval p becomes zero, but in such a way that $p \, . \, \kappa^{-\frac{1}{2}}$ approaches to the limit, d, which is given by

$$d = \{(xz' - x'z)^2 + (yz' - y'z)^2\}^{\frac{1}{2}} / zz' \, ;$$

the interval, θ, however, remains finite, being, save for sign, such that

$$\cos \theta = (uu' + vv')/mm', \quad \sin \theta = (uv' - u'v)/mm',$$

where $m^2 = u^2 + v^2$, $m'^2 = u'^2 + v'^2$. When κ becomes infinitely great the absolute conic becomes the line pair given by $x \pm iy = 0$. The interval p becomes such that

$$\cos p = (xx' + yy')/nn', \quad \sin p = (xy' - x'y)/nn',$$

where $n^2 = x^2 + y^2$, $n'^2 = x'^2 + y'^2$; and the interval θ becomes zero, but in such a way that $\theta \, . \, \kappa^{\frac{1}{2}}$ approaches to the limit ϕ given by

$$\phi = [(uw' - u'w)^2 + (vw' - v'w)^2]^{\frac{1}{2}} / ww'.$$

Consider the former case, obtained by supposing κ to vanish. If we regard the line $z = 0$ as consisting of those inaccessible points which we ordinarily speak of as being at infinity, this case may be held to include the familiar metrical geometry of the Greeks, on the basis of which measuring instruments at present in use are constructed. This usual metrical geometry evidently assumes that when two points are given, the segment which joins them is without ambiguity, being that segment of the line joining them which does not contain a point of the line $z = 0$. It will be natural enough to make that convention; for the degenerate conic of reference may be regarded as arising from an imaginary conic, $(\kappa > 0)$, and such conic contains no real point of any line considered; or may be regarded as arising from a real conic, $(\kappa < 0)$; and we have considered, in that case, only points lying within the conic. When that convention is made, only triangles being considered whose segments contain no point of the line $z=0$, by putting $\alpha=\kappa^{\frac{1}{2}}a$, $\beta=\kappa^{\frac{1}{2}}b$, $\gamma=\kappa^{\frac{1}{2}}c$, we may deduce from the formulae, previously given (above, pp. 177, 178), using α', β', γ' in place of A, B, C,

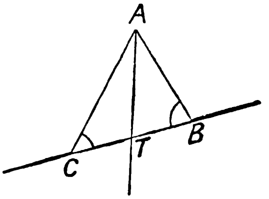
$$\sin\alpha'/\sin\alpha=\sin\beta'/\sin\beta, \quad \cos\alpha=\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos\alpha',$$

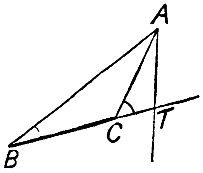
$$\cos\alpha'=-\cos\beta'\cos\gamma'+\sin\beta'\sin\gamma'\cos\alpha,$$

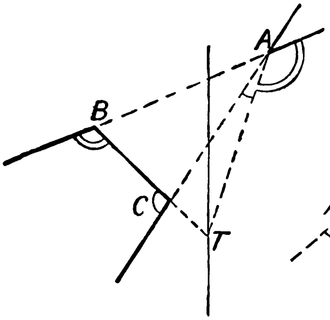
the respective results

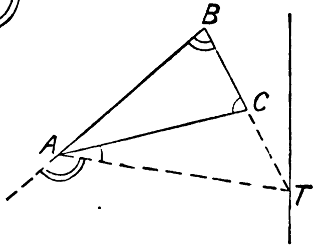
$$a^{-1}.\sin\alpha'=b^{-1}.\sin\beta', \quad a^2=b^2+c^2-2bc\cos\alpha', \quad \cos\alpha'=-\cos(\beta'+\gamma').$$

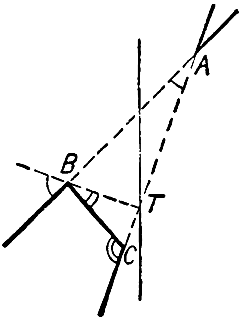
These may also be deduced independently. The last of these follows from $\alpha'+\beta'+\gamma'=\pi$, which can be proved, as by Euclid, from the fact that two lines BA, CA, which meet $z=0$ in the same point A, are at equal angular intervals, ABT, ACT, with a line BC meeting them in B and C, respectively. As before, this assumes that the segment BC contains no point of $z=0$; the point T is the point of the line $z=0$, on the line BC, on the *other* segment BC.

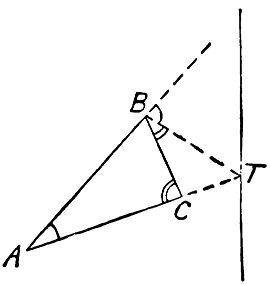












It follows from this* that if we consider an (improper) triangle of which the segments AB, AC contain points of $z=0$, but the segment BC contains no such point, and A, B, C denote the angular intervals BAC, CBA, ACB, then $A+\pi-B+\pi-C=\pi$, or $B+C=\pi+A$. In particular when $A=\frac{1}{2}\pi$, that is, when the points of intersection of AB and AC with $z=0$ are harmonic in regard to the absolute points, I, J, lying on $z=0$, we have $A+B+C=2\pi$.

It appears sufficiently clear from this that any statement of the ordinary Euclidean geometry, even one which has reference to measurement, continues to hold when the so-called line at infinity is replaced by any line of the plane, and the so-called circular points at infinity are replaced by any two conjugate imaginary points lying thereon†. [Add.]

The arbitrariness and comparison of the several methods of measurement. When intervals, between points and between lines, are measured by means of an imaginary conic, the figure considered is sometimes said to lie in an *elliptic* plane; when the conic is real, in an *hyperbolic* plane; and, when the conic of reference is a point pair, in a *parabolic*, or Euclidean plane. The phraseology is convenient; but it is apt to be misconceived. *The plane and the geometry* (though not the symbols, or numbers, associated with intervals) *are the same in all the three cases.* Statements in regard to the intervals are to be considered as abbreviated statements in regard to particular combinations of the algebraic symbols merely; and we have sought to shew that the symbolism is not a necessary, though it is sometimes a convenient, support of the geometrical theory. No deduction of a really geometrical kind can be legitimately based on statements of which any particular conception of distance forms an indispensable part; such statements are equivalent

---

* In this deduction the fundamental fact that the angles ACT, ABT (in the first two diagrams) are equal, follows from the formulae for $\cos\theta$, $\sin\theta$ on page 184, two lines meeting on $z=0$ having equal ratios u'/v'. The last four diagrams illustrate the proof of $\alpha'+\beta'+\gamma'=\pi$ in the various cases, similarly marked angles being equal.

† But a difference is necessary in the definition of movement when we pass to the case in which the Absolute conic becomes a point-pair. The linear transformation of the plane leaving an undegenerate conic unaltered is uniquely determined by its effect upon the range of points on the conic; but the pair of points $z=0$, $x^2+y^2=0$ is unaltered by every transformation $x_1=mx$, $y_1=my$, $z_1=z$. For the former case the self-transformation of the conic involves that the distance of two points is unaltered (cf. p. 169); for the latter case it is necessary to introduce this condition. Thereby, as may be shewn directly (p. 184), the formulae for a movement takes the forms (still involving three arbitrary constants)

$$x_1=px-qy+r,\quad y_1=qx+py+s,\quad z_1=z,$$

where $p^2+q^2=1$. These may be obtained from the formulae of p. 169, if therein we first put $z\kappa^{-\frac{1}{2}}$, $z_1\kappa^{-\frac{1}{2}}$ in place of z, z_1, so that the formulae may relate to the self-transformation of $z^2+\kappa(x^2+y^2)=0$, then divide the coefficients on the right by $\frac{1}{2}(D^2+C^2-A^2-B^2)$, then put $m\kappa^{\frac{1}{2}}$, $n\kappa^{\frac{1}{2}}$ in place of A, B, and then, regarding the ratios of C, D, m, n as definite, put $\kappa=0$.

only to statements in regard to the behaviour of particular measuring instruments, which must rest on physical hypotheses.

In illustration of this general remark, we may, if we please, apply two of the systems of measurement simultaneously. Thus, let A, B, C be any three points, and consider a triangle formed with these, in the sense explained above when discussing the reference to an imaginary conic. Let the segment intervals, BC, CA, AB, with reference to a certain imaginary absolute conic, be denoted, respectively, by α, β, γ, and the angle intervals, BAC, CAB, ABC, corresponding thereto, be denoted, respectively, by α', β', γ'. If we take any three points as reference points for coordinates, and two points I, J which, relatively to these reference points, are given by $z=0$, $x \pm iy = 0$, we can also measure the segments and angles, as explained above, with reference to the point pair I, J; let them be denoted then, respectively, by a, b, c, A, B, C. Then *both* the equations

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\alpha',$$

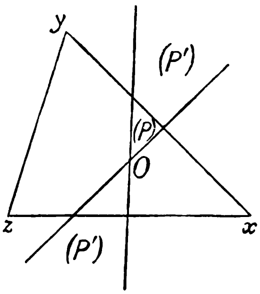
and

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

are true.

An interesting application of this liberty, which has some importance of its own, may be made. This expresses the interval, measured with reference to the imaginary conic $x^2+y^2+z^2=0$, between two lines, by means of an integral which is to be evaluated with measurements of distance and angle that are made relatively to the point pair, I, J, given by $z=0$, $x \pm iy = 0$. Represent x/z, y/z, respectively, by X and Y; an element of area, with the Euclidean reference, is, we may assume, on the basis of familiar notions, given by $dXdY$, or by $rdrd\theta$, where r is the Euclidean distance from a point, and θ the Euclidean angle from a fixed line. If two lines, intersecting in O, be given, of equations $ux+vy+wz=0$, $u'x+v'y+w'z=0$, these may be regarded as defining a region, (P) and (P'), over which the integral

$$\iint \frac{dXdY}{(1+X^2+Y^2)^{\frac{3}{2}}}$$



may be taken (every element $dXdY$ being positive). It will be found that the *integral is* then *equal to twice the interval between the lines*, measured in regard to the conic $x^2+y^2+z^2=0$. Of this statement we may give indications of a straightforward proof. Let d denote the Euclidean distance from the origin z, or $(0, 0, 1)$, to the intersection, O, of the lines, r denote the Euclidean distance from O to a point, P, of the region of integration, and θ the Euclidean angle between the line zO, and the line OP. The integral is then

$$\int_{\phi}^{\phi'} d\theta \int_0^{\infty} rdr\{(1+d^2+r^2+2dr\cos\theta)^{-\frac{3}{2}}+(1+d^2+r^2-2dr\cos\theta)^{-\frac{3}{2}}\},$$

where ϕ is the angle between the line zO and one of the given lines, ϕ' the angle between zO and the other given line. This form of the integral is obtained by pairing together points, P and P', on a line through O, in regard to which the harmonic conjugate of O is on the line $z=0$; the factor $r\cos\theta$, arising for P, corresponds to a factor $r\cos(\theta+\pi)$, or $(-r)\cos\theta$, arising for P'. To carry out the integration, we may notice that

$$\int xdx\,[(x-a)^2+b^2]^{-\frac{3}{2}} = [ax-(a^2+b^2)]\,b^{-2}\,[(x-a)^2+b^2]^{-\frac{1}{2}},$$

and utilise the identity

$$\cos^{-1}\frac{u'V-v'U}{s'\Delta} - \cos^{-1}\frac{uV-vU}{s\Delta} = \cos^{-1}\frac{uu'+vv'+ww'}{ss'},$$

where

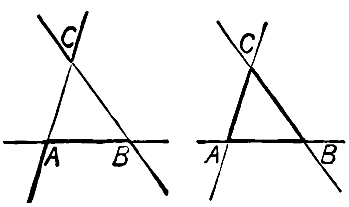
$$U=vw'-v'w,\quad V=wu'-w'u,\quad s^2=u^2+v^2+w^2,$$
$$s'^2=u'^2+v'^2+w'^2,\qquad \Delta^2=U^2+V^2.$$

The integral will then be found to become

$$2\cos^{-1}[(uu'+vv'+ww')/ss'].$$

On the basis of this result, we may speak of the *extent*, relatively to the imaginary conic, of a region of the plane, meaning thereby the value of the integral, evaluated, with the parabolic measurements, as here, over this region. In this sense, any two lines enclose a finite extent, equal to $2A$, where A is the interval between the lines, measured by the imaginary conic. The two lines enclose two regions, which are complementary; the sum of the extents of these two regions is that of the whole plane, which is thus finite. The angular interval, in regard to the imaginary conic, for the other region, is $\pi-A$, and the extent of this is $2\pi-2A$. Thus 2π is the extent of the whole plane.

Thus also the extent of a triangle, in terms of the angular intervals of it, with reference to the imaginary conic, is $A+B+C-\pi$. For, if the extent of the triangle be denoted by Δ, the whole plane is made up of four regions whose respective extents are $2A-\Delta$, $2B-\Delta$, $2C-\Delta$, and Δ; putting the sum of these equal to 2π, we find



$$\Delta = A+B+C-\pi.$$

Conversely, we may avoid the use of the integral, by defining the extent of a triangle as being $A+B+C-\pi$, allowing this to apply also to the triangle whose angular intervals are C, $\pi-A$, $\pi-B$, and assuming that the extent of a composite region is the sum of the extents of its component regions. For the extent between the lines CA, CB we should thus obtain

$$[A+B+C-\pi]+[C+\pi-A+\pi-B-\pi], \text{ or } 2C.$$

Ex. 1. Shew that the extent of a *circle*, relatively to the imaginary conic, that is of the conic whose equation is

$$(xx'+yy'+zz')^2=(x^2+y^2+z^2)(x'^2+y'^2+z'^2)\cos^2 r,$$

is $2\pi(1-\cos r)$.

Ex. 2. If σ be the interval, relatively to the imaginary conic, between two points, prove that the integral $\int d\sigma$, extended along the circle, has the value $2\pi\sin r$. (Cf. p. 200.)

The remarks in the present section are largely anticipatory; it will be understood that they are intended to be suggestive, and are not complete.

Plane metrical geometry by projection from a quadric surface. Spherical geometry. In Chap. III we have considered the theory of the circle, or of geometry with two absolute points of reference. In the present chapter we have considered the theory with reference to an absolute conic. These are in pursuance of the plan we have adopted, of devoting the present volume to geometry in one plane. We shall see in detail, in a subsequent volume, that a plane in which two absolute points are given is more properly to be regarded as the projection of a surface, called a quadric, existing in space of three dimensions, which has the property of being met in two points by an arbitrary line. The centre of projection lies on this surface. Similarly, a plane in which an absolute conic is given is naturally suggested by the projection of the points of a quadric, from a centre of projection not lying on the quadric. It appears, therefore, to be necessary to make some remarks here in regard to a quadric surface; as we are to deal with geometry in three dimensions in the next volume, these remarks will be brief, and incomplete.

We have already, in Volume I, p. 82, constructed, from four points A, B, C, D which do not lie in a plane, a point whose symbol, in terms of the symbols of these points, is

$$P=\lambda A+\mu B+\lambda\mu C+D;$$

if we denote this by $P=xA+yB+zC+tD$, we have $xy=zt$; conversely, any point whose *coordinates*, x, y, z, t, relatively to A, B, C, D, satisfy this equation, is such a point as P, for suitable values of λ and μ. It is the aggregate, or locus, of all such points which we speak of here as a quadric surface. For our purpose it will be convenient to use ξ, η instead of x, y, subject to $x=\xi+i\eta$, $y=\xi-i\eta$, so that the equation takes the form $\xi^2+\eta^2=zt$. We can shew that the surface contains an infinite number of lines, which break up into two systems, there being one line of each system through every point of the surface. Every line of either system, in fact, meets every line of the other system, but two lines of the same system do not intersect. For, first, whatever θ may be, every point (ξ, η, z, t) which is such that $\xi+i\eta=\theta t$, $\xi-i\eta=\theta^{-1}z$, is evidently such that $\xi^2+\eta^2=zt$, and is, therefore, a point of the surface; such a point has the symbol $\theta tA+\theta^{-1}zB+zC+tD$, and is a point of the line joining the points whose symbols are $\theta^{-1}B+C$, $\theta A+D$; the two equations taken together thus represent this line. By taking different values of θ, we thence obtain a system of lines. If $(\xi_0, \eta_0, z_0, t_0)$ be any point of the surface, the equal values, $(\xi_0+i\eta_0)\,t_0^{-1}$, $z_0(\xi_0-i\eta_0)^{-1}$, give a value of θ for which the line contains this point. It is easy, moreover, to verify that two lines of this system, for the respective values θ, ϕ of the parameter, given by $\xi+i\eta=\theta t$, $\xi-i\eta=\theta^{-1}z$ and $\xi+i\eta=\phi t$, $\xi-i\eta=\phi^{-1}z$, do not meet.

Similarly, the surface contains another system of lines, of which one is given by $\xi-i\eta=\phi t$, $\xi+i\eta=\phi^{-1}z$, one of which passes through every point of the surface. No two of these intersect. But this line meets that, of the former system, given by $\xi+i\eta=\theta t$, $\xi-i\eta=\theta^{-1}z$, in the point given by $\xi+i\eta=\theta$, $\xi-i\eta=\phi$, $z=\theta\phi$, $t=1$.

It is easy to prove that any line of the threefold space contains two points of the quadric, unless it consist wholly of points belonging thereto; for the substitution in $xy=zt$ of values $x_1+\sigma x_2$, $y_1+\sigma y_2$, $z_1+\sigma z_2$, $t_1+\sigma t_2$, respectively for x, y, z, t, leads to a quadric equation for σ. Further, that the points of an arbitrary plane, which lie on the quadric, are the points of a conic; for, let any line of this plane meet the quadric in the two points P and R, of coordinates (x_1, y_1, z_1, t_1) and (x_3, y_3, z_3, t_3); and let Q (x_2, y_2, z_2, t_2) be another point of this plane, not on the line PR: if, then, in $xy=zt$, we replace x, y, z, t, respectively, by $x_1+\theta x_2+\phi x_3$, and similar expressions, we find that ϕ is expressible in terms of θ; by choosing Q suitably, this leads to a symbol, for any point of the quadric lying in the plane, of the form $P+m\theta Q+n\theta^2R$. This shews that all such points lie on a conic. In particular, the plane containing the two lines of the quadric which meet in any point, P, of the surface, contains no other points of the quadric than the points of these two lines. This plane is called the tangent plane of the quadric at P. The lines are called the generating lines, or generators, of the surface, at P.

Any equation which is linear (and homogeneous) in the coordinates, ξ, η, z, t, of a point, expresses that this point lies on a definite plane; this equation may then be called the equation of the plane. If the equation be $-ct+2f\eta+2g\xi+hz=0$, the point $(\xi+i\eta)A+(\xi-i\eta)B+zC+tD$, being, in virtue of this equation, the same as

$$c(\xi+i\eta)A+c(\xi-i\eta)B+czC+(2f\eta+2g\xi+hz)D,$$

or $$\xi(cA+cB+2gD)+\eta(icA-icB+2fD)+z(cC+hD),$$

is a point of the plane containing the three points whose symbols are, respectively,

$$c(A+B)+2gD,\quad ic(A-B)+2fD,\quad cC+hD;$$

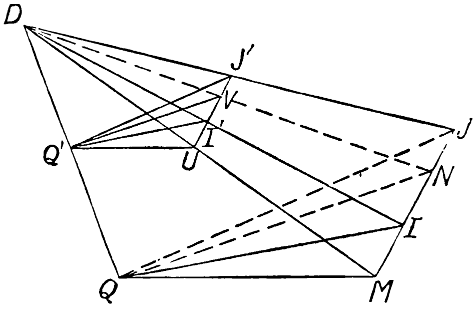
by proper choice of c, f, g, h, this is any plane whatever.

In particular, the three equations $\xi=0$, $\eta=0$, $z=0$ represent, respectively, three planes. And these intersect in the point of coordinates $(0, 0, 0, 1)$, that is, the point D. This is a point of the quadric, since these coordinates satisfy the equation $\xi^2+\eta^2=zt$. The line which joins this point, D, to any point, of coordinates (ξ, η, z, t), meets the plane whose equation is $t=0$, in the point $(\xi, \eta, z, 0)$. This point is the projection, from D, of the point (ξ, η, z, t). Any point of the quadric, other than D itself, can thus be made to correspond, by projection, to a point of the plane $t=0$. But the line which is expressed by the two equations $\xi+i\eta=0$, $z=0$, is wholly on the quadric; and this line passes through D; every point of the quadric lying on the line is, therefore, projected from D into the same point of the plane $t=0$. So, likewise, is every point of the quadric lying on the line expressed by the two equations $\xi-i\eta=0$, $z=0$. Every point of the plane $t=0$, other than the two points so obtained, is the projection of a definite point of the quadric. These two exceptional points of the plane $t=0$ are, necessarily, themselves points of the quadric; if we denote them by I and J, we may regard the point D of the quadric as projecting into the line IJ.

In this projection, the points of the quadric which lie on any plane passing through D project into the points of a line, in the plane $t=0$, and conversely. A plane not passing through D is intersected by the generators of the surface at D (whose equations are $\xi+i\eta=0$, $z=0$ and $\xi-i\eta=0$, $z=0$) in two points, say, I' and J'; the points of the quadric lying on such a plane project, therefore, into the points of a curve, in the plane $t=0$, which passes through the points I and J. If the plane in question be that given by the equation $-ct+2f\eta+2g\xi+hz=0$, we see, eliminating t by means of $\xi^2+\eta^2=zt$, that the equation of this curve is $-c(\xi^2+\eta^2)+2f\eta z+2g\xi z+hz^2=0$. The curve is that we have, in Chap. III, called a circle, relatively to the points I, J. In particular, when the plane contains the point D, we have $c=0$, and the circle degenerates into the line, in the plane $t=0$, given by $z=0$, together with another line.

Thus the geometry in a plane, relative to two absolute points, which we have considered in Chap. III, may be regarded as the geometry dealing with the points of a quadric surface. A circle in the plane is thereby replaced by a plane section of the quadric surface.

Consider, from this point of view, the measurement of angle between two lines in the plane $t=0$, relatively to the points I, J. Let the lines intersect in Q, and meet the line IJ, respectively, in the points M, N. The measurement, we have explained (above, p. 167), is by means of the range M, N, I, J. The four lines QM, QN, QI, QJ, of the plane $t=0$, are projected from D by means of four planes which intersect in the line DQ. Any plane, not containing the line DQ, is met by these four planes in a flat pencil of four lines which is related to the pencil $Q(M, N, I, J)$, in the sense explained in Vol. I; in particular we may take the tangent plane of the quadric at D, which is met by these four planes in the pencil $D(M, N, I, J)$. Let DQ meet the quadric surface in Q' (beside D); there is, at Q', one generator which meets the generator DI, say in I', and another generator which meets the generator DJ, say in J'. It is easy to verify that the tangent line of any plane section of the quadric, at any point, lies in the tangent plane of the surface at that point. The planes DQM, DQN give rise to two conics lying on the quadric surface, which have the points Q' and D for common points; the tangent at Q' of the former conic meets the tangent of this at D, in a point, say, U, which is on the line DM in which the plane DQM meets the tangent plane, DIJ, at D; similarly, the tangent at Q' of the latter conic meets the tangent of this at D, in a point V lying on DN. These points U, V are in the tangent plane of the quadric at Q'. Thus the pencil $Q'(U, V, I', J')$ is related to the pencil $Q(M, N, I, J)$. Two tangent planes can be drawn to the quadric surface from an arbitrary line, as we may see by taking the two points where the line meets the quadric, and considering the intersections of the generators of the surface at one of these points with the generators of the surface at the other of these two points. In our figure, as $I'D$, $I'Q'$ are generators, and $J'D$, $J'Q'$ are generators, the planes $DI'Q'$ and $DJ'Q'$ are the tangent planes of the surface which can be drawn through the line DQ; thus the axial pencil of planes which has



been spoken of, consists of the two planes through DQ containing the lines QM and QN, together with the two tangent planes to the surface from the line DQ. It is by considering the two former planes, with respect to the two latter, that the angle between the lines QM, QN is measured.

It can be shewn that if, through any point E, a line be drawn to meet the quadric surface in R and S, and the point F be taken thereon harmonic to E in regard to R and S, then the locus of F, for all such lines drawn through E, is a plane. This plane, say e, is called the polar plane of E in regard to the quadric. For the quadric $\xi^2+\eta^2=zt$, its equation is $2(\xi\xi'+\eta\eta')=zt'+z't$, where (ξ', η', z', t') are the coordinates of E. If the polar plane of one point, E, contain a point, F, then the polar plane of F contains the point E; two such points, E, F, are said to be conjugate in regard to the quadric. To any arbitrary plane there is one point, called its pole, of which the plane is the polar plane; if the pole of one plane lie on another plane, the pole of this other lies on the former; two such planes are said to be conjugate. To any plane drawn through a given line, there corresponds another plane passing through the line, conjugate to the former plane; the pairs of planes so obtainable form an axial pencil in involution, of which the double elements are the two tangent planes to the quadric which can be drawn through the given line.

With this, we can state in another way the condition adopted in Chap. III for the lines QM, QN to be at right angles to one another in regard to I and J, which was that M and N should be harmonic conjugates in regard to I and J. It requires that the planes DUQ', DVQ' should be harmonic in regard to the tangent planes, $DI'Q'$, $DJ'Q'$, drawn to the quadric from the line DQ. Thus, from what is said above, the condition is that the planes DQM and DQN should be conjugate to one another in regard to the quadric.

Now suppose we take two circles in the plane $t=0$, intersecting in the point Q, and in another point, R, in addition to I and J. We may consider the pencil of four lines, consisting of QI, QJ and the two tangents of these circles at Q. Let DQ and DR meet the quadric again, beside D, in Q' and R', respectively. The pencil of four lines, spoken of, passing through Q, is, we have seen, related to a pencil of four lines through Q', lying in the tangent plane of the quadric at Q'; these lines are the generators of the quadric at Q' (corresponding to QI, QJ), and the tangents at Q' of the two plane sections of the quadric which, as we have seen, project, from D, into the two given circles of the plane $t=0$. Through these four lines, in the tangent plane at Q', we can draw four planes all passing through the line $Q'R'$, so determining an axial pencil of planes related to the plane pencil in question. Of these planes, one will be a plane containing one of the generators at Q' and, also, that generator at R' which intersects this; this plane will be a tangent plane to the quadric drawn through the line $Q'R'$; another of the four planes will be that containing the other generator at Q', of the quadric, together with the generator at R' which intersects this, this being the other tangent plane to the quadric drawn from the line $Q'R'$. The other two planes of the axial pencil through $Q'R'$ are the planes of the two sections of the quadric which project, from D, into the two given circles. In accordance with what is said at the beginning of this chapter, the angle between the tangents at Q, of the two given circles, may be measured by the plane pencil, of centre Q, by taking these tangents with respect to QI and QJ. By what we have now said, it may also be measured by taking the plane sections of the quadric, through the line $Q'R'$, which project from D into the given circles, and considering these planes with respect to the two tangent planes of the quadric which can be drawn from the line $Q'R'$. Incidentally, it appears that the circles cut at the same angle at both their common points, Q and R. In particular, the circles cut at right angles if the two plane sections of the quadric, which project from D into these circles, are in planes which are conjugate to one another in regard to the quadric.

In these discussions, the lines and circles whose inclinations have been considered lie in the plane $t = 0$, which is in fact a tangent plane of the quadric, namely at the point $(0, 0, 1, 0)$. It is true, however, that a plane section of the quadric projects, from a point D of the quadric, into a circle upon any plane whatever, if the two absolute points for this plane are taken to be the points in which the generators of the quadric, at D, meet this plane.

Let us now replace the coordinates z and t by coordinates ζ, τ by means of $z = \tau + \zeta$, $t = \tau - \zeta$; the equation of the quadric then takes the form $\xi^2 + \eta^2 + \zeta^2 = \tau^2$. There is in the plane $\tau = 0$ a conic, lying on the quadric, given by $\xi^2 + \eta^2 + \zeta^2 = 0$. This conic meets an arbitrary plane, ϖ, of the threefold space in two points. Let us agree to take for the two absolute points of this plane, ϖ, in the sense discussed at length in Chap. III, these two points of this quadric. For a tangent plane of the quadric, say, at Q', these two points, lying on the quadric and on this plane, are the two points in which the plane $\tau = 0$ is met by the generators of the quadric at Q'. The effect is, then, that the angle between the tangent lines of the two plane sections of the quadric, at a point Q' where these sections intersect, as we have been led to consider it, is in fact obtainable by the rule, applied to the tangent plane of the quadric at Q', which we originally (p. 167) adopted for geometry in a plane; but, at the same time, the suggestion clearly arises of considering the interval between two points, of the plane $\tau = 0$, by a reference to a fixed conic lying in this plane.

Metrical geometry in regard to an absolute conic. We proceed now to consider this measurement in regard to a fixed conic from the point of view we have reached by introducing the quadric surface.

Denote the point $\xi = 0$, $\eta = 0$, $\zeta = 0$, which does not lie on the quadric, whose equation is $\xi^2 + \eta^2 + \zeta^2 = \tau^2$, by O. It is easily seen that the plane $\tau = 0$ is the polar plane of O in regard to the quadric. Any point, say, Q', of the quadric, may be projected from O on to the plane $\tau = 0$, giving a point, say, Q_1. All points, Q', of the quadric which lie on a plane passing through O, give rise, then, to points, Q_1, lying on a line of the plane $\tau = 0$. Consider two plane sections of the quadric whose planes pass through O; the common points, say Q' and R', of these two sections of the quadric, will then lie in line with O. We have seen that the angle between the tangent lines, at Q', of these sections, obtained by considering these lines with respect to the generators of the quadric at Q', may be obtained also by considering the planes of the two sections with respect to the two tangent planes of the quadric which can be drawn through the line $Q'R'$. Now consider the four lines in which these planes meet the plane $\tau = 0$; as $OQ'R'Q_1$ are in line, these four lines meet in the point Q_1 of the plane $\tau = 0$. Two of these lines, corresponding to the tangent planes of the quadric drawn through $Q'R'$, are the intersections with $\tau = 0$ of the planes joining O to the generators of the quadric at Q'; the other two of the four lines are the intersections with $\tau = 0$ of the planes of the two given sections. We can shew that the two former lines are the tangent lines, from Q_1, to the conic of the plane $\tau = 0$ given by $\xi^2 + \eta^2 + \zeta^2 = 0$. For let one of the generators at Q', of the quadric, meet the plane $\tau = 0$, say, in H; this point is on the section of the quadric by the plane $\tau = 0$, that is, on the conic $\xi^2 + \eta^2 + \zeta^2 = 0$. The plane $\tau = 0$ being the polar plane of O, it is easy to see that the tangent plane of the quadric at H passes through O; this tangent plane contains the generator $Q'H$. Thus the plane joining O to this generator meets $\tau = 0$ in a line which touches the conic $\xi^2 + \eta^2 + \zeta^2 = 0$ at H, as was said. We are thus led from the measurement of the angle, at Q', between two lines, meeting at this point, lying in the tangent plane at Q', by means of two absolute points of this plane, to the measurement of the angle, at Q_1, between two lines, meeting at this point, lying in the plane $\tau = 0$, by means of the tangent lines drawn from this point to the conic $\xi^2 + \eta^2 + \zeta^2 = 0$.

Next consider any two points of the quadric, say, Q' and K'. Let them be projected from O into the two points, Q_1 and K_1, of the plane $\tau = 0$. Suppose that we measure the interval between the points Q' and K', as points of the quadric surface, by the angle between the lines OQ' and OK', taken in respect to the two absolute points of the plane $Q'OK'$, these being, as before, the points where this plane meets the conic $\tau = 0$, $\xi^2 + \eta^2 + \zeta^2 = 0$. We are thus led to measure the interval between the points, Q_1 and K_1, of the plane $\tau = 0$, by considering these with respect to the two points in which their joining line meets the absolute conic of their plane.

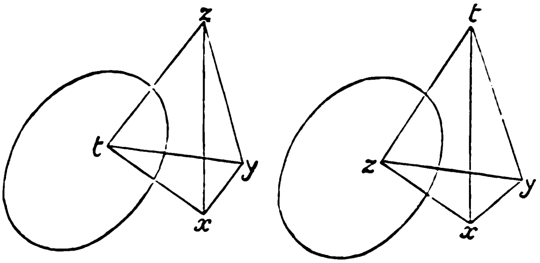
Thus, finally, the plane metrical geometry we have explained in this chapter, relatively to a fixed absolute conic, which gives an interval between two lines, and an interval between two points, is seen to be obtainable by projection of a geometry upon the quadric surface, the centre of projection, O, not being on the quadric. Lines of the plane are replaced by sections of the quadric, whose planes pass through the centre of projection, O. The interval between two points on the quadric surface is understood to be the interval between the lines joining these to O, taken with respect to the absolute points of the plane containing these joining lines; the interval between two lines of the plane is replaced by the interval between two lines lying in a tangent plane of the quadric surface, again taken with respect to the absolute points of this plane.

But this reduction of the metrical plane geometry, with respect to an absolute conic, to metrical geometry on a quadric surface, with respect, in the case of every plane concerned, to two absolute points in that plane, calls for a further important remark. While any point, Q', of the quadric, gives rise, by projection from O, to a single point, Q_1, of the plane, this last gives rise, conversely, not only to Q', but also to the second point, R', in which the line OQ_1 meets the quadric. Thus, whereas two points of the plane coincide when the interval between them, in regard to the absolute conic, is π (or any odd multiple of π), the corresponding points of the quadric surface do not then coincide, but are two points in line with O. Two points of the quadric surface coincide only when the interval between them is 0, or 2π, or an even multiple of π. The geometry of the points of a plane, in this book, is in fact that of lines, extended in both directions, passing through a point outside the plane. In the geometry on the quadric surface, which we may call *spherical* geometry, any such line gives rise to *two* points. In the plane two lines have one point in common; on the quadric surface, two plane sections are curves having *two* points in common. With proper conventions, the metrical formulae we have found for a triangle, when the measurements are with respect to an imaginary conic, continue to hold on the quadric given by the equation $\xi^2 + \eta^2 + \zeta^2 = \tau^2$, in which the coordinates ξ, η, ζ, τ are real; the arcs between two points being taken on planes through the point $(0, 0, 0, 1)$, and measured, as explained, with respect to the two points in which the planes meet the conic $\xi^2+\eta^2+\zeta^2=0$, $\tau=0$; while the angles between two such arcs are measured, on the tangent plane of the quadric, at the point of intersection of the arcs, with respect to the two points in which this plane meets the same conic.

But there are many cases in which the distinction between the plane geometry and the spherical geometry is of importance. As a single example we may refer to the following: On the quadric $\xi^2+\eta^2+\zeta^2=\tau^2$, consider the four points, A, B, C, D, of coordinates, respectively, $(-a, b, c, 1)$, $(a, -b, c, 1)$, $(a, b, -c, 1)$, $(-a, -b, -c, 1)$, where a, b, c are arbitrary real symbols such that $a^2+b^2+c^2=1$. These can be joined in pairs by six arcs, lying on the quadric in planes which pass through the point $(0, 0, 0, 1)$, forming four triangles ABC, BCD, CAD, ABD, in such a way that if the angles of the first of these be called A, B, C, those of the others are, respectively, C, B, A; A, C, B and B, A, C; the four triangles, which together exhaust the whole surface of the quadric, are in fact equal to one another, in angles and in arcs. The three angles A, B, C are such that $A+B+C=2\pi$. When, however, this figure is projected from O, the point $(0, 0, 0, 1)$, on to the plane $\tau=0$, while we get, from A, B, C, the triangle A_1, B_1, C_1, for which the angles are such that $A_1+B_1+C_1=2\pi$, the point D_1, arising from D, is an interior point of this, namely is such that every line through it meets two of the segments B_1C_1, C_1A_1, A_1B_1. (See below, p. 207.)

Representation of the original plane upon another plane. In the preceding section we have passed from a point, Q, of the plane, by projection from a point D, lying on a quadric surface, to a point, Q', of this surface. And then, by projection from a point, O, not lying on the quadric, to another point Q_1, lying on another plane (that denoted by $\tau=0$). If, with O and D given, we consider the reverse process of passing from Q_1 to Q, the point Q_1 will give rise, by projection from O, to two points of the quadric surface, say, Q' and R', and each of these will give, by projection from D, a single point of the original plane; so that we get two points, Q and R, corresponding to Q_1. But the results of the process will be found to be interesting, nevertheless. In particular, it will appear that the *distance* used by Riemann in his celebrated Dissertation of 1854, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen* (Werke, 1876, p. 254), which at first sight appears to require that a system of metrical coordinates has been established beforehand, is in fact identical with the interval with respect to a conic which we have considered in the present chapter. Further, the origin of one particular representation of Lobatschewsky's Geometry (used, for example, by Poincaré, in his remarkable papers on Automorphic Functions, *Acta Math.* I, 1882, p. 7) will be made clear.

The absolute conic being supposed to be given by one of the two equations $z^2+x^2+y^2=0$, $z^2-x^2-y^2=0$, and, in the latter case, limiting ourselves to real values of x, y, z for which $z^2-x^2-y^2$ is positive, as in the earlier part of this chapter, let us associate, with each point, (x, y, z), of the plane the positive square root, $(z^2+x^2+y^2)^{\frac{1}{2}}$, or $(z^2-x^2-y^2)^{\frac{1}{2}}$. If we denote this square root by t, this comes to considering the point (x, y, z, t), of the quadric surface expressed, respectively, in the two cases, by the first or second of the equations $x^2+y^2+z^2=t^2$, $z^2-x^2-y^2=t^2$. This point, (x, y, z, t), is one of the two points in which the quadric is met by the line joining the point (x, y, z), of the plane, to the point $(0, 0, 0, 1)$. For the former quadric surface, this point is an interior point, in the sense that every line through it meets the quadric in two real points. For the latter quadric, the point $(0, 0, 0, 1)$ is an exterior point, only those lines through it which contain points for which $z^2-x^2-y^2>0$ meeting it in real points.



In the case of points of the quadric $x^2+y^2+z^2=t^2$, if we define ξ, η, respectively, by $\xi=2x/(z+t)$, $\eta=2y/(z+t)$, we may take, for the homogeneous coordinates, x, y, z, t, of a point of the quadric, respectively,

$$x=\xi,\quad y=\eta,\quad z=1-\tfrac{1}{4}(\xi^2+\eta^2),\quad t=1+\tfrac{1}{4}(\xi^2+\eta^2);$$

if we take $\lambda=\frac{1}{2}(\xi+i\eta)$, $\mu=\frac{1}{2}(\xi-i\eta)$, these are the same as

$$x=\lambda+\mu,\quad y=-i(\lambda-\mu),\quad z=1-\lambda\mu,\quad t=1+\lambda\mu.$$

Noticing that

$$\xi=x+t\,.\,0,\quad \eta=y+t\,.\,0,\quad 2=z+t\,.\,1,\quad 0=t+t(-1),$$

we may, if we wish, regard $(\xi, \eta, 2, 0)$ as the coordinates of a point, in the original plane of x, y, z, obtained by projecting the point, (x, y, z, t), of the quadric, from the point $(0, 0, 1, -1)$; this is a point of the quadric, being one of the points in which it is met by the line given by the two equations $x=0$, $y=0$. It is clear that, while any pair (ξ, η) gives rise only to one point (x, y, z), this point, being the same as $(-x, -y, -z)$, gives rise, conversely, beside (ξ, η), to the pair

$$\xi'=\frac{-2x}{-z+t},\quad \eta'=\frac{-2y}{-z+t},$$

which are $\xi' = -4\xi/(\xi^2+\eta^2)$, $\eta' = -4\eta/(\xi^2+\eta^2)$. Interpreting $(\xi, \eta, 2)$ as rectangular coordinates, so that the equation $\xi^2+\eta^2=4$ represents a circle, when (ξ, η) is within this circle, then (ξ', η') is without it, being such that $(-\xi', -\eta')$ and (ξ, η) are inverse points in regard to this circle (see pp. 67, 109, above). While, therefore, the complete plane (ξ, η) corresponds to all the points of the quadric $x^2+y^2+z^2=t^2$, the complete plane (x, y, z) is represented by restricting ourselves, for example, to the interior of the circle $\xi^2+\eta^2=4$ (together with, say, the points of the circumference of this circle for which η is positive).

With these formulae, any line of the original plane, with the equation $ux+vy+wz=0$, becomes, in terms of ξ and η, the locus represented by the equation

$$u\xi + v\eta + w\left[1 - \tfrac{1}{4}(\xi^2+\eta^2)\right] = 0,$$

which, if $(\xi, \eta, 2)$ be coordinates, represents a circle cutting at right angles the fixed (imaginary) circle whose equation is $1+\frac{1}{4}(\xi^2+\eta^2)=0$. Any such circle meets the real circle given by the equation $\xi^2+\eta^2-4=0$ at the ends of a diameter of this, and is, negatively, self-inverse in regard to this circle. Two such circles have two intersections of which only one is interior to the circle

$$\xi^2+\eta^2-4=0.$$

Further, if (x', y', z') be any other point of the original plane, and we introduce corresponding symbols, ξ', η', and λ', μ', the interval, θ, between the points (x, y, z) and (x', y', z'), relatively to the conic $x^2+y^2+z^2=0$, which, as we have seen, is given by

$$\cos\theta = \frac{xx'+yy'+zz'}{tt'},$$

is equally such that

$$\sin^2\tfrac{1}{2}\theta = \frac{(\lambda'-\lambda)(\mu'-\mu)}{(1+\lambda\mu)(1+\lambda'\mu')},$$

as we at once verify. In terms of ξ, η and ξ', η', this leads to

$$\sin\tfrac{1}{2}\theta = D/PP',$$

where

$$D^2 = \tfrac{1}{4}\left[(\xi'-\xi)^2+(\eta'-\eta)^2\right],\quad P^2 = 1+\tfrac{1}{4}(\xi^2+\eta^2),\quad P'^2 = 1+\tfrac{1}{4}(\xi'^2+\eta'^2).$$

If we suppose $\lambda'-\lambda$, $\mu'-\mu$ to be both small, and replace θ by ds, we thence obtain, for the determination of the interval between the points (ξ, η) and $(\xi+d\xi, \eta+d\eta)$, the form [Add.]

$$ds^2 = \frac{d\xi^2+d\eta^2}{\left[1+\frac{1}{4}(\xi^2+\eta^2)\right]^2}.$$

This may, of course, be obtained directly from the value of ds^2 for the interval between the points (x, y, z) $(x+dx, y+dy, z+dz)$; this latter is found at once to be

$$ds^2 = \frac{(ydz - zdy)^2 + (zdx - xdz)^2 + (xdy - ydx)^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{dx^2 + dy^2 + dz^2 - dt^2}{t^2}, \; = \left[d\left(\frac{x}{t}\right)\right]^2 + \left[d\left(\frac{y}{t}\right)\right]^2 + \left[d\left(\frac{z}{t}\right)\right]^2.$$

The value obtained for ds, in terms of ξ, η, is precisely that found by Riemann (*loc. cit.* p. 264) for the element of length appropriate to a manifoldness which, in his sense, is of constant curvature, given by the absolute coordinates ξ, η. His result is for a metrical manifoldness of any number, n, of absolute coordinates. But it may be shewn, reversing the procedure here followed, that any such manifoldness is in correspondence with a manifoldness to which $(n+1)$ homogeneous coordinates are appropriate; and that his measure of length is then obtained by considering the interval between two points with respect to an enveloping cone of this, as in the preceding work, where $x^2+y^2+z^2=0$ is the equation of the cone formed by the lines drawn from $(0, 0, 0, 1)$ to touch the quadric $x^2+y^2+z^2=t^2$. The intermediate parameters λ, μ, which are convenient, but not necessary, are not available in the general case.

If in the formula for $\sin^2\frac{1}{2}\theta$, in terms of λ, μ, λ', μ', we replace λ, μ, respectively, by $-\mu^{-1}$, $-\lambda^{-1}$, and make the corresponding replacements for λ', μ', the form remains the same. This also happens in the changes by which λ and $-\mu^{-1}$ become $(A\lambda+B)/(C\lambda+D)$ and $(-A\mu^{-1}+B)/(-C\mu^{-1}+D)$, wherein A, B, C, D are arbitrary constants. This last change corresponds to a linear transformation of the coordinates x, y, z which is such that the conic $x^2+y^2+z^2=0$ remains unchanged. (See p. 169.) The former change, in terms of ξ, η, is of the form $\xi' = -4\xi/(\xi^2+\eta^2)$, $\eta' = -4\eta/(\xi^2+\eta^2)$, which has been noticed above. The expression for $\sin^2\frac{1}{2}\theta$ is thus unaltered by a transformation obtained by combining these two transformations.

Further, it is easy to compute that, if we measure the angle between two circles, of equations

$$u\xi + v\eta + w\left[1 - \tfrac{1}{4}(\xi^2+\eta^2)\right] = 0, \quad u'\xi + v'\eta + w'\left[1 - \tfrac{1}{4}(\xi^2+\eta^2)\right] = 0,$$

by taking the tangents at one of their common points, (ξ_0, η_0), with respect to the two lines whose equations are $\xi - \xi_0 \pm i(\eta - \eta_0) = 0$, as explained above, then this angle between the circles is equal to the angular interval between the lines, of the plane (x, y, z), which correspond to these circles, this last interval being taken relatively to the conic $x^2+y^2+z^2=0$. That this must be so is obvious geometrically, as in the preceding section.

And, it may be verified, by the formulae $X=\xi[1-\frac{1}{4}(\xi^2+\eta^2)]^{-1}$, $Y=\eta[1-\frac{1}{4}(\xi^2+\eta^2)]^{-1}$, that the integral introduced above (p. 187), to measure the extent of a region, becomes, in terms of ξ, η,

$$\iint\frac{d\xi\,d\eta}{[1+\frac{1}{4}(\xi^2+\eta^2)]^2}.$$

This is unaltered by the substitutions

$$\xi'=-4\xi/(\xi^2+\eta^2),\quad \eta'=-4\eta/(\xi^2+\eta^2).$$

When applied to a region which does not lie wholly within the circle $\xi^2+\eta^2=4$, it is to be taken only over that part of the region which does lie within this circle.

The case when the equation of the quadric surface is of the form $z^2-x^2-y^2=t^2$ is similar to the preceding. It is convenient, however, in this case, to make a substitution of the form

$$x=\lambda+\mu,\quad y=1-\lambda\mu,\quad z=1+\lambda\mu,\quad t=-i(\lambda-\mu);$$

if we take $\lambda=\frac{1}{2}(\xi+i\eta)$, $\mu=\frac{1}{2}(\xi-i\eta)$, this gives $\xi=2x/(y+z)$, $\eta=2t/(y+z)$, and $x=\xi$, $y=1-\frac{1}{4}(\xi^2+\eta^2)$, $z=1+\frac{1}{4}(\xi^2+\eta^2)$, $t=\eta$.

By these formulae, any point of the original plane (x, y, z), with the associated positive value of t, is made to correspond to a point of the plane in which $(\xi, \eta, 2)$ are coordinates, only the points of this latter plane for which η is positive being considered. The separating line $\eta=0$ plays, however, a different part from that played by the circle $\xi^2+\eta^2=4$ in the former case: the line $\eta=0$ consists of points each corresponding only to one point of the quadric, and is a natural boundary; the circle $\xi^2+\eta^2=4$ was generated as a locus of pairs of points, and was a conventional boundary, convenient but not unique. Further, the formulae give $z^2-x^2-y^2=\eta^2$; thus points of the original plane (x, y, z) which are outside the conic $z^2-x^2-y^2=0$, as they would correspond to a negative value of η^2, are not represented by real points of the plane $(\xi, \eta, 2)$; the curves of the plane $(\xi, \eta, 2)$ corresponding to two lines of the plane (x, y, z), which do not intersect within the conic $z^2-x^2-y^2=0$, will be curves with no real intersection. In fact a line whose equation is $ux+vy+wz=0$ is represented in the plane $(\xi, \eta, 2)$ by the locus whose equation is

$$u\xi+v[1-\tfrac{1}{4}(\xi^2+\eta^2)]+w[1+\tfrac{1}{4}(\xi^2+\eta^2)]=0;$$

this is a circle whose centre is on the line $\eta=0$, of which, as we have said, we consider only the points for which η is positive. All such circles arise, if all lines of the original plane be considered, and conversely. When two such circles intersect in two points, only one will be such that η is positive; this will correspond to an intersection of two lines, in the original plane, at a point interior to the conic $z^2-x^2-y^2=0$; when two such circles meet (and therefore touch) at a point for which $\eta=0$, they correspond to lines of the original plane which meet on the absolute conic, and may, therefore, be said to be parallel. It is easy to see that, through an arbitrary point, of the plane $(\xi, \eta, 2)$, for which η is positive, there can be drawn two circles, with centres on the line $\eta = 0$, each of which shall touch (on $\eta = 0$) a given circle whose centre is on $\eta = 0$. Particular (degenerate) circles of the system are the lines, of equations of the form $\xi =$ constant, which are at right angles to the line $\eta = 0$; these correspond to the lines of the original plane which meet at the point $(0, 1, -1)$ of the conic $z^2 - x^2 - y^2 = 0$. It is easy to compute that the angle between any two circles of the system, estimated as before by considering the tangents at the intersection, (ξ_0, η_0), of the circles, with respect to the two lines

$$\xi - \xi_0 \pm i(\eta - \eta_0) = 0,$$

is equal to the angular interval between the lines of the original plane which they represent, estimated with respect to the conic $z^2 - x^2 - y^2 = 0$. For the case of this conic, instead of the original interval, θ, between two points, we introduced an interval σ such that $\theta = i\sigma$. It can be shewn that, for the plane $(\xi, \eta, 2)$, this is such that

$$\sinh^2 \tfrac{1}{2}\sigma = \frac{(\lambda' - \lambda)(\mu' - \mu)}{-(\lambda - \mu)(\lambda' - \mu')}, \quad = \frac{(\xi' - \xi)^2 + (\eta' - \eta)^2}{4\eta\eta'},$$

becoming infinite when one of the points is on $\eta = 0$. When $\xi' = \xi + d\xi$, $\eta' = \eta + d\eta$, this gives [Add.]

$$d\sigma = \frac{(d\xi^2 + d\eta^2)^{\frac{1}{2}}}{\eta};$$

and, by the ordinary methods of the Calculus of Variations, it will be found that the curve for which the integral

$$\int (d\xi^2 + d\eta^2)^{\frac{1}{2}}/\eta,$$

taken between two fixed points, has a stationary value, is the circle through these points whose centre is on the line $\eta = 0$. We can interpret the interval σ, between two points P, P', by drawing, through these, the circle whose centre is on $\eta = 0$, and considering the points P, P' with respect to the two points, say M and N, in which this circle meets the line $\eta = 0$. For the pencil which these four points subtend at any point of the circle we then have, supposing that N, P, P', M are in order upon the portion of the circle considered, for which η is positive, the equation (p. 166)

$$(P, P'; M, N) = e^{-\sigma}.$$

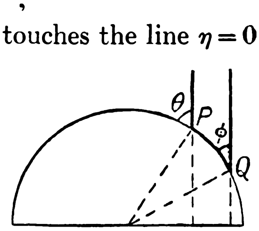
The curve, called by Lobatschewsky a circle, of centre (x', y', z'), whose equation is (above, p. 182)

$$(zz' - xx' - yy')^2 - \lambda^2(z^2 - x^2 - y^2) = 0,$$

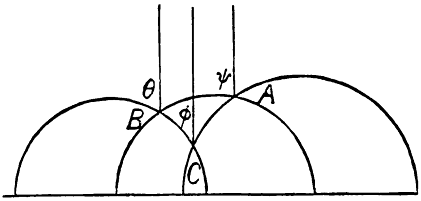
where λ is supposed real and positive, is represented, in the part of the $(\xi, \eta, 2)$ plane for which η is positive, by the circle whose equation is

$$(\xi - \xi')^2 + (\eta - \lambda)^2 = \lambda^2 - \eta'^2;$$

this circle has its centre at $\xi = \xi'$, $\eta = \lambda$. It touches the line $\eta = 0$ and becomes a horocycle if $\eta' = 0$, that is, if (x', y', z') lie on $z^2 - x^2 - y^2 = 0$.

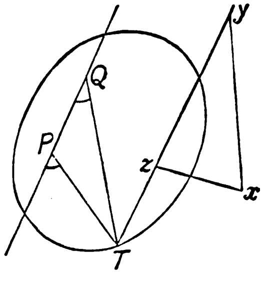


By means of any two points P, Q, a region is defined, bounded by the lines through P, Q with equations of the form $\xi = \text{constant}$, and by the circle through P, Q whose centre is on $\eta = 0$, but lying entirely outside this circle. If θ, ϕ be the angles between this circle and the two lines spoken of, respectively at P and Q, this being measured in the Euclidean way, by means of lines $\xi - \xi_0 \pm i(\eta - \eta_0) = 0$, it is easy to shew, if $\theta - \phi$ be positive, that the integral $\iint \eta^{-2}\, d\xi\, d\eta$, extended over this region, has the value $\theta - \phi$. Suppose now that we have three circles, with centres on $\eta = 0$, whose intersections in pairs are A, B, C. Any two of these points, say A and B, determine such a region as we have just considered, which we may denote momentarily by (A, B). The points A, B, C determine a triangle, whereof the joins are arcs of these circles; and, to limit ourselves to one case, this triangle may be supposed to be obtained by the addition of the two regions (B, C) and (C, A) and the subsequent subtraction of the region (B, A). With appropriate angles θ, ϕ, ψ (explained by the annexed diagram), the integral $\iint \eta^{-2} d\xi d\eta$, taken over the triangle, is then equal to



$$[\theta - \phi] + [(\pi - C + \phi) - (A + \psi)] - [(B + \theta) - \psi]$$

which is $\pi - A - B - C$, where A, B, C are the measures of the angles of the triangle. We have, above (p. 180), given a proof that this expression is necessarily positive.



The forms for x, y, z in terms of ξ, η, suggest, putting $X = x/z$, $Y = y/z$, that we consider the transformation

$$X = \xi[1 + \tfrac{1}{4}(\xi^2 + \eta^2)]^{-1},$$
$$Y = [1 - \tfrac{1}{4}(\xi^2 + \eta^2)]\,[1 + \tfrac{1}{4}(\xi^2 + \eta^2)]^{-1};$$

it is found that, then, the above integral, $\iint \eta^{-2} d\xi d\eta$, becomes

$$\iint (1 - X^2 - Y^2)^{-\frac{3}{2}}\, dX dY.$$

Thus we infer that if, in the original plane (x, y, z), the point

$(0, 1, -1)$ be T, and P, Q be two points of any line, interior to the absolute conic, so that TPQ is a triangle for which the angular interval between TP and TQ is zero, then this integral, extended over the triangle TPQ, is equal to the difference of the angular intervals between PQ and, respectively, PT and QT.

Comparison of the two cases considered. The distinction between the two cases considered above, of an imaginary conic of reference, and a real conic, has arisen by respecting the distinction between real and imaginary points. If, in the former case, we put $t_1 = -it$, and, in the latter case, put $z_1 = -iz$, as well as λ', μ' in place of λ, μ, the formulae are

$$x = \lambda + \mu, \quad y = -i(\lambda - \mu), \quad z = 1 - \lambda\mu, \qquad t_1 = -i(1 + \lambda\mu),$$
$$x = \lambda' + \mu', \quad y = 1 - \lambda'\mu', \quad z_1 = -i(1 + \lambda'\mu'), \quad t = -i(\lambda' - \mu'),$$

and the equations of the quadric surface, in these two cases, are $x^2 + y^2 + z^2 + t_1^2 = 0$ and $x^2 + y^2 + z_1^2 + t^2 = 0$, respectively. If we now replace z_1, t_1, respectively, by z and t, the comparison of the formulae leads to

$$\lambda' = -i\frac{1+\lambda}{1-\lambda}, \quad \mu' = i\frac{1-\mu}{1+\mu}.$$

Corresponding to the twenty-four orders in which the four coordinates x, y, z, t may be taken, there follow twenty-four such ways in which we may replace x, y, z, t by functions of two parameters λ, μ, so as to satisfy, identically, the equation of the quadric surface. These twenty-four sets of formulae can all be obtained from any one of them by replacing λ and μ by proper linear (fractional) functions, respectively of λ and μ, or of μ and λ. It is not difficult to see that these twenty-four pairs of replacements are combinations of the three pairs

$$(\alpha),\ \lambda' = \lambda^{-1},\ \mu' = -\mu; \quad (\beta),\ \lambda' = i\mu,\ \mu' = -i\lambda;$$
$$(\gamma),\ \lambda' = (1-\mu)(1+\mu)^{-1}, \quad \mu' = (1-\lambda)(1+\lambda)^{-1};$$

for instance the replacement above is $\alpha\beta\gamma$, obtained by first carrying out γ, then β, and then α. Neglecting the distinction between real and imaginary, we thus have, by putting $\lambda = \frac{1}{2}(\xi + i\eta)$, $\mu = \frac{1}{2}(\xi - i\eta)$, twenty-four such representations of the original plane (x, y, z), upon a plane (ξ, η).

Examples of the preceding considerations. *Ex.* 1. *Delambre's formulae in spherical trigonometry.* Let the absolute conic, which we suppose to be an imaginary conic, when referred to three real points X, Y, Z, have the equation

$$ax^2 + 2fyz + \ldots = 0,$$

containing six terms, of which the coefficients, $a, f, \ldots$, are real. No one of a, b, c is zero, and no one of $bc - f^2$, $ca - g^2$, $ab - h^2$, for the (imaginary) conic does not contain any of the points X, Y, Z, nor touch any of the lines YZ, ZX, XY. As usual, let $A = bc - f^2$, and $F = gh - af$, etc., and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$. The left side of the equation being

$$a\left(x + \frac{h}{a}y + \frac{g}{a}z\right)^2 + \frac{C}{a}\left(y - \frac{F}{C}z\right)^2 + \frac{\Delta}{C}z^2,$$

it follows that a, C, Δ, and, therefore, similarly, all of b, c, A, B, may be taken to be positive (and not zero). We can then define intervals, YZ, ZX, XY, which we denote, respectively, by α, β, γ, and also angular intervals between the pairs of these lines, which we denote by α', β', γ', the first being that between XY and XZ, and so on, by the equations (following from pp. 175, 176)

$$\cos\alpha = \frac{f}{(bc)^{\frac{1}{2}}}, \quad \cos\beta = \frac{g}{(ca)^{\frac{1}{2}}}, \quad \cos\gamma = \frac{h}{(ab)^{\frac{1}{2}}},$$

$$\sin\alpha = \left(\frac{A}{bc}\right)^{\frac{1}{2}}, \quad \sin\beta = \left(\frac{B}{ca}\right)^{\frac{1}{2}}, \quad \sin\gamma = \left(\frac{C}{ab}\right)^{\frac{1}{2}},$$

$$\cos\alpha' = \frac{-F}{(BC)^{\frac{1}{2}}}, \quad \cos\beta' = \frac{-G}{(CA)^{\frac{1}{2}}}, \quad \cos\gamma' = \frac{-H}{(AB)^{\frac{1}{2}}},$$

$$\sin\alpha' = \left(\frac{a\Delta}{BC}\right)^{\frac{1}{2}}, \quad \sin\beta' = \left(\frac{b\Delta}{CA}\right)^{\frac{1}{2}}, \quad \sin\gamma' = \left(\frac{c\Delta}{AB}\right)^{\frac{1}{2}};$$

and it is then easy to see, comparing the relations previously obtained (above, p. 177), that these belong to a triangle of which X, Y, Z are the vertices. It is understood that, for example, $(bc)^{\frac{1}{2}}$ means $b^{\frac{1}{2}} . c^{\frac{1}{2}}$, both the factors being real, and so in each case. The signs of the square roots are to be determined consistently with the formulae referred to.

Now, if ϵ be $+1$ or -1, and σ be $+1$ or -1, we have, identically,

$$\frac{A\left[(BC)^{\frac{1}{2}} + \epsilon F\right]}{\Delta\left[(bc)^{\frac{1}{2}} - \sigma f\right]} = \frac{a\left[(bc)^{\frac{1}{2}} + \sigma f\right]}{(BC)^{\frac{1}{2}} - \epsilon F};$$

hence, as $AF = GH - f\Delta$, or directly, we see that the function

$$\left\{1 + \epsilon\frac{G}{(CA)^{\frac{1}{2}}}\frac{H}{(AB)^{\frac{1}{2}}} - \sigma\epsilon\left(1 - \frac{G^2}{CA}\right)^{\frac{1}{2}}\left(1 - \frac{H^2}{AB}\right)^{\frac{1}{2}}\right\} \div \left\{1 + \epsilon\frac{F}{(BC)^{\frac{1}{2}}}\right\}$$

is equal to the function obtained from this by replacing A, B, C, F, G, H, respectively, by a, b, c, f, g, h, and interchanging ϵ and σ. Wherefore

$$\{1 + \epsilon\cos\beta'\cos\gamma' - \sigma\epsilon\sin\beta'\sin\gamma'\} \div \{1 - \epsilon\cos\alpha'\}$$

is equal to

$$\{1 + \sigma\cos\beta\cos\gamma - \sigma\epsilon\sin\beta\sin\gamma\} \div \{1 + \sigma\cos\alpha\}.$$

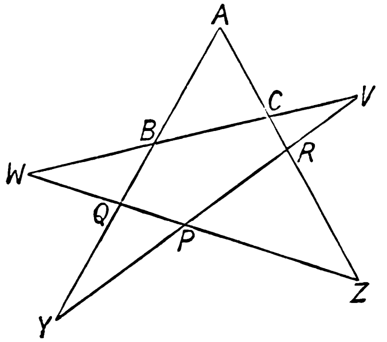
For instance, when $\epsilon = \sigma = 1$, we have

$$\left\{\frac{\cos\frac{1}{2}(\beta'+\gamma')}{\sin\frac{1}{2}\alpha'}\right\}^2 = \left\{\frac{\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}\alpha}\right\}^2,$$

and similarly in the other cases.

Ex. 2. *Napier's analogies.* Let A, B, C be a triangle, in which AB, AC are conjugate lines, in regard to an imaginary conic of reference, so that the angular interval BAC is $\frac{1}{2}\pi$. Let the polar of B meet AB, AC, BC, respectively, in Y, R and V, and the polar of C meet AC, AB, BC, respectively, in Z, Q and W, while these polars meet in P, which is thus the pole of BC.

It can then be shewn that, as at A, the pairs of lines meeting in Y, Z, V, W are, in each of these four cases, conjugate to one another; and that, in the pentagon $PRCBQ$, each vertex is the pole of the opposite side.



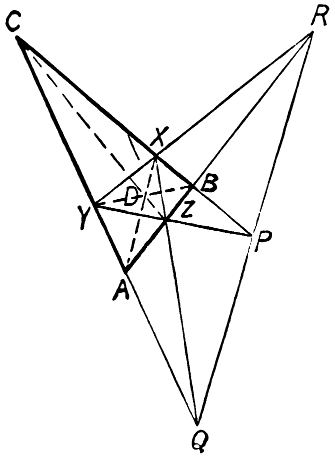
Each of the four triangles CRV, BQW, PQY, PRZ has its elements, three segments and three angles, expressible in a simple way in terms of those of the original triangle A, B, C. Obtain these expressions. Hence shew, in particular, that the relation $\sin RV = \sin CR \sin (VCR)$, which holds for the triangle CRV (see above, p. 176), involves, for the triangle ABC, the relation

$$\cos B = \cos b \sin C;$$

similarly, shew that the relation $\sin BW = \sin BQ \sin (BQW)$ involves, for ABC, the relation

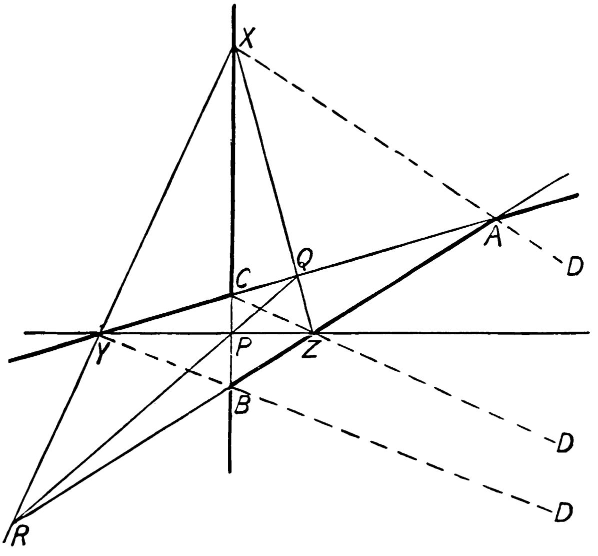
$$\cos a = \cos b \cos c;$$

and so on. (John Napier, *Mirifici logarithmorum canonis descriptio*, Edinburgh, 1614, *Lib.* ii, *Cap.* iv. Cf. Gauss, *Ges. Werke*, iii, p. 481.) [Add.]



Ex. 3. *A particular triangle.* Let X, Y, Z be a self-polar triad, of real points, in regard to the imaginary conic $x^2 + y^2 + z^2 = 0$. Let any line, PQR, meet YZ, ZX, XY, respectively, in P, Q, R, the lines YQ, ZR meeting in A, the lines ZR, XP in B, and the lines XP, YQ in C. The lines XA, YB, ZC will then meet in a point, say D. If the equation of PQR be

$a^{-1}x + b^{-1}y + c^{-1}z = 0$, the coordinates of A, B, C, D will respectively be $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, (a, b, c). Let d be the positive quantity $(a^2 + b^2 + c^2)^{\frac{1}{2}}$. In regard to $x^2 + y^2 + z^2 = 0$, the pole of the line XP lies in that segment YZ which does not contain P; the line through B, at an angular interval with the line BXC equal to $\frac{1}{2}\pi$, thus contains a point of this segment; so that the angular interval between BXC and BAR is greater than $\frac{1}{2}\pi$. Similarly the angular interval between AYC and ABR is greater than $\frac{1}{2}\pi$, and the angular interval between CXB and CYA is greater than $\frac{1}{2}\pi$.



This is true for all positions of P, Q, R in the segments YZ, ZX, XY. We may consider two cases, that in which a, b, c are all positive, and that in which a, b are positive, but c is negative. Let the angular intervals referred to, at A, B, C, be denoted, respectively, by α', β', γ'. We then have (see above, p. 174), the equations of XP, YQ, ZR being, respectively, $b^{-1}y + c^{-1}z = 0$, $c^{-1}z + a^{-1}x = 0$, $a^{-1}x + b^{-1}y = 0$, for the case in which a, b, c are all positive,

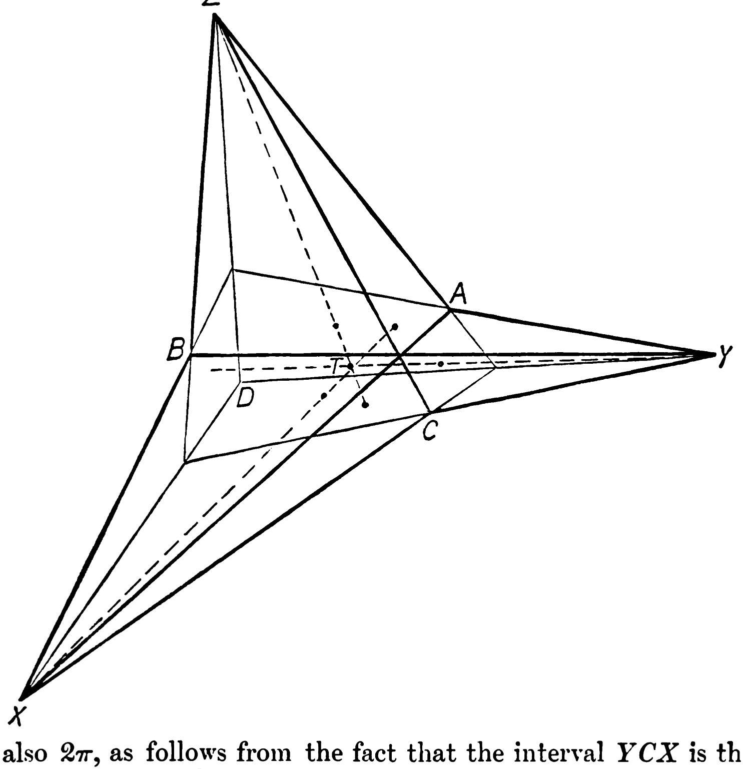
$$\cos\alpha' = -bc\,(a^2 + b^2)^{-\frac{1}{2}}(a^2 + c^2)^{-\frac{1}{2}},$$

and similar expressions, $\sin\alpha' = ad\,(a^2 + b^2)^{-\frac{1}{2}}(a^2 + c^2)^{-\frac{1}{2}}$, etc. .

From these we find $\cos\alpha' = \cos(\beta' + \gamma')$, and similar equations, and infer that $\alpha' + \beta' + \gamma' = 2\pi$. (Cf. p. 197.)

The same follows when c is negative; the values of $\cos\alpha'$, $\cos\beta'$, $\sin\gamma'$ are then changed in form, by the substitution of $-c$ for c.

If α, β, γ be the respective intervals BC, CA, AB for the segments which are appropriate to α', β', γ', in order to form a triangle, according to the definition above given (p. 172), it follows from the formulae such as $\cos\alpha\sin\beta'\sin\gamma' = \cos\alpha' + \cos\beta'\cos\gamma'$, that $\cos\alpha=(a^2-b^2-c^2).d^{-2}$, $\cos\beta=(b^2-c^2-a^2).d^{-2}$, $\cos\gamma=(c^2-a^2-b^2).d^{-2}$, while $\sin\alpha = 2a(b^2+c^2)^{\frac{1}{2}}d^{-2}$, and so on, the form for $\sin\gamma$ being $-2c(a^2+b^2)^{\frac{1}{2}}d^{-2}$ when c is negative. Of the expressions, $a^2-b^2-c^2$, $b^2-c^2-a^2$, $c^2-a^2-b^2$, two at least must be negative, as we easily prove, remarking that the sum of every two is negative; evidently, $1+\cos\alpha+\cos\beta+\cos\gamma$ is zero. It is interesting to remark also that if the angular intervals BCA, CAB, ABC be measured with reference to the two points $z=0$, $x\pm iy=0$, the sum of these intervals



is also 2π, as follows from the fact that the interval YCX is then $\frac{1}{2}\pi$ (cf. p. 167 preceding). The figure can be obtained by taking, upon the quadric $x^2+y^2+z^2=t^2$, the points $(-a, b, c, d)$, $(a, -b, c, d)$, $(a, b, -c, d)$, and $(-a, -b, -c, d)$, and projecting from the point $(0, 0, 0, 1)$ upon the plane XYZ, given by $t=0$. The planes TAB, TAC intersect in the line ATA', where A' is $(a, -b, -c, d)$, and are at an angular interval α'. Upon the quadric the four triangles

ABC, *BCD*, *CAD*, *ABD* have, every two, equal angles and segments (p. 197).

The relations for the angles and segments of the triangle *ABC* may be obtained also by remarking (see above, Chap. III, Ex. 14) that the tangents to the absolute conic from *A*, *B*, *C* meet, respectively, the lines *BC*, *CA*, *AB*, in points which lie in threes upon two lines. If we put $\xi = b^{-1}y + c^{-1}z$, $\eta = c^{-1}z + a^{-1}x$, $\zeta = a^{-1}x + b^{-1}y$, the equation of the absolute conic referred to *A*, *B*, *C* is

$$a^2(-\xi + \eta + \zeta)^2 + \ldots = 0, \text{ or } \xi^2 + 2\cos\alpha \,.\, \eta\zeta + \ldots = 0,$$

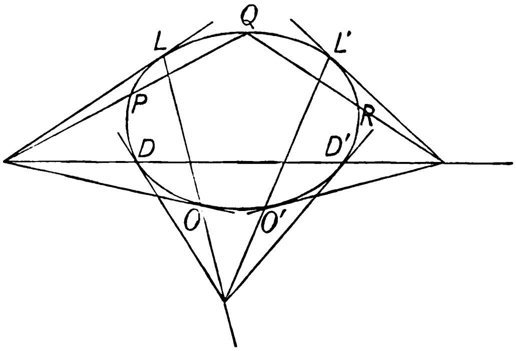
whose tangential form is $\sin^2\alpha \,.\, u^2 - 2\sin\beta\sin\gamma\cos\alpha' \,.\, vw + \ldots = 0$. The condition in question, writing l, m, n respectively for $u\sin\alpha$, $v\sin\beta$, $w\sin\gamma$, is, that there should be a line (l, m, n) for which each of the equations $m^2 + n^2 - 2mn\cos\alpha' = 0$, $n^2 + l^2 - 2nl\cos\beta' = 0$, $l^2 + m^2 - 2lm\cos\gamma' = 0$ is satisfied; which is so if $\alpha' + \beta' + \gamma' = 2\pi$.

Ex. 4. *General form of the theorem for the sum of the focal distances of a point of a conic.* If the common tangents of any conic, Ω, and the absolute conic, be taken, and *S*, *H* be a point pair through which all these tangents pass, we may prove that the intervals *PS*, *PH*, in regard to the absolute conic, for any point *P* of the conic Ω, have a constant sum (or difference). When the absolute conic is an imaginary point pair, this gives the familiar theorem of metrical geometry, for the sum (or difference) of the focal distances of a point of a conic.

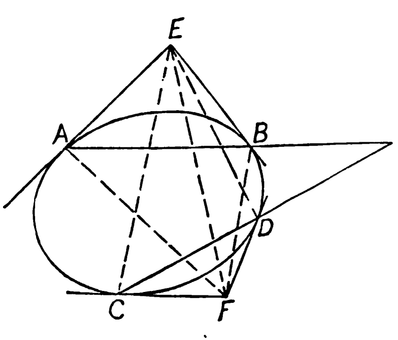
This result may be deduced from the following: Let *A*, *B*, *C*, *D* be four variable points of the absolute conic, such that *AB* passes through a fixed point, *S*, while *BC* passes through a fixed point, *M*, and *CD* through a fixed point, *H*, the points *S*, *H*, *M* being in line. Then *DA* passes through a further fixed point, *N*, of this line. And the locus of the intersection, *P*, of the lines *AB*, *CD*, is a conic, which touches the tangents to the absolute conic from the points *S* and *H*, and passes through the points of contact of this absolute conic with the tangents drawn to it from *M* and *N*.

Although an anticipation, it seems proper to mention, in connexion with this, two theorems given by Chasles, *Géom. Supér.* 1880, pp. 517, 528. If the points of the conic Ω be joined by lines to a point *O* not lying in the plane of the conic, the points of these lines are said to lie on a quadric cone; the points of the cone lying in any plane are points of a conic. If any plane be drawn through a common chord of Ω and the absolute conic, the conic in which this plane meets the cone is called a circular section of the cone. A plane through the complementary common chord of the two conics meets the cone in a circular section complementary to any one of the former system. Now let the angle between any two planes be measured by the interval between the lines in which these planes meet the plane of the absolute conic, in respect to this conic. Then we have the result that the sum (or difference) of the angles between any tangent plane of the cone and a pair of complementary circular sections, is constant. Again, if O be joined to one of the point pairs through which pass the common tangents of the conic Ω and the absolute conic, the two joining lines are called a pair of focal lines of the cone. If the angle between two lines, not lying in the plane of the absolute conic, be measured by the interval between the points in which these meet the plane of the absolute conic, taken with respect to this, we have the result that the angles which any edge of the cone makes with a pair of focal lines have a constant sum (or difference).

Ex. 5. If on a conic there be taken, arbitrarily, the points D, D', O, O', P, and then there be determined in succession, L, L', Q, R, by means of $(LO, DD') = -1$, $(L'O', DD') = -1$, $(QP, OL) = -1$, $(RQ, O'L') = -1$, so that L is the harmonic conjugate of O, on the conic, in regard to D and D', and so on, prove that (RP, DD') is equal to $(O'O, DD')^2$.



Ex. 6. If the tangents at the points C, D of a conic meet in F, and the tangents at the points A, B meet in E, prove that, measured with respect to this conic, the angular intervals AFE, EFB, CEF, FED are equal. Denoting this interval by α, and the angular interval between the lines AB and CD by β, prove that $\sin\alpha \sin\beta = \pm 1$.



Ex. 7. If PQ, PR be two tangents to a conic which touches the absolute conic in two points, the intervals PQ, PR are equal; and are the same as for any other point, P', lying on a conic through P which touches the absolute conic at the same two points as the former.

Ex. 8. Let V and W be two conics each of which touches the absolute conic in two points, not necessarily the same, and U be a conic touching each of V and W in two points; from any point, P, of U let a tangent PQ be drawn to V, and a tangent PR be drawn to W; prove that the intervals PQ, PR have a constant sum (or difference) as P varies on U. Shew that this includes the result given in Ex. 4.

Ex. 9. From a point, P, of a conic $f=0$, a tangent PT is drawn to a conic $\psi=0$, and PT meets $f=0$ again in P'. Let $f_1=\partial f/\partial x$, etc., and c_1, c_2, c_3 be arbitrary. Prove that the differential, taken along the conic $f=0$, at P, which is expressed by

$$\begin{vmatrix} dx, & dy, & dz \\ x, & y, & z \\ c_1, & c_2, & c_3 \end{vmatrix} \div (c_1 f_1 + c_2 f_2 + c_3 f_3)(\psi)^{\frac{1}{2}},$$

has the same value as the corresponding differential at P'.

The conics being in general relation, the form of the integral shews that there is no loss of generality in supposing $\psi=\xi^2+\eta^2-1$, $f=a\xi^2+b\eta^2-1$. Let $F(\xi)$ denote what f becomes by putting $\eta=\operatorname{cosec}\alpha\,.\,(1-\xi\cos\alpha)$, and $F'(\xi)=\partial F/\partial\xi$; by this substitution we have $\partial\eta/\partial\alpha=\operatorname{cosec}^2\alpha\,.\,(\xi-\cos\alpha)=\pm\operatorname{cosec}\alpha\,.\,\psi^{\frac{1}{2}}$. From $F(\xi)=0$, the integral $-d\xi/\eta\psi^{\frac{1}{2}}$ becomes $2b\,.\,d\alpha\,.\,[F'(\xi)\,.\,\psi^{\frac{1}{2}}]^{-1}\partial\eta/\partial\alpha$, or

$$\pm 2b\,.\,d\alpha\,.\operatorname{cosec}\alpha/F'(\xi);$$

and $F'(\xi)$ has opposite signs at P and P'.

Similarly for other forms of f. (Cf. Ex. 25, p. 147.)

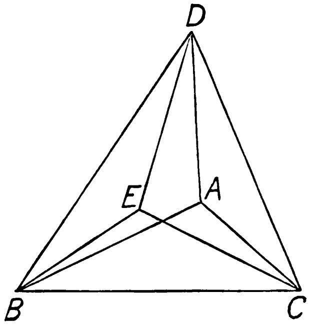
NOTE I

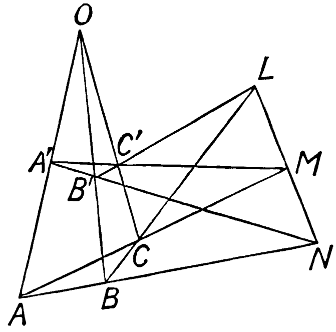
ON CERTAIN ELEMENTARY CONFIGURATIONS, AND ON THE COMPLETE FIGURE FOR PAPPUS' THEOREM

The present note deals in an incomplete way with certain configurations which present themselves naturally, especially in the consideration of the complete figure for Pascal's theorem which is considered in the succeeding note.

1. **The simpler Desargues' figure.** If A,B,C and A',B',C' be two triads of points in perspective, from a point O, and the lines BC, $B'C'$ meet in L, while CA, $C'A'$ meet in M, and AB, $A'B'$ meet in N, then L, M, N lie in line.

Regarded as a plane figure, there are here, in all, ten points, each lying on three lines, and ten lines, each containing three points; and the dual of the figure is of the same character. To express the incidences it may be denoted by $10_3 10_3$ (or simply 10_3), or by $10(\,.\,3)\,.\,10(3\,.\,)$.





Regarded as a figure in three dimensions, the figure is formed by the intersections of five planes, the ten points being the intersections of threes of these planes, and the ten lines the intersections of twos of these planes. Each point now lies in three lines and in three planes, each line contains three points and lies in two planes, and each plane contains six points and four lines. These incidences we may summarise by denoting the figure by

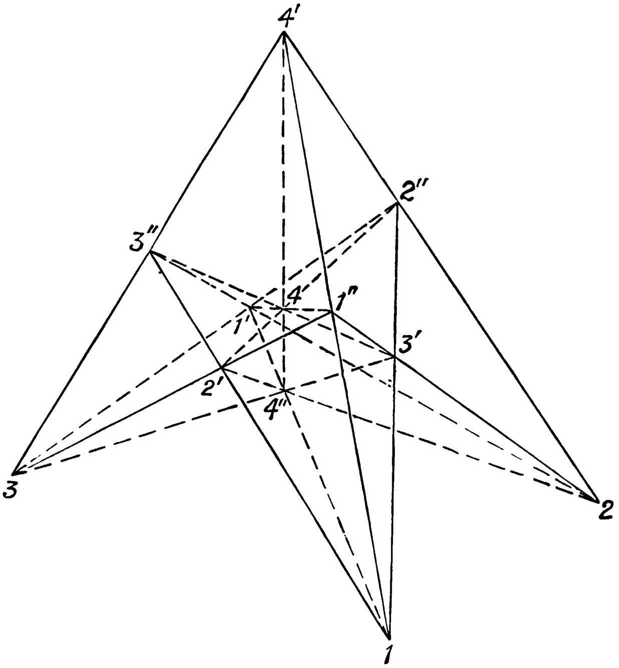
$$10(\,.\,3,3)\,.\,10(3\,.\,2)\,.\,5(6,4\,.\,).$$

The figure, however the points be taken, necessarily lies in a threefold space. Its dual, in this space, is formed by five points, the lines joining the pairs of these, and the planes containing the threes of these. But this dual figure may evidently be regarded as obtained by projection from a figure formed by taking five points in a space of four dimensions. For this figure, we shall naturally take into account, beside the lines and planes, also the threefold spaces defined by the fours of the five given points. The representation of the incidences will then be by a symbol

$$5\,(\,.\,4, 6, 4)\,.\,10\,(2\,.\,3, 3)\,.\,10\,(3, 3\,.\,2)\,.\,5\,(4, 6, 4\,.\,).$$

And the dual of this figure in four dimensions will be a figure of the same character.

2. **The figure of three desmic tetrads of points in space of three dimensions.** Two triads of points, A, B, C and A', B', C', which are in perspective, enable us to define two sets each of four planes, namely ABC, $AB'C'$, $BC'A'$, $CA'B'$ and $A'B'C'$, $A'BC$, $B'CA$, $C'AB$; these planes have the property that the sixteen lines in which any one of the first four planes meets the second four planes, lie by fours in four other planes.



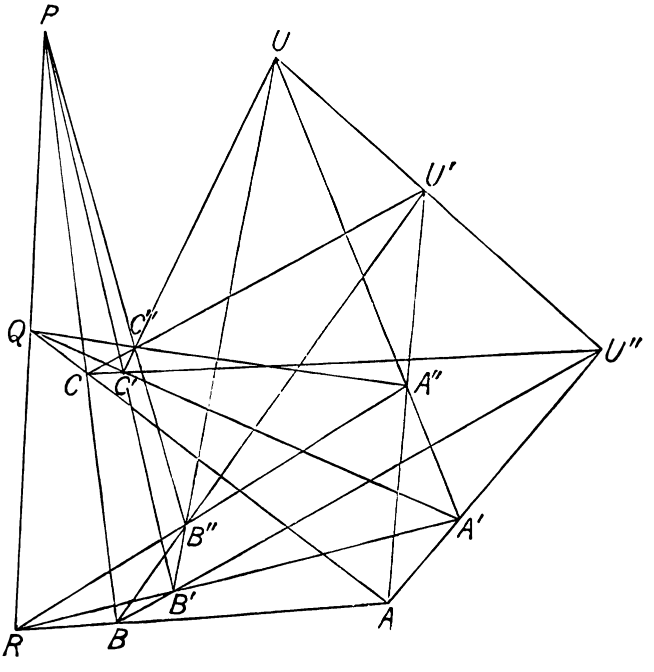
We consider the dual of this figure in three dimensions. We take six planes meeting in pairs in three lines which lie in a plane; namely the planes $234'$, $234''$; $314'$, $314''$; $124'$, $124''$; then the planes 314′, 124′ meet in a line, 14′; the intersection of this with the plane 234″ is a point, 1″; similarly the planes 314″, 124″ meet in a line, 14″; the intersection of this with the plane 234′ is a point, 1′. Let the points 2″, 2′ and 3″, 3′ be similarly defined. It can be shewn that the lines 1′1″, 2′2″, 3′3″, 4′4″ meet in a point, 4. Thus the twelve points lie in threes on sixteen lines in the following way:

11′4″, 12′3″, 12″3′, 11″4′,
23″1′, 22′4″, 23′1″, 22″4′,
31′2″, 31″2′, 33′4″, 33″4′,
41′1″, 42′2″, 43′3″, 44′4″,

and any two of the four tetrads of points 1, 2, 3, 4; 1′, 2′, 3′, 4′; 1″, 2″, 3″, 4″ are in perspective in four ways, the centres of perspective being the points of the other tetrad.

Beside the twelve points, and the sixteen lines, the figure contains twelve planes, consisting of the six from which we started, together with the three 14 2′3′2″3″, 24 3′1′3″1″, 34 1′2′1″2″, and the three 14 1′4′1″4″, 24 2′4′2″4″, 34 3′4′3″4″. The description of the complete figure is given by 12 (. 4, 6) 16 (3 . 3) 12 (6, 4 .). The dual of this figure is a figure of the same character.

3. **The figure formed from three triads of points which**



are in perspective in pairs, with centres of perspective in line. This figure, which is that used in the proof of Desargues' theorem for two triads in one plane (Vol. I, Frontispiece, and p. 7), may be described as consisting of two tetrads of points, Q, C, C', C'' and R, B, B', B'', in perspective from P, together with the six points of intersection of the corresponding joins in these tetrads, such as QC and RB, or $C'C''$ and $B'B''$, namely, the points A, A', A'', U, U', U'', which lie in threes in four lines lying in a plane. Any point of the figure may be taken, in place of P, as such a centre of perspective.

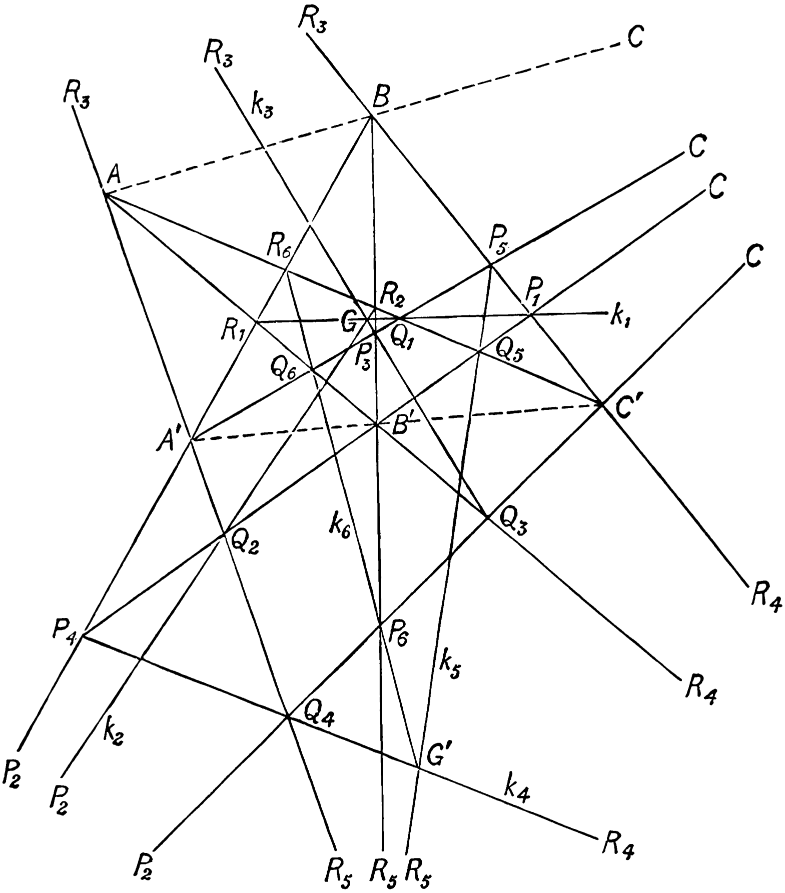
The figure may be in four dimensions, but not in higher dimensions. So considered it contains fifteen points, twenty lines and fifteen planes, namely the three planes, ABC, $A'B'C'$, $A''B''C''$, passing through the line PQR, and three similar planes through the line $UU'U''$, containing, respectively, the triads $AA'A''$, $BB'B''$, $CC'C''$, together with nine other planes, of which for instance there are three through P, namely, $PB'C'B''C''$, which contains the point U; $PB''C''BC$, which contains the point U'; and $PBCB'C'$, which contains the point U''. The figure also contains six spaces of three dimensions. One of these, for instance, contains the ten points $P, Q, R, A, B, C, A', B', C', U''$; another contains the ten points $U, U', U'', B, B', B'', C, C', C'', P$; the former of these contains two planes through the line PQR, the two planes which join the line $AA'U''$ to Q and R, and the plane $PBB'CC'$, five in all, and is, in fact, such a figure as considered above, in § 1. The whole description of the incidences of the figure is then given by the scheme

$$15\,(\,.\,4, 6, 4)\ 20\,(3\,.\,3, 3)\ 15\,(6, 4\,.\,2)\ 6\,(10, 10, 5\,.\,).$$

If this figure be projected into a plane, it becomes the dual of a generalisation of the complete figure for Pappus' theorem. We shall however obtain this by taking a section, by a plane, of the dual of this figure in four dimensions, which will appear to have great simplicity. But first we consider the complete figure of Pappus' theorem.

4. **The complete figure for Pappus' theorem.** The points A, B, C being in line, and the points A', B', C' in another line intersecting the former, the pairs of lines BC', $B'C$; CA', $C'A$ and AB', $A'B$ intersect in three points lying on a line, say k_1. This is Pappus' theorem referred to. We may however, retaining the order of A, B, C, take A', B', C' in six different orders, and so obtain in all six lines such as k_1, each containing three points of intersection. It is found however that the three lines, k, corresponding to the orders A', B', C'; B', C', A' and C', A', B' meet in a point, say G; and the three lines, k, corresponding to the orders A', C', B';

B', A', C'; C', B', A' also meet in a point, say G'. This was remarked by Steiner (*Werke*, I, pp. 451, 452). If now we omit the lines ABC, $A'B'C'$, and the points A, B, C, A', B', C', and add to


R_3
R_3
R_3
C
B
k_3
C
A
C
C
R_6
P_5
P_1
R_2
k_1
R_1
G
Q_1
P_3
Q_5
Q_6
C'
A'
B'
Q_3
k_6
Q_2
R_4
P_4
P_6
k_5
R_4
k_2
Q_4
G'
k_4
P_2
P_2
R_4
P_2
R_5
R_5
R_5

the points the two points G and G', we have a figure containing in all twenty points, each being an intersection of three lines, and containing fifteen lines (the nine lines originally arising as the joins of A, B, C with A', B', C', and the six lines k) upon each of which lie four of the points. The description of the incidences of the figure will then be 20 (. 3) 15 (4 .).

5. **The figure formed from six points in four dimensions.** Consider in four dimensions the dual of the figure described in § 3. We may construct it by the dual procedure to that before employed;

but it will be simpler to take the scheme of incidences which we can infer from that for the other figure. This will be

$$6(\,.\,5, 10, 10)\ 15(2\,.\,4, 6)\ 20(3, 3\,.\,3)\ 15(4, 6, 4\,.\,).$$

From this it can be seen at once that the figure consists of six points, the lines joining them in pairs, the planes containing threes of them, and the threefold spaces defined by fours of them.

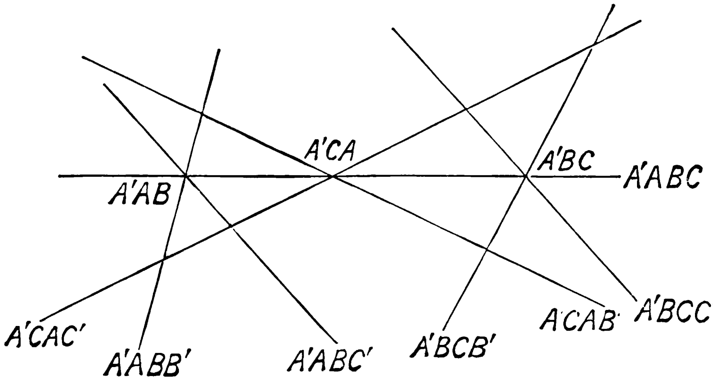
Consider now this figure, and the section of it by an arbitrary plane, ϖ. The figure being in four dimensions, a line of it will not in general have any intersection with ϖ; a plane of it will meet ϖ in a point; and a threefold space of it will meet ϖ in a line. Thus, the lines in which ϖ is met by two threefold spaces of the figure will meet in the point in which ϖ is met by the plane in which the two threefold spaces intersect; and the points in which ϖ is met by three planes which belong to the same threefold space of the figure, lie in a line, in which this threefold space meets ϖ.

Though the figure is symmetrical, we shall grasp the relations of the plane figure better, perhaps, if we choose out, from the figure in four dimensions, two planes which have no point in common, and the three spaces which contain each of these planes. Denoting the six fundamental points by A, B, C and A', B', C', we take the planes ABC and $A'B'C'$, and the six threefold spaces

$$A', A, B, C;\quad B', A, B, C;\quad C', A, B, C,$$
$$A, A', B', C';\quad B, A', B', C';\quad C, A', B', C'.$$

Save for the planes ABC and $A'B'C'$, no two of these six spaces have a plane in common.

With each of these six spaces we can now associate six other spaces—as we explain immediately. But of the thirty-six spaces so derived, each occurs four times in the enumeration. The nine different ones, together with the original six, are the whole fifteen spaces of the figure.



Take, for instance, the space A', A, B, C. Beside the plane ABC, this contains the three planes $A'BC$, $A'CA$, $A'AB$. Each of these lies in two other spaces beside $A'ABC$, namely, respectively,

	$A'BC$,	$A'CA$,	$A'AB$
lie in	$A'BCB'$,	$A'CAC'$,	$A'ABB'$
and in	$A'BCC'$,	$A'CAB'$,	$A'ABC'$.

We thus have, by section with the plane ϖ, a figure such as that occurring in Pappus' theorem, but in a generalised form. The two original triads of points of Pappus' theorem, each in line, are here replaced by triads of points which are not generally in line. The reader will easily find the specialisation of the six points sufficient to secure that they shall be.

Ex. 1. If two sets, each of n points, $A_1 \ldots A_n$ and $B_1 \ldots B_n$, in $(n-1)$ fold space, be in perspective, from a point O, there is an unique quadric $(n-2)$ fold, in regard to which any point, say A_1, of one set, is the pole of the $(n-2)$ fold containing the corresponding points, $B_2 \ldots B_n$, of the other. If C_{rs} be the intersection of the lines A_rA_s, B_rB_s, the points C_{rs} lie in a $(n-2)$ fold, which is the polar of O in regard to the quadric. Further, the figure is symmetrical, everyone of the $\frac{1}{2}(n+1)(n+2)$ points being a centre of perspective of two n-ads, and the pole of a corresponding $(n-2)$ fold, in regard to the same quadric.

Ex. 2. Shew that the figure is obtainable by projection of the points of intersection of $(n+2)$ arbitrary $(n-1)$ folds in a space of n dimensions, from an arbitrary point, T, of this. Further, that $(n+2)$ points $P_1 \ldots P_{n+2}$ can be found in this n-fold space (in ∞^{n+1} ways), such that the $\frac{1}{2}(n+1)(n+2)$ points are of symbols $P_r - P_s$ $(r, s = 1 \ldots n+2)$, where $P_1 \ldots P_{n+2}$ are the symbols of these points.

Ex. 3. Shew that the figure is also obtainable by section, with a $(n-1)$ fold, from that formed by the joins (lines, planes, etc.) of $n+2$ arbitrary points of a space of n dimensions.

Ex. 4. In Ex. 2, if the $(n-1)$ folds be $x_1 = 0, \ldots, x_{n+2} = 0$, the centre of projection, T, being such that $x_1 = x_2 = \ldots = x_{n+2}$, where $\Sigma a_r x_r = 0$, $\Sigma a_r = 0$, shew that there is polarity in regard to the cone $\Sigma a_r x_r^2 = 0$, or $\Sigma a_r u_r^2 = 0$, where $u_1, u_2, \ldots$ are such that any $(n-1)$ fold through T is $\Sigma a_r u_r x_r = 0$. Further, that the point P_m is $(h_1 \ldots h_{m-1} \sigma_m h_{m+1} \ldots h_{n+2})$, where $\Sigma a_r h_r = a_m (h_m - \sigma_m)$, and $h_1 \ldots h_{n+2}$ are arbitrary. Consider the cases $n = 2$, $n = 3$, $n = 4$.

Ex. 5. The aggregate of the points $P_1 \ldots P_{n+2}$, and the points $P_r - P_s$, is a figure, in space $[n]$, of the same kind as the original figure in $[n-1]$. [Add.]

NOTE II

ON THE *HEXAGRAMMUM MYSTICUM* OF PASCAL

If we have in a plane two triads of points, U, V, W and U', V', W', which are in perspective, the joins UU', VV', WW' meeting in a point, and the intersections of the pairs of lines, VW and $V'W'$, WU and $W'U'$, UV and $U'V'$ being in line, then the remaining intersections of the lines VW, WU, UV with the lines $V'W'$, $W'U'$, $U'V'$ are six points which lie on a conic, as we know from what we have in the text called the converse of Pascal's theorem; and, if we take these six points in any order, the triad formed by one set of alternate joins of these six points is in perspective with the triad formed by the other set of alternate joins, as we know from Pascal's theorem. The axes of perspective of these various pairs of triads are called Pascal lines; they are sixty in number. They cointersect in sets of three (or of four) in various ways, and these points of intersection lie in sets upon lines, as has gradually appeared from the work of many mathematicians, Steiner, Kirkman, Cayley, Salmon, Veronese, Cremona, and others. The proof of the properties in the plane, though interesting, is intricate. It depends upon successive applications of Desargues' theorem, which we have learnt to regard as deduced from Propositions of Incidence in three dimensions; and it is the fact that all the results which have elicited attention are such as may be obtained by projection from a figure existing in such a space; this manner of proof is much simpler than the proof in the plane. Applied by Cayley in a particular way (*Coll. Papers*, VI, pp. 129–134 (1868)), it was applied by Cremona to a figure more general than Pascal's (*Memorie d. r. Acc. d. Lincei*, I, 1877, pp. 854–874). Three distinct points of Cremona's treatment of the figure in three dimensions are, (*a*) That it depends upon a configuration of fifteen lines lying in threes in fifteen planes, of which three pass through each line, (*b*) That this configuration is deducible from a figure of six planes, forming, in Cremona's phrase, the *nocciolo* of the whole, (*c*) That when the equations of these planes are introduced, only one of the identical relations connecting these comes into consideration. It follows from the last that we may regard the six loci as being in a space of four dimensions. This procedure, which Richmond has used, leads to a very simple treatment of the whole matter, which is the more interesting because the figure in four dimensions is one which, as we shall see in a subsequent volume, is of fundamental importance. Cremona's figure of fifteen lines lying by threes in fifteen planes is an example of the (3, 3) correspondence which it is possible to set up (in various ways) between any two sets of fifteen things; this had been considered by Sylvester, in connexion with the problem of naming a function of six letters capable only of six values under permutation of the letters (Sylvester, *Coll. Papers*, I, p. 92 (1844); II, p. 265 (1861)); it is pertinent and instructive for the problem in hand to consider this. In the following note we have (1) dealt with this problem of arrangements, (2) shewn how this can be represented by figures in four, three and two dimensions, (3) pointed out the properties of the figures which lead to the classical properties of the Pascal figure, (4) explained the exact figure, relating to a cubic surface with a node, by which Cremona was led to the generalisation, (5) given some examples of the proof in the plane of the properties of Pascal's figure, and of the configurations arising from some selected portions of the figure in three dimensions, and (6) made reference to the more important original authorities. The reader may prefer to read (4) before the other sections.

(1) Consider six elements, which we denote, at present, by 1, 2, 3, 4, 5, 6. These can be arranged in three pairs, for example 12, 34, 56, wherein, in each pair, the order of the elements is indifferent, and the order of the pairs, in each such set of three pairs, is also indifferent. Such a set of three pairs, involving all the elements, is what was called by Sylvester a *syntheme*, each of the pairs being what he called a *duad*. The total number of possible duads is fifteen; this is also the total number of possible synthemes. It is possible to choose a set of five synthemes which contain, in their aggregate, all the fifteen duads. Such a set of synthemes is called a system. The total number of possible systems is six; if these be all taken, each containing five synthemes, every syntheme will occur twice, in different systems; as has been said, each duad occurs once in each system. Such an arrangement was given by Sylvester (see below). If we assign names to the systems, say P, Q, R, P', Q', R', each syntheme, as occurring in two systems, will correspond to two of these six letters. Thus every one of the fifteen pairs which can be formed by two of these six letters, say, a letter-duad, will correspond to three number-duads, chosen from the possible fifteen duads of numbers, forming a syntheme. The converse is also true; any duad of the numbers being taken, the remaining four numbers can be taken in pairs in three ways; thus any duad occurs in three synthemes, and each of these synthemes enters in two of the systems. The duad thus corresponds to three pairs of systems; or any one of the fifteen number-duads corresponds to three letter-duads, chosen from the possible duads of P, Q, R, P', Q', R'. Therefore, as any

fifteen things can be identified by naming them after the pairs which can be formed from six arbitrary symbols, we can have a (3, 3) correspondence between any two sets of fifteen things. As has been said, the importance of this for the present purpose was emphasized by Cremona (*loc. cit.* pp. 854, 866, 870). A further remark which is of use is that two synthemes which have no duad in common occur together in only one system, since two systems have only one syntheme in common. Two such synthemes thus serve to identify a system. [Add.]

Of the various possible ways of assigning the names $P, Q, \ldots, R'$ to the systems, one example is given in the following scheme, which can be read either in rows or columns.

	P	Q	R	P'	Q'	R'
P		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
Q	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
R	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
P'	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
Q'	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
R'	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

With this table, the correspondence of the pairs of systems with the synthemes can be enumerated at once; for instance, (Q, R) is associated with (15.26.34), and (R, P') with (14.23.56). Conversely, the duads of the numbers are each associated with three duads of letters; for example, (14) with $(PQ \,.\, RP' \,.\, Q'R')$, since (14) occurs in the same syntheme in P and Q, in the same, other, syntheme in R and P', and in the same, still other, syntheme in Q' and R'. This set of three pairs of letters forms a syntheme, and the synthemes of the letters can similarly be arranged in six systems, with the names 1, 2, 3, 4, 5, 6.

We shall also denote the number-synthemes by the letters $a, b, c, d, a', b', c', d', e, l, m, n, p, q, r$, by the rule which is found by identifying the preceding scheme with the scheme subjoined:

	P	Q	R	P'	Q'	R'
P		e	d	a	b	c
Q	e		d'	a'	b'	c'
R	d	d'		l	m	n
P'	a	a'	l		r	q
Q'	b	b'	m	r		p
R'	c	c'	n	q	p	

Perhaps the easiest way to describe these notations is by the diagram given below in (2), p. 225, or, in another form, in the Frontispiece of the volume.

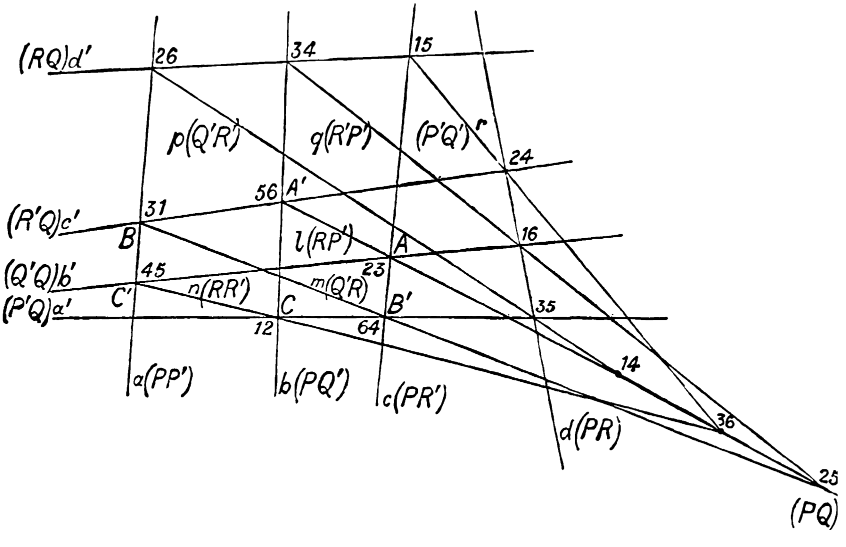
Now let us consider four of the six systems, considered as two pairs, say, for definiteness Q, R' and Q', R. The synthemes common to the two pairs, that is the syntheme common to Q and R', and the syntheme common to Q' and R, will necessarily have a duad in common. Further this aggregate of two letter-duads, Q, R' and Q', R, may be identified by the common number-duad of the two synthemes, taken with one duad from the first syntheme, and one duad from the second syntheme, so chosen as to have one of its numbers the same as one of the numbers of its companion duad from the first syntheme. In the particular case chosen, Q and R' have common the syntheme 13 . 24 . 56, while Q' and R have common the syntheme 13 . 25 . 46. These synthemes have the duad 13 common; we say that the combination $QR' . Q'R$ may be identified by the symbol 13 . 24 . 25, or by 13 . 42 . 46, or by 13 . 56 . 52, or by 13 . 65 . 64. All this is quite easy to see. For first, two synthemes, that have two duads identical, are themselves identical; and second, two synthemes that have no duad in common both occur in the same system, and nowhere else, as we have remarked. The two synthemes chosen, the former common to one pair of systems, the latter common to another pair, must therefore have a duad common. Using $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ to denote the numbers $1, 2, \ldots, 6$, in some order, let these synthemes be $\beta'\gamma' . \alpha\beta . \gamma\alpha'$ and $\beta'\gamma' . \alpha\gamma . \beta\alpha'$. From these we can form the symbol $\beta'\gamma' . \alpha\beta . \alpha\gamma$ (as well as three others). Conversely, if this be given, the formation of the two synthemes having $\beta'\gamma'$ in common and containing, respectively, also $\alpha\beta$ and $\alpha\gamma$, is without ambiguity; so that the symbol identifies the two pairs of systems from which it was formed. Such a symbol, or any one of the other three symbols which identify the same pair of systems, we speak of as defining a T-element. The total number of T-elements is thus $\frac{1}{4} . 15 . 4 . 3$, or forty-five. Now take the six systems in any particular order, say, $PQ'RP'QR'$, where it is to be understood that this is considered equivalent with its reverse order, $R'QP'RQ'P$, and may be named beginning with any one of its letters, as for instance by $RP'QR'PQ'$; thus the total number of orders is $(6!)/2 . 6$, or sixty. Then take, with each pair of letters in this order, that pair which is not contiguous with it on either side; thus, with QR' take $Q'R$, with RP' take $R'P$, and with $P'Q$ take PQ'. For each of these three sets of two pairs, form the corresponding T-element. It will be found that these can be represented by symbols of the respective forms $\beta'\gamma' . \alpha\beta . \alpha\gamma$, $\gamma'\alpha' . \alpha\beta . \alpha\gamma$, $\alpha'\beta' . \alpha\beta . \alpha\gamma$. For instance, with the order $PQ'RP'QR'$, the two pairs $QR', Q'R$ give a symbol 13 . 64 . 65, the two pairs $RP', R'P$ give a symbol 23 . 64 . 65, and the two pairs

PQ', $P'Q$ give a symbol $12 . 64 . 65$. And these three symbols have in common the duads $64 . 65$, which have the number 6 in common. Conversely consider the aggregate of two duads $\alpha\beta . \alpha\gamma$, having the number α in common; we shew that this leads to a particular order of the symbols $P, Q, \ldots, R'$. This aggregate belongs to only three symbols of T-elements, namely $\beta'\gamma' . \alpha\beta . \alpha\gamma$, $\gamma'\alpha' . \alpha\beta . \alpha\gamma$ and $\alpha'\beta' . \alpha\beta . \alpha\gamma$. Of these the first arises by combining the two synthemes $\beta'\gamma' . \alpha\beta . \gamma\alpha'$, $\beta'\gamma' . \alpha\gamma . \beta\alpha'$, and there is no ambiguity as to these; for the moment let these synthemes be denoted, respectively, by p and p'. The symbol $\gamma'\alpha' . \alpha\beta . \alpha\gamma$ similarly arises from the two synthemes $\gamma'\alpha' . \alpha\beta . \gamma\beta'$, $\gamma'\alpha' . \alpha\gamma . \beta\beta'$, which we denote, respectively, by q and q'. Lastly, the symbol $\alpha'\beta' . \alpha\beta . \alpha\gamma$ arises from the two synthemes $\alpha'\beta' . \alpha\beta . \gamma\gamma'$, $\alpha'\beta' . \alpha\gamma . \beta\gamma'$, which we denote, respectively, by r and r'. We have, however, remarked that two synthemes, which have no duad in common, determine a particular system. Thus the synthemes q and r', namely $\gamma'\alpha' . \alpha\beta . \gamma\beta'$ and $\alpha'\beta' . \alpha\gamma . \beta\gamma'$, determine one of the systems $P, Q, \ldots, R'$; this system we may, for a moment, denote by (qr'). It is easy to see that the whole of the six systems are thus determined by such pairs of the six synthemes, which we may arrange in the order $pq' . q'r . rp' . p'q . qr' . r'p$, there being no other pairs of these synthemes which have no duad in common. So arranged, every consecutive pair of these systems determines, with the pair which is not contiguous with it on either side, a T-element. For instance, if we take the systems pq', $q'r$, $p'q$, qr', the first pair have common the syntheme q' or $\gamma'\alpha' . \alpha\gamma . \beta\beta'$, the second pair have common the syntheme q or $\gamma'\alpha' . \alpha\beta . \gamma\beta'$, and these together determine the T-element represented by $\gamma'\alpha' . \alpha\beta . \alpha\gamma$.

We see then that the operation which we carry out when, in Pascal's figure, we form the Pascal line for six points $P, Q, \ldots, R'$ of a conic, taken in a particular order, can be carried out exactly with the symbols, the systems of synthemes taking the places of the points of the conic. The join of any two of the six Pascal points is replaced by one of the fifteen number-synthemes; the intersection of two of these joins is replaced by a T-element, represented by such a symbol as $\beta'\gamma' . \alpha\beta . \alpha\gamma$; and the Pascal line containing three of these intersections is replaced by a symbol, such as $\alpha\beta . \alpha\gamma$, consisting of two duads having a number (α) in common. We have seen that any such symbol leads back to a particular order of the six letters $P, Q, \ldots, R'$. It is in accordance with this, that the number of possible symbols $\alpha\beta . \alpha\gamma$ is $6 . 10$ or sixty. It is at once seen in the Pascal figure that through the intersection of two opposite sides of the hexagon there pass four Pascal lines, the two pairs QR', $Q'R$, for example, being non-contiguous in each of the four hexagons $PQ'RP'QR'$, $PQ'RP'R'Q$, $PRQ'P'QR'$, $PRQ'P'R'Q$; this corresponds to the fact remarked that the T-element $\beta'\gamma' . \alpha\beta . \alpha\gamma$ is capable also of the representations $\beta'\gamma'.\gamma\alpha'.\alpha\gamma$, $\beta'\gamma'.\alpha\beta.\beta\alpha'$, $\beta'\gamma'.\alpha'\gamma.\alpha'\beta$. The total intersections of the fifteen sides in the Pascal hexagon, in number $\frac{1}{2}15.14$ or 105, consist in fact of ten intersections at each of the six vertices $P, Q, \ldots, R'$, together with an intersection at each of forty-five T-points.

(2) We now consider a geometrical interpretation of the relations we have described. First, in four dimensions, we can set up a figure of fifteen points lying in threes on fifteen lines, of which three pass through each of the points, beginning in various ways. If we take six general points, say F, G, H, R, S, T, of which the symbols will be subject to one relation, which we write in the form $F+G+H+R+S+T=0$, there will be fifteen points of symbols each the sum of two of $F, G, \ldots, T$, and, for instance, the points $F+G$, $H+R$, $S+T$ will be in line; of such lines there will be fifteen. Three of these will pass through each point; for instance, the lines $(F+G, H+R, S+T)$, $(F+G, H+S, R+T)$, $(F+G, H+T, R+S)$ pass through the point $F+G$. Denoting the six points $F, G, \ldots, T$ by 1, 2, ..., 6, each duad, such as 12, may be supposed to represent one of these points, and each syntheme, such as 12.34.56, to represent one of these lines. Or, we may take four arbitrary lines of general position, say, a, b, c, d; every three of these will have, in four dimensions, a single transversal; let the transversal of a, b, c be denoted by d', that of b, c, d by a', and so on, the four transversals being a', b', c', d'. It is then easy to shew that the line, say n, joining the points (a, b'), (a', b) intersects the line, say r, joining the points (c, d'), (c', d); so, the line joining the points (c, a'), (c', a), say m, meets the line joining the points (b, d'), (b', d), say q; and likewise the line joining the points (b, c'), (b', c), say l, meets the line joining the points (a, d'), (a', d), say p. And further that these three points of intersection are in a line, say e. Six fundamental points $F, G, \ldots, T$, from which these statements can be justified, are obtained by regarding the points (b', c), (c', a), (a', b), (b, c'), (c, a'), (a, b') as being, respectively, the points 23, 31, 12, 56, 64, 45. These six points may in fact be taken arbitrarily to obtain such a figure; denoting them, respectively, by A, B, C, A', B', C', the six consecutive joins of the hexagon $BC'AB'CA'$ are the lines a, b', c, a', b, c'; the lines d, d' are then, respectively, the common transversals of a', b', c', and of a, b, c, and the figure can be completed. The fundamental points are then $F=\frac{1}{2}(B+C-A)$, $R=\frac{1}{2}(B'+C'-A')$, etc. The various relations and notations are represented by the diagram annexed; the fundamental importance of the figure justifies this lengthy description. We shall for clearness denote this figure by Ω. It depends on twenty-four constants. We do not now consider the dually corresponding figure in four dimensions. We consider however the figure obtained by projecting the figure Ω into space of

three dimensions, which we denote by S. It can be constructed, as in the diagram annexed, or as in the diagram given in the Frontispiece, by taking four points which lie in a plane, those denoted by 23, 35, 56, 62; then drawing, through 23, the two arbitrary lines b' and c, and, through 56, the two arbitrary lines b and c'; then, from 35, drawing the line d to meet b' and c', and the line a' to meet b and c, and also, from 62, drawing the line d' to meet b and c, and the line a to meet b' and c'. The remaining incidences of the figure then follow necessarily[1], as the reader may easily see. This figure depends on nineteen constants; so that there are ∞^1 possible figures when the six fundamental points, from which it can be constructed, are given. The figure in three dimensions which is the dually corresponding to S will be denoted by S'; it is the figure mainly considered by Cremona (*loc. cit.*), containing fifteen lines lying in threes in fifteen planes, of which three pass through each line. Finally we consider the figure obtained by projecting the figure S' on to a plane, which we denote by ϖ; it is in this figure, ϖ, that the various incidences which arise from Pascal's hexagram are found, and for a figure more general than Pascal's. This plane figure consists of two quadrilaterals, say $ABCD$, $A'B'C'D'$, the intersections of corresponding sides being X of AB and $A'B'$, Y of BC and $B'C'$, Z of CD and $C'D'$ and U of DA and $D'A'$, which are such that the points of intersection, O of XZ and YU, P of AC' and $A'C$ and



---

[1] A proof of this is given, *Proc. Roy. Soc.* LXXXIV, 1911, p. 599.

Q of BD' and $B'D$, are in line. This figure depends on fifteen constants; so that there are ∞^3 possible figures when the six fundamental points, from which it can be constructed, are given.

The correspondence between the figures and the Pascal configuration can now be described precisely, as in the preceding section. If we take a particular order of the systems, say $PQ'RP'QR'$, and consider either the figure Ω, or the figure S, it is clear that the lines $Q'R$, QR', or m and c', are those which join the points 64, 65 to the point 13, the lines RP', $R'P$, or l and c, are those which join the points 65, 64 to the point 23, and the lines $P'Q$, PQ', or a' and b, are those which join the points 64, 65 to the point 12. Thus we have three T-planes passing through the line 64 . 65. In the figure S', dually corresponding to S, we have correspondingly three T-points lying on a line. Projection of S' into the figure ϖ gives then three T-points lying on a line, as in the Pascal configuration. More particularly, each of the systems P, Q', ... consists of five lines, both in the figure S and the figure S'; the systems Q, R' have a definite line in common, as also have the systems Q', R. In order to project S' into ϖ, we must take a definite centre of projection, say O; from this, a transversal can be drawn to these two lines QR' and $Q'R$; it is the intersection of this with the plane of ϖ which gives the T-point. The consecutive lines obtained by the projections of the lines PQ', $Q'R$, RP', $P'Q$, QR', $R'P$ on to the plane of ϖ, similarly give six lines forming a hexagon; the three points of intersection of the pairs of opposite sides of this lie on a line. The vertices of the hexagon in the plane of ϖ therefore lie on a conic. If we take another order of the systems, for instance, $PQ'RP'R'Q$, and project from the same point O, we obtain another hexagon, whose vertices lie on a conic, with its Pascal line. This hexagon will not coincide with the former; in the particular instance, we shall have, instead of the vertex obtained by the intersection of the lines which are the projections of RP' and $P'Q$, the vertex obtained by the projections of the lines RP' and $P'R'$; and so on. As we shew in section (4) below, it is however possible so to specialise the figure as to obtain always the same conic in the plane ϖ.

(3) Dealing now with the figure Ω, or S, we have the fifteen points such as 12, which we may call the Cremona points; we also have the fifteen lines, each containing three of these points whose symbols form a syntheme, such as 12 . 34 . 56. Any two of the Cremona points whose symbols have a number in common may be joined by a line; for instance the line joining the points 12 and 13 is such a line. These lines we call the Pascal lines; their number is sixty. A plane containing three Cremona points whose duad symbols are formed with five of the six numbers, as for instance the plane 23 . 64 . 65, is called a T-plane; it contains four Pascal lines (64 . 65;

46 . 41 ; 56 . 51 ; 14 . 15), and three such planes pass through any Pascal line. In all there are forty-five such planes. There are however also sets of four Pascal lines which meet in a point. For instance the points 64, 61 are those which, in terms of the six fundamental points F, G, H, R, S, T, have the symbols $T+R$, $T+F$. The line joining these contains the point $R-F$; this we denote by [14]. Evidently there are fifteen such points, which we call Plücker points; and, for instance, through the point [14], there pass the four Pascal lines 21 . 24, 31 . 34, 51 . 54 and 61 . 64. The Plücker points lie in threes upon twenty lines; for instance the points [23], [31], [12], whose symbols are, respectively, $G-H$, $H-F$, $F-G$, lie upon a line. Such a line is called a *g*-line, or a Cayley-Salmon line. The Plücker points also lie in sixes upon fifteen planes. For instance the points [63], [64], [65], [45], [53], [34] lie on a plane. Such a plane is called an *I*-plane, or a Salmon plane. The Pascal lines lie in threes in planes, in two distinct ways. A plane containing three Cremona points whose duad symbols contain only three of the numbers, for instance the plane of the points 23, 31, 12, evidently contains the three Pascal lines 12 . 13, 23 . 21, 31 . 32. Such a plane is called a *G*-plane, or a Steiner plane; the total number of such planes is twenty. Again a plane containing three Cremona points whose duad symbols all contain one number in common, for instance the plane of the points 41, 42, 43, evidently contains the three Pascal lines 42 . 43, 43 . 41, 41 . 42. Such a plane is called a *K*-plane, or a Kirkman plane; the total number of such planes is sixty. Lastly, it is necessary to refer to fifteen lines, each joining a Cremona point to the Plücker point whose symbol is formed with the same numbers, for instance the line joining 12 to [12]. Such a line is called an *i*-line, or a Steiner-Plücker line. The Pascal lines will also be called *k*-lines. There are certain theorems of incidence among the various elements in addition to those which have been referred to. For instance, a Pascal line, or *k*-line, lies in one *G*-plane, and in three *K*-planes; this fact will be denoted, in the summary we now give, by writing *k*...*G*, 3*K*. Conversely a *K*-plane contains, not only three *k*-lines, as we have seen, but also one *g*-line; this fact will be denoted by writing *K*...*g*, 3*k*. We may then summarise the definitions, and the incidences referred to, which will be immediately proved, as follows: [Add.]

60 *k*-lines, Pascal lines, 12, 13,
60 *K*-planes, Kirkman planes, 14, 15, 16;

20 *g*-lines, Cayley-Salmon lines, [12], [13],
20 *G*-planes, Steiner planes, 56, 64, 45;

15 *i*-lines, Steiner-Plücker lines, 12, [12],
15 *I*-planes, Salmon planes, [63], [64], [65].

$k \ldots G, 3K$; $g \ldots G, 3K, 3I$; $i \ldots 4G$,
$K \ldots g, 3k$; $G \ldots g, 3k, 3i$; $I \ldots 4g$.

The proof of these last theorems of incidence is immediate:

(1) The k-line 12, 13 lies in the G-plane 23, 31, 12; and in the K-planes (12, 13, 14), (12, 13, 15), (12, 13, 16).

(2) The K-plane 14, 15, 16 contains the g-line ([56], [64], [45]), and the k-lines 15 . 16, 16 . 14, 14 . 15.

(3) The g-line [12], [13] lies in the G-plane (23, 31, 12), it lies in the K-planes (41, 42, 43), (51, 52, 53), (61, 62, 63), and it lies in the I-planes ([12], [13], [14]), ([12], [13], [15]), ([12], [13], [16]).

(4) The G-plane 56, 64, 45 contains the g-line [56], [64], [45], the k-lines 45 . 46, 56 . 54, 64 . 65, and also the i-lines (56, [56]), (64, [64]), (45, [45]).

(5) The i-line (12, [12]) lies in the G-planes (12, 23, 31), (12, 24, 41), (12, 25, 51), (12, 26, 61).

(6) The I-plane ([63], [64], [65]) contains the g-lines ([63], [64], [34]), ([64], [65], [45]), ([65], [63], [35]), ([45], [53], [34]).

It is interesting, however, further, to remark, that the relations expressed by

$$K \ldots g, 3k; \quad G \ldots g, 3k, 3i; \quad I \ldots 4g$$

can be verified by considering only a single threefold space which forms part of Ω, there being in Ω fifteen such spaces; and the relations expressed by

$$k \ldots G, 3K; \quad g \ldots G, 3K, 3I; \quad i \ldots 4G$$

can be verified by considering only elements passing through a single point of Ω, there being fifteen such points, which correspond dually in Ω to the threefold spaces just spoken of. For if we consider the space determined by four of the six fundamental points, say by G, H, S, T, which contains the points 23, 25, 26, 35, 36, 56, [23], [25], [26], [35], [36], [56], it is at once seen that this contains

four K-planes, four G-planes, one I-plane;

it contains also

twelve k-lines, four g-lines, six i-lines.

Correspondingly, consider elements passing through the Plücker point [14]; there are of such (cf. the Frontispiece of the volume)

four k-lines, four g-lines, one i-line;

any plane of the complete figure Ω which contains one of these lines necessarily contains the point [14]. There pass in all through this point

twelve K-planes, four G-planes, six I-planes;

one such K-plane is (21, 24, 23); one such G-plane is (21, 24, 14); one such I-plane is ([14], [15], [16]).

And in fact, if we introduce the equations to the spaces and planes of the figure Ω, we can make the duality suggested by the notation apparent analytically.

If now we suppose that in place of the figure S we consider the figure S', dually corresponding to S in three dimensions, we shall have therein k-lines, g-lines and i-lines, but we shall have K-*points*, instead of K-planes, G-points and I-points. If then we project on to a plane, obtaining the figure ϖ, we shall have therein lines from the lines of the figure S', and points from the points of S'. And therein the six theorems of incidence which we have obtained will continue to hold.

It is this fact which is the outcome of the many investigations made for Pascal's figure, of which the figure ϖ is a generalisation.

(4) We shew now how the exact Pascal's figure can be obtained from a figure in three dimensions; and it will appear at once that this figure is a particular case of the figure S' which we have considered.

Let x, y, z, t be coordinates in three dimensions. Let U be a homogeneous polynomial of the second order in x, y, z only, and M be a homogeneous polynomial of the third order in x, y, z only. Thus $U=0$ may be regarded as the equation to a conic, in the plane whose equation is $t=0$; similarly $M=0$ may be regarded as the equation to a curve in this plane which has the property of being met, by an arbitrary line of this plane, in three points, and is therefore said to be a curve of the third order, or a cubic. This cubic meets the conic in six points, as follows from the elements of the theory of elimination. These points we name, in some order, P, Q', R, P', Q, R'; a cubic curve can be drawn through nine arbitrary points, so that, by proper choice of M, we may regard these six points as arbitrary points of the conic $U=0$. Now consider the equation $M-tU=0$, which is homogeneous in the four coordinates x, y, z, t. It represents a surface which is met by an arbitrary line of the threefold space in three points, say, a cubic surface. But, in particular, any line of the threefold space which is drawn through the point of coordinates (0, 0, 0, 1), say, the point D, meets the surface in two coincident points at D, and in a further point; for this reason the point D is called a node of the cubic surface. More particularly, however, there are lines which lie entirely upon the cubic surface: for instance, whatever θ may be, if $(x, y, z, 0)$ be one of the six points for which $U=0$ and $M=0$, the point (x, y, z, θ) lies on the cubic surface, since its coordinates satisfy $M-\theta U=0$, and, for different values of θ, this point is any point of the line joining D to $(x, y, z, 0)$. Let the six lines so obtained, joining D to the six points $P, Q', \ldots,$ be named, respectively, p, q', r, p', q, r'; it is easy to see that these are the only lines passing through D which lie on the cubic surface. Then, further, it is at once clear that any plane meets the cubic surface in a cubic curve; thus a plane containing two of the lines $p, q', \ldots, r'$, will contain a further line lying entirely on the cubic surface. Thereby we obtain fifteen further lines of the surface, any one of which may be denoted by a duad symbol, such as qr'. There are however no other lines lying on the surface. For we have enumerated all those which pass through D; suppose then one which does not pass through D, and consider the curve in which the cubic surface is met by the plane joining D to this line; this consists of the line, and a curve of the second order; but this curve of the second order has the property of being met in two points coinciding at D by every line in the plane drawn through D, and must therefore itself consist of two lines intersecting at D. The plane is therefore one of those before considered, drawn through two of the lines $p, q', \ldots, r'$. Now consider the three lines of the surface, q, r' and qr', which lie in a plane. The first line, q, is evidently met by the four lines qp, qr, qp', qq', in addition to qr'; the other line, r', is evidently met by the four lines $r'p$, $r'r$, $r'p'$, $r'q'$, in addition to $r'q$. Beside the four lines which meet both q and r' at the point D, there remain then, of the total twenty-one lines of the surface, just six, namely those whose symbols are the duads from p, r, p', q'. These six lines do not lie in the plane of q, r' and qr'; each must meet, then, either q, or r', or qr'; but each meets two lines, other than q and r', of the lines of the surface which pass through D; it cannot, then, for example, meet also the line q, since else three of the lines through D would lie in a plane, and the conic $U=0$ would break into two lines, which we suppose not to be the case. Thus these six lines all meet the line qr'.

Consider, for instance, rp'; the plane containing qr' and rp' must then meet the cubic surface in another line, by an argument applied above; this other line, meeting both qr' and rp', will have for symbol a duad not containing either q or r', or r or p'; this symbol is, then, pq', the symbols of the three lines qr', rp', pq' containing all the six symbols p, q', r, p', q, r'. In this way we see that there are, beside the plane containing q and r', three planes through the line qr' each containing two other lines with duad symbols. Whence we see that the fifteen lines of the cubic surface, other than those through D, lie in threes in fifteen planes, of which three planes pass through every one of these lines.

If desired, the equations of the fifteen lines can be obtained, when the six points of intersection of the conic $U=0$, and the cubic $M=0$ are given. For if $ax+by+cz=0$, with $t=0$, be the equation of the join of two of these six points, and we suppose, for instance, that c is not zero, as the plane joining D to this line contains a line of the cubic surface, the substitution for z, in the quotient M/U, of $-(ax+by)/c$, must reduce this to a form $lx+my$; the line in question will then be given by the two equations $ax+by+cz=0$, $t-lx-my=0$.

Having obtained the figure of fifteen planes, and the lines lying in threes of these, we can denote any one of the planes by a duad formed from two of the six numbers 1, 2, 3, 4, 5, 6, and the line of intersection of three of these planes by a syntheme of three duads, such as 12.34.56, formed from the duads which represent the three planes passing through the line. We shall then have a figure which, combinatorially, has the same properties as the figure S' considered above, specialised however by the fact that the fifteen lines are arranged in sets of five all of which have a common transversal passing through D, any of the lines meeting two of the transversals. These sets of five are the systems of synthemes considered above in (1). But the incidence theorems obtained above will hold for the specialised figure; and, after projection on to the plane $t=0$, these give the properties of the Pascal figure which are the occasion of the present note.

It may be remarked that on a general cubic surface, which does not possess a node, there are sets of fifteen lines such as those here denoted by the duad symbols qr', together with twelve others, each of the six lines here found passing through D being replaced by a pair of skew lines. Moreover, we shall find that the general figure Ω in four dimensions, here considered, is of fundamental importance for the theory of the general cubic surface.

(5) We now give some particular examples of the theory.

Ex. 1. *The pairs of conjugate Steiner planes.* Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ denote the numbers 1, 2, ..., 6, in any order. In the figure Ω, there pass, as we have seen, through each of the three Cremona points, $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$, three of the fifteen lines which are fundamental in the figure (beside two k-lines joining this point to the other two); and the nine lines so obtained meet in threes in the points $\beta\gamma, \gamma\alpha, \alpha\beta$. The two Steiner planes $(\beta'\gamma', \gamma'\alpha', \alpha'\beta')$ and $(\beta\gamma, \gamma\alpha, \alpha\beta)$ are called conjugate. In the Pascal figure there are two corresponding Steiner points, which are in fact conjugate to one another in regard to the fundamental conic. In the figure S', we have two Steiner points; through each of these there pass three Cremona planes, intersecting in pairs in three Pascal lines which pass through the point. The three planes through one of the two Steiner points meet the three planes through the other Steiner point in three triads of lines, of which any two are then in perspective, the axis of perspective being a Pascal line. By projection to the plane ϖ we obtain then a similar result, of which the elementary proof is indicated in more detail below. (Ex. 5. See also v. Staudt, *Crelle*, LXII, p. 142.)

Ex. 2. *The Steiner planes and Steiner-Plücker lines.* The plane determined by the Cremona points 23, 31, 12, is the same as that determined by the three points F, G, H, of the six fundamental points. The Steiner-Plücker line determined by joining the Cremona point 12 to the Plücker point [12], is that joining the two fundamental points F, G. The twenty Steiner planes are thus all the planes each containing three of the six fundamental points, and the fifteen Steiner-Plücker lines are all the lines each joining two of these points. In the figure S', the Steiner points are then the intersections in threes, and the Steiner-Plücker lines the intersections in twos, of six planes, which, so far as the configuration is concerned, may be taken arbitrarily. The corresponding points and lines in the Pascal figure are then the projection of this very simple configuration. (Cf. p. 218.)

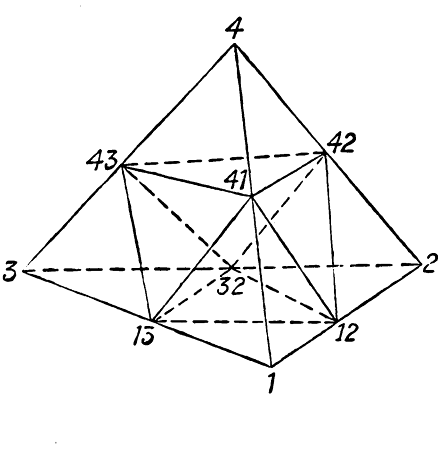
Ex. 3. *The separation of the complete figure into six figures.* In the figure Ω, the notation at once suggests that we consider together the set of five Cremona points such as 12, 13, 14, 15, 16; and there will be six such sets, each point belonging to two sets. The five points of a set are such that the join of any two is a Pascal line, and the plane of any three is a Kirkman plane. In the figure S', we have then six sets of five Cremona planes, of which any two meet in a Pascal line, and any three in a Kirkman point. In the plane ϖ, the sixty Pascal lines are thus divided into six sets of ten, meeting in threes in ten Kirkman points, each of the Pascal lines containing three of the Kirkman points. The configuration, in each of the six partial figures is that of two triads in perspective, with their centre and axis of perspective. These partial figures were considered by Veronese.

Ex. 4. *Tetrads of Steiner points each in threefold perspective with a tetrad of Kirkman points.* In the figure Ω, the six Cremona points 23, 31, 12, 41, 42, 43, which lie in the threefold space of the four fundamental points F, G, H, R, give four Steiner planes,

$\alpha=(24,43,32)$, $\beta=(34,41,13)$, $\gamma=(14,42,21)$, $\delta=(23,31,12)$, and four Kirkman planes,

$\alpha'=(12,13,14)$, $\beta'=(23,21,24)$, $\gamma'=(31,32,34)$, $\delta'=(41,42,43)$.

The planes α, α' meet in the



g-line ([23], [24], [34]), the planes β, β' meet in the g-line ([31], [34], [14]), the planes γ, γ' meet in the g-line ([12], [14], [24]), and the planes δ, δ' meet in the g-line ([12], [13], [23]); and these are the four g-lines which lie in the I-plane ([41], [42], [43]).

Hence if we consider the figure S', we have two tetrads of points, one formed by Steiner points, and the other by Kirkman points, which are in perspective, the lines joining corresponding points of the two tetrads being g-lines, and the centre of perspective being an I-point.

But in fact, in this figure S', the two tetrads are in perspective in three other ways. In this figure, $\alpha, \beta, \ldots, \alpha', \beta', \ldots$ are points; the lines (α, δ'), (β, γ'), (γ, β'), (δ, α') also meet in a point, namely the point (12, 13, [14]), or (12, 13, 56), a T-point. This fact we may denote by writing

$$12, 13, [14] \left\{\begin{matrix} 24, 43, 32 & 34, 14, 31 & 14, 24, 12 & 23, 31, 12 \\ 41, 42, 43 & 31, 32, 34 & 23, 21, 24 & 12, 13, 14 \end{matrix}\right.$$

The joins of corresponding points of these two tetrads are, respectively, the Pascal lines 42 . 43, 31 . 34, 21 . 24, 12 . 13.

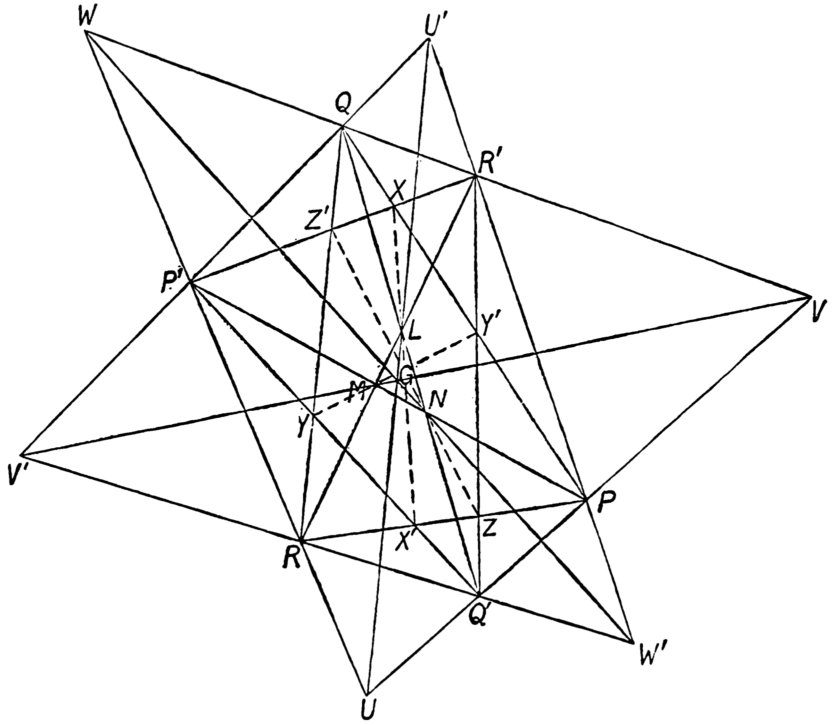
Similarly the tetrads $(\alpha, \beta, \gamma, \delta)$, $(\gamma', \delta', \alpha', \beta')$ are in perspective from the point (23, 21, [24]), or (23, 21, 56), and the tetrads $(\alpha, \beta, \gamma, \delta)$, $(\beta', \alpha', \delta', \gamma')$ are in perspective from (31, 32, [34]), or (31, 32, 56). The two given tetrads, and that formed by the four centres of perspective, form, therefore, what is known as a desmic system of three tetrads, of which the points can be supposed to have symbols of the form (cf. p. 213)

$$(P,Q,R,S),(-P+Q+R+S, P-Q+R+S, P+Q-R+S, P+Q+R-S),$$
$$(P+Q+R+S,\ P-Q-R+S,\ -P+Q-R+S,\ -P-Q+R+S).$$

Ex. 5. *Relation of the Cayley-Salmon lines and the Salmon planes.* The fifteen Plücker points in the figure Ω lie in threes on the twenty Cayley-Salmon lines, g-lines, and in threes also on the fifteen Salmon planes, or I-planes, forming a figure such as that occurring in the proof of Desargues' theorem for two triads in one plane (cf. p. 214).

The figure is evidently dual with itself, in threefold space, and has the same description in S' and in ϖ. Through each of the fifteen points there pass, in S or S', six planes and four lines, each of the twenty lines contains three points and lies in three planes, each plane contains four of the lines and six of the points. The figure has been considered in Note I (§ 3).

Ex. 6. Let U, V, W and U', V', W' be two triads of points in a plane, the lines UU', VV', WW' meeting in G; let the points $(UV, W'U')$, $(VW, U'V')$, $(WU, V'W')$ be denoted, respectively, by P, Q, R, and the points $(U'V', WU)$, $(V'W', UV)$, $(W'U', VW)$ be denoted, respectively, by P', Q', R'; so that, by the converse of Pascal's theorem, the six points P, Q', R, P', Q, R' lie on a conic. Prove, (1), that QQ', RR', UU' meet in a point, say, L; and, similarly, RR', PP', VV' meet in a point, say, M; and, likewise, PP', QQ', WW' meet in a point, say, N. Let the points $(PQ, R'P')$, $(QR, P'Q')$, $(RP, Q'R')$ be denoted, respectively, by X, Y, Z; and the points $(P'Q', RP)$, $(Q'R', PQ)$, $(R'P', QR)$ be denoted, respectively, by X', Y', Z'. Prove, (2), that X, L, X' are in line, as also Y, M, Y' and Z, N, Z'; and that



the lines XLX', YMY', ZNZ' meet in a point. Prove also, (3), that the lines YZ, PP', QR' meet in a point, as do the lines $Y'Z'$, PP', $Q'R$; that the lines ZX, QQ', RP' meet in a point, as do $Z'X'$, QQ', $R'P$; and that the lines XY, RR', PQ' meet in a point, as do the lines $X'Y'$, RR', $P'Q$. Thus any two of the triads U, V, W; L, M, N; X, Y, Z are in perspective, as also are any two of the triads U', V', W'; L, M, N; X', Y', Z', the lines XU, YV, ZW meeting in a point, as do $X'U'$, $Y'V'$, $Z'W'$.

Shew also, (4), that the lines UU', VV', WW' are Pascal lines each for a proper order of the six points P, Q', etc., on the conic, their point of meeting, G, being a Steiner point; and, (5), that the axis of perspective of the triads U, V, W and U', V', W' is a Pascal line;

that the axis of perspective of the triads U, V, W and L, M, N is a Pascal line, and that the axis of perspective of the triads U', V', W' and L, M, N is a Pascal line; and that these three Pascal lines meet in a Steiner point. Shew also, (6), that the lines XLX', YMY', ZNZ' are Pascal lines, meeting in a Kirkman point, that the lines XU, YV, ZW are Pascal lines, meeting in a Kirkman point, and that the lines $X'U'$, $Y'V'$, $Z'W'$ are Pascal lines, meeting in a Kirkman point.

These facts may all be obtained by application of Desargues' theorem.

They may also be obtained by shewing that, with that association of P, Q', R, etc. with the systems which has been adopted in this note, the points U, V, W, U', V', W' are the respective T-points (56, 21, 23), (56, 12, 13), (56, 31, 32), (64, 21, 23), (64, 12, 13), (64,31,32), so that the lines UU', VV', WW' are the respective Pascal lines (23,21), (12,13), (31, 32). Then that the points L, M, N are the respective T-points (45,23,21), (45,12,13), (45,13,23). Then that the points X, Y, Z are the respective T-points (25, 63, 61), (15, 62, 63), (35, 61, 62), and the points X', Y', Z' are the respective T-points (24, 61, 63), (14, 62, 63), (34, 61, 62), so that the lines XX', YY', ZZ' are, respectively, the Pascal lines (63, 61), (62, 63), (61, 62) and the lines YZ, $Y'Z'$ are, respectively, the Pascal lines (24, 26), (25, 26), with similar results for ZX, $Z'X'$, XY, $X'Y'$. The points (QR', PP'), (RP', QQ'), (PQ', RR') are, respectively, the T-points (31, 56, 54), (23, 56, 54), (12, 56, 54); the lines XU, $X'U'$, YV are, respectively, the Pascal lines (41, 43), (51, 53), (42, 43), and so on.

(6) For the literature of the matter, the reader may consult:

Pascal, *Essai pour les coniques* (1640), *Oeuvres*, Brunschvigg et Boutroux, Paris, 1908, i, p. 245.

Brianchon, *Mémoire sur les lignes du 2^e ordre* (1806).

Steiner, *Gergonne's Ann.* xviii (1828), *Ges. Werke*, i, p. 451 (1832).

Plücker, *Ueber ein neues Princip der Geometrie*, *Crelle*, v (1830), p. 274. Also *Crelle*, xxxiv (1847), p. 337.

Hesse, *Crelle*, xxiv (1842), p. 40; *Crelle*, xli (1851), p. 269; *Crelle*, lxviii (1868), p. 193, and *Crelle*, lxxv (1873), p. 1.

Cayley, various notes, beginning 1846 (*Coll. Papers*, i, pp. 322, 356); see particularly *Papers*, vi (1868), p. 129.

Grossmann, *Crelle*, lviii (1861), p. 174.

Von Staudt, *Crelle*, lxii (1863), p. 142.

Bauer, *Abh. der k. bayer. Ak. d. Wiss.* (Munich) xi (1874), p. 111.

Kirkman, *Manchester Courier*, June and August, 1849; *Camb. and Dub. Math. Jour.* v (1850), p. 185.

Veronese, *Memorie d. r. Acc. d. Lincei*, i (1877), pp. 649–703; and Cremona, in the same volume, pp. 854–874.

Sylvester, *Coll. Papers*, i, p. 92, and ii, p. 265 (1844 and 1861).

NOTE III

IN REGARD TO THE LITERATURE FOR NON-EUCLIDEAN GEOMETRY

The following references may be of use, as indications of ways in which the introduction given in Chapter v may be amplified:

R. Bonola, *Non-Euclidean geometry, a critical and historical study*, English by H. S. Carslaw (Open Court Publishing Co., Chicago, 1912, pp. 1–268).

R. Bonola, *Sulla teoria delle parallele e sulle geometrie non-euclidee*, in Vol. i, pp. 248–362, of Enriques' *Questioni riguardanti le matematiche elementari*, 1912.

D. M. Y. Sommerville, *The elements of non-Euclidean geometry*, 1914, pp. 1–274; and, by the same author, *Bibliography of non-Euclidean geometry*, St Andrews, 1911, pp. 1–403.

Simon Newcomb, *Elementary theorems relating to...space...of uniform...curvature in the fourth dimension*, Crelle's Journal, lxxxiii, 1877, pp. 293–299.

H. Helmholtz, *Ueber die Thatsachen die der Geometrie zu Grunde liegen*, 1868, Ges. Wiss. Abhand., Band ii.

B. Riemann, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, 1854, Ges. Werke.

F. Enriques, *Prinzipien der Geometrie*, Enzykl. der Math. Wiss. iii. 1. 1.

F. Klein, *Nichteuklidische Geometrie*, Göttingen, 1913; *Elementarmathematik*, 1913; many papers in the *Math. Annal.*; and *Mathematical Works*, in particular Vol. i, 1921, p. 252.

D. Hilbert, *Grundlagen der Geometrie* (First Edit. 1899).

E. Schur, *Grundlagen der Geometrie*, 1909.

N. I. Lobatschewsky, *Geometrische Untersuchungen zur Theorie der Parallellinien*, 1840 (facsimile, Mayer u. Müller, 1887).

Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles, 1855 (reprint, Ostwald's Klassiker, 1902).

Geometrical works of Lobatschewsky (Kasan, 1883–1886).

F. Engel, *N. I. Lobatschewsky, zwei geometrische Abhandlungen..., mit einer Biographie*, 1898, pp. 1–476.

W. Bolyai, *Tentamen juventutem studiosam in elementa matheseos purae* (1832), with appendix by J. Bolyai, *Appendix scientiam spatii absolute veram exhibens*.

J. Frischauf, *Absolute Geometrie nach Johann Bolyai* (1872).

P. Stäckel, *W. u. J. Bolyai, geometrische Untersuchungen*, 1913, two parts, each pp. 1–281 (see also Stäckel, *Math. u. Naturw. Ber. aus Ungarn*, xvii, 1901, and xviii, 1902. Also Stäckel u. Engel, *Math. Annal.* xlix, 1897, p. 149).

F. Engel u. P. Stäckel, *Die Theorie der Parallellinien von Euclid bis auf Gauss*, 1895, pp. 1–325.

C. A. F. Peters, *Briefwechsel zwischen K. F. Gauss u. H. C. Schumacher* (6 vols. 1860–1865). See Gauss, *Ges. Werke*.

Cayley, *Papers*, ii, p. 583; xiii, p. 480.

Beltrami, *Saggio di interpretazione*...Napoli, 1868 (*Giorn. d. Mat.* vi); *Teoria fond....Ann. d. Mat.* ii (1868), p. 232; *Opere Mat.* (1902) i.

In what is said in the text, references to non-Euclidean geometry in three dimensions are excluded, as belonging later. Also questions

NOTE III

IN REGARD TO THE LITERATURE FOR NON-EUCLIDEAN GEOMETRY

The following references may be of use, as indications of ways in which the introduction given in Chapter v may be amplified:

R. Bonola, *Non-Euclidean geometry, a critical and historical study,* English by H. S. Carslaw (Open Court Publishing Co., Chicago, 1912, pp. 1–268).

R. Bonola, *Sulla teoria delle parallele e sulle geometrie non-euclidee,* in Vol. i, pp. 248–362, of Enriques' *Questioni riguardanti le matematiche elementari,* 1912.

D. M. Y. Sommerville, *The elements of non-Euclidean geometry,* 1914, pp. 1–274; and, by the same author, *Bibliography of non-Euclidean geometry,* St Andrews, 1911, pp. 1–403.

Simon Newcomb, *Elementary theorems relating to...space...of uniform...curvature in the fourth dimension,* Crelle's Journal, lxxxiii, 1877, pp. 293–299.

H. Helmholtz, *Ueber die Thatsachen die der Geometrie zu Grunde liegen,* 1868, Ges. Wiss. Abhand., Band ii.

B. Riemann, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen,* 1854, Ges. Werke.

F. Enriques, *Prinzipien der Geometrie,* Enzykl. der Math. Wiss. iii. 1. 1.

F. Klein, *Nichteuklidische Geometrie,* Göttingen, 1913; *Elementarmathematik,* 1913; many papers in the *Math. Annal.*; and *Mathematical Works,* in particular Vol. i, 1921, p. 252.

D. Hilbert, *Grundlagen der Geometrie* (First Edit. 1899).

E. Schur, *Grundlagen der Geometrie,* 1909.

N. I. Lobatschewsky, *Geometrische Untersuchungen zur Theorie der Parallellinien,* 1840 (facsimile, Mayer u. Müller, 1887).

Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles, 1855 (reprint, Ostwald's Klassiker, 1902).

Geometrical works of Lobatschewsky (Kasan, 1883–1886).

F. Engel, *N. I. Lobatschewsky, zwei geometrische Abhandlungen..., mit einer Biographie,* 1898, pp. 1–476.

W. Bolyai, *Tentamen juventutem studiosam in elementa matheseos purae* (1832), with appendix by J. Bolyai, *Appendix scientiam spatii absolute veram exhibens.*

J. Frischauf, *Absolute Geometrie nach Johann Bolyai* (1872).

P. Stäckel, *W. u. J. Bolyai, geometrische Untersuchungen,* 1913, two parts, each pp. 1–281 (see also Stäckel, *Math. u. Naturw. Ber. aus Ungarn,* xvii, 1901, and xviii, 1902. Also Stäckel u. Engel, *Math. Annal.* xlix, 1897, p. 149).

F. Engel u. P. Stäckel, *Die Theorie der Parallellinien von Euclid bis auf Gauss,* 1895, pp. 1–325.

C. A. F. Peters, *Briefwechsel zwischen K. F. Gauss u. H. C. Schumacher* (6 vols. 1860–1865). See Gauss, *Ges. Werke.*

Cayley, *Papers,* ii, p. 583; xiii, p. 480.

Beltrami, *Saggio di interpretazione*...Napoli, 1868 (*Giorn. d. Mat.* vi); *Teoria fond.*...*Ann. d. Mat.* ii (1868), p. 232; *Opere Mat.* (1902) i.

In what is said in the text, references to non-Euclidean geometry in three dimensions are excluded, as belonging later. Also questions of Transformation and Group-Theory are not considered (see S. Lie, *Transformationsgruppen*, III, 1893, which gives an exhaustive analytical study of the Riemann-Helmholtz problem). Further, the arbitrary limitation of the symbols adopted in Chapters IV and V leads to the omission of many general logical questions. Cf., for example, Dehn, *Math. Annal.* LIII (1900), pp. 404–439, and *Math. Annal.* LX (1905), pp. 166–174; R. L. Moore, *Tr. Amer. Math. Soc.* VIII (1907) (and Hilbert's *Grundlagen*).

ADDITIONS

p. 27. In Ex. 4, it follows, by Ex. 3, that the director conic of the given conic, in respect of B' and C', touches $A'B'$ and $A'C'$, at B' and C', and passes through the point, X, where CC' meets $A'B'$, and through the point, M, where BB' meets $C'A'$. From this it follows that AA' contains the point of intersection of $B'M$ and $C'X$, that is, of BB' and CC'. This simplification was suggested by Mr R. J. Lyons, of the University of Sydney, to whom also several other of the following remarks are due.

We may remark also that if the lines from three points A, B, C, respectively conjugate, in regard to a conic, to the joins of three other points A', B', C', be concurrent, the same is true of the lines from A', B', C' respectively conjugate to the corresponding joins of A, B, C. When the conic is a point-pair, the result is remarked by Steiner, *Werke*, I, p. 157.

And compare Moebius's theorem (Vol I, p. 61) that if a line meet $B'C'$, $C'A'$, $A'B'$ in P', Q', R', so that AP', BQ', CR' are concurrent, and meet BC, CA, AB in P, Q, R, then $A'P$, $B'Q$, $C'R$ are concurrent. (See also the Addition to p. 218.)

p. 29. The result in Ex. 7 suggests the consideration of a theorem with wide bearings, relating to three triads of tangents of a conic. See the addition to p 145.

p. 30. The proof of Ex. 8 may be given as follows (R. J. Lyons): By considering conics through the four points A, B, C, O, which determine an involution on IJ, we see that the four pairs of rays (OA, OL), (OB, OM), (OC, ON), (OI, OJ), are in involution. Hence it follows that the four pairs of rays (OA, OL'), (OB, OM'), (OC, ON'), (OI, OJ) are in involution. Whence, considering the ranges on AB, AC, by BC, OI, OJ, $M'N'$, we see that $M'N'$ touches the conic touching BC, CA, AB, OI, OJ, IJ, and, therefore, by symmetry, passes through L'. (Cf. Ex. 3, p. 82.)

p. 32. For the theorem of Ex. 10 see Hesse, *De Curvis...2. Ord.*, *Crelle*, XX (1840), p. 301. Another view of the proof that X, X' are conjugate is as follows (R. J. Lyons): Let y, z be the polars of Y, Z, meeting in T, and let a variable line through X' meet these in Y_1 and Z_1; then the point of intersection (X_1) of YZ_1 and ZY_1 describes the line XT. But, if P be the point of contact of one tangent from X', and ZP, YP meet y and z in Q and R, it can be proved that X', Q, R are in line. Thus P lies on XT; and X lies on the polar of X'.

p. 34. To prove, in Ex. 11, that QMN is a self-polar triad, we may argue as follows (R. J. Lyons): As two lines which are conjugate, in regard to the conic Σ, meet the polar of any point of either in two conjugate points, it follows that the involution on MN, by conics through the four points Q, R, Q', R', contains two pairs (by QR, $Q'R'$, and QQ', RR') which both consist of points conjugate in regard to Σ. Hence M, N are likewise conjugate.

p. 45. For the theorem of Ex. 20, it is easy to prove that the six tangents of two conics at three of their intersections are touched by a conic. The tangents at three points of a conic meet the respective joins of the other two points in three collinear points; and, if two lines be drawn intersecting the joins of three points, the six lines obtained by joining each of the three points to the two points where the opposite join is met by the two lines, touch a conic (cf. p. 121 below). Having so determined a conic, it is to be shewn that any tangent of this meets the two given conics in harmonically conjugate pairs of points.

But there is a simple view of the theorem, associating it with the theory of the director conic, which involves only such elements of the theory of quadric surfaces as are explained in Chap. v (p. 189):—Given two conics in a plane, of which I, J are two intersections, take an arbitrary point O, not lying in the plane of the conics, and describe any quadric surface containing the lines OI, OJ. Then the two given conics may be regarded as the projections from O of two plane sections, α, β, of the quadric. Any line l, of the original plane, meeting either of the given conics in two points which are conjugate in regard to the other conic (and therefore cutting the conics in a harmonic range), lies on a plane, γ, through O. The section of the quadric surface by the plane γ is met by the plane α in two points P, P' and by the plane β in two points Q, Q', such that the rays OP, OP' are harmonically separated by OQ, OQ'; the lines PP', QQ' are thus conjugate to one another with respect to the conic γ. Now let A, B, C be the respective poles of the planes α, β, γ in regard to the quadric surface, so that CA, CB are respectively the polar lines of PP' and QQ' in regard to this surface. It easily follows, by the reciprocity of pole and polar, that CA, CB are conjugate lines in regard to the enveloping cone of the surface drawn with vertex C, which touches the quadric surface along the conic γ; or that the segment AB is divided harmonically by this cone.

Now the locus of a point D such that the enveloping cone to the given quadric surface, Ω, divides the given segment AB harmonically, is easily seen to be another quadric surface, Φ, whose section by any plane through AB is the director conic of the section of Ω by this plane, in regard to A and B; namely the conic which is the locus of a point, D', for which $D'A$, $D'B$ are conjugate lines in regard to the section of Ω. The quadric surface Φ contains A and B and also the conic sections, α, β, of Ω, by the polar planes of A, B in regard to Ω.

The point C, spoken of, thus lies on this quadric Φ; and, as the plane γ passes through O, the point C describes the conic section in which Φ is met by the tangent plane of Ω at O. Wherefore the plane γ touches a quadric cone with vertex at O (the polar reciprocal, in regard to Ω, of the section of Φ by this tangent plane). This shews that the line l, in the original plane, touches a conic; which is the proposition.

If $f_x^2=0$ be the equation of Ω, and (y), (z) be A and B, the equation of Φ is $f_x^2 . f_{yz}=f_{xy}f_{xz}$. The equation of the envelope of the line l is given below in the Addition to p. 142. (Cf. also the Addition to p. 126.)

p. 49. The proposition of Ex. 25 is really stated in the second paragraph of p. 44 preceding (R. J. Lyons). For any conic, Φ, outpolar to Ω, meets the polar of J, in regard to Ω, in points A, B, conjugate in regard to Ω, and this polar contains the pole, O, of IJ in regard to Ω. (Cf. also p. 80 below.)

The proposition is in Gaskin, *The Geometrical Construction of a Conic Section...*, Cambridge, 1852, p. 33. See also Faure, *Nouv. Ann. de Math.* xix (1860), p. 234.

The following is a generalisation of Gaskin's theorem: Let $U=0$, $V=0$, $W=0$ be any three conics, and Σ be a conic which is inpolar to all these; let Φ be the director conic of Σ in regard to two arbitrary points I, J. There is then a definite conic $\lambda U+\mu V+\nu W=0$ which passes through I and J, in which λ, μ, ν are certain constants. The generalised proposition is that this conic cuts Φ at right angles with respect to I and J (*v.* p. 63). Gaskin's theorem is the particular case of this in which U, V, W are the three line-pairs containing a triad of points self-polar in regard to Σ.

p. 65. It may be noticed that the theorem of the centroid does not require Pappus's theorem, but the theorem of the orthocentre, as involving six points in involution, does require this (Vol. i, p. 60). Another related theorem is that the lines through the mid-points (in respect to I, J) of the joins of a triad of points, each at right angles to the join, meet in a point. If BI, CJ meet in P, and BJ, CI meet in P', the line PP' is at right angles to BC, and passes through the mid-point of BC, as is at once seen. If also CI, AJ meet in Q, and CJ, AI meet in Q', while AI, BJ meet in R and AJ, BI meet in R', the fact that PP', QQ', RR', meet in a point is a direct application of Pappus's theorem for the collinear triads QJR' and RIQ'. Or, otherwise, assuming the result of Ex. 1, p. 23, for any conic through B, C, I, J, the pole of PP' is the intersection of BC and IJ, so that the pole of IJ lies on PP'. Whence, taking the conic through A, B, C, I, J, the lines PP', QQ', RR' meet in the pole of IJ, the so-called *circumcentre* of A, B, C.

p. 70. We may remark, in Ex. 2 (R. J. Lyons), that we have the following sequence of related pencils of rays:

$$F(O, A, I, J),\quad F(O, B, I, J),\quad D(O, B, I, J),\quad D(O, C, I, J),$$
$$E(O, C, I, J),\quad E(O, A, I, J),$$

and the comparison of the first and last shews that a conic contains A, E, O, F, I, J.

p. 79. As an additional simple example, it may be proved that: the common tangents of any two circles whose centres are harmonic conjugates in regard to the foci of a given conic, touch a confocal conic.

p. 81. The theorem that the director circles are coaxal is ascribed to Plücker, *Analytisch-geometrische Entwicklungen* (1831), II, p. 198.

p. 83. The rectangular hyperbola which is also a parabola is also an equiangular spiral. For a conic touching the lines JI and JO at I and O, if the tangent at any point P, and the chord OP, meet IJ in T and Q, respectively, the angle TPQ, relatively to I and J (see p. 167), is constant as P varies. Using the symbols of points, we can suppose that $Q=I+J$, $T=2I+J$.

p. 90. We may remark also: (*a*) The rectangular hyperbolas which have a given self-polar triad of points all pass through four points, and their centres describe the circle through the points of the triad. (*b*) Given four points there is, corresponding to any point O, a definite other point, O', such that O, O' are conjugate in regard to all conics through the four points (Ex. 15, p. 37). Hence prove that, if OT, OT' be the tangents from a fixed point O to one of a system of confocal conics, the circle OTT' passes through another fixed point.

p. 96. We remark also

(*a*) If the perpendiculars from a fixed point S, to two fixed lines AM, AN, meet these in M and N, the conic with S as focus which touches AM, AN, MN, is a parabola. More generally: If the joins of a fixed point S to the poles, in regard to a given conic Ω, of two lines m, n, meet these in M and N, the conic touching the lines m, n, MN, and the tangents to Ω from S, also touches the tangents of Ω at the points where MN meets it.

(*b*) A variable chord of a conic which subtends a right angle at a fixed point, envelopes a conic having one focus at this point. The foot of the perpendicular from the fixed point to the chord describes a circle (cf. p. 85). This circle cuts at right angles the unique circle, with centre at the fixed point, which is inpolar to the given conic; and reduces to two lines if the given point be on the director circle of the given conic. If the fixed point be on the given conic, the chords of the conic subtending a right angle at this point, all pass through the same point on *the* normal to the conic at this point. (The Frégier point. *Gergonne's Annales*, VI, 1816, p. 231.)

p. 107. For a function of a real variable having a finite differential coefficient, there is at least one zero of this between two zeros of the function. The following extension to a polynomial function of a complex variable (in the first instance of the third order) seems to deserve mention: Let $(a_1, b_1, 1)$, $(a_2, b_2, 1)$, $(a_3, b_3, 1)$ be the coordinates of any three points, not lying on the line $z=0$. The tangential equations of these points will then be $ua_1+vb_1+w=0$, etc., say $U_1=0$, etc. It follows that the equation $U_2U_3+U_3U_1+U_1U_2=0$ is the tangential equation of a conic touching the three joins of the points. In particular the point of contact with the join of the points U_2, U_3, obtainable by the operator $u\partial/\partial u'+v\partial/\partial v'+w\partial/\partial w'$, is the point $U_2+U_3=0$. This is the harmonic of the point $U_2-U_3=0$, in which the join meets $z=0$ (whose coordinates are $u=0$, $v=0$, $w=1$), with respect to the two points U_2, U_3. Thus there exists a conic touching the three joins of the given points, at the mid-points of these joins with respect to $z=0$ (Ex. 29, p. 53).

Now the coordinates of the line joining a point (x, y, z) to the point $(1, i, 0)$, on $z=0$, are $(-z, -iz, x+iy)$; if these be substituted for u, v, w in the tangential equation of the conic, the result will express that the tangent from (x, y, z) passes through the point $(1, i, 0)$. The substitution in ua_1+vb_1+w leads to $x+iy-(a_1+ib_1)z$. Hence, if we write ζ for $(x+iy)z^{-1}$, and γ_1 for a_1+ib_1, and also write ζ' for $(x-iy)z^{-1}$ and γ_1' for a_1-ib_1, etc., the foci of the conic (cf. p. 119) are the intersections of the two lines expressed by the single equation

$$(\zeta-\gamma_2)(\zeta-\gamma_3)+(\zeta-\gamma_3)(\zeta-\gamma_1)+(\zeta-\gamma_1)(\zeta-\gamma_2)=0,$$

with the two lines expressed by the result of substituting $\zeta', \gamma_1', \ldots$ for $\zeta, \gamma_1, \ldots$ in this equation. This equation is obtainable by differentiating, in regard to ζ, the equation

$$(\zeta-\gamma_1)(\zeta-\gamma_2)(\zeta-\gamma_3)=0.$$

We may thus say, if on the plane representing a complex variable ζ, a triad of points be given by $f(\zeta)=0$, the foci of the conic touching the joins of the pairs of the triad at their mid-points are given by $df(\zeta)/d\zeta=0$.

A precisely similar treatment determines the tangential equation of a curve of *class* $(n-1)$ touching the joins of n arbitrary points, taken in order, each at its mid-point; and determines the foci of this curve.

p. 114. For the final statement in Ex. 8, see Morley, *Bull. Amer. Math. Soc.* I (1895), p. 116, and the Note, *Proc. Camb. Phil. Soc.* XX (1920), p. 119.

p. 126. It may be verified that the tangents at the common points of the two circles

$$x^2+y^2-2axz-h^2z^2=0, \quad x^2+y^2+2bxz-h^2z^2=0$$

touch the conic $\qquad A\,[x-\tfrac{1}{2}(a-b)z]^2+By^2=z^2,$

where $A^{-1}=h^2+\frac{1}{4}(a-b)^2$, $B^{-1}=h^2-ab$. This conic has foci at the centres of the two circles; its tangents cut the two circles harmonically. (Cf. the Addition to p. 45.)

p. 142. The simplicity of the theory of the invariants of conics is readily seen with the use of a certain symbolical notation, first introduced by Cayley (*Papers*, I, p. 95 (1846), etc.), but systematised by Clebsch (*Bin. alg. Formen*, 1872).

Consider first an ordinary binary quadratic form, $ax^2+2hxy+by^2$. Change the notation, writing x_1, x_2 for x and y, and a_{11}, a_{12}, a_{22} for a, h, b. Now introduce two symbols, a_1, a_2, subject to the ordinary laws of algebra, but capable of interpretation as numbers only in products of two, according to the rule $a_ia_j=a_{ij}$. Then the form $a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2$ can be represented by $(a_1x_1+a_2x_2)^2$, which is expressed by a_x^2. And a symbolic form a_xa_y, meaning

$(a_1x_1+a_2x_2)(a_1y_1+a_2y_2)$, being $a_1^2x_1y_1+a_1a_2(x_1y_2+x_2y_1)+a_2^2x_2y_2$, represents $a_{11}x_1y_1+a_{12}(x_1y_2+x_2y_1)+a_{22}x_2y_2$. This vanishes when the point (x_1, x_2) is the harmonic conjugate of the point (y_1, y_2) in regard to the pair of points given by the vanishing of the original form a_x^2. If we have a second quadratic form $b_{11}x_1^2+\ldots$, or b_x^2, the condition that the pair of points given by the vanishing of one of these forms should be harmonic in regard to the pair of points given by the other, which is

$$a_{11}b_{22}-2a_{12}b_{12}+a_{22}b_{11}=0,$$

can be represented symbolically by $(a_1b_2-a_2b_1)^2=0$, which can be written $(ab)^2=0$. We cannot, however, without further convention, express in the symbolic form, for instance, the discriminant $a_{11}a_{22}-a_{12}^2$ of the original quadratic, since $a_1^2a_2^2-(a_1a_2)^2$ vanishes. We thus replace the discriminant by $\frac{1}{2}(a_{11}a_{22}'+a_{22}a_{11}'-2a_{12}a_{12}')$, where $a_{11}', a_{12}', a_{22}'$, are ultimately to be a_{11}, a_{12}, a_{22}, and introduce symbols a_1', a_2', to represent a_{11}', etc., which are ultimately equivalent with a_1, a_2. Then we have the symbolic form

$$\tfrac{1}{2}(a_1^2\,.\,a_2'^2+a_2^2\,.\,a_1'^2-2a_1a_2\,.\,a_1'a_2'),$$

or $\frac{1}{2}(a_1a_2'-a_2a_1')^2$, and this can be written $\frac{1}{2}(aa')^2$.

When we pass to a quadratic form in three homogeneous variables x_1, x_2, x_3, if we similarly introduce three symbols a_1, a_2, a_3, the equation of a conic may be represented by $(a_1x_1+a_2x_2+a_3x_3)^2=0$, which may be written $a_x^2=0$. Then the condition that two points, (x_1, x_2, x_3) and (y_1, y_2, y_3), or say (x) and (y), should be conjugate in regard to this conic is represented by a_xa_y. This is obvious at once on interpretation. But also it may be remarked that the points, on the line joining (x) and (y), with coordinates $(\lambda x_1+y_1, \lambda x_2+y_2, \lambda x_3+y_3)$, which lie on the conic $a_x^2=0$, correspond to the two values of λ given by the equation

$$\lambda^2a_x^2+2\lambda a_xa_y+a_y^2=0,$$

which is $(\lambda a_x+a_y)^2=0$, and, in order that these points should be harmonic in regard to (x) and (y), it is necessary that the two values of λ should be equal and of opposite sign. This leads to the equation $a_xa_y=0$. A similar method at once gives the condition for a line, of coordinates u_1, u_2, u_3, to meet two conics $a_x^2=0$, $b_x^2=0$, in two pairs of mutually harmonic points. For if (y), (z) be any two points of this line, and therefore

$$u_1=y_2z_3-y_3z_2,\quad u_2=y_3z_1-y_1z_3,\quad u_3=y_1z_2-y_2z_1,$$

the condition that the roots of one of the quadratic equations in λ

$$\lambda^2a_y^2+2\lambda a_ya_z+a_z^2=0,\quad \lambda^2b_y^2+2\lambda b_yb_z+b_z^2=0,$$

should be harmonic conjugates in regard to the roots of the other is (p. 111)

$$a_y^2\,.\,b_z^2-2a_ya_z\,.\,b_yb_z+a_z^2\,.\,b_y^2=0,$$

which is $(a_yb_z-a_zb_y)^2=0$, or

$$[(a_2b_3-a_3b_2)u_1+(a_3b_1-a_1b_3)u_2+(a_1b_2-a_2b_1)u_3]^2=0.$$

This we write $(abu)^2=0$. It is the tangential equation of the conic of Ex. 20, p. 44.

The condition that the line (u) should touch the conic $a_x^2=0$, found as the condition that the former of the two quadratics in λ should have equal roots, is $a_y^2\,.\,a_z^2-(a_ya_z)^2=0$; but as this involves the coefficients in the equation of the conic to the second degree, it can be expressed symbolically only by introducing other symbols a_1', a_2', a_3', ultimately equivalent with a_1, a_2, a_3; namely by writing it as

$$\tfrac{1}{2}(a_y^2a_z'^2-2a_ya_z\,.\,a_y'a_z'+a_z^2a_y'^2)=0,$$

or $\frac{1}{2}(aa'u)^2=0$. This is the tangential equation of the conic. It is very usual to write $\alpha_1, \alpha_2, \alpha_3$ respectively for $a_2a_3'-a_3a_2'$, $a_3a_1'-a_1a_3'$, $a_1a_2'-a_2a_1'$; then the tangential equation is written $u_\alpha{}^2=0$.

To express symbolically the discriminant of the conic, which is of the third degree in the coefficients, it is similarly necessary to use three ultimately equivalent sets of symbols, say (a), (a'), (a''). The methodical manner of introducing these is to operate on the real cubic form of the discriminant

$$a_{11}a_{22}a_{33}+2a_{23}a_{31}a_{12}-\text{etc.},$$

in succession by

$$a_{11}'\frac{\partial}{\partial a_{11}}+\ldots+a_{23}'\frac{\partial}{\partial a_{23}}+\ldots,\quad a_{11}''\frac{\partial}{\partial a_{11}}+\ldots+a_{23}''\frac{\partial}{\partial a_{23}}+\ldots,$$

so reducing it to a form linear in each of

$$(a_{11},\ldots, a_{23},\ldots), (a_{11}',\ldots, a_{23}',\ldots), (a_{11}'',\ldots, a_{23}'',\ldots),$$

and then to replace each coefficient by the symbols, a_{ij}' by $a_i' a_j'$, a_{ij}'' by $a_i'' a_j''$. The result is that the discriminant is represented by $\frac{1}{6}(aa'a'')^2$, where $(aa'a'')$ means, as before, the determinant of three rows and columns of which the diagonal term is $a_1a_2'a_3''$.

Hence we may express the discriminant of the conic $\lambda a_x{}^2+b_x{}^2=0$, formed from two different conics $a_x{}^2=0$, $b_x{}^2=0$, in the form

$$\tfrac{1}{6}[\lambda^3(aa'a'')^2+3\lambda^2(aa'b)^2+3\lambda(abb')^2+(bb'b'')^2],$$

where b, b', b'' are also equivalent symbols; so that, in the notation of p. 142 of the text,

$$\Delta=\tfrac{1}{6}(aa'a'')^2,\quad \Theta=\tfrac{1}{2}(aa'b)^2,\quad \Theta'=\tfrac{1}{2}(abb')^2,\quad \Delta'=\tfrac{1}{6}(bb'b'')^2.$$

Whence, using as above $\alpha_1=a_2a_3'-a_3a_2'$, etc., $\beta_1=b_2b_3'-b_3b_2'$, etc., we may write $\Theta=\frac{1}{2}b_\alpha{}^2$, $\Theta'=\frac{1}{2}a_\beta{}^2$. And *the condition that the conic $a_x{}^2=0$ should be outpolar to the conic $u_\beta{}^2=0$ is expressed by $a_\beta{}^2=0$.*

The equation of the director conic (p. 26) of a given conic $a_x{}^2=0$, in regard to two given points $(y),(z)$, being the condition that the lines joining the variable point (x) to (y) and (z) should be conjugate in respect to the conic envelope $u_\alpha{}^2=0$, is expressed, in terms of the coordinates $(v),(w)$ of these joining lines, by $v_\alpha w_\alpha=0$, or, at fuller length, by $(xy\alpha)(xz\alpha)=0$. This last, in terms of the equivalent symbols $(a), (a')$ for the point-equation of the conic, α_1 being $a_2a_3'-a_3a_2'$, etc., as before, is the same as

$$(a_xa_y'-a_ya_x')(a_xa_z'-a_za_x')=0;$$

this, in unsymbolical form, is

$$a_x{}^2 \,.\, a_y a_z - a_x a_y \,.\, a_x a_z=0,$$

and is then easily verified to be the condition that the segment joining (y) to (z) should be divided harmonically by the two tangents from (x) to the conic, whose equation, with (ξ) as current coordinate, is $a_x{}^2 . a_\xi{}^2-(a_xa_\xi)^2=0$.

It is easy also to verify (a), that the condition that the line (u) should meet the conics $a_x{}^2=0$, $b_x{}^2=0$, $c_x{}^2=0$ in three pairs of points in involution, is expressed by

$$(bcu)(cau)(abu)=0;$$

this is then the tangential equation of the cubic envelope considered below, in the Addition to p. 145, and called the *Cayleyan* of the three conics; and (b), that the Jacobian of the three conics is expressed by $(abc)\,a_x b_x c_x=0$.

p. 145. The theorem, proved in Ex. 7, p. 29, that the three intersections of a triad of tangents of a conic lie, with the intersections of another triad of tangents, upon a conic, is capable of generalisation. If three triads of tangents of the same conic be taken, and we consider the three conics, each containing the six intersections of the pairs in two of the triads, then these three conics meet in a point. More precisely, if $BC, CA, AB, B'C', C'A', A'B'$, and IJ, JS, SI be tangents of a conic, the conics containing respectively (A, B, C, S, I, J), (A', B', C', S, I, J), (A, B, C, A', B', C') meet in a point; or, in metrical language, if BC, CA, AB as well as $B'C', C'A', A'B'$, be tangents of a parabola, the circles A, B, C and A', B', C' have, beside the focus, S (p. 81), another common point H; this lies on the conic containing A, B, C, A', B', C'. This can be proved (C. Taylor, *Anct. and Mod. Geom. of Conics*, 1881, p. 360) by shewing that in metrical language, the angle subtended at H by any two of the six points A, B, C, A', B', C' is equal to the angle between the two joins respectively opposite to these points. Of this statement a non-metrical proof is added below.

The theorem converse to the general theorem is also true, namely, if three conics have a common point, the nine common chords of pairs of these, other than the common chords through the common point, all touch a conic.

The theorems however arise from the result referred to in the preceding Addition, that a line which meets three conics in pairs of points belonging to the same involution, touches a curve of the third class. For, if this be assumed, it is clear that the eighteen common chords of the pairs of the conics belong to the envelope; and that, if the three conics have a common point, all lines through this point equally belong to the envelope. In this case, then, the envelope degenerates into the pencil of lines through the common point of the conics (expressed by a linear equation in the coordinates of the variable line), and a conic envelope, which is the conic of the converse theorem just stated (Picquet, *Étude Géom....de sections coniques*, Paris, 1872, p. 27, etc.).

We prove the theorem of the cubic envelope in a manner different from that of the preceding Addition: If a line $ux+vy+wz=0$ meet the conics $U=0$, $V=0$, $W=0$ in pairs of an involution (to which will then belong the pair in which the line cuts any conic $\lambda U+\mu V+\nu W=0$, in which λ, μ, ν are any constants), there must be, of the involution determined on the line by the conics $\lambda U+\mu V=0$, one pair lying on $W=0$; and, therefore, for proper values of $\nu, l, m, n, \lambda, \mu$, there must exist an identity

$$\lambda U+\mu V+\nu W+(lx+my+nz)(ux+vy+wz)=0.$$

Equating to zero the coefficients of $x^2, y^2, z^2, yz, zx, xy$ in this, and eliminating the constants $\lambda, \mu, \nu, l, m, n$ from the resulting six equations, we find for u, v, w a homogeneous cubic relation expressible by the vanishing of the determinant of six rows

$$\begin{vmatrix} u & 0 & 0 & 0 & w & v \\ 0 & v & 0 & w & 0 & u \\ 0 & 0 & w & v & u & 0 \\ a & b & c & 2f & 2g & 2h \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

in which the last three rows contain the coefficients in U, V and W.

If the three conics have a common point, we may suppose this taken for $y=0$, $z=0$; then the three coefficients a vanish, and the determinant divides by u. The locus then breaks up into a conic envelope, and the common point, whose tangential equation is $u=0$. If the three conics have two common points, we may suppose, beside the three coefficients a, that also the three coefficients b are zero. The determinant is then the product of uv and a factor linear in u, v, w; the envelope then consists of three pencils of lines, two with centres at the common points, and another. If, in metrical terms, we speak of the conics as circles, in this case, this is the theorem that any line through the radical centre of the circles, in which the three radical axes of the circles cointersect (p. 70 above), cuts the circles in three pairs of points in involution. If the three conics be such that there is a point whose polar line is the same for all, we may suppose the coefficients g, h in all the conics to be zero; the determinant then again divides by u, and the envelope contains the pencil of lines through the common pole. In metrical terms we may say that the twelve common chords of three conics which have a common centre, other than the six common diameters, all touch the same conic. Conversely, by considering the lines of the envelope through the intersections of two of the conics, we prove that if the cubic envelope contain a pencil of lines through a point, then this point is either common to the three conics, or has the same polar line in regard to all; the former requires only one condition for the conics, while the latter requires two.

Now, returning to the hypothesis of three triads of tangents of a given conic, we may take for the conics U, V, W those of which each contains the six intersections arising from two triads of tangents. The cubic envelope belonging to U, V, W must then contain all the nine tangents of the original conic, each of which is a common chord of two of the conics U, V, W. As a cubic envelope contains in general only six tangents of the original conic envelope, we infer that the cubic envelope in this case consists of this conic envelope, together with a pencil of lines through a point, and hence, in general, that the three conics U, V, W have a common point. Conversely, the theorem stated as to the nine common chords of pairs of three conics which have a common point (other than the chords through this point)—that these nine chords touch a conic—follows at once.

In regard now to the proposition referred to above, from which a proof of this result may be constructed independently of the cubic envelope: Let the lines BC, CA, AB, as well as the lines $B'C', C'A', A'B'$, and the lines IJ, JS, SI, be nine tangents of a conic Φ. Let f denote the conic containing A, B, C, S, I, J, and f' the conic containing A', B', C', S, I, J; and let H be the fourth common point of the conics f, f'. Further, let BC, $B'C'$ meet IJ in X and X', respectively, and HA, HA' meet IJ in U, U', respectively, while HS meets IJ in K, and the conic Φ touches IJ in F. Then the involution determined by the two pairs I, J and K, F contains the pairs X, U and X', U'; and hence contains four other pairs obtained by replacing A by B or C and A' by B' or C'. For the triads A, B, C and S, I, J are (Ex. 21, p. 45) both self-polar in regard to the same conic, with respect to which f and Φ are polar reciprocals (Ex. 24, p. 47); thus the range of points A, B, C, S, I, J on the conic f, as seen from any point O of this conic, is related to the range, on any tangent of the conic Φ, obtained by the intersections of this with the respectively opposite tangent lines of Φ, BC, CA, AB, IJ, JS, SI. We consider this range on IJ; if $BC, CA, AB, OA, OB, OC, OS$ meet IJ respectively in X, Y, Z, P, Q, R, T, these last four varying as O varies on f, and F be the point where Φ touches IJ, we infer that the range P, T, I, J is related to the range X, F, J, I, which shews that X, P belong to the involution determined by I, J and T, F. In particular when O is at H, the points P, T become those above denoted by U, K, respectively, and we have the result stated above. To complete the theorem under consideration, it is then to be shewn, from the relations obtained, that the conic containing A, B, C and A', B', C' has H for its fourth intersection with f or f'.

It can be shewn that any three conics $U=0$, $V=0$, $W=0$ may be regarded as the polar conics of three properly chosen points with respect to a single cubic curve, say F. Any conic $\lambda U+\mu V+\nu W=0$ is then the polar conic of another point, in regard to the same cubic F. The cubic envelope we have discussed is then called the *Cayleyan* of F. It is clear from our deduction that any line of the Cayleyan makes up, with another line, a degenerate conic of the system $\lambda U+\mu V+\nu W=0$ (and the other line is then evidently also a line of the Cayleyan). If we now consider the conic envelopes which are inpolar to all of $U=0$, $V=0$, $W=0$, these, expressed in tangential coordinates, will equally form a linear system of three terms, say $\rho\Theta+\sigma\Phi+\tau\Psi=0$, in which ρ, σ, τ are parameters (cf. p. 142); and we may, dually to what has preceded, consider the locus of the points of degenerate point-pairs belonging to this latter system. But a point-pair inpolar to a conic consists (pp. 36, 144) of two points which are conjugate in regard thereto. Thus the locus whose consideration is suggested is that of pairs of points which are conjugate to one another in regard to all the conics of the system $\lambda U+\mu V+\nu W=0$. Such points arise naturally from what has been said. For on any tangent of the Cayleyan there is a pair of points which are the double points of the involution cut thereon by all the conics of this system; and these are conjugate to one another in regard to all these conics. Thus the locus suggested is that of points whose polar lines in regard to $U=0$, $V=0$, $W=0$ are concurrent; as we see at once by forming the determinantal equation which expresses the condition, the locus is a cubic curve. It is that known as the Jacobian of the three conics. Another definition of the curve is possible, namely the locus of the point of intersection of the two lines of a line-pair of a degenerate conic of the system $\lambda U+\mu V+\nu W=0$—as we may easily prove; at a double point of the involution, on a tangent of the Cayleyan, by these conics, there are two coincident intersections of this line with a conic of the system; there is in fact a line-pair, of two tangents of the Cayleyan, which intersect at this point. From this it follows that we may regard the Cayleyan as the Jacobian of the system of conic envelopes, $\rho\Theta+\sigma\Phi+\tau\Psi=0$, inpolar to the original system; or the Jacobian of the original system as being the locus of points from which the pairs of tangents to Θ, Φ, Ψ form a pencil in involution.

In regard to the matters here dealt with reference may be made to Hesse, *Ueber Curven dritten Ordn.*, *Crelle*, XXXVI (1848); Cayley, *Curves of the third order* (1856), *Papers*, II, p. 381; Cremona, *Introduzione ad una teoria geometrica delle curve piane*, Bologna, 1861 (*Opere*, t. I); Salmon, *Higher Plane Curves*; Picquet (*loc. cit.*). Several of the following examples are suggested by this last.

(*a*) In a system of conics $\lambda U+\mu V+\nu W=0$ are three whose equations can be taken to have the forms

$$x^2+2myz=0,\ y^2+2mzx=0,\ z^2+2mxy=0,$$

where m is a constant; and the system of conics inpolar to this system is based upon three expressed by

$$mu^2-vw=0,\ mv^2-wu=0,\ mw^2-uv=0.$$

(*b*) Three conics having two common points can be supposed to have equations

$$z^2+2xy+2g_ixz+2f_iyz=0, \qquad (i=1, 2, 3),$$

and the Jacobian of these has the equation

$$z(z^2-2xy)=0.$$

The conic $z^2-2xy=0$ cuts all three given conics at right angles in respect of the two common points.

(*c*) Three conics with a common pole and polar line are capable of the forms

$$x^2+b_iy^2+c_iz^2+2f_iyz=0, \qquad (i=1, 2, 3)$$

and their Jacobian consists of the common polar line $x=0$, together with a pair of lines through the common pole.

(*d*) Through the four common points of any two of a set of three conics can be drawn a conic to have two given points as conjugate points. The three such conics have four common points.

(*e*) If conics which are rectangular hyperbolas in regard to two arbitrary points I, J (p. 83) be drawn through two given points A, B, any line which is at right angles to AB, in regard to I, J, cuts these conics in pairs of points of an involution.

(*f*) In a system of conics $\lambda U+\mu V+\nu W=0$ there is one which is a circle, say Ω, with respect to two arbitrary points I, J. The circle, with centre on any tangent of the Cayleyan of U, V, W, which contains the two double points of the involution on this tangent, cuts the circle Ω at right angles (p. 66). Cf. the generalisation of Gaskin's theorem given above in the Addition to p. 49.

(*g*) Given any four conics, there are three lines each cutting the conics in four pairs of points of the same involution. There are in fact two linearly independent conic envelopes, say Θ, Φ, which are inpolar to all the four given conics (p. 142); and in the system $\lambda\Theta+\mu\Phi=0$ there are three degenerating into a point-pair.

p. 147. The form to which a pair of quadratic polynomials U, U', homogeneous in n variables, can be simultaneously reduced by linear transformation of the variables, depends upon the character of the discriminantal equation in λ of the quadratic $U-\lambda U'$. We suppose here that the discriminant of both U and U' is other than zero. The various possibilities are then all included in the statement following: Form a symbol $\vartheta_p\phi_q\ldots\psi_r$, where the number of ϑ, $\phi, \ldots, \psi$ may be n, or $n-1, \ldots$, or only 1, and the positive integers $p, q, \ldots, r$ have a sum equal to n. Corresponding to ϑ_p take p variables, say $\xi_1, \ldots, \xi_p$; corresponding to ϕ_q take q variables, say $\eta_1, \ldots, \eta_q$; ..., and finally, corresponding to ψ_r take r variables, say $\zeta_1, \ldots, \zeta_r$. The n variables $\xi, \eta, \ldots, \zeta$ are the transformed variables. Then, corresponding to ϑ_p write down a pair of quadratic polynomials in $\xi_1, \ldots, \xi_p$, of forms given immediately, say Θ_p and Θ_p'; corresponding to ϕ_q write down a pair of quadratic polynomials in $\eta_1, \ldots, \eta_q$, say Φ_q and Φ_q'; ...; finally, corresponding to ψ_r write down a pair of quadratic polynomials in $\zeta_1, \ldots, \zeta_r$, say Ψ_r and Ψ_r'. The rule for the forms which these polynomials may have is the same in each case, and may be stated for Θ_p, Θ_p' only; these are respectively

$$\Theta_p=\xi_1\xi_p+\xi_2\xi_{p-1}+\ldots+\xi_p\xi_1,$$
$$\Theta_p'=\vartheta\,\Theta_p+\xi_1\xi_{p-1}+\xi_2\xi_{p-2}+\ldots+\xi_{p-1}\xi_1.$$

The reduced forms for the two original polynomials are then simply

$$U=\Theta_p+\Phi_q+\ldots+\Psi_r,$$
$$U'=\Theta_p'+\Phi_q'+\ldots+\Psi_r';$$

these contain explicitly only the constants $\vartheta, \phi, \ldots, \psi$, whose number may be n, or $n-1, \ldots$ or only 1. These constants need not all be different, and, in the extreme case, may all be equal.

For the case of two conics considered in the text, the symbols $\vartheta_2\phi_1$, ϑ_3 correspond to the cases of two, or three, coincident intersections, respectively. For the other cases given in the text the respective symbols, in order, are $\vartheta_1\phi_1\psi_1$, $\vartheta_1\vartheta_1\phi_1$, $\vartheta_2\vartheta_1$.

The reduction of two quadratic polynomials, as depending on the discriminantal equation, was initiated by Sylvester, *Math. Papers,* I (1904), pp. 119, 139, 219. The classical account is by Weierstrass, *Werke,* II (1895), p. 19. It may be remarked that the form for Θ_p is arbitrary, and may for instance be a sum of squares, subject to a suitable change for Θ_p'. For the determination of a quadratic in regard to which the two given quadratics are polar reciprocals, reference may be made to *Proc. Camb. Phil. Soc.* XXIII (1926), p. 22, where references are given.

p. 152. We add the three following examples:

(*a*) Anticipating the definitions of angle and distance given in Chap. v, we may associate with a circle whose equation is as in Ex. 5, p. 111, a *radius,* r, such that $g^2+f^2-c=r^2$. Then we may prove that the cosine of the angle between the lines drawn from the centres of two circles to one of their common points is the sum of the products of corresponding elements of the two sets

$$(g_1/r_1,\ f_1/r_1,\ \tfrac{1}{2}c_1/r_1,\ -1/r_1),\quad (g_2/r_2,\ f_2/r_2,\ -1/r_2,\ \tfrac{1}{2}c_2/r_2).$$

This sum is then zero when the circles cut at right angles; and is $+1$ or -1 when the circles touch; we may say that they *touch internally* when the sum is $+1$, and *externally* when it is -1.

We may denote this sum by $(1, 2)$. If we take five circles, by combining them in pairs there will arise sums (i, j), for $i, j=1, 2, \ldots, 5$. Prove *that the determinant of five rows and columns, of which (i, j) is the element in the i-th row and j-th column, is identically zero.*

As a particular application prove that if every two of the circles (1), (2), (3) touch externally, and all these circles touch (4) internally, while (5) cuts (1), (3) and (4) at right angles, then $(25)^2=4$. And interpret this when (5) is a line (not meeting the circle (2) in real points). This particular problem goes back to Pappus (*Coll. Math.* IV, xii). See Steiner, *Ges. Werke,* I (1881), p. 47, who suggests (p. 51) a general problem which may be solved by the determinantal equation suggested. See also Cayley, *Papers,* I (1841), p. 4; Darboux, *Ann. d. l'École norm.* I, 1872; and Frobenius, *Crelle,* LXXIX, 1875. An account of Pappus's results is given by Heath, *History of Greek Math.* II (1921), p. 376. A more general result, including the above, is this: Circles (1), (2), (3) touch one another in pairs externally; circle (4) touches (1) and (2), and surrounds all three circles (1), (2), (3). Circle (5) cuts (3) and (4) at right angles. If x, y be the cosines of the angles made by (5) with (1) and (2), prove that

$$(x-y)^2+m(x+y)^2=4,$$

where m depends on the angle between (3) and (4), vanishing when these touch (cf. Steiner, *Werke,* I, pp. 43, 135).

(*b*) Let A, B, C be any three points of a conic and H the pole of the Hessian line of A, B, C, so that AH, BC are conjugate lines, etc.; and let any line through H meet the conic in O and O'. Next let P, Q, R be other three points of the conic which are apolar with A, B, C. Prove that the conics through O, P, Q, R cut the same involution on the Hessian line of A, B, C as do the conics through the four points O, A, B, C; and that the double points of this involution are on the conic through O', H, A, B, C (cf. Ex. 26, p. 147).

Prove that there are three points, A, B, C, of a conic whereat the circle of curvature passes through any given point O of the conic—the circle of curvature at A being that circle having three intersections with the conic coincident at A, etc.; further that the four points O, A, B, C lie on a circle, and the normals of the conic at A, B, C meet in a point, while the centroid of the triad A, B, C is the centre of the conic (cf. p. 93, above. See Steiner, *Werke,* II, pp. 377, 691, and Joachimsthal, *Crelle,* XXXVI, p. 95).

(*c*) We may have two sets of *four* points, or more, on a line or on a conic, apolar to one another, the condition, as for two sets of three points, being the vanishing of the bilinear invariant of the two equations which determine the sets. Prove, for example, that any conic dividing harmonically one join of a triad of points which are self-polar in regard to a given conic Ω, meets Ω in four points which are apolar with the four intersections of Ω with the other two joins of the self-polar triad.

p. 155. A discussion of non-Archimedean geometry will be found in Enriques, *Enzyklopädie d. Math. Wiss.* III, 1, pp. 117–129; Hilbert, *Grundlagen d. Geom.*; K. Th. Vahlen, *Abstrakte Geom.* See also the treatise of Veronese, *Fondamenti di geometria*, Padova, 1891.

p. 157. For exposition of some of the many proofs of the existence of the roots of an algebraic equation, reference may be made to Netto's *Algebra*, I (1896), pp. 25, 173; and to Enriques, *Le Matematiche elementari*, II (1925), p. 186.

p. 165. If we can construct the polar of a given point in regard to a given conic, we can find the point P' which is the intersection of the polars of a point P in regard to two given conics. As P moves on an arbitrary line, P' describes a conic through the triad of points self-polar in regard to both conics (see p. 38). This triad is then determinable by taking three arbitrary lines. The reader should consult the detailed discussion in Poncelet, *Traité des Propriétés Projectives* (1865), I, p. 193, etc.

p. 178. For an entirely different discussion see Hilbert, *Grundlagen der Geom.* 1923, p. 117.

p. 181. See also Bonola, in Enriques, *Le Matematiche elementari*, II (1925), p. 309; K. Th. Vahlen, *Abstrakte Geom.* (1905), pp. 258, 261; J. Hjelmslev, *Math. Ann.* LXIV (1907), p 472; Hilbert, *Grundlagen*, p. 33; and the general references in Note III, p. 237, above. The quadrangle for which three angles are right angles is sometimes called a Lambert quadrilateral.

p. 186. A notable deduction that the three possibilities for the Absolute conic which are discussed in the text arise logically from the properties which seem to characterise the space of our experience is found in Lie, *Transformationsgruppen*, III, 1893 (in particular, Theor. 28, p. 353). See also Poincaré's discussions in his four volumes of essays (Paris, Flammarion); and the general references in Note III, p. 237, above.

That, in the Greek efforts to construct a logical theory of our conception of rigid solid bodies, Euclid's postulate of the unique parallel was recognised as the adoption of one alternative, seems clear from the words of Geminus (? 70 B.C.) quoted in Proclus's *Commentary on Euclid*, Book I (? 450 A.D.). See Heath's *History of Greek Math.* (Oxford, 1921), II, p. 227: "The fact that the straight lines converge is true and necessary, but the statement...they will sometime meet...is not necessary."

In the view adopted in this volume, full clarity is obtainable only by giving up, beside the postulate of the unique parallel, also the postulate of absolute "length," this being regarded as only one of possible correspondences between one pair of points and another pair of points.

Bare reference may be made here to Clifford's suggestion for parallel lines, defined in regard to one of the two systems of generators of an absolute quadric surface; the line through a given point parallel, in this sense, to a given line, is the transversal from this point to the two generators of the quadric which meet the given line. In this connexion also the reader may prove that, considered relatively to the imaginary surface $x^2+y^2+z^2+t^2=0$, the angle, a, between any two intersecting generators of the surface $x^2+y^2-m^2(z^2+t^2)=0$ is the same, given by $\tan^2\frac{1}{2}\alpha=m^2$; and the interval between any two neighbouring points of the surface is given by

$$ds^2=du^2+dv^2+2du\,dv\cos\alpha,$$

where u, v are constant along generators. Regarded relatively to the former, this latter is that called Clifford's surface; its whole extent is finite, and the geometry upon it is Euclidean. See Klein, *Abhandl.* I (1921), p. 362.

p. 199. For a fuller discussion the reader may see Bianchi, *Geometria differenziale*, II (1923), pp. 434, 506, etc.

p. 202. The $d\sigma^2$ can also be expressed by

$$\begin{aligned}d\sigma^2&=t^{-2}(dt^2-dz^2+dx^2+dy^2)\\&=(z^2-x^2-y^2)^{-2}[(z\,dx-x\,dz)^2+(z\,dy-y\,dz)^2-(x\,dy-y\,dx)^2]\\&=[d(x/t)]^2+[d(y/t)]^2-[d(z/t)]^2.\end{aligned}$$

p. 206. In regard to the history of the star pentagon see Heath, *Hist. of Greek Math.* (Oxford, 1921), I, p. 161.

p. 218. For configurations in general there is an Article, E. Steinitz, *Konfigurationen der proj. Geom., Enzykl. der Math. Wiss.* III, 1, pp. 481-516.

For the particular figure of p. 218, see Veronese, *Behandlung der proj. Verhältn....Projizierens und Schneidens, Math. Ann.* XIX, 1882. This figure may be regarded from many points of view. For instance, for the figure of ten lines and ten points in a plane we may state the theorem: Omitting four lines of the figure of which no three meet in a point, and their six intersections, there remains a set of four points; and there are, in all, five such sets of four points. If an arbitrary point, O, be taken in the plane, the five conics, each through O and one set of four points, have two further points, P, Q, in common; and O, P, Q is a self-polar triad in regard to the conic with respect to which the whole figure is self-polar. Of this result (A. C. Dixon, *Quart. J. of Math.* XXIII, 1889, p. 343), a simple proof can be given by projection (cf. Vol. IV of this book, p. 11; also Vol. III, p. 202).

For the general figure, a simple notation for the points and spaces may be explained as follows: Let the aggregate of $(k+1)$ independent points which determine a space $[k]$, of k dimensions, be called a *simplex*, and a flat space of $(k-1)$ dimensions lying therein be called a *prime*. Suppose, in space of $(n-1)$ dimensions, we have two simplexes, each of n points, in perspective by n lines from a point O, every line containing a vertex of each simplex. Denote the vertices of one simplex by the pairs of symbols $(a1)$, $(a2)$, ..., (a, n), and the corresponding vertices of the other simplex by $(b1)$, $(b2)$, ..., (b, n), the centre of perspective, O, being denoted by (ab). The edge joining the vertices (ar), (as) of one simplex will meet the edge containing the vertices (br), (bs) of the other simplex, say in the point (rs). The $\frac{1}{2}n(n-1)$ points (rs) lie in a prime, which we may denote by $[ab]$. In all, including O, we thus have

$$1+2n+\tfrac{1}{2}n(n-1),\ =\tfrac{1}{2}(n+1)(n+2),$$

points $(\rho\sigma)$, in which ρ (or σ) has the range of values a, b, 1, ..., n; and every three points of such symbols as $(\rho\sigma)$, $(\sigma\tau)$, $(\tau\rho)$ are in line. If, omitting the four points $(a, n-1)$, (a, n), $(b, n-1)$, (b, n) for a moment, we consider the prime containing O and the $(n-2)$ points $(a1)$, $(a2)$, ..., $(a, n-2)$, this will contain the points $(b1)$, $(b2)$, ..., $(b, n-2)$, and, also, all the $\frac{1}{2}(n-2)(n-3)$ intersections (pq) in which neither p nor q is $(n-1)$ or n. This prime also will thus contain

$$1+2(n-2)+\tfrac{1}{2}(n-2)(n-3),$$

or $\frac{1}{2}n(n-1)$ points; and we may denote it by $[n-1, n]$. We may likewise denote the prime containing all the points of the original first simplex, other than (ar), by $[br]$, and similarly denote the primes of the second simplex by symbols $[ar]$. The whole figure thus contains the prime $[ab]$ and the primes $[ar]$, $[br]$, $[rs]$; in all

$$1+2n+\tfrac{1}{2}n(n-1), \text{ or } \tfrac{1}{2}(n+1)(n+2),$$

primes $[\rho\sigma]$. The prime $[\rho\sigma]$ then contains all points (rs) in which neither r nor s is one of ρ or σ; and it may be proved that it contains all the $\frac{1}{2}(n-1)(n-2)$ points of the figure which are common to the two primes $[\rho\tau]$, $[\sigma\tau]$, whatever τ may be $(\tau=a, b, 1, \ldots, n)$. If now we replace a, b by $(n+1)$ and $(n+2)$, the notation expresses that the figure is completely symmetrical, and may be initiated by taking any one of the $\frac{1}{2}(n+1)(n+2)$ points as centre of perspective of two simplexes properly taken in the figure; and it is not only thus self-dual, but there exists a quadric locus, expressed by a single quadratic equation connecting n homogeneous coordinates of the space, in regard to which the prime $[\rho\sigma]$ is the polar of the point $(\rho\sigma)$, for every one of the $(n+2)$ values of ρ and σ.

A remark may be added in regard to a particular configuration, not referred to in the text, especially as this can be used to illustrate the theory, explained in pp. 165–173 of Vol. I, of the representation of imaginary points and lines by means of triads of real elements. The figure referred to is that of nine points in a plane lying by threes on twelve lines, four of which pass through every one of the points; so that the join of any two of the nine points contains another of them. Take any four of these points of which no three lie in a line, so that the join of two of them meets the join of the other two in a fifth point of the nine; denote these four by the symbols A, B, C, D, subject to $A+B+C+D$, so that the fifth point on the pair of joins is $A+C$ or $B+D$. Then it can be proved that the nine points are those represented in the scheme

$$\begin{vmatrix} A, & D_1, & D \\ A_1, & A+C, & C_1 \\ B, & B_1, & C \end{vmatrix}$$

in which, if ω denote one of the two roots of $\theta^2+\theta+1$, we have

$$A_1=A-\omega B, \quad B_1=B-\omega C, \quad C_1=C-\omega D, \quad D_1=D-\omega A;$$

these lead to

$$A+B_1-\omega^2C_1=0, \quad B+C_1-\omega^2D_1=0, \quad C+D_1-\omega^2A_1=0, \quad D+A_1-\omega^2B_1=0,$$

and the twelve lines of the figure are those containing the triads of points in the six lines and columns of the scheme, with the six containing triads of the scheme in which no two elements are in the same row and column (corresponding to the terms in the expansion of a determinant).

Thus also the figure consists of two plane tetrads of points, each of four points joined in order, which are inscribed and also circumscribed to one another, these tetrads having the same diagonal point which is the ninth point of the figure. The twelve lines consist then of the four joins of both tetrads together with the two diagonals of both tetrads.

Recurring to the representation explained in Vol. I, p. 165, it follows that the four imaginary points of the configuration, other than the points A, B, C, D, $A+C$, are representable by triads of points on the cross joins of A, C with B and D, requiring no additional points but the intersections of these cross joins; and that the six imaginary lines of the configuration, other than the six joins of A, B, C, D, are likewise representable by triads of lines already drawn. This interesting remark is due to Mr T. G. Room.

By proper choice of coordinates the nine points may be given as the intersection of three lines represented by $u \equiv (x^2-y^2)(az+y)=0$, with three lines represented by $v \equiv (x^2-z^2)(ay-z)=0$, where $a^2=-3$. The Hessian of the cubic curve $u+\lambda v=0$ is a curve of the same form, $u+\mu v=0$.

For this Configuration, see also Steinitz, *Enzykl. Math. Wiss.* III, 1, p. 491; and for the group of possible interchanges of its points, Weber, *Algebra*, II, 1896, p. 345, and Dickson, *Linear Groups*, 1901, p. 77. Also Steiner, *Werke*, II, p. 435 and Netto, *Combinatorik* (1901), p. 202.

p. 221. Cf. also Hudson, *Kummer's Quartic Surface* (1905), p. 43.

p. 227. Reference may be made also to Veronese, *Math. Ann.* XIX (1882), p. 177, as well as to the paper by the same referred to on p. 235 of the text.

INDEX

(The numbers refer to pages. See also p. 237 and the Additions, p. 239.)

Absolute conic, properties relative to, 166; by projection from a quadric, 196

Angle between two circles interpreted on a quadric, 193

Angle properties for a circle, 75

Angular intervals for a triangle, 177, 180, 181

Anharmonic, or cross-ratio, symbol for four points, 166; example of, 210

Apolar triads, 114, 117, 124, 147

Asymptotes of a conic, 77

Auxiliary circle of a conic, 85, 88, 119, 133

Axes and foci of a conic, 77, 160

Axis of relation, or axis of homography, 16

Bauer, 235

Bipunctual conic, in regard to a triad of points, 123

Bolyai-Lobatschewsky geometry, angle of parallelism, 182; circle, 182; equidistant curve, 183; horocycle, 183; interpretation with minimal lines as circles, 203

Boscovich, 88

Brianchon's theorem, 25, 235

Burnside, 186

Caporali, 236

Castelnuovo, 236

Cayley's measure of interval deduced from Laguerre's, 195–197; Cayley-Salmon lines, 227

Centre and axes of a conic, equation referred to, 118

Centres of similitude of two circles, 112

Centroid of three points, 64

Circle: definition and centre, 65; circles at right angles, or cutting orthogonally, 66; coaxial circles, 66; limiting points and radical axis, 66; inversion in regard to, 67; Miquel's theorem, 70; circles determined by threes of four points meet in a point, and their centres lie on a circle, 71, 72; angle properties for, 76; common chords of circle and conic, 79; through three points whose joins touch a parabola, contains the focus, 81; auxiliary circle of a conic, 85, 88, 119, 133; equation of, 107, 108; circle through centres of similitude of two circles, with centre on line of centres, 132; after Lobatschewsky, 182; angle between two circles interpreted on a quadric, 193

Cone, circular sections and focal lines, 209

Configurations, some examples of, 212; a primary theorem, 218; see also Note II

Confocal conics, 79, 119; polars of fixed point in regard thereto, touch a parabola, 90; intersection of tangents at right angles describes a circle, 133

Conic, Absolute, 166, 196

Conic constructed: having two triads, lying on another conic, as self-polar triads, 45; given self-polar triad and pole and polar, 46; given self-polar triad and two points, 46; given three points and two pairs of conjugate points, 165

Conic, first properties: definition, 10; degenerate, 10; tangent of, 11, 14; Maclaurin's definition of, 13; Pascal's theorem, 13, 16, Note II; Related ranges and involution on, 14; symbols applied to, 16, 97; polar lines, 20; dual definition, 22; two through four points to touch a line, 24; four through two points to touch three lines, 39; dual of this, 41; conics through two points containing the threes of four given points, 61; conic outpolar to another, 33; triangularly circumscribed, 48

Conics associated with a triad, 53, 121; conic harmonically conjugate to a given one, 122; see also *Feuerbach's theorem*, *Nine points circle*, and *Bipunctual*

Conics through four points; their common self-polar triad, 23; cut

an arbitrary line in pairs of points in involution, 12; conjugate points in regard thereto, 38; poles of a line in regard thereto, 41; two such, whose eight tangents at the common points meet in two points, 42; Poncelet's porism, 55
Conics touching four lines: tangents from a point are in involution, 25; director circles are coaxial, 80
Conics, two, reduced equations, 146
Conjugate diameters of a conic, 77
Conjugate lines, 26, 31
Conjugate points, 28, 31; given three points of a conic, line conjugate to one join meets other joins in conjugate points, 28; conjugate points in regard to conics through four points, 38; conjugate points, lying on two conics, envelope of join of, 127; conjugate imaginary points, 158
Contact, point of, of conic, with one of five given tangents, 25
Coordinates of a point, 97; of a line, 99
Correspondence of two points in regard to conics through four points, 39; of foci of conic touching three lines, 77; of points of a conic, 135
Cremona, 219, 235, 236
Cross-ratio, or anharmonic ratio, for four points, 166; example of, 210
Cubic surface with node, applied to Pascal's figure by Cremona, 229
Curve of third order, 229

Delambre's formulae in spherical trigonometry, 205
Desmic tetrads, 213, 233
Director conic of a conic, 26; director circle, 80, 95, 125, 127, 151
Directrix of a parabola, 81
Discussion of a turning point in the logical theory, 153
Distance, Riemann's formula in space of constant curvature, 199
Double points of an involution, 3, 158
Duality in regard to a conic, 21, 22; dual of Pascal's theorem, 25

Eleven-point conic, 41
Elliptic, hyperbolic and parabolic plane, 186
Elliptic integral, from two conics, 211
Envelope of a line cutting two conics in harmonic conjugates, 44, 125; of polar lines of a point in regard to confocal conics, 90, 120; of line meeting two conics in points conjugate in regard to a third conic, 127; the same generalised, 134; of line joining points of two circles bisected by radical axis, 132
Equation of a line, 98; of a point, 100; of a conic, 101; of tangent, and polar, 104; of a circle, 107, 108; of a conic referred to self-polar triad, 107; of coaxial circles, 109; of conic referred to centre and axes, 118; of confocal conics, 119; of a plane, 190, 191
Equidistant curve, 183
Euclid's distance, and triangle, 184, 185
Exponential function, 157
Extent, between two lines, in elliptic plane, equal to twice their angular interval, 188; of a plane, 188, 201; of a triangle, 188, 203; of a circle, 189
Exterior and interior points of a real conic, 159, 164

Feuerbach's theorem, 59, 75, 89, 113, 122; Hamilton's extension of, 59, 117
Fifteen points on fifteen lines, three on each, Note II
Figure determined by $(n+2)$ points, or hyperplanes, in n dimensions, 218
Focal distances for a conic, in general form, 209, 211
Focal lines of a cone, 209
Foci and axes of a conic, 77; of a parabola, 81
Fontené, 75
Four lines in four dimensions, figure derived from, 225

Gaskin's theorem, for conics through two points outpolar to another conic, 48
Gauss, 154, 237
Grossmann, 235

Hamilton's modification of Feuerbach's theorem, 59, 117
Harmonic, condition for two pairs of points to be, 111; harmonic images, 37
Hatton, 62

Helmholtz, 237
Hesse's theorem, 31
Hessian line of three points of a conic, 30, 124; of a circle, 148; Hessian point of three tangents, 30; of a parabola, 149
Horocycle, construction of, 183; another view of, 203
Hyperbolic, parabolic and elliptic plane, 186; see *Lobatschewsky*

Imaginary points, 9, 158, 161; imaginary and real symbols, 154; imaginary conic, 163
Inpolar conic, 33-36; algebraic condition for, 142; expressed by sum of four squares, 144, 218
Interior and exterior points of a real conic, 159, 164
Intersections of two conics, number assumed, 9
Interval of two points, or two lines, after Laguerre, 166, 167; interval in regard to an Absolute conic, 168; interval for circumference of circle, 189
Invariants of two conics, 141; their geometrical significance, 142
Inversion in regard to a circle, 67; inverse of a circle is a circle or line, 68; circles at right angles invert into circles at right angles, 69; circle and two inverse points invert into similar figure; inverse of centre of circle, 69; Feuerbach's theorem by inversion, 74
Involution, 2; common pairs of two involutions, 3, 158; composition of a projectivity from two involutions, 6, 50; on a line by conics through four points, 4, 12; dual of this, 4, 25; on a conic, 14; symbolic representation of, 111; reality of double points, 158; general involution, 135; determined by two sets, 136; determined by conics through four points, 138; apolar triads as particular case, 147

Kirkman, 227, 235
Klug, 236

Laguerre interval, 167; angle interpreted on a quadric, 192, 196
Lambert's quadrilateral, 181
Length, and movement, 169
Limiting points of coaxial circles, 66
Lobatschewsky-Bolyai geometry, angle of parallelism, 182; circle, 182; equidistant curve, 183; horocycle, 183; interpretation with minimal lines as circles, 203
Locus of point from which tangents to two conics form a harmonic pencil, 125; of intersection of tangents at right angles of two confocals, 133; of poles of line in regard to conics through four points, 4
Logarithmic function, 157

Maclaurin's definition of a conic, 13
Magnitude, order of, for real symbols, 156
Measurement in regard to two points, 166; in regard to a conic, 170, 178, 184; comparison of the several methods, 186, 187
Metrical properties, 63; metrical geometry in a plane with two Absolute points deducible from geometry on quadric surface, 189
Middle point of two points, 63
Miquel's theorem, for a triad of points and three circles, 70
Morley, 114
Moulton, 186
Movement, 169

Nanson, 61
Napier's analogies, 206
Nine points circle, see *Feuerbach's theorem*; of self-polar triangle of parabola contains the focus, 83; is locus of middle point of orthocentre and point of circumcircle, 89; point of contact with inscribed circle, 114
Non-Archimedean (non-Eudoxian) symbols, 155
Normal of a conic, 91; four can be drawn from an arbitrary point, 91; feet lie on a rectangular hyperbola, 91; three of the feet and the opposite of the other lie on a circle, 93, 119; normals to a parabola from a point, their feet lie on a circle through the vertex, 94; condition three normals meet in a point, 119

Orthocentre of three points, 64
Orthogonal circles, condition for, 110; line through centre of either cuts

circles harmonically, 66, 126; invert into orthogonal circles, 69
Outpolar conics, 33-36; algebraic condition for, 142; and triangularly circumscribed conic, 146

Pappus' theorem, complete figure, 215
Parabola, 81; obtained as envelope of polars of a point in regard to confocal conics, 90
Parabolic, or Euclidean, plane, 186
Parameter expression for a general conic, 103; for conic referred to its centre and axes, 118
Pascal's theorem, 13; and related ranges of points on a conic, 15; converse of, 16; dual of, 25; Pascal's *hexagrammum mysticum*, 219; Pascal line, symbol for in general figure, 223, 227; Pascal's figure derived from four dimensions, 224
Pedal line of three points of a conic, 29; of a circle, 71; pedal of conic from focus is auxiliary circle, 85; of parabola is tangent at vertex, 82; pedal circle of a point for three lines, 87; when point moves on a line through circumcentre, 88; pedal line for a triad in involution, 139
Perpendicular lines, 63
Plane, equation of, 190, 191
Plücker, 227, 235
Poincaré, 197
Polar line and point, in regard to a conic, 20; self-polar triad, 23; polars of three points give a triad in perspective with original, 27, 31, 123; polar lines of a point in regard to conics through four points meet in a point, 37; poles of a line in regard to conics through four points lie on a conic, 41; dual of this, 42, 90; polar reciprocal of one conic in regard to another, 47, 91, 140, 147; polars of a point in regard to conics through fives of six points touch a conic, 61; polar reciprocal of a certain parabola gives the rectangular hyperbola of Apollonius, 91
Polynomial, roots of, 110, 135, 157
Poncelet's theorem, for porism of tangents of conics through four points, 55, 96
Positive and negative real symbols, 155
Projectivity on a conic resoluble into succession of two involutions, 50

Quadratic system of conics, inpolar to another conic, 145
Quadric surface, elementary properties of, 190
Quadrilateral with three right angles, 181

Radical axis of two circles, 66; radical axes of three circles meet in a point, 70, 112
Range of points on a conic, 14, 52; on variable line, by conic and joins of three points of this, 28
Real and imaginary points, 9, 160; real and imaginary symbols, 154; real and imaginary conic, 163
Reality of common harmonic conjugates with respect to two given pairs, 158; reality a relative term, 160; reality of common self-polar triad of two conics, 164
Reciprocal polar of one conic in regard to another, 47, 140, 147
Rectangular hyperbola, definition, 83; contains orthocentre of three of its points, 84; conjugate diameters of, examples of, 84; through three points, locus of centres, 88
Related ranges on the same line, 1; condition two ranges be related, abbreviated argument for, 8; relation of two ranges reduced to two involutions, 6, 50; envelope of line joining corresponding points of two related ranges on a conic, 52
Richmond, 219, 236
Riemann's space, as derived from projective space, 199
Right angles, lines at, 63, 110; circles cutting at, 110, 111

Salmon, 59, 227, 236
Schoute, 236
Segre, 236, 238
Self-polar triad, in regard to a conic, 23; of two conics, algebraic determination of, 140; self-conjugate tetrad, Hesse's theorem, 31
Similitude, centres of, for two circles, 112; circle through, with centre on line of centres, 112, 132
Soal, 62

Sphere, division of surface into four equal and congruent triangles, 197, 207
Spherical and elliptic geometry, 178; spherical geometry and plane geometry, 196, 197
Staudt, von, 232, 235; theorem of Seydewitz, 28
Steiner, 72, 216, 231, 232, etc.
Sylvester, 220, 235
Symbols applied to involutions, 4; applied to initial theorems for a conic, 16; symbol of a line, 99, 100; applied to general theory of conics, Chap. III; applied to measurement of intervals, Chap. V
Syntheme, 220

Tangent of a conic, 11, 14; when tangents at intersections of two conics meet in two points, 42, 66; tangents of two conics, at intersections with a line, meet on another conic, 54; general tangent of a conic which touches three lines at given points, 58; tangents common to pairs of three conics which touch three lines, 60; tangents of confocal conics which are at right angles meet on a circle, 133
Tangential coordinates, and tangential equation, 100, 105, 106; of confocal conics, 119
Taylor, H. M., 61; J. P., 75; C., 88
Tetrads of points of a conic whose joins touch another conic, 96, 127; see also *Poncelet's theorem*
Three conics related in a particular way, 149
Triads, two, of points of a conic: their joins touch another conic, 29; two self-polar triads are points of another conic, 35, 45, 48; three triads of points of a conic are in perspective with another triad, 52; relation of a conic to a triad of general points, 53; centroid and orthocentre of a triad, 64
Triangle, definition of, 173; trigonometrical formulae for, 176, 177, 179
Triangularly circumscribed conic, 48, 145
Two conics, enumeration of reduced equations in all cases, 146
Two-two relation on a conic, condition it be derived from a general involution, 137

Veronese, 235
Vertex of a parabola, 82

Wallace line (pedal line), 29, 71
Weill, 139
White, 236

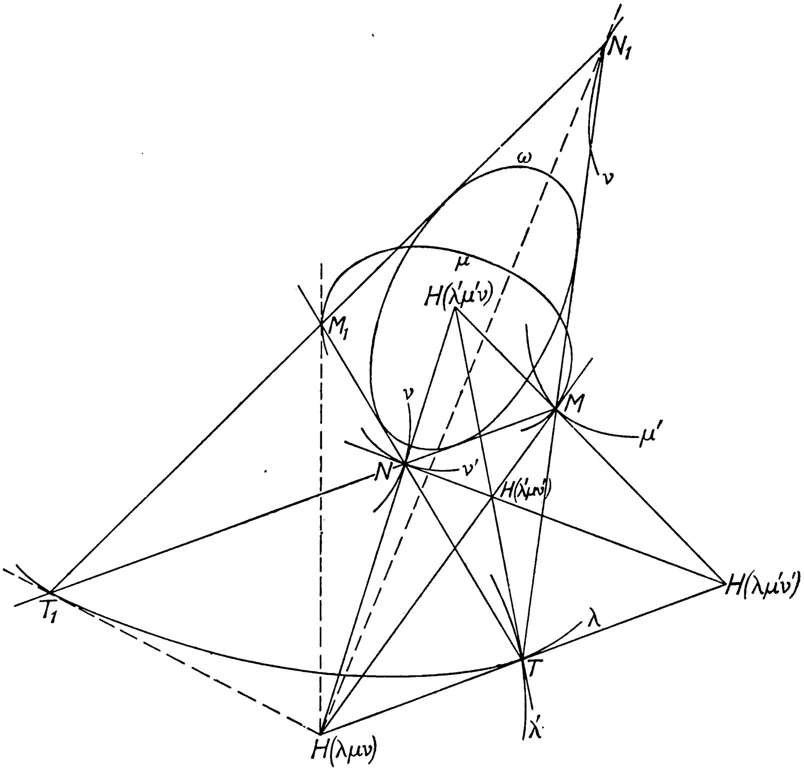
CAMBRIDGE: PRINTED BY
W. LEWIS, M.A.
AT THE UNIVERSITY PRESS

PRINCIPLES OF GEOMETRY

CAMBRIDGE UNIVERSITY PRESS
C. F. CLAY, Manager
LONDON : FETTER LANE, E.C. 4

LONDON : H. K. LEWIS AND CO., Ltd.,
136, Gower Street, W.C. 1
NEW YORK : THE MACMILLAN CO.
BOMBAY
CALCUTTA } MACMILLAN AND CO., Ltd.
MADRAS
TORONTO : THE MACMILLAN CO. OF CANADA, Ltd.
TOKYO : MARUZEN-KABUSHIKI-KAISHA

ALL RIGHTS RESERVED



N_1
ν
ω
μ
$H(\lambda'\mu'\nu)$
M_1
ν
M
μ'
N
ν'
$H(\lambda'\mu\nu')$
$H(\lambda\mu'\nu')$
T_1
λ
T
λ'
$H(\lambda\mu\nu)$

PONCELET'S PORISM AND CONFOCAL QUADRICS
(see p. 116)

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., LL.D., F.R.S.,
LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY, AND FELLOW OF
ST JOHN'S COLLEGE, IN THE UNIVERSITY OF CAMBRIDGE

VOLUME III

SOLID GEOMETRY

QUADRICS, CUBIC CURVES IN SPACE, CUBIC SURFACES

CAMBRIDGE
AT THE UNIVERSITY PRESS
1923

PRINTED IN GREAT BRITAIN

PREFACE

THE present volume is devoted to geometry in three dimensions. The discussion of the logical standpoint, to which sufficient space has been given in the preceding volume, is left aside; and, from a desire to limit the size of the volume, many things are omitted which might well have been included. What is given may, however, be regarded as essential to any student who professes to have received a mathematical education. The aptitude for geometrical construction in space, important as it is in the applications of mathematics to physics and engineering, receives, in our educational system at present, less training than it deserves. It is the writer's hope that this volume may help to emphasize this; and may convey to readers something of the fascination and freedom which belongs to the reduction of intricate geometrical relations to the properties of a constructed figure. Only by such methods, moreover, can progress be made beyond the first principles of the subject.

Up to the end of Chapter III, this volume was in type when death severed an association to which the writer owed more help than he can well express. In business, James Bennet Peace was clear and honest; in friendship, constant and self-regardless; many beside the writer deplore his loss. To him, and to the co-operation of the other members of the Staff of the University Press, great acknowledgment is due.

H. F. BAKER.

14 *July* 1923.

“TEODORO REYE, ..., *che avevo cominciato ad ammirare fin da studente, leggendo la sua classica* Geometrie der Lage; *e col quale poi non avevo tardato ad entrare in relazione scientifica, ed anche personale, sì da poter apprezzare, oltre al valore del matematico, la grande bontà d'animo dell'uomo: vero gentiluomo!*

Nato a Cuxhaven il 20 giugno 1838, ..., era passato verso il 1864 ad insegnare nel Politecnico di Zurigo. ......Aveva esordito nella scienza con lavori di Fisica matematica e di Meteorologia. Ma, poichè a Zurigo il corso del Culmann, fondatore della Statica grafica, si basava sulle teorie della Geometria di posizione, e il classico trattato di Staudt era troppo difficile per gli studenti; Reye fu condotto ad insegnare quelle teorie e ad esporle in un nuovo trattato, che uscì in due parti nel 1866 e nel 1868. ....

....

Artista non meno che scienziato, Reye ha molto contribuito a quella grandiosa e pure snella costruzione scientifica che è la Geometria di posizione, introducendo o svolgendo idee semplici e geniali; studiando, com' è carattere di essa, svariate figure in tal maniera da illuminarne di vivida luce le proprietà più profonde, e i legami che le uniscono. Non solo ci ha fatto conoscere nuovi veri; ma ci ha procurato squisiti godimenti estetici, quali solo può dare il bello. Onore e gratitudine a Lui!”

Corrado Segre, *Rendiconti...dei Lincei,* 2 Aprile 1922.

TABLE OF CONTENTS

CHAPTER I. INTRODUCTION TO THE THEORY OF QUADRIC SURFACES

PAGES

Preliminary remark. Equation of a quadric surface . . . 1
Definition of a quadric surface by means of its lines . . . 1—3
Definition of a quadric cone 4
The duality of the definition of a quadric 4
Application of the symbols; equation of a quadric . . . 5—7
Quadric defined by two conics having two points in common, and a line meeting these conics . . . . 7—10
Three conics of which every two have two points in common lie on a quadric 10
Two conics which have two points in common lie on two quadric cones 10
Quadric defined by two dually related central systems . . . 11—13
Quadric representing a plane with two fundamental points . . 13, 14
General homogeneous quadric polynomial in four variables . . 15, 16
Equation of a quadric cone 17
Condition for quadric form to represent a cone 18
Conditions for quadric form to represent two planes . . . 19
Conditions for quadric form to represent a repeated plane . . 20
General properties of a quadric given by its equation . . . 20—26
Tangent cone, or enveloping cone, of a quadric 26
Polar plane, and tangent plane, descriptively defined . . . 27
Polar line of a line in regard to a quadric 28
Polar point of a plane in regard to a quadric 29, 30
Polar point of a plane in regard to a quadric, expressed by the symbols 31
Examples of the general properties of quadrics (Exx. 1—19) . 31—56
Ex. 1. *The self-dual character of a quadric, in particular form* . 31
Ex. 2. *The same in general* 32
Ex. 3. *Two conjugate sections have tangent lines at a common point harmonic in regard to the generators; the dual of this* . . 32, 33
Ex. 4. *Self-polar tetrad in regard to a quadric* 33, 34
Ex. 5. *Tetrad of points whose opposite joins are conjugate lines; self-conjugate pentad; construction from two tetrads in perspective; analytical treatment; expression of a quadric by five squares* 34—40
Ex. 6. *A general self-polar figure in space of any number of dimensions* 40, 41

a 5

PAGES

Ex. 7. *Joins of two reciprocal tetrads are generators of a quadric* 41, 42

Ex. 8. *Symmetry of two particular quadrics arising in preceding example* . . . . 42—44

Ex. 9. *Dandelin's figure of generators; its application to Steiner points in Pascal's figure* . . . . 44—47

Ex. 10. *Construction of a self-conjugate hexad. Expression of a quadric by six squares* . . . . 47—49

Ex. 11. *Construction of an incomplete tetrad, or pentad, or hexad, of points of a quadric* 49, 50

Ex. 12. *Condition for one quadric to be outpolar to another* . . 50—52

Ex. 13. *A property of two self-polar tetrads* 52

Ex. 14. *Quadric for which two proper tetrads are reciprocal* . 53

Ex. 15. *A theorem of Chasles for the intersections of a quadric with the joins of four points* 53, 54

Ex. 16. *A further theorem for the intersections of a quadric with the joins of four points* 54

Ex. 17. *Bipunctual quadric, the harmonic conjugate of two planes* 54

Ex. 18. *The equation of a quadric touching the four planes of reference* 55

Ex. 19. *A birational transformation of a quadric into itself. Its geometrical meaning* 55, 56

The coordinates and equations of a line in three dimensions. Elements of the theory of a linear complex . . . . 56—69

Ex. 1. *The analytical conditions for a plane to contain a line* . 57

Ex. 2. *Common element of a line and point, or of a line and plane* 57, 58

Ex. 3. *The condition of intersection of two lines* 58

Ex. 4. *Lines of a quadric as a linear system* 58

Ex. 5. *The tangent planes from a line to a quadric* . . . 58

Ex. 6. *Coordinates of the polar line of a line in regard to a quadric* 58, 59

Ex. 7. *Condition a line should touch a quadric* 59

Ex. 8. *Change of line coordinates by change of reference points* . 59, 60

Ex. 9. *Generators of a quadric, of one system, belong to three linear complexes* 60

Ex. 10. *First properties of a focal system and linear complex. Analytical definition. Derivation from generators of a quadric. Sylvester's derivation. Polar lines. Linear complex consists of those lines of a focal system which intersect, and coincide with, their polars. Invariant of two linear complexes* . . . 60—66

Ex. 11. *Pole of a plane in a focal system* 66

Exx. 12—14. *Geometrical theorems. A theorem of Sylvester's* . 66

Ex. 15. *Chasles' generation of a linear complex* 66, 67

Exx. 16—22. *The focal system and Moebius tetrads* . . . 67, 68

Ex. 23. *The two lines common to a linear complex and a quadric* . 68

Ex. 24. *Theorem for the mutual perpendiculars of lines* . . 68

PAGES

Ex. 25. *The tangent lines of a cubic curve belong to a linear complex* 68
Ex. 26. *A skew pentagon determines a focal system* . . . 68
Ex. 27. *The polar line of a line in regard to a general quadric* . 69
Ex. 28. *The tangent lines of the curve of intersection of two quadrics; eight of these meet an arbitrary line* 69

CHAPTER II. RELATIONS WITH A FIXED CONIC. SPHERES, CONFOCAL SURFACES; QUADRICS THROUGH THE INTERSECTION OF TWO GENERAL QUADRICS

Parallel lines and planes. Middle points. Lines and planes at right angles 70, 71
Exx. 1—3 71
Spheres and Circles 72, 73
Exx. 1—5. *Plane sections of a sphere* 73
Relations of two spheres. Coaxial spheres . . . . . . 74—76
Ex. 1. *Two circles with two common points lie on a sphere* . . 77
Ex. 2. *The radical planes of the pairs of three spheres meet in a line* 77
Ex. 3. *Any three circles are met in two points by another properly taken circle* 77
Ex. 4. *Analytic treatment of the relations of two spheres. Centres of similitude* 77, 78
Ex. 5. *A sphere can be constructed of given centre to pass through a given point* 78
Ex. 6. *Sphere determined by perpendiculars from a fixed point to planes through another point* 78
Ex. 7. *Theorems for an orthogonal tetrad* 78
Relations of a quadric in general with the absolute conic. The circular sections 79
Umbilici of a quadric. Generators through these 80
Relations of a quadric cone with the absolute conic. Cone of revolution; or right circular cone 81
Ex. *Analytical conditions for a right circular cone* . . . 82
Quadric touching the absolute plane 82, 83
Ex. 1. *Umbilici of a paraboloid* 83
Ex. 2. *Two circular sections of a quadric not in parallel planes lie on a sphere* 83
Ex. 3. *Condition a plane should give a circular section* . . 83
Ex. 4. *Centres of plane sections of a quadric* 83, 84
Ex. 5. *Locus of line of intersection of two perpendicular tangent planes of a quadric* 84
Ex. 6. *Locus of normals of a quadric at points of a generator* . 84

PAGES

Line of striction on a quadric 84

Exx. 1—6. *Properties of the line of striction* 85, 86

The director sphere of a quadric. Analogue of Gaskin's theorem . 86—89

Ex. 1. *The intersection of the director sphere with the quadric* . 89, 90

Ex. 2. *Condition two generators should be at right angles* . . 90

Ex. 3. *Point on quadric and director sphere at which three confocals coincide* 90

Ex. 4. *Generalisation of director sphere* 90, 91

Ex. 5. *Locus of intersection of three perpendicular tangent lines of a quadric* 91

Confocal quadrics, defined algebraically 91, 92

Points through which confocals are not distinct 92

Ex. 1. *The umbilicar generators lie on a quartic surface* . . 92, 93

Ex. 2. *Polar reciprocal of a confocal in regard to original* . . 93

Axis of a plane in regard to a system of confocals. Plane touching all the confocals 93

Alternative ways of initiating the system of confocal quadrics. Dual of quadrics through common curve of two quadrics. By common tangent planes of two quadrics 94

Ex. *Confocal paraboloids* 94, 95

The relations of the four focal conics. Locus of vertex of right circular cone circumscribing a quadric. Section of quadric by plane at right angles to a focal conic has a focus on this conic 95

Ex. 1. *Equation of enveloping cone of quadric drawn from point of focal conic* 96

Ex. 2. *Generalisation of focal distance theorem for a conic. The geometrical meaning* 96

Ex. 3. *Planes through a point whose sections with the quadric have the point as focus* 96

Ex. 4. *A system of right circular cones with given vertex* . . 96

The axes of a confocal system of quadrics 97

Ex. *The condition for a line to be one of these axes* . . . 97, 98

The axes of the confocal system lying in a plane touch a conic; those through a point lie on a quadric cone 98

Ex. *The axes are the lines which are met by the principal planes in related ranges* 99

The tetrahedral complex. The associated conics and cones . . 99

Various geometrical conditions for an axis 99, 100

Ex. 1. *Condition two polar lines should be at right angles* . . 100

Ex. 2. *Axes of the confocals lying in a plane* 100

Ex. 3. *The parabola touched by the axes lying in a plane is inpolar to the section of a confocal by this plane* 101

Ex. 4. *An axis of the confocals meets one of the surfaces in points whereat the normals intersect* 101

PAGES

Ex. 5. *Axis of the confocals as axis of a plane section of one of the confocals* 101
Ex. 6. *Tangent planes of a surface where polar line of an axis meets it* 102
Ex. 7. *Plane of which a line of the tetrahedral complex is the axis* 102
Ex. 8. *Normal of one confocal meeting another of the surfaces* . 102
Ex. 9. *The locus of the conics which are touched by the axes intersecting a line is a rational quartic surface* 102
The normals of a quadric, and of confocal quadrics, which pass through a point . . . . 102—104
The polar planes of a point in regard to the confocals determine a cubic developable 103
Ex. 1. *The cubic curve defining the six normals from a point to a quadric* 104
Ex. 2. *General definition of a cubic curve in space illustrated in this case* 104
Exx. 3, 4. *The planes containing two triads of feet of normals from a point to a quadric* 104, 105
Ex. 5. *Six points determining the cubic curve* 105
Ex. 6. *A property of the six feet of normals* 105
Ex. 7. *The normals of a quadric at the points of a plane section determine a ruled quartic surface with a cubic curve as double curve* 105
The normals of the confocals at right angles to a given line; and the polar lines of this line 106, 107
Ex. 1. *Equation of locus of polar lines of a given line in regard to the confocals* 107
Ex. 2. *Equations of the cubic curve which is the locus of points of contact of tangent planes to the confocals from a given line* . 107
Special positions for the line 107, 108
Ex. *Points of contact of parallel tangent planes of the confocals lie on a conic* 108
The enveloping cones drawn from a point to the surfaces of the confocal system; and the tangent planes drawn from a line. Two of the confocals touch a line, with their tangent planes at right angles. The generators of three intersecting confocals are focal lines of enveloping cones. 108, 109
Ex. 1. *Equation of enveloping cone to a quadric, and of the quadric itself, referred to normals at vertex. Normals lie in threes on four planes* 110
Ex. 2. *Chasles' theorem of correspondence between two systems of confocals* 110, 111
Ex. 3. *A property for a line, touching two confocals, leading to the definition of geodesics* 111, 112

PAGES

Ex. 4. *The angle between two geodesic tangents of a line of curvature reduced to the angle between two tangents of a conic* . . 112, 113

Ex. 5. *Quadric of rigid generators transformable into a confocal* . 113—116

Ex. 6. *The generators, of one system, of confocal surfaces, at points of curve of intersection of two other fixed confocals, generate a conic on a principal plane. Proof of Poncelet's porism for conics* 116—118

Ex. 7. *Ivory's theorem, in generalised form* 118

Ex. 8. *The developable of common tangent planes of the confocals*. 118, 119

Dual method of developing the theory of confocals . . . 119, 120

Ex. 1. *Equation of quadric locus of polar lines of a fixed line in regard to quadrics through the intersection of two* . . . 121

Ex. 2. *Equation of cone touched by tangent planes of the family at points where surfaces are met by a line in a principal plane* . 121

CHAPTER III. CUBIC CURVES IN SPACE. THE INTERSECTION OF TWO OR MORE QUADRICS

The curve of intersection of two quadrics. Enumeration of possible cases 122, 123

Exx. 1—9. *Equations of quadrics having the respective intersections* 123—125

The cubic curve in three dimensions 125

Definition by two quadric cones. Tangent line and osculating plane of curve 126, 127

Definition by related axial pencils of planes 127

General quadric containing the curve 128

Cubic curve in space determined by six points 129

Cubic curves, lying on a quadric, passing through five points . 129

Intersections with generators, of cubic curve lying on a quadric . 129

Cubic developable dual to a cubic curve 130

The cubic curve as a self-dual construct 131

Surface formed by tangent lines of cubic curve 132

Four tangents of cubic curve meet any line 132

Concrete illustration of cuspidal edge of a developable . . . 132

Cubic developable considered initially 133

Examples in regard to the cubic curve in space (Exx. 1—31) . 134—149

Ex. 1. *Tangent line and osculating plane of curve at a point belong to cone projecting curve from this point* 134

Ex. 2. *Points and osculating planes of curve define a focal system* 134

Ex. 3. *Tetrahedral complex of chords depending upon four points of the curve* 134

Ex. 4. *Equation of general quadric containing the curve. Coordinates of chord of the curve through an arbitrary point* . . 134, 135

Ex. 5. *A* (1, 2) *correspondence of generators of a quadric determining a cubic curve on the quadric* 135

PAGES

Ex. 6. *Quadrics through six points of the curve determine an involution on any chord* 135
Ex. 7. *Various generations of a cubic developable* . . . 135
Ex. 8. *Cone touched by planes joining a point of the curve to the chords which meet an arbitrary line* 135, 136
Ex. 9. *Planes meeting six lines and a common transversal of these in points lying on a conic belong to a cubic developable* . . 136
Ex. 10. *The form apolar to any system of binary forms of the same order* 136
Ex. 11. *The Hessian pair of three points of the curve* . . . 136, 137
Exx. 12, 13. *Four mutually inscribed tetrads arising for the curve* 137, 138
Exx. 14, 15. *General case of four mutually inscribed tetrads* . 138
Ex. 16. *The general involution of sets of four points of the curve* . 138, 139
Ex. 17. *Number of common points of two cubic curves lying on a quadric* 139
Ex. 18. *A cubic curve to pass through five points and have a given chord* 139
Ex. 19. *A cubic curve to pass through four points and have two given chords when possible* 139, 140
Ex. 20. *A cubic curve to pass through three points and have three given chords* 140
Ex. 21. *A cubic curve to pass through two points and have four given chords* 140
Ex. 22. *A cubic curve to pass through one point and have five given chords* 140, 141
Ex. 23. *Two general cubic curves in space have ten common chords* 141, 142
Ex. 24. *There are six cubic curves having six given lines, of general position, as chords* 142
Ex. 25. *A cubic transformation giving another proof of preceding results. The condition that four lines should be tangents of a cubic curve* . . . . 142—145
Ex. 26. *A cubic curve to meet each of four lines in a pair of points of an involution given thereon* 145
Ex. 27. *The condition for a cubic curve to be outpolar to a quadric. Self-polar tetrads, self-conjugate pentads and hexads thereon* . 145—147
Ex. 28. *Cubic developable inpolar to a quadric* 147
Ex. 29. *Cubic curve containing six points of two self-polar tetrads has join of other two for chord. Three independent quadrics contain the eight points. Associated points* 147, 148
Ex. 30. *Squares of eight points of a cubic curve linearly related* . 148
Ex. 31. *A theorem of Lie, for* complex curves, *in generalised form* 148, 149
The rational curve of order n in space of n dimensions . . 149, 150
Examples thereon (Exx. 1—4). *Number of conics meeting eight lines* 150, 151

PAGES

The curve of intersection of two general quadric surfaces; its irrationality 151

Examples (Exx. 1—3). *The common self-polar tetrad* . . . 151, 152

The determination of a quadric by given conditions . . . 152, 153

Ex. 1. *The condition for two quadrics to intersect in two conics* . 153

Ex. 2. *Three quadrics intersect in eight points* 153

Ex. 3. *A detailed analytical consideration* 153, 154

Ex. 4. *Number of remaining common points of three quadrics with a common curve* 154

Ex. 5. *Weddle surface as locus of vertex of quadric cone containing six points* 154

Ex. 6. *Hesse's sextic curve, locus of vertex of quadric cone containing seven points* 154

Theorems for eight associated points . . . . 154—158

Exx. 1—3. *Four lines, determined by the eight points, which lie on a quadric (in analogy with Pascal's theorem)* 154, 155

Ex. 4. *A second construction for eight associated points* . . 155

Ex. 5. *Two determinations of the eighth point when seven are given* 155, 156

Ex. 6. *Determination of the ninth intersection of two plane cubic curves when eight are given* 156, 157

Ex. 7. *Associated points interpreted in space of four dimensions* . 157, 158

CHAPTER IV. THE GENERAL CUBIC SURFACE; INTRODUCTORY THEOREMS

The theorem of a double-six of lines . . . . 159—161

Reciprocity of the figure. Schur's quadric . . . . . 161—163

Another determination of the double-six, by a porismatic relation 163, 164

The Steiner trihedral pairs . . . . 164—166

Examples of the figure of twenty-seven lines (Exx. 1—5) . . 166—170

Ex. 1. *Numbers of sets of lines of which no two meet* . . . 166

Ex. 2. *The sets of three coplanar lines* 166

Ex. 3. *The number of constants on which the figure depends, and the general cubic polynomial in four coordinates* . . . 166—168

Ex. 4. *Another proof of the reciprocity of the figure* . . . 168

Ex. 5. *A double-six determined by five arbitrary points* . . 168—170

Geometrical definitions of a cubic surface . . . . . . 170—174

Cubic curves having as chords five lines with a common transversal 172

Deduction of a double-six of lines 173

Ex. 1. *Pairs of ranges, of four points, intersections of lines, which are in perspective from a point of intersection of two lines* . . 174

Ex. 2. *Deduction of a pair of Steiner trihedrals* 174

Ex. 3. *The number of such pairs is one hundred and twenty* . 174

PAGES

The definition of a cubic surface in general. The equation of the surface 174, 175

The cubic curves of the surface which belong to a particular double-six . . . . 175—177

Each curve is determined by two points. Two curves of the same system have one common point 175

The cubic curves determined by quadrics through three (suitable) lines 176

A curve of either system lies on a quadric containing a curve of the other system 176

Two cubic curves of complementary systems meet in five points . 176

The quadrics containing the cubic curves are outpolar to the Schur quadric associated with the double-six 177

The five common points of two complementary cubics form a self-conjugate pentad in regard to the Schur quadric . . . 177

Ex. 1. *Consideration of two cubic curves of the surface with one common point* 177, 178

Ex. 2. *A property of plane cubic curves: if conics be drawn through four given points of the curve, the line joining the two remaining intersections passes through a fixed point of the curve. General theorem of coresiduation for a plane curve* 178, 179

Ex. 3. *Illustration of the theorem of coresiduation for curves on the cubic surface* 179, 180

Ex. 4. *Statement of the theorem in general* 181

Ex. 5. *The conics lying in planes through a line of the surface meet the line in pairs of points of an involution* 181

Ex. 6. *All polar quadrics meet a line of the surface in pairs of points of an involution* 182

Ex. 7. *Definition of the cubic surface from two cubic curves with a common point. Number of lines meeting four curves of given orders* 183, 184

Definition of a cubic surface by three related star-figures, or central-systems . . . . 184—186

Meaning of related star-figures 184

Resulting equation of the cubic surface 186

Ex. 1. *The cubic curves, and the quadrics containing these, arising from this form of the equation* 186

Ex. 2. *All Schur quadrics are inpolar to the polar quadrics; there are six linearly independent Schur quadrics in tangential form; they may be expressed by six point-pairs of the Hessian* . . 187

Ex. 3. *The equation of the Hessian for a certain form of the equation of the cubic surface* 187, 188

The representation of the cubic surface on a plane. Representation of the systems of curves of the surface by means of seven systems. . . . . 189—192

PAGES

Ex. 1. *A further way of obtaining the representation upon a plane* 192, 193
Ex. 2. *Representation of complementary systems of cubic curves* . 193
Ex. 3. *Representation of conics of the cubic surface* . . . 194
Ex. 4. *Representation of all cubic curves of the cubic surface* . 194
Ex. 5. *Number of intersections, freedom and genus, for systems of curves of the cubic surface* 194
Ex. 6. *Converse derivation of the cubic surface from six arbitrary points of the plane* 194, 195
Ex. 7. *Given seven lines with a common transversal, there is a point from which the eight lines are projected by planes touching a quadric cone. Corresponding theorems given six lines, or five lines, with a common transversal* 195, 196
Derivation of a plane quartic curve from a cubic surface . . 196—201
The inflexional lines at a point of the cubic surface. They are the generators of the polar quadric of the point 197
The sextic curve of contact of tangent planes from a point of the cubic surface. Projects into a plane quartic curve . . . 198
The sixty-three systems of conics, each touching the plane quartic curve in four points; each system representable by a pair of double tangents of the curve 198
Statement of the facts for a degenerate contact cone containing the tangent plane at the point of projection. Steiner system of bitangents of a plane quartic curve 199, 200
Proof of the results for the case of planes drawn to two skew lines of the cubic surface. The contact cones are those containing the cubic curves, through the point of projection, associated with the two lines. Relation with the Schur quadric. The transversals from a point of the surface to the six pairs of a double-six of lines lie on a quadric cone 200, 201
Ex. 1. *The quadric cone containing the points of contact of five tritangent planes of the cubic surface through a line. A property of cubic curves through these five points* 201, 202
Ex. 2. *The algebraic treatment of the case of two skew lines of the surface* 202, 203
The equation of the cubic surface referred to two Steiner trihedrals. Deduction of the lines of the surface . . . . . . 203—206
The Cremona form of the equation of the cubic surface . . 206
The Sylvester form of the equation of a general cubic surface . 206—208
The polars in regard to a cubic surface. The Hessian surface . 208—210
Ex. 1. *The vertices of a pair of Steiner trihedrals are corresponding points of the Hessian* 210
Ex. 2. *The joins of the eight points of which a given plane is the polar plane* 210

PAGES

Ex. 3. *The remaining intersections with the Hessian of the join of a pair of corresponding points* 211
Ex. 4. *The cubic surface enveloped by the polar planes, in regard to the cubic surface, of the points of a plane, touches the Hessian along a sextic curve* 211
Ex. 5. *A construction for the expression of a cubic surface by six cubes* 211
Ex. 6. *Every line of the cubic surface touches the Hessian in two points. The twenty-four intersections with the Hessian of the lines of a double-six* 211
Ex. 7. *The equation of the Hessian when the cubic surface is expressed by six cubes. The question of the generality of Sylvester's form* 211, 212
The representation of a general cubic surface upon a quadric surface. The twenty-seven elements on the quadric corresponding to the twenty-seven lines of the cubic surface . . . 212—214
Note on the cubic surface with four double points . . . 214—218
The representation upon a plane. The fundamental cubic curves. 215
The tangent cone from any point of the surface breaks up into two quadric cones 216
The inflexional curves of the surface. One touches every fundamental cubic curve 216, 217
Ex. *The envelope sextic curve associated with a point. Proof of a theorem leading to a generalisation of the theorem of the nine-point circle* . . . . 218—221
The Steiner quartic surface. Representation upon a plane . . 221, 222
Ex. 1. *Envelope of polar planes, in regard to a cubic surface, of points of a plane in connexion with Steiner's surface* . . . 222
Ex. 2. *The inflexional curves of Steiner's surface* . . . 222
Ex. 3. *These curves are rational quartics. A property of these* . 222, 223
Ex. 4. *A formula for inflexional curves which includes the cases of the cubic surface with four double points and of Steiner's surface* 223
Ex. 5. *Veronese's surface* 223
Ex. 6. *The general cubic surface derived from two tetrads in perspective. The cubic surface with four double points as a pedal locus. Kronecker and Castelnuovo's theorem* 223

CORRECTIONS FOR VOLUMES I AND II 224

INDEX . . . . 225—228

CHAPTER I

INTRODUCTION TO THE THEORY OF QUADRIC SURFACES

Preliminary remark. We have, in previous volumes, given trouble to emphasizing the view that the use of the algebraic symbols is not necessary to the geometrical theory; and that the use of symbols of any particular system is equivalent to the adoption of definite geometrical restrictions.

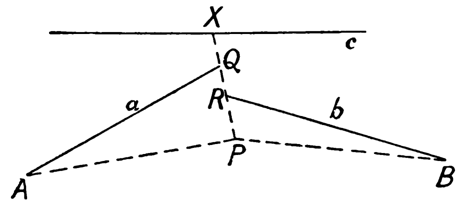
It is, however, often conducive to clearness and brevity, to employ symbols; and it is usual to suppose that the symbols have the same laws of operation as the numbers of ordinary Analysis. Accordingly in the present volume we shall employ such symbols, whenever it seems desirable. The arithmetic notion of the *magnitude* of the symbols, and especially of *infinite* values, remains excluded—as indeed it is in a logical theory of Analysis; and, as heretofore, the *length* and *congruence* of geometrical lines are not employed, save in the conventional sense explained in Volume II (Chap. V).

It will be seen that the utility of the symbols arises chiefly from the use of the *equation of a quadric* (or other) *surface*. We here deduce this from the geometrical definition of the surface. It becomes therefore an interesting problem to replace proofs depending on this equation by direct geometrical deductions from the definition.

Not much space is given in this Volume to the proof, which has already been given in Volume II, p. 191, that a plane is represented by a single equation which is linear in the coordinates; that this is so will be readily understood from the explanation given of the equation of a line, in a plane, in Volume II. A line, in this chapter, is generally given as the join of two points, or the intersection of two planes. The theory of the coordinates of a line is discussed at length in a subsequent section (pp. 56 ff.).

Definition of a quadric surface by means of its lines. We may define a quadric surface, or, as we shall often briefly say, a quadric, in several ways.

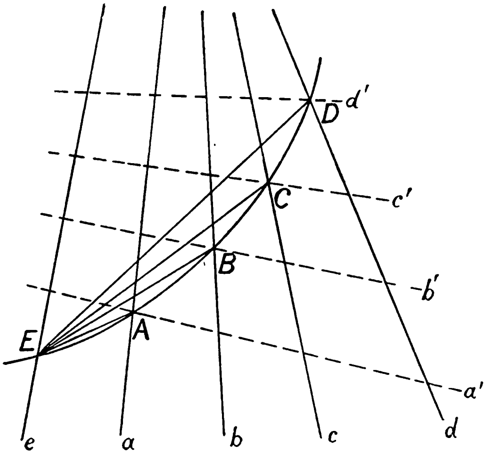
Take three arbitrary lines, in space of three dimensions, of which no two intersect. From any point of one of these can be drawn a transversal to meet both the other lines. Consider the aggregate of all the points on all the transversals so drawn. This does not include all points of space, since it is not in general possible to draw from an arbitrary point a transversal to meet three given lines.



The points of the aggregate which lie in an arbitrary plane lie on a conic. For denote the given lines by a, b, c; let an arbitrary plane meet the lines a, b, respectively, in A and B, and let the transversal drawn from an arbitrary point, X, of the third line, c, meet the lines a, b, respectively, in Q and R, and meet the arbitrary plane in P. Then the two axial pencils of planes, having, respectively, a and b as axes, given by the planes $a(X)$ and $b(X)$ as X varies on c, are related. On the arbitrary plane these give therefore the two related pencils of lines, $A(P)$ and $B(P)$. The locus of P is therefore a conic, passing through A and B; this conic equally passes through the point, say C, where the line c meets the arbitrary plane, for evidently a transversal of a and b can be drawn from this point.

If the points A, B, C are in line, the ray AC of the pencil $A(P)$ is AB, and coincides with the ray BC of the pencil $B(P)$. Then the locus of P consists, beside the line ABC, of a straight line. In other words, the aggregate now being considered, of points lying on all transversals of the three lines a, b, c, contains, in a plane which contains a transversal ABC of a, b, c, beside the points of this line, also the points of another line, say d, which does not meet a or b or c. Every point of this new line, d, is thus a point from which a transversal can be drawn to meet the given lines a, b and c. It is thus clear that the aggregate of points under consideration consists of the points of a system of lines a, b, c, d, ..., infinite in number, of which no two intersect, together with the points of another system of lines, every one of which meets all those of the first system, but of which no two intersect.

This aggregate of points constitutes a quadric. We see that through every point of the locus there pass two lines, consisting of points belonging to the locus, and the plane of these two lines contains no other points of the locus. But a general plane meets the quadric in the points of a conic. Through points $E, A, B, C, D, \ldots$ of this conic will pass lines e, a, b, c, d, ..., no two of which meet, of which the points are points of the quadric, and also lines e', a', b', c', d', ..., also skew to one another, of which every one meets all those of the first system. On the conic, the flat pencil of lines, $E(A, B, C, D)$, is a section of the axial pencil of planes ea', eb', ec', ed', which is met, by any particular one of the lines a, b, c, d, in the points where this line meets the lines a', b', c', d'. Thus we see that these lines a', b', c', d' meet the lines of the other system in related ranges, which are also related to the range, on the conic in a plane section, determined by the points where these lines a', b', c', d' meet the conic. Similarly the lines a, b, c, d determine, on any of the lines e', a', b', c', d', a range related to that of A, B, C, D on the conic.



Conversely, if, on two lines, a, b, which do not intersect one another, we have two related ranges, the joins of corresponding points of these ranges are lines which are all met by an infinite number of other lines; the points of these lines are, therefore, those of a quadric surface. For, if P, Q, R be points of the line a, to which correspond the points P', Q', R', respectively, of the line b, and a transversal, c, be drawn from any point of the line PP' to meet QQ' and RR', it follows at once, from what we have seen, that a line drawn from any point of c, to meet a and b, meets these in corresponding points of the two given related ranges thereon. The construction assumes, by hypothesis, that the lines, a, b, of the two related ranges, are not in one plane; if they are, the joins of corresponding points of the two ranges are, as we have seen in Volume II, the tangents of a conic lying in that plane, having the two given lines also as two tangents. The points of a plane, regarded as lying on the tangents of a conic in that plane, may thus be regarded as the points of a degenerate quadric, each tangent line of the conic being taken twice over as a line of the quadric, as a line of each of the systems lying on the quadric.

This account has been obtained by considering the aggregate of the points lying on all the lines which meet three given skew lines. But, in three dimensions, a line is self-dual, and two lines which have a point in common lie also in a common plane; it is proper to consider, then, also the aggregate of all the planes which pass through all the transversals of three given lines. As we have found that through any point of the quadric there pass two lines lying entirely thereon, so we have incidentally found that in any plane passing through a transversal of three given skew lines, *a*, *b*, *c*, there is another line, *d*, not meeting *a*, *b*, *c*, which meets all transversals of these.

To describe the dual correspondence in further detail it is necessary to speak first of what is meant by a quadric cone. We have seen that the quadric surface is the locus of points on lines joining corresponding points of two related ranges on skew lines, these joining lines degenerating into the tangents of a conic when the skew ranges are replaced by ranges on two intersecting lines. Similarly the aggregate of planes we are now considering, as the dual of the aggregate of points of a quadric, may be obtained by taking two skew lines as axes of two related axial pencils of planes; a line of intersection of a plane of one pencil, with the corresponding plane of the other pencil, is then a line meeting both the axes, and, as may easily be seen, in points of two related ranges thereon. The planes of the aggregate are the planes through such lines of intersection. But, now, if we take two intersecting lines, and have two related axial pencils of planes with these as axes, the line of intersection of a plane of one pencil with the corresponding plane of the other pencil, is a line through the point of intersection of the axes; the aggregate of these lines, passing through a point, constitutes what is called a quadric cone, and is the dual of the aggregate of lines in a plane which touch a conic. As two tangents of the conic meet in a point, so two of the lines forming the cone lie in a plane; as the point of intersection of two tangents of the conic becomes a point of the conic when the two tangents coincide (Vol. II, p. 25), so, if the two lines of the cone coincide, the plane containing them is replaced by a definite plane, called a tangent plane of the cone. The section of a cone by an arbitrary plane is evidently a conic, of which one point is determined by a line of the cone, and a tangent line by a tangent plane of the cone; conversely from any conic we obtain a quadric cone by projection from an arbitrary point not lying in the plane of the conic.

This being understood, consider the dual of the statement that a plane section of a quadric is a conic, through every point of which there passes a line of each of the two systems lying on the quadric. Let an arbitrary point of space be joined to every line of one system of lines lying on the quadric, by a plane. Each of these planes will, as we have seen, contain another line lying on the quadric, of the other system. The aggregate of these planes is then that of the tangent planes of a quadric cone. This is obvious from the fundamental duality of the figure. But it is clear, too, by considering that a variable line, a', of one system, of the quadric surface, meets two lines, a, b, of the other system, in related ranges of points. The plane Oa', joining the line a' to a fixed point O, thus meets the planes Oa, Ob in two related flat pencils of lines, with O as common centre. By taking a section by an arbitrary plane, the planes Oa' give rise to lines in this plane meeting two fixed lines, which are the sections of the arbitrary plane by the planes Oa and Ob, in two related ranges. These lines, therefore, are tangents of a conic in that plane; and the planes Oa' are tangent planes of a quadric cone of which all the lines pass through O. This point O is called the *vertex* of the cone, and its lines are called its *generators*; the lines of a quadric surface are also called the *generators* of the quadric.

A particular consequence is that, as there are two points of the quadric locus lying on an arbitrary line, so there are two of the aggregate of planes, passing through an arbitrary line. These may be obtained, directly from the definition, as the two common corresponding planes of two related axial pencils, just as in the case of the locus of points.

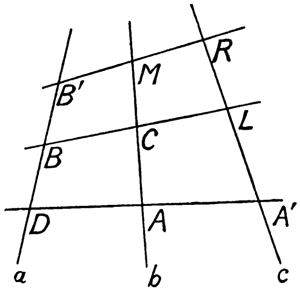
Ex. 1. If A, B, C, D, O be five general points in space, the pairs of planes joining O to the opposite pairs of joins of A, B, C, D, such as OAD, OBC, meet an arbitrary line in three pairs of points which are in involution.

Ex. 2. If an axial pencil of planes be drawn through a line, l, and a related axial pencil of planes be drawn through another line, m, which does not meet l, the lines of intersection, of a plane of the first pencil with the corresponding plane of the second pencil, define a quadric surface, upon which the lines l and m also lie.

Ex. 3. A line l meets the planes BCD, CAD, ABD, ABC, which contain the triads of four arbitrary points A, B, C, D, respectively in P, Q, R, S; another line, l', meets these planes, respectively, in P', Q', R', S'; and the four lines PP', QQ', RR', SS' all lie on the same quadric surface. Prove that the four transversals drawn from A, B, C, D, each to meet both the lines l and l', all lie on a quadric surface. (Cf. Vol. I, p. 30.)

Ex. 4. Let D, P, Q, R be four points in line, and A, B, C be three points whose plane does not contain the line. Let the transversal, l, be drawn from A to meet the lines BQ, CR; the transversal, m, be drawn from B to meet the lines CR, AP; and the transversal, n, be drawn from C to meet the lines AP, BQ. Prove that a line can be drawn from D to meet all of l, m, n.

The representation of this definition by means of the algebraic symbols. We may represent the matter very simply by means of the algebraic symbols. Let a, b, c be three skew lines, of which DAA', BCL, $B'MR$ are any three transversals. Taking D, A, C, B, which do not lie in a plane, as fundamental points, we



can choose the symbols of the points A, A', relatively to that of D, so that $A' = D + A$, and, similarly, the symbols of C, L, relatively to that of B, so that $L = B + C$. Then the symbol of R may be supposed to be

$$R = A' + \mu L = D + A + \mu(B + C)$$
$$= D + \mu B + A + \mu C,$$

where $D + \mu B$ is the symbol of some point on the line a, and $A + \mu C$ of some point on the line b. Hence we infer, for the symbols of B' and M, which lie on a line through R, respectively,

$$B' = D + \mu B, \quad M = A + \mu C.$$

Any other point of the transversal $B'M$ has thus a symbol $B' + \lambda M$, namely

$$\lambda A + \mu B + \lambda\mu C + D,$$

which, then, for different symbols λ, μ, is the general point of the general transversal of the fundamental lines a, b, c. It is thus the general point of the quadric surface under consideration. The symbol can also be written

$$D + \lambda A + \mu(B + \lambda C),$$

and the point lies on a transversal of the lines DA, BC, this transversal meeting these, respectively, in the points $D + \lambda A$, $B + \lambda C$.

We see that every point of the quadric is characterised by two particular algebraic symbols, λ and μ; and further that all points of the quadric for which λ is the same are on a line, not intersecting a, or b, or c; and all points for which μ is the same are on a line, meeting a, b, c but not meeting DA, BC, $B'M$.

The two points of the quadric which lie on an arbitrary line, say on the line joining the points whose symbols are

$$a_1A + b_1B + c_1C + D, \quad a_2A + b_2B + c_2C + D,$$

would then be found by choosing σ, so that the point of this line given by

$$(a_1 + \sigma a_2)A + (b_1 + \sigma b_2)B + (c_1 + \sigma c_2)C + (1 + \sigma)D$$

is the same, for proper values of λ and μ, as the point

$$\lambda A + \mu B + \lambda\mu C + D;$$

this requires

$$(1 + \sigma)(c_1 + \sigma c_2) = (a_1 + \sigma a_2)(b_1 + \sigma b_2).$$

We can express the character of the quadric also by representing any point of it by a symbol

$$XA + YB + ZC + TD,$$

so that X, Y, Z, T are the *coordinates* of this point relatively to the points A, B, C, D; then the sole condition for X, Y, Z, T, is that

$$XY = ZT;$$

this is called the *equation of the quadric*. If we take new points of reference A_1, B_1, C_1, D_1, such that

$$A_1 = C - D, \quad B_1 = A + B, \quad C_1 = A - B, \quad D_1 = C + D,$$

any point of space given by

$$xA_1 + yB_1 + zC_1 + tD_1,$$

is $$x(C - D) + y(A + B) + z(A - B) + t(C + D),$$

and is a point of the quadric if

$$X = y + z, \quad Y = y - z, \quad Z = t + x, \quad T = t - x,$$

that is, if $$x^2 + y^2 - z^2 = t^2.$$

This, then, is another form for the equation of the quadric. Whatever θ may be, this equation is satisfied by

$$y + z = \theta^{-1}(t + x), \quad y - z = \theta(t - x);$$

also, whatever ϕ may be, it is satisfied by

$$y + z = -\phi^{-1}(t - x), \quad y - z = -\phi(t + x).$$

It is, however, easy to see that a single *linear* equation connecting the coordinates, (x, y, z, t), of a point, implies that this point lies on a certain plane; for instance, an equation $t = lx + my + nz$ shews that the point, $xA_1 + yB_1 + zC_1 + (lx + my + nz)D_1$ is on the plane through the three points $A_1 + lD_1$, $B_1 + mD_1$, $C_1 + nD_1$. Thus the points for which $y + z = \theta^{-1}(t + x)$, $y - z = \theta(t - x)$ are those of a line; and, by taking different values of θ, we thus obtain a system of lines lying on the quadric surface; it is easy to see that no two of these have a point in common. Another system of mutually non-intersecting lines is given by the equations $y + z = -\phi^{-1}(t - x)$, $y - z = -\phi(t + x)$, for different values of ϕ. It is easily verified that, whatever θ and ϕ may be, the θ-line, of the first system, has a point in common with the ϕ-line, of the second system, this being, in fact, that given by

$$x = \theta + \phi, \quad y = 1 - \theta\phi, \quad z = 1 + \theta\phi, \quad t = \theta - \phi.$$

These expressions then satisfy identically the equation

$$x^2 + y^2 - z^2 - t^2 = 0.$$

Quadric surface defined by two conics in space having two points in common, and a line which meets these conics. We have seen that the points of a quadric which are on a plane lie on a conic. If two points, A, B, of the quadric be taken, and two planes be drawn through these two points, we thus obtain two conics lying on the quadric. Considering only the lines of the quadric belonging to one system of generators, there will be one line of this system through an arbitrary point, P, of one of these conics; this line will meet the plane of the other conic in a point, say P', which, as the line lies entirely on the quadric, will be on the other conic. There is thus a (1, 1) correspondence between the points of the two conics, whereby to every point, P, of one of these conics, there corresponds a definite point, P', of the other, while P' equally determines P. By what has been proved above, the range of points, P', on one conic, is related to the range of points, P, on the other.

Conversely, let two arbitrary conics be given, lying in different planes, but having two points, A, B, in common; let a further arbitrary fixed point, C, be taken on one conic, and a point, C', on the other; then, we can, to any variable point, P, of the former conic, make correspond a definite point, P', of the latter conic, by the condition that the range of points, A, B, C, P, of the former conic is related to the range of points, A, B, C', P', of the latter. The aggregate of all the points lying on all the lines PP' is then a quadric surface. And when the conics only are given, there is an infinite number of quadrics so obtainable, since we can choose the point C', which is to correspond to C, in an infinite number of ways. A definite quadric is determined by the two conics together with the line CC'. In particular, if the tangents of one conic at the points A, B meet in the point T, and the tangents of the other conic at the points A, B meet in the point U, and the line CC' intersect the line TU, say in O, it can be shewn that the conics are in perspective from the point O. For if the lines which join the point O to the points, P', of one of the two given conics, σ', be allowed to meet the plane of the other conic, σ, in points, Q, it follows, because the pencils of lines, $A(P')$, $B(P')$, are related, that the pencils of lines, $A(Q)$, $B(Q)$, are also related, so that Q describes a conic, τ, in the plane of the conic σ. As O, A, T, U are in one plane, the conic τ will touch the conic σ at A; similarly it will touch it at B; and the point C, lying on OC', will be common to the conics τ and σ. Two conics which touch at two points, and have another point in common, are however coincident. Thus the lines, PP', joining corresponding points of the two conics, all pass through O. The quadric determined by this particular correspondence of C' to C thus reduces to a quadric cone, of vertex O.

In symbols, referred to A, B, T, U, the corresponding points P, P' of the two conics may be taken to be

$$P = \theta^2 A + \theta T + B, \quad P' = \phi^2 A + \phi U + B,$$

where θ, ϕ, by their variation, are to give two related ranges on the two conics, with the condition that the values $\theta^{-1} = 0$, $\theta = 0$, at

A, B, are to correspond to the values $\phi^{-1}=0$, $\phi=0$; thus ϕ is of the form θa^{-1}, where a is the same for all points P, P', but depends upon the initial correspondence adopted, of C' to C. A general point $\rho P+aP'$, of the line PP', is then

$$\theta^2(\rho+a^{-1})A+\theta\rho T+\theta U+(\rho+a)B,$$

or, say, $xA+tT+zU+yB$, so that

$$x=\theta^2(\rho+a^{-1}),\quad t=\theta\rho,\quad z=\theta,\quad y=\rho+a;$$

eliminating ρ we have, for the equation satisfied by the coordinates of any point of the line PP',

$$xy=z^2+(a+a^{-1})zt+t^2;$$

if we put

$$\xi=x+y,\quad \eta=x-y,\quad \zeta=(a^{\frac{1}{2}}+a^{-\frac{1}{2}})(z+t),\quad \tau=(a^{\frac{1}{2}}-a^{-\frac{1}{2}})(z-t),$$

this equation is the same as

$$\xi^2-\eta^2-\zeta^2+\tau^2=0,$$

which is another form of the equation of the quadric.

But, without assigning the correspondence of the two points C, C', we may suppose the two conics to be given, respectively, with the coordinates here used, by the equations

$$z=0,\ xy-t^2=0,\quad \text{and}\quad t=0,\ xy-z^2=0;$$

then, whatever k may be, the equation

$$xy=z^2+2kzt+t^2,$$

is evidently that of a locus containing both the given conics. And k can be chosen so that the locus contains an arbitrary point of space which does not lie in the plane of either of the two conics. A plane drawn through such an arbitrary point, say O, will meet each of the two conics in two points; through the four points so found, and the point O, a definite conic can be drawn. The points of all such conics, obtained by drawing different planes through the same point O, do in fact constitute a quadric determined by the two conics and the point O. To prove this, it is sufficient, after what has preceded, to shew that a quadric can be found to contain two conics which have two points in common, and to contain also an arbitrary point not lying on the plane of either of the two conics; and this follows from the equation just put down. But we can deduce the result at once without the symbols, from the preceding theory, by remarking that a line (two lines, indeed) can be drawn from the arbitrary point to meet the given conics, say in C and C' respectively, and these points C, C' can then be used to establish a (1, 1) relation of points of the two conics. That such a transversal of the two conics can be drawn from the arbitrary point, is clear by projecting one of the conics, from the point, on to the plane of the other conic; thereby a conic is obtained having, beside A and B, two points of intersection with the given conic on that plane. Thereby two such transversals are obtained; but these lead to the same quadric, being the generators of the two systems of the quadric which pass through the arbitrary point.

When k is real, in the quadric given by the equation

$$xy = z^2 + 2kzt + t^2,$$

there is a difference according to the value of k, that is, according to the position of the arbitrary point, which is given in addition to the two conics. When $k^2 > 1$, there exists a *real* number a such that $2k = a + a^{-1}$; then the quadric contains real lines, as we have seen. When $k^2 < 1$, the lines are imaginary. When $k^2 = 1$, the equation of the quadric surface is one of the two represented by

$$xy = (z+t)^2, \quad xy = (z-t)^2,$$

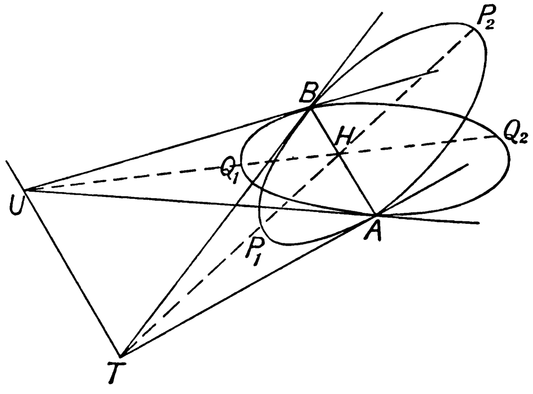
and the surface is, in fact, a cone. In this case the corresponding points, P, P', of the two conics, are

$$P = \theta^2 A + \theta T + B, \quad P' = \theta^2 A \pm \theta U + B,$$

and the line PP' contains one of the two points $T \mp U$, whatever P and P' may be.

An interesting simple result follows from what we have said; *if we have three conics in space of which every two have two points in common, then a quadric surface exists containing all these conics.* This remark is made by Poncelet, *Traité des propriétés projectives des figures*, 1865, I, p. 378, § 606.

Ex. 1. There are in fact just two quadric cones which can be constructed to contain two conics in space which have two points in common, as will appear also in another way. We may construct these, if T, U be, as above, the intersections of the tangents of the



two conics, respectively, at their common points A, B, by taking an arbitrary point, say H, upon the line AB, then joining TH, to meet the conic of the plane TAB in P_1 and P_2, and joining UH, to meet the conic of the plane UAB in Q_1 and Q_2. It is then clear that the lines P_1Q_1, P_2Q_2 meet in a point, say O, of the line TU; and the lines P_1Q_2, P_2Q_1 meet in another point, say O', of this line. The lines joining O to the points of one of the conics meet the plane of the other conic in this conic itself, and constitute one of the two quadric cones, with O as vertex; and O' is the vertex of the other cone. The same points, O and O', are found whatever be the point H taken on AB.

Ex. 2. Shew that the equations of three conics in space of which every two have two points in common, can be taken to be, respectively,

$$x = 0,\ y^2 + 2fyz + z^2 = t^2; \quad y = 0,\ z^2 + 2gzx + x^2 = t^2;$$
$$z = 0,\ x^2 + 2hxy + y^2 = t^2.$$

All these then lie on the locus represented by the equation

$$x^2 + y^2 + z^2 + 2fyz + 2gzx + 2hxy = t^2.$$

Definition of a quadric by two related central systems of lines and planes. We have in Vol. I (pp. 148, 149) explained what is meant by two related plane systems of points and lines, of which the correspondence is determined by assigning four points of one plane to correspond respectively to four points of the other, no three of the four points being in line. In this relation, to any point of one plane corresponds a point of the other, the points of a range, on a line of one plane, corresponding to the points of a related range on a corresponding line of the other plane; and, consequently, to a pencil of lines through a point in one plane, corresponds a related pencil of lines through the corresponding point of the other plane. We may, however, have two dually related plane systems, in which to a point of one plane corresponds a line of the other plane, and, to the points of a range of the former plane, correspond the lines of a pencil of lines through the corresponding point of the latter plane; then to the lines of a pencil in the former plane will correspond the points of a range in the latter plane. As we have established the theory of related ranges and pencils, in Vol. I, by means of incidences, coupled with Pappus' theorem, which is a self-dual theorem, it follows at once that when a range in the former plane corresponds to a pencil in the latter plane, these will be related to one another; and, hence, that when a pencil of lines in the former plane corresponds to a range of points in the latter plane, these will also be related.

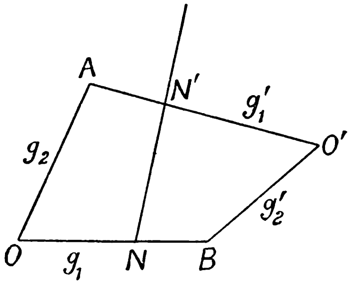
When we have, in space of three dimensions, two related plane systems, ϖ and ϖ', and, also, two points, O and O', of which O does not lie in the plane ϖ, and O' does not lie in the plane ϖ', we may join the points and lines of ϖ to O, by lines and planes, respectively, and the points and lines of ϖ' to O', by lines and planes, respectively. Thereby we obtain two systems of lines and planes, passing respectively through O and O', which we may call ***star systems***, or ***central systems***. When the plane systems ϖ and ϖ' are related, we may speak of these central systems as being related.

But then, equally, if we have two dually related plane systems, ϖ and σ', and two points, O and O', of which O is not in the plane of ϖ, and O' is not in the plane of σ', we may join the points and lines of ϖ to O by lines and planes, respectively, and the lines and points of σ' to O' by planes and lines, respectively. Then we obtain two central systems which are dually related. To a line of the system of centre O, will correspond a plane of the system of centre O'; to a plane through O, containing two lines of the system O, will correspond the line through O', which is the intersection of the planes through O' corresponding to the lines through O; to a pencil of lines through O, lying in a plane through O, will correspond an axial pencil of planes through O', all passing through the line, through O', which corresponds to the plane of the flat pencil through O; and this axial pencil of planes will be related to the flat pencil of lines. And the same may be said of the correspondence, of planes and lines through O', to the lines and planes through O.

Taking now two such dually related central systems, of centres O and O', we consider the locus of the intersection of any line of the system O with the corresponding plane of the system O'. It is the fact that this locus is a quadric surface containing O and O'; conversely, on a given quadric surface we may take two points O and O', arbitrarily, and, in an infinite number of ways, regard the surface as the locus of the intersection of lines of a central system of centre O, with the corresponding planes of a dually related system of centre O'. The theory arises naturally in a later section, devoted to the $(1, 1)$ correspondence of two spaces, and we shall limit ourselves here to the proof of the first part of the statement we have made. To every line, l, through O, there corresponds a plane, λ', through O'; and to every line in the plane λ', passing through O', there corresponds a plane λ, passing through the line l, and through O; and the same for O' and O as for O and O'. In particular, to the line $O'O$, of the system O', there corresponds a plane τ, through O; to every line, l, through O, in this plane τ, there corresponds then a plane through the line $O'O$. This plane, which corresponds to the line l, will meet the plane τ in a line, say m, passing through O. We shall thus have two related pencils of lines, l and m, in the plane τ, all passing through O. These two related pencils will, in the case we regard as general, have two common corresponding rays.

There are, therefore, two lines l, through O, in the plane τ, such that the corresponding planes of the central system O', respectively, contain them; we may call these lines g_1 and g_2, the planes through O' to which they correspond being called, respectively, γ_1' and γ_2'. As every point of the line g_1 lies in the corresponding plane γ_1', every point of g_1 is a point of the locus; and, similarly, so is every point of g_2. Then, to every plane through g_1 (and hence, through O), will correspond a line through O' lying in the plane γ_1'; in particular, to the plane g_1O', will correspond a line g_2', through O', which, as it lies in this plane g_1O', will meet g_1. And there will, similarly, be a line g_1', through O', in the plane g_2O', and, therefore, meeting g_2. The lines g_2', g_1' will be the lines through O' which lie, respectively, in the planes through O which correspond to them.

Now consider a line meeting g_1, say in N, and meeting g_1', say in N'. Then, to the lines through O in the plane NON', correspond planes through O', all passing through a line, and forming an axial pencil related to the pencil of lines through O. The lines in question, and these planes, will meet the line NN' in two related ranges; in the most general case these ranges will have two corresponding points in common; these are the points N, N', and no other point of the locus lies on the line NN'. The locus, however, contains a point on any line drawn through O; let P be such a point, not lying on g_1 or g_1'; let PNN' be the transversal from P to g_1 and g_1'. Then, upon this line, there are two related ranges, as explained, which now have three common corresponding points, and therefore coincide entirely. The line PNN' thus lies entirely on the locus. By taking various points P we can thus obtain an infinite number of lines such as PNN', lying entirely on the locus. If three such lines be taken, it can be proved that any transversal of these lies entirely on the locus, each of these transversals containing three corresponding points of two related ranges in common.



The locus is thus identified with that of all points on all the transversals of three skew lines, namely as a quadric.

Quadric as representing a plane on which two fundamental points are given. The expression above given of the points of a quadric by means of two parameters

$$x = \theta + \phi, \quad y = 1 - \theta\phi, \quad z = 1 + \theta\phi, \quad t = \theta - \phi,$$

is capable of being regarded from a point of view to which brief reference may now be made. We have already, in Vol. II (p. 191), come to regard the points of a plane on which two absolute points are given as arising from the points of a quadric. Without repeating what is there said, we refer here to a general point of view.

If for the parameters θ, ϕ, in the formulae just cited, we write, respectively, ξ/ζ and η/ζ, we may suppose the points of the quadric to be given by

$$x = \zeta(\xi + \eta), \quad y = \zeta^2 - \xi\eta, \quad z = \zeta^2 + \xi\eta, \quad t = \zeta(\xi - \eta);$$

if we regard ξ, η, ζ as homogeneous coordinates in a plane, there will correspond, in this plane, to a condition $Ax + By + Cz + Dt = 0$ for the points of the quadric, the equation

$$A\zeta(\xi + \eta) + B(\zeta^2 - \xi\eta) + C(\zeta^2 + \xi\eta) + D\zeta(\xi - \eta) = 0;$$

this is easily seen to be that of the general conic, in the plane of ξ, η, ζ, which passes through the two points $(1, 0, 0)$, $(0, 1, 0)$. The condition $Ax + \ldots + Dt = 0$ expresses, as we have remarked above, that the point (x, y, z, t) lies on a certain plane; the section of the quadric by this plane, then, corresponds to a particular conic in the plane ξ, η, ζ, passing through two definite points, which are the same for all such plane sections. More generally, take two absolute points in the plane of ξ, η, ζ, and let S_1, S_2, S_3, S_4 be any four homogeneous quadratic functions of ξ, η, ζ, such that the general conic passing through these two absolute points has an equation of the form

$$AS_1 + BS_2 + CS_3 + DS_4 = 0;$$

then, to any point (ξ, η, ζ) of the plane, determine a point in three-fold space whose coordinates are given by

$$\frac{x}{S_1} = \frac{y}{S_2} = \frac{z}{S_3} = \frac{t}{S_4};$$

the coordinates x, y, z, t will then be connected by a rational homogeneous equation, obtainable by elimination of the two ratios ξ/ζ, η/ζ from the three equations by which the ratios of x, y, z, t are defined. This equation may be regarded as the equation of a locus. Any general line will meet this locus in two points; for such a line consists of points, whose coordinates satisfy the equations of two planes, $Ax + \ldots + Dt = 0$, and $A'x + \ldots + D't = 0$; and these correspond to points (ξ, η, ζ) lying on two conics $AS_1 + \ldots + DS_4 = 0$, $A'S_1 + \ldots + D'S_4 = 0$; as these conics have two points in common, which are independent of $A, \ldots, D, A', \ldots, D'$, they will have two other points in common. It is in fact the case that this locus is a quadric surface, as will appear abundantly below. To any point of this, which lies on an infinite number of planes, will correspond a single point of the plane (ξ, η, ζ).

Consideration of the most general homogeneous equation of the second order connecting the space coordinates. It will add considerably to clearness to prove at once that the most general homogeneous quadratic equation connecting x, y, z, t implies that the point x, y, z, t lies upon a quadric surface, or upon a locus which we may regard as a particular case of this.

Write this equation in the form

$$ax^2 + by^2 + cz^2 + dt^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt = 0,$$

which we may abbreviate into either of the forms

$$(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0, \quad (x, y, z, t)^2 = 0\,;$$

we prove, first, that, in an infinite number of ways, this can be put into the form

$$A_1(a_1x + b_1y + c_1z + d_1t)^2 + A_2(a_2x + b_2y + c_2z + d_2t)^2$$
$$+ A_3(a_3x + b_3y + c_3z + d_3t)^2 + A_4(a_4x + b_4y + c_4z + d_4t)^2 = 0,$$

wherein $A_1, A_2, A_3, A_4, a_1, b_1, c_1, d_1, \ldots, a_4, b_4, c_4, d_4$ are all rational real functions of the coefficients, $a, b, c, d, \ldots, w$, which occur in the original equation.

1°. A quadratic form in two variables, x and y, say $ax^2+2hxy+by^2$, if the coefficient b be not zero, is

$$b^{-1}(ab - h^2)\,x^2 + b^{-1}(hx + by)^2\,;$$

and similarly if a be not zero. If a and b be both zero it is

$$\tfrac{1}{2}h\,[(x+y)^2 - (x-y)^2].$$

In any case, therefore, it is of the form $A\xi^2 + B\eta^2$, where A, B are rational real functions of a, b, h, and ξ, η are linear functions of x, y with coefficients which are real rational functions of a, b, h.

2°. A quadratic form in three variables, x, y, z, say

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

if one of the coefficients a, b, c be not zero, for example, if c be not zero, can be written

$$c^{-1}(gx + fy + cz)^2 + c^{-1}[(ca - g^2)\,x^2 - 2\,(fg - ch)\,xy + (bc - f^2)\,y^2],$$

wherein the terms after the square constitute a quadratic form in the variables x, y only; if a, b, c be all zero, and none of f, g, h be zero, the form is

$$\tfrac{1}{2}fg^{-1}h^{-1}(gz + hy + 2f^{-1}ghx)^2 - 2f^{-1}ghx^2 - \tfrac{1}{2}fg^{-1}h^{-1}(gz - hy)^2\,;$$

if a, b, c be all zero, and one of f, g, h be zero, or two of these, for example either $f = 0$, or $f = g = 0$, the form is, respectively, $2x\,(gz + hy)$ or $2hxy$, that is, respectively,

$$\tfrac{1}{2}\,[(x + gz + hy)^2 - (x - gz - hy)^2], \quad \tfrac{1}{2}\,h\,[(x+y)^2 - (x-y)^2].$$

In any case then the form can be written as $A\xi^2+B\eta^2+C\zeta^2$, where A, B, C are real rational functions of the original coefficients $a, b, \ldots, h$, of which one, or two, may be zero, and ξ, η, ζ are independent linear functions of x, y, z, with coefficients which are rational real functions of $a, b, \ldots, h$.

3°. The quadratic form in four variables,

$$ax^2+by^2+cz^2+dt^2+2fyz+2gzx+2hxy+2uxt+2vyt+2wzt,$$

(i) if not all of a, b, c, d be zero, for example if d be not zero, is

$$d^{-1}(ux+vy+wz+dt)^2+\phi,$$

where ϕ is a quadratic form in x, y, z only, whose coefficients are rational real functions of the original coefficients $a, b, \ldots, w$.

(ii) if all of a, b, c, d be zero, but none of f, g, h be zero, we can choose l, m, n so that

$$hm+gn+u=0, \quad hl+fn+v=0, \quad gl+fm+w=0,$$

namely

$$l=\tfrac{1}{2}g^{-1}h^{-1}(fu-gv-hw), \quad m=\tfrac{1}{2}h^{-1}f^{-1}(-fu+gv-hw),$$
$$n=\tfrac{1}{2}f^{-1}g^{-1}(-fu-gv+hw);$$

then, substituting in the quadratic form, for x, y, z, respectively,

$$x=x'+lt, \quad y=y'+mt, \quad z=z'+nt,$$

this form reduces to

$$2fy'z'+2gz'x'+2hx'y'+2t^2(mnf+nlg+lmh+ul+vm+wn),$$

wherein the first part is a quadratic form in x', y', z' only, and the last part contains only the term in t^2.

(iii) if all of a, b, c, d be zero, and f be zero, but not both of g and h, the form is

$$2(hy+gz)x+2(ux+vy+wz)t;$$

here it may happen that $ux+vy+wz$ is of the form $ux+\lambda(hy+gz)$, namely that $v/h=w/g$, $(=\lambda)$; then the form is

$$2(hy+gz)x+2[ux+\lambda(hy+gz)]t,$$

and is a quadratic form in the *three* variables $x, t, hy+gz$; if $ux+vy+wz$ is not of the form in question, taking ξ, η, ζ, τ so that

$$hy+gz=\eta+\zeta, \quad ux+vy+wz=\tau+\xi,$$
$$x=\eta-\zeta, \qquad t=\tau-\xi,$$

we can, from these, determine x, y, z, t as linear functions of ξ, η, ζ, τ, with coefficients which are real and rational in the coefficients of the original form, which then is

$$2(\eta^2-\zeta^2)+2(\tau^2-\xi^2).$$

(iv) if all of a, b, c, d be zero, and all of f, g, h be zero, the form is

$$2(ux + vy + wz)\, t,$$

and is a quadratic form in the *two* variables $ux + vy + wz$, t.

Thus in all cases the reduction is possible, to the form

$$A_1(a_1x + b_1y + c_1z + d_1t)^2 + \dots + A_4(a_4x + b_4y + c_4z + d_4t)^2,$$

wherein the four linear forms are linearly independent.

Consideration of particular cases. The case of a cone. We regard the case in which none of the coefficients A_1, A_2, A_3, A_4 is zero as the general case. Before coming to this we consider the particular cases, and, first, the case in which one of these coefficients, and only one, say A_4, is zero. Then if we put

$$\xi = a_1x + b_1y + c_1z + d_1t, \quad \eta = a_2x + b_2y + c_2z + d_2t,$$
$$\zeta = a_3x + b_3y + c_3z + d_3t, \quad \tau = a_4x + b_4y + c_4z + d_4t,$$

which is equivalent to choosing for the points of reference, for the space, the four points of intersection of the four planes which are represented by the equations $\xi = 0$, $\eta = 0$, $\zeta = 0$ and $\tau = 0$, the fundamental quadratic form becomes $A_1\xi^2 + A_2\eta^2 + A_3\zeta^2$. We consider then what is represented by the equation

$$A_1\xi^2 + A_2\eta^2 + A_3\zeta^2 = 0.$$

Let $(\xi, \eta, \zeta, 1)$ be any point which satisfies this equation; the general point on the line which joins this to the point $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\tau = 1$, has coordinates of the form $(\xi, \eta, \zeta, 1 + \lambda)$; this point then, also, satisfies the equation. The equation is then satisfied by all the points of an aggregate of lines passing through the point $(0, 0, 0, 1)$. These lines evidently meet the plane $\tau = 0$ in the points of a conic, of which the equations are $\tau = 0$, $A_1\xi^2 + A_2\eta^2 + A_3\zeta^2 = 0$. The equation therefore represents what we have called a quadric cone. Conversely, from the definition of such a cone given above, it follows that its equation must be capable of this form.

Thus we see that the necessary and sufficient condition that the general quadratic polynomial, in the space coordinates x, y, z, t, should represent a quadric cone, when equated to zero, is that this polynomial should be capable of being expressed as a function of only three linear functions of x, y, z, t.

We can hence obtain the condition for this in terms of the coefficients in the original form. For, if this be $f(x, y, z, t)$, and be capable of being expressed in terms of the three linear functions, ξ, η, ζ, of x, y, z, t, in the form $f(x, y, z, t) = F(\xi, \eta, \zeta)$, we have four identities such as

$$\frac{\partial f}{\partial x} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial F}{\partial \xi} + \frac{\partial \eta}{\partial x} \cdot \frac{\partial F}{\partial \eta} + \frac{\partial \zeta}{\partial x} \cdot \frac{\partial F}{\partial \zeta},$$

wherein the factors $\partial\xi/\partial x$, $\partial\eta/\partial x$, etc., are functions of the original coefficients alone, and do not depend on x, y, z, t. Thus the four linear functions of x, y, z, t given by $\partial f/\partial x$, $\partial f/\partial y$, etc., are all expressible as linear functions of the three forms $\partial F/\partial\xi$, $\partial F/\partial\eta$, etc. which are linear in ξ, η, ζ and therefore, also, in x, y, z, t. There exist, however, values of x, y, z, t reducing to zero these three linear forms $\partial F/\partial\xi$, etc. There exist, therefore, values of x, y, z, t reducing to zero the four forms $\partial f/\partial x$, etc. These forms are, respectively,

$$2(ax+hy+gz+ut), \quad 2(hx+by+fz+vt),$$
$$2(gx+fy+cz+wt), \quad 2(ux+vy+wz+dt),$$

and the condition necessary and sufficient that there should be values of x, y, z, t for which these all vanish is that the determinant

$$\Delta, = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix},$$

should be zero.

Conversely, if this determinant is zero, and (x_0, y_0, z_0, t_0) be values, then existing, for which all of $\partial f/\partial x$, $\partial f/\partial y$, etc., vanish, while (x, y, z, t) are any values for which $f(x, y, z, t) = 0$, then every point on the line joining the point (x_0, y_0, z_0, t_0) to (x, y, z, t) is a point for which $f(x, y, z, t) = 0$. For such a point has coordinates of the form $(x_0 + \lambda x, y_0 + \lambda y, z_0 + \lambda z, t_0 + \lambda t)$, and we have

$$f(x_0 + \lambda x, \ldots) = f(x_0, y_0, z_0, t_0)$$
$$+ \lambda\left(x\frac{\partial f}{\partial x_0} + y\frac{\partial f}{\partial y_0} + z\frac{\partial f}{\partial z_0} + t\frac{\partial f}{\partial t_0}\right) + \lambda^2 f(x, y, z, t);$$

now, by hypothesis, $f(x, y, z, t) = 0$; and $\partial f/\partial x_0 = 0$, $\partial f/\partial y_0 = 0$, etc., and, therefore, also $f(x_0, y_0, z_0, t_0) = 0$, since

$$f(x_0, y_0, z_0, t_0) = \frac{1}{2}\left(x_0\frac{\partial f}{\partial x_0} + \ldots + t_0\frac{\partial f}{\partial t_0}\right).$$

It follows from this that the equation $f(x, y, z, t) = 0$ is satisfied by all points of an aggregate of lines passing through the point (x_0, y_0, z_0, t_0). It is also easy to see directly that, with the hypothesis made, namely that the four forms $\partial f/\partial x$, $\partial f/\partial y$, etc., are all annulled by the values $x = x_0$, $y = y_0$, etc., the quadratic form $f(x, y, z, t)$ is a function of three forms, linear in x, y, z, t; which, when t_0 is not zero, may be taken to be $xt_0 - x_0t$, $yt_0 - y_0t$, $zt_0 - z_0t$. We find at once, indeed, when we utilise this hypothesis, that

$$a(xt_0 - x_0t)^2 + b(yt_0 - y_0t)^2 + c(zt_0 - z_0t)^2 + 2f(yt_0 - y_0t)(zt_0 - z_0t)$$
$$+ 2g(zt_0 - z_0t)(xt_0 - x_0t) + 2h(xt_0 - x_0t)(yt_0 - y_0t)$$

is equal to $t_0^2 f(x, y, z, t)$. This supposes that t_0 is not zero; but we can suppose that one at least of x_0, y_0, z_0, t_0 is not zero, and so, in any case, obtain a similar formula.

The case of two planes. Consider now the more particular case when the quadratic form in x, y, z, t, $f(x, y, z, t)$, reduces to only two squares,

$$A_1(a_1x + b_1y + c_1z + d_1t)^2 + A_2(a_2x + b_2y + c_2z + d_2t)^2,$$

say to $A_1\xi^2 + A_2\eta^2$. The equation $f(x, y, z, t) = 0$ is then equivalent to

$$(\sqrt{A_1}\,\xi + \sqrt{-A_2}\,\eta)(\sqrt{A_1}\,\xi - \sqrt{-A_2}\,\eta) = 0,$$

so that, as ξ and η are linear in x, y, z, t, it represents two planes. These are imaginary when A_1 and A_2 are real and both positive, or negative. In this case, by identities such as those used in the last case, the four linear forms $\partial f/\partial x$, $\partial f/\partial y$, etc., are linearly expressible by *two* linear forms; thus every first minor, of three rows and columns, in the determinant

$$\Delta, = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix},$$

vanishes. Conversely, denote respectively by U, V, W, P the linear forms in x, y, z, t, $\frac{1}{2}\partial f/\partial x$, $\frac{1}{2}\partial f/\partial y$, $\frac{1}{2}\partial f/\partial z$, $\frac{1}{2}\partial f/\partial t$, and assume that the three particular first minors of this determinant which are the cofactors respectively of c, d and w, are all zero (from which, as will appear, all the first minors are zero), and that the second minor $ab - h^2$ is not zero; then it is easy to verify that, for all values of x, y, z, t, the forms W and P are expressible in terms of U and V, by means of the equations

$$-(ab - h^2)W = (hf - bg)U + (gh - af)V,$$
$$-(ab - h^2)P = (hv - bu)U + (hu - av)V;$$

from these it follows that $(ab - h^2)f(x, y, z, t)$, which is equal to

$$(ab - h^2)(xU + yV + zW + tP),$$

is equal to $U[(ab - h^2)x + (bg - hf)z + (bu - hv)t]$
$+ V[(ab - h^2)y + (af - gh)z + (av - hu)t]$

and hence to $\quad U(bU - hV) + V(aV - hU)$,

or to $\quad aV^2 - 2hUV + bU^2$.

Wherefore, $f(x, y, z, t)$ is expressible in the form $A_1\xi^2 + A_2\eta^2$, where ξ, η are linear in x, y, z, t. This same conclusion follows, in a similar way, whenever all the first minors of the determinant Δ are zero, if there be one second minor which is not zero.

The case of two coincident planes. For the last particular case suppose that $f(x, y, z, t)$ reduces to the single term

$$A_1(a_1x + b_1y + c_1z + d_1t)^2.$$

Then, we see, as in the preceding cases, that the four linear forms $\partial f/\partial x$, $\partial f/\partial y$, etc., are identical save for a factor independent of x, y, z, t. When this is so, every second minor of the determinant Δ vanishes. Conversely suppose that this is so, but $f(x, y, z, t)$ does not vanish identically. The forms U, V, W, P, with the notation just employed, are then the same, save for a factor independent of x, y, z, t. The coefficients a, b, c, d, respectively of x^2, y^2, z^2, t^2, in $f(x, y, z, t)$, cannot all vanish, since then, from $bc - f^2 = 0$, $ad - u^2 = 0$, would follow $f = 0$, $u = 0$, and similarly all other coefficients in $f(x, y, z, t)$ would vanish. Suppose then, for example, that a is not zero; then from the equivalence of the forms U, V, W, P, it follows that $U/a = V/h = W/g = P/u$, and hence

$$f(x, y, z, t) = xU + yV + zW + tP = a^{-1}U(ax + hy + gz + ut) = a^{-1}U^2,$$

so that $f(x, y, z, t)$ is the square of a single linear form, and the equation $f(x, y, z, t) = 0$ represents a plane taken twice over.

After Sylvester (*Coll. Papers*, I, p. 147), the vanishing of six, suitably chosen, second minors of Δ, is sufficient to ensure that they all vanish.

General properties of a quadric given by its equation. After what we have seen we may agree to understand by a quadric the locus determined by the vanishing of any homogeneous quadratic form, $f(x, y, z, t)$, in x, y, z, t. When the contrary is not referred to, we suppose that the determinant Δ does not vanish, so that $f(x, y, z, t)$ can be expressed in the form $A\xi^2 + B\eta^2 + C\zeta^2 + D\tau^2$, where A, B, C, D are rational real functions of the original coefficients, all different from zero, and ξ, η, ζ, τ are linear functions of x, y, z, t, with coefficients which are equally rational real functions of the original coefficients. The coefficients A, B, C, D, if real, may be all positive, or all negative; then, clearly, there is no point, of which the coordinates are all real, for which $f(x, y, z, t)$ vanishes; it is, then, usual, as in the case of a conic (Vol. II, p. 163), to speak of $f(x, y, z, t) = 0$ as representing an imaginary quadric. Or, the coefficients A, B, C, D may consist of two which are positive and two which are negative. In this case, as appears from what we have seen above (p. 7), there are two systems of real lines of which every point is a point of the quadric $f(x, y, z, t) = 0$. Or, finally, three of the coefficients A, B, C, D may be of one sign, and the remaining one of the opposite sign. In this case, the locus $f(x, y, z, t) = 0$ contains real points; for example, if A, B, C be of one sign and D be of the opposite sign, two real points are given by $\eta = 0$, $\zeta = 0$, $A\xi^2 + D\tau^2 = 0$. But the locus does not contain any real lines. For example, with the particular signs just taken, a real line of the locus would contain a real point lying on the plane represented by $\tau = 0$; but $\tau = 0$ involves, for points of the locus, $A\xi^2 + B\eta^2 + C\zeta^2 = 0$, and, as A, B, C are of the same sign, there is no set of real values of ξ, η, ζ for which this can be satisfied. We see thus that the three cases are distinct, if we take account of the reality of the points of the locus; and, though the reduction to a sum of four squares can be made in an infinite number of ways, yet, when made by rational real substitutions only, as here, the number of positive and negative signs obtained for the coefficients A, B, C, D will be the same, for any particular form $f(x, y, z, t)$ of *real* coefficients, in whatever way the reduction is made.

Certain general properties of the quadric locus may now be enumerated: (1) If a linear transformation be made, from variables x, y, z, t to variables ξ, η, ζ, τ, by equations of the forms

$$\xi = a_1x + b_1y + c_1z + d_1t, \quad \eta = a_2x + b_2y + c_2z + d_2t,$$
$$\zeta = a_3x + \dots, \quad \tau = a_4x + \dots,$$

whereby a function $f(x, y, z, t)$ becomes $F(\xi, \eta, \zeta, \tau)$, and if $\xi', \eta', \zeta', \tau'$ be the same linear functions of variables x', y', z', t' as are ξ, η, ζ, τ of x, y, z, t, then

$$\xi'\frac{\partial F}{\partial \xi} + \eta'\frac{\partial F}{\partial \eta} + \zeta'\frac{\partial F}{\partial \zeta} + \tau'\frac{\partial F}{\partial \tau} = x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z} + t'\frac{\partial f}{\partial t}.$$

For we have
$$\frac{\partial f}{\partial x} = a_1\frac{\partial F}{\partial \xi} + a_2\frac{\partial F}{\partial \eta} + a_3\frac{\partial F}{\partial \zeta} + a_4\frac{\partial F}{\partial \tau},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{\partial f}{\partial t} = d_1\frac{\partial F}{\partial \xi} + d_2\frac{\partial F}{\partial \eta} + d_3\frac{\partial F}{\partial \zeta} + d_4\frac{\partial F}{\partial \tau};$$

if these equations be multiplied in turn by x', y', z', t', and then added, the result follows at once.

It is often convenient to represent the original equations of transformation by

$$(\xi, \eta, \zeta, \tau) = \left|\begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{matrix}\right| (x, y, z, t);$$

the array of coefficients is then called a *matrix*. The matrix obtained from this, by writing as columns what are here written as rows, is then called the *transposed* matrix. Denoting the original matrix by M, and the transposed matrix by $\bar{M}$, we have

$$(\xi, \eta, \zeta, \tau) = M(x, y, z, t),$$

and $$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial t}\right) = \overline{M}\left(\frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial \zeta}, \frac{\partial F}{\partial \tau}\right),$$

where the new variables, ξ, η, ζ, τ, occur on the left in the first equation, but the new differential coefficients $\partial F/\partial \xi$, $\partial F/\partial \eta$, etc., occur on the right in the second equation. This is often expressed by saying that the differential coefficients $\partial f/\partial x$, $\partial f/\partial y$, etc., are transformed *contragrediently* from the variables x, y, z, t.

(2) A general quadric, expressed by the equation $f(x, y, z, t) = 0$, is met in two points by the line joining any two given points $(x, y, z,$ and (x', y', z', t'). The necessary and sufficient condition that the two points of meeting should be harmonic conjugates of one another in regard to the given points (x, y, z, t), (x', y', z', t'), is

$$x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z} + t'\frac{\partial f}{\partial t} = 0\,;$$

this equation, when $f(x, y, z, t)$ is a quadratic function, is symmetrical in regard to the two points (x, y, z, t), (x', y', z', t'); with the notation originally suggested for the coefficients in $f(x, y, z, t)$, it is (after division by 2)

$$\begin{aligned} &axx' + byy' + czz' + dtt' + f(yz' + y'z) + g(zx' + z'x) \\ &\qquad + h(xy' + x'y) + u(xt' + x't) + v(yt' + y't) + w(zt' + z't). \end{aligned}$$

For the point $(x+\lambda x', y+\lambda y', z+\lambda z', t+\lambda t')$ lies on $f(x, y, z, t) = 0$ if

$$f(x, y, z, t) + \lambda\left(x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z} + t'\frac{\partial f}{\partial t}\right) + \lambda^2 f(x', y', z', t') = 0;$$

this equation in λ has two roots, and they are equal, and of opposite sign, only if the coefficient of λ vanish.

(3) Particular applications of the result in (2) may be referred to: It appears that if, from an arbitrary point, (x', y', z', t'), or O, variable lines be drawn to meet the quadric, each in two points, P and Q, the locus of a point, R, which is the harmonic conjugate of O in regard to P and Q, is a plane, with the equation

$$x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z} + t'\frac{\partial f}{\partial t} = 0.$$

This plane is called the *polar plane* of O in regard to the quadric.

In the particular case in which the quadric is a cone, and all the differential coefficients $\partial f/\partial x$, $\partial f/\partial y$, etc., vanish at the point which is the vertex of this cone, this polar plane passes through the vertex. In the still more particular case when the quadric is a pair of planes, the polar plane passes through the line in which these two planes intersect. Finally, when the quadric is a pair of coincident planes, the polar plane coincides with these.

It appears, also, that if (x', y', z', t') be a point of the quadric $f(x, y, z, t) = 0$, which we now suppose not to be a cone or a particular case of this, so that one of the two roots of the above quadratic equation for λ^{-1} is zero, and if (x, y, z, t) be taken on the polar plane of (x', y', z', t'), but not on $f(x, y, z, t) = 0$, then the second value of λ^{-1} is also zero, so that the two points, where the line joining the points (x', y', z', t'), (x, y, z, t) meets the quadric, both coincide with (x', y', z', t'); while if, with (x', y', z', t') on the quadric, the point (x, y, z, t) be both on the polar plane of (x', y', z', t'), and also such that $f(x, y, z, t) = 0$, then the quadratic equation for λ is satisfied for every value of λ, namely every point of the line joining (x', y', z', t') to (x, y, z, t) lies on the quadric. In general the intersection of the quadric surface with a plane is a conic, being a locus met by a line of the plane in two points; in the case spoken of, when the plane is the polar plane of a point of the quadric, this conic, we see, contains a line, and, therefore, also another line, as part of itself, and consists of these two lines. Thus for a general quadric, the polar plane of any point lying on the quadric, which for such a point is called the *tangent plane* of the quadric at the point, meets the quadric in two lines intersecting at the point; any other line, in the tangent plane through this point, has two coincident intersections with the quadric, lying at this point. The point itself is called the *point of contact* of the tangent plane with the quadric. When the quadric considered is a cone, the tangent plane at any point passes through the vertex of the cone, and the two lines in which the tangent plane meets the cone coincide, in the line joining the vertex to the point of contact, as is easy to see; at the vertex itself there is no proper tangent plane. When the quadric considered consists of two planes, the tangent plane at any point, not on the line of intersection of the two planes, coincides with one of these.

(4) The condition that a plane, represented, suppose, by the equation

$$lx + my + nz + pt = 0,$$

should be a tangent plane of the quadric $f(x, y, z, t) = 0$, is that its equation should be capable, for values of x', y', z', t' which satisfy the two equations

$$f(x', y', z', t') = 0, \quad lx' + my' + nz' + pt' = 0,$$

of being written in the form appropriate to that of the tangent plane at (x', y', z', t'), which is

$$x(ax' + hy' + gz' + ut') + \ldots + t(ux' + vy' + wz' + dt') = 0.$$

For this it is necessary and sufficient that the coefficients of x, y, z, t, in the last equation, should be proportional to those in the given equation, namely that there should be a value of σ for which we have the four equations

$$ax' + hy' + gz' + ut' + \sigma l = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$ux' + vy' + wz' + dt' + \sigma p = 0,$$

while, also, $$lx' + my' + nz' + pt' \qquad = 0.$$

These equations, as we see by adding the first four, after multiplication, respectively, by x', y', z', t', involve $f(x', y', z', t') = 0$. Thus the necessary and sufficient condition, for the given plane to be a tangent plane of the given quadric, is that l, m, n, p should satisfy the equation

$$\begin{vmatrix} a, & h, & g, & u, & l \\ h, & b, & f, & v, & m \\ g, & f, & c, & w, & n \\ u, & v, & w, & d, & p \\ l, & m, & n, & p, & 0 \end{vmatrix} = 0;$$

if the cofactors of a, b, c, etc., in the determinant Δ, of four rows and columns, which is obtained by omitting the last row and column of this determinant, be denoted, respectively, by A, B, C, etc., this condition is

$$Al^2 + Bm^2 + Cn^2 + Dp^2 + 2Fmn + 2Gnl + 2Hlm$$
$$+ 2Ulp + 2Vmp + 2Wnp = 0.$$

As the equation $f(x, y, z, t) = 0$ is the condition that the point (x, y, z, t) should be a point of the quadric, so the condition just obtained is the condition that the plane (l, m, n, p) should be a tangent plane of the quadric. Thus it appears that, as the quadric is the aggregate of points satisfying the most general homogeneous quadratic relation connecting the point coordinates x, y, z, t, so the aggregate of the tangent planes of a quadric is obtained by subjecting the coefficients, l, m, n, p, in the equation of a plane, to the most general homogeneous quadratic relation. For, supposing the determinant Δ not to be zero, we may regard, not the coefficients a, b, c, etc., as fundamental, but the coefficients A, B, C, etc., as fundamental; taking the determinant

$$\begin{matrix} A, & H, & G, & U \\ H, & B, & F, & V \\ G, & F, & C, & W \\ U, & V, & W, & D \end{matrix}$$

the coefficients a, b, c, etc., each multiplied by Δ^2, are the cofactors of A, B, C, etc., in this last determinant. The case of $\Delta = 0$ will be considered immediately below.

The equation $Al^2 + Bm^2 + \ldots + 2Wnp = 0$, may then be spoken of as the *tangential equation* of the quadric, and l, m, n, p as the *coordinates* of a plane, the whole being analogous to the case of a conic dealt with in Chap. III of Vol. II.

In particular, if $F(l, m, n, p) = 0$ be the tangential equation of a quadric, the point of contact of a particular tangent plane, whose coordinates are l_0, m_0, n_0, p_0, has for its equation

$$l_0 \frac{\partial F}{\partial l} + m_0 \frac{\partial F}{\partial m} + n_0 \frac{\partial F}{\partial n} + p_0 \frac{\partial F}{\partial p} = 0,$$

so that, if $F(l_0, m_0, n_0, p_0)$ be denoted by F_0, the coordinates of the point of contact are $\partial F_0/\partial l_0$, $\partial F_0/\partial m_0$, etc.

When the determinant Δ vanishes, the tangential equation above written, regarded as deduced from the point equation, is not general, being in fact a perfect square. For, by the theorem of reciprocal determinants referred to above, the determinants of two rows and columns such as

$$BC - F^2, \quad AD - U^2, \quad GH - AF, \quad HU - AV,$$

which are equal respectively to $ad - u^2$, $bc - f^2$, $fd - vw$, $fw - vc$, multiplied by Δ, all vanish; thus, when A is not zero, the tangential equation is the same as

$$(Al + Hm + Gn + Up)^2 = 0,$$

and so in other cases.

Taking, however, the general tangential equation, we may shew, as for the point equation above, that it can be written in the form

$$A_1(a_1 l + b_1 m + c_1 n + d_1 p)^2 + \ldots + A_4(a_4 l + b_4 m + c_4 n + d_4 p)^2 = 0.$$

In the most general case none of A_1, A_2, A_3, A_4 vanishes. When one of these vanishes, the planes conditioned by this equation are all those which pass through the tangent lines of a certain conic. When two of the coefficients $A_1, \ldots, A_4$ vanish, the planes conditioned by this equation are all those which pass through one of two particular points. Finally, when only one of the coefficients is present, the planes are all those passing through a particular point. This follows from the principle of duality, the tangential *equation of a point*, of the form

$$x_0 l + y_0 m + z_0 n + t_0 p = 0,$$

being exactly analogous to the equation of a plane. The matter has been explained in detail for a plane, in Chap. III of Vol. II.

(5) Denoting $f(x, y, z, t)$ by $f(x)$, and $f(x', y', z', t')$ by $f(x')$ and

$$\frac{1}{2}\left(x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + z' \frac{\partial f}{\partial z} + t' \frac{\partial f}{\partial t}\right)$$

by $f(x, x')$, we have remarked above that the points

$$(x+\lambda x', \quad y+\lambda y', \quad z+\lambda z', \quad t+\lambda t'),$$

of the line joining (x', y', z', t') to (x, y, z, t), which lie on the quadric $f(x, y, z, t) = 0$, are given by the quadratic equation

$$f(x) + 2\lambda f(x, x') + \lambda^2 f(x') = 0.$$

The line will then meet the quadric in two coincident points if, for an assigned point (x', y', z', t'), the point (x, y, z, t) be such that

$$f(x) f(x') - [f(x, x')]^2 = 0.$$

Regarded as a condition for (x, y, z, t), this equation represents a quadric. It is, however, evidently satisfied by any point of the line joining (x', y', z', t') to (x, y, z, t). Thus all the lines from (x', y', z', t') which meet the quadric in two coincident points lie on a quadric cone. This is called the *tangent cone*, or the *enveloping cone*, to the given quadric, from the point, and the lines are said to *touch* the quadric. It can in fact be directly verified, (a), that the equation represents a cone; (b), that every one of its tangent planes is equally a tangent plane of the given quadric.

(6) As the general equation $f(x, y, z, t) = 0$ contains ten, homogeneously entering, coefficients, and the condition that the quadric should contain an assigned point is a single linear condition for these coefficients, it follows that a quadric can be found containing nine points arbitrarily chosen. For instance, if three non-intersecting lines be given, a quadric can be found containing three points on each of these lines; and then, as each line meets this quadric in more than two points, the line lies entirely upon the quadric. Or again, if three conics be given in space, of which every two have two points in common, by taking another point upon each of the conics, we have nine points in all; the quadric through these nine points will meet the plane of any one of the three given conics in a conic having five points lying thereon, and, therefore, coinciding with it. The quadric thus contains the three given conics.

It has seemed desirable to give the preceding analysis in order to make clear the general conception of a quadric surface; the notion of the equation of a quadric is undoubtedly of great power, and as we have deduced this from the geometrical definition of the surface, it would be pedantic to avoid the use of it. The descriptive theory, however, often gives a better insight into the geometrical figure under consideration. This theory we now resume.

The polar plane of a point in regard to a quadric. Given any quadric, take an arbitrary point, O, not lying thereon. Draw an arbitrary plane through O, meeting this quadric in a conic. Let a line, passing through O, lying in this plane, meet this conic in the points A and B, and let P be the harmonic conjugate of O

in regard to A and B. As the line varies in this plane, passing through O, the locus of P is a line, the polar line of O in regard to this conic. Now let another plane be drawn through O, meeting the former plane in a line l, passing through O, and let the points in which the line l meets the quadric be U and V. If W be the harmonic conjugate of O in regard to U and V, the point W is on the polar lines of O in regard to both the conics, in which the quadric is met by the two planes drawn through O; these two polar lines, therefore, intersect one another. Hence, if a third plane, not containing the line l, be drawn through O, the polar line of O, in regard to the conic in which this third plane cuts the quadric, will intersect the polar lines of O taken in regard to the two former conics, and not pass through the point of intersection of these lines, which is on l. The third polar line will thus lie in the plane of the first two. From this it is clear that, if an arbitrary line be drawn through O, to meet the quadric in the points Q and R, and P be the point which is the harmonic conjugate of O in regard to Q and R, then the locus of P is a plane. This is called the *polar plane* of O in regard to the quadric. From the definition it is manifest that if the polar plane of O pass through P, the polar plane of P passes through O.

The tangent plane of the quadric at any point. The polar plane of O meets the quadric in a conic. Let H be any point of this. Then the second point, H', in which the line OH meets the quadric, coincides with H; for H is the harmonic conjugate of O in regard to H and H'. We therefore say that the line OH *touches* the quadric at the point H. Any plane drawn through OH meets the quadric in a conic; this conic is met by OH in two points which coincide at H, that is, HO is the tangent at H of this conic. Again, the conic, also passing through H, in which the quadric is met by the polar plane of O, has a tangent at H, say t, unless this conic consists of two lines. This line, t, meets the conic in two points which coincide at H, and, therefore, meets the quadric in two points which coincide at H. Consider now the conic in which the quadric is met by the plane containing the two lines HO and t; this conic is met in two points coinciding at H, both by the line HO and by the line t. It cannot, therefore, be a proper conic, but must consist of two lines meeting at H. The plane of these lines is called the *tangent plane* of the quadric at H. It is to be regarded as the polar plane of the point H; it passes through O, and the polar plane of O passes through H. By choosing O suitably, H may be supposed to be any general point of the quadric. But there may be exception if the polar plane of O meets the quadric in two lines, as happens when the quadric degenerates into a cone, or into two planes; the polar plane of O then passes, respectively, through the vertex of the cone, or through the line of intersection of the two planes; in the former case it meets the cone in lines, in the latter case it meets the planes only in their line of intersection; in the former case there is still a tangent plane of the cone, which passes through the vertex, unless H be actually the vertex.

The polar line of any line in regard to a quadric. Let O_1, O_2 be any two points, not lying on the quadric. Their polar planes will meet in a line, say m. Draw through the line O_1O_2 any plane, meeting the line m in a point K. As the polar planes of O_1 and O_2 both pass through K, it follows that the polar plane of K passes through both O_1 and O_2, and therefore contains any point, O_3, of the line O_1O_2. The polar plane of O_3, therefore, passes through K. The lines O_1O_2 and m are thus in reciprocal relation, the polar plane of any point of either of these containing every point of the other, and, therefore, containing the other. Either of these lines is called the polar line of the other in regard to the quadric. A plane drawn through one of these lines will meet the quadric in a conic and meet the polar line in a point; in regard to this conic, this point is the pole of the first line, as follows from the harmonic relation for points on a transversal. The polar planes, in regard to the quadric, of any range of points on one of the lines, form an axial pencil, of which the polar line is the axis, and, as follows from what has just been remarked, and the theory of conics, this axial pencil of planes is related to the range of points.

Let one of these two lines meet the quadric in the two points F and G; then, as the polar plane of any point, O, of the other line, contains the line FG, and hence contains F and G, it follows, as we have seen above, that the tangent plane of the quadric at F, as also the tangent plane at G, both contain O. In other words, as O is any point of the second line, the tangent planes of the quadric at F and G intersect in this line, the polar line of FG. From this it follows, also, that the lines, lying on the quadric, which pass through F, being in the tangent plane of the quadric at F, both meet the polar line of FG. The points of meeting, being points on lines which lie on the quadric, are themselves on the quadric; they are, therefore, the two points in which the quadric is intersected by the polar line of FG. The lines on the quadric which pass through G must, similarly, pass through these two points. Calling these points F' and G', the two lines, on the quadric, which meet at F', determine the tangent plane of the quadric at F', and this, we see, contains FG; similarly the two lines, on the quadric, which meet at G', determine the tangent plane of the quadric at G', and this plane also passes through the line FG. When the quadric is a cone, the lines on the cone, at any point F, coincide in a line which passes through the vertex; the tangent plane of the cone at any point of this line is the same plane as the tangent plane at any other point of this line, and passes through the vertex. The polar line of a line FG, where F, G are points of the cone other than the vertex, is thus a line through the vertex.

Employing a symbolism introduced in connexion with the original definition of a quadric (above, p. 6), we may suppose, in general, that the points F, G are, respectively, of symbols

$$\lambda A + \mu B + \lambda\mu C + D, \quad \lambda' A + \mu' B + \lambda'\mu' C + D,$$

so that the parameters λ, μ are associated, respectively, with the two lines of the quadric which pass through F, and the parameters λ', μ' with those through G. The parameters associated with the two lines through F' may, therefore, be taken to be (λ, μ'), and those for G' to be (λ', μ); thus the symbols for F' and G' will, respectively, be

$$\lambda A + \mu' B + \lambda\mu' C + D, \quad \lambda' A + \mu B + \lambda'\mu C + D.$$

The coordinates of the points F, G, F', G', relatively to A, B, C, D, will thus be, respectively,

$$(\lambda, \mu, \lambda\mu, 1); \quad (\lambda', \mu', \lambda'\mu', 1); \quad (\lambda, \mu', \lambda\mu', 1); \quad (\lambda', \mu, \lambda'\mu, 1).$$

The coordinates of the three points F, F', G', thus, all satisfy the equation

$$x\mu + y\lambda = z + t\lambda\mu;$$

by the theory we have just given this is, therefore, the equation of the tangent plane of the quadric at F. If we put

$$x_0 = \lambda, \quad y_0 = \mu, \quad z_0 = \lambda\mu, \quad t_0 = 1,$$

for the coordinates of F, this is the same as

$$xy_0 + yx_0 - zt_0 - tz_0 = 0,$$

or
$$\left(x_0 \frac{\partial}{\partial x} + y_0 \frac{\partial}{\partial y} + z_0 \frac{\partial}{\partial z} + t_0 \frac{\partial}{\partial t}\right)(xy - zt) = 0,$$

which is in accordance with preceding results, the equation of the quadric being $xy - zt = 0$.

The polar point, or pole, of a plane in regard to a quadric. There are two senses in which we may very naturally speak of the pole of a plane in regard to a quadric; we shew here that these lead to the same point.

(*a*) Given any plane ϖ, if any three points of this be A, B and C, and we take their polar planes in regard to the quadric, meeting, suppose, in the point O; then, as the polar plane of A passes through O, therefore the polar plane of O passes through A, and this plane similarly passes through B and C; it is, thus, the plane ϖ. The polar plane of any other point of the plane ϖ, since this point lies in the polar plane of O, also passes through O. The pole of the plane ϖ, in this sense, is thus the point common to the polar planes of all points of ϖ; and the plane is the polar plane of this point.

(*b*) As, however, we defined the polar plane of a point when the quadric is regarded as a locus of points, so we may define the pole of a plane, in a dual way, when the quadric is generated by an aggregate of planes. When regarded in the former way, the quadric is the aggregate of points lying on all transversals of three non-intersecting lines. We may, however, consider the aggregate of all planes passing through the transversals of three given non-intersecting lines. We have proved above that these planes are in fact the tangent planes of a quadric defined as a locus of points, each plane containing points of the quadric lying on two lines. As we defined the polar plane of a point, O, in regard to the quadric locus of points, by drawing a line through O, and taking the points Q, R of this line which are points of the quadric, and then the locus of the point P of this line which is the harmonic conjugate of O in regard to Q and R, so we may proceed for the aggregate of planes: given a plane ϖ, let l be a line on this plane; through this line l two planes of the aggregate can be drawn, which are in fact tangent planes of the quadric locus; we may take the plane, through the line l, which is the harmonic conjugate of ϖ in regard to these two planes. All these fourth harmonic planes, when l takes different positions on the plane ϖ, have a point, O', in common. This is the point which we might naturally call the pole of the plane ϖ, when the figure is regarded in this dual way. We shew that this point O' is the same as the point O obtained above, of which ϖ is the polar plane.

For the polar line of l, containing all points whose polar planes pass through l, passes through O; and, as we have seen above, this polar line meets the quadric locus in two points, Q and R, whereat the tangent planes of the quadric are such as to pass through l; as the point in which this line meets the plane ϖ is the harmonic conjugate of O, in regard to Q and R, it follows that the plane ϖ is the harmonic conjugate of the plane Ol in regard to the tangent planes Ql and Rl. This being so for every line l of the plane ϖ, the result follows, that O' coincides with O.

Alternative deduction of the polar point of a plane with the help of the symbols. If we are given three arbitrary non-intersecting lines, we have seen that two points of a transversal of these may, by taking suitable points of reference, be supposed to be of coordinates

$$(\lambda, \mu, \lambda\mu, 1), \quad (\lambda, \mu', \lambda\mu', 1),$$

respectively, where λ refers to the particular transversal chosen, and μ, μ' belong to the generators of the opposite system which determine the two points of this. The conditions that a plane, of equation $lx + my + nz + pt = 0$, should contain these two points, are

$$l\lambda + p + \mu(m + \lambda n) = 0, \quad l\lambda + p + \mu'(m + \lambda n) = 0;$$

the conditions that the plane should contain the transversal λ are, therefore,

$$l\lambda + p = 0, \quad m + \lambda n = 0;$$

these are the only conditions for the coordinates (l, m, n, p) of a plane containing a transversal of the three given lines, which may thus be expressed in terms of λ and another variable parameter, σ, in the forms

$$l = -\frac{1}{\lambda}, \quad m = -\frac{1}{\sigma}, \quad n = \frac{1}{\lambda\sigma}, \quad p = 1,$$

which are of the same forms as are the coordinates $(\lambda, \mu, \lambda\mu, 1)$ of a point of the locus. They satisfy the equation $lm - np = 0$; and the pole of an arbitrary plane, $Lx + My + Nz + Pt = 0$, in regard to the aggregate of planes whose coordinates satisfy this relation, calculated in the precisely dual way, will be the point whose equation is

$$\left(L\frac{\partial}{\partial l} + M\frac{\partial}{\partial m} + N\frac{\partial}{\partial n} + P\frac{\partial}{\partial p}\right)(lm - np) = 0,$$

or

$$Lm + Ml - Np - Pn = 0,$$

that is, the point whose coordinates are $(M, L, -P, -N)$. The polar plane of this point in regard to the quadric locus, of which the points satisfy the equation, $xy - zt = 0$, is the plane whose equation is

$$\left(M\frac{\partial}{\partial x} + L\frac{\partial}{\partial y} - P\frac{\partial}{\partial z} - N\frac{\partial}{\partial t}\right)(xy - zt) = 0;$$

this is exactly the original plane $Lx + My + Nz + Pt = 0$.

Examples of some general properties of quadrics. *Examples* 1—19. We may now usefully apply these ideas to deduce some general properties of a quadric.

Ex. 1. The plane, whose equation is

$$-\frac{x}{\lambda} - \frac{y}{\mu} + \frac{z}{\lambda\mu} + t = 0,$$

is at once verified to contain the three points

$$(\lambda, \mu, \lambda\mu, 1), \quad (\lambda, \mu', \lambda\mu', 1), \quad (\lambda', \mu, \lambda'\mu, 1),$$

whatever λ' and μ' may be, that is, it contains the two intersecting generators λ and μ. Correspondingly, if this plane be called $[\lambda, \mu]$, it is at once verified that the point $(\lambda, \mu, \lambda\mu, 1)$ lies on the three planes $[\lambda, \mu]$, $[\lambda, \mu']$, $[\lambda', \mu]$. The plane $[\lambda, \mu]$ is, in fact, the tangent plane of the quadric $xy - zt = 0$ at the point (λ, μ).

Ex. 2. The pole of the plane $Lx + My + Nz + Pt = 0$ in regard to the plane-aggregate which is represented by the general equation $(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0$, has a tangential equation

$$L(Al + Hm + Gn + Up) + \dots + P(Ul + Vm + Wn + Dp) = 0,$$

and this point, therefore, has the coordinates

$$AL + HM + GN + UP, \dots, UL + VM + WN + DP.$$

The polar plane of this point in regard to the locus represented by $(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0$ has the equation

$$(AL + HM + GN + UP)(ax + hy + gz + ut) + \dots$$
$$+ (UL + VM + WN + DP)(ux + vy + wz + dt) = 0.$$

If A, B, C, etc., be the cofactors of a, b, c, etc. in the determinant Δ, as before, this equation can be verified to reduce to

$$\Delta(Lx + My + Nz + Pt) = 0.$$

Ex. 3. Two points are said to be *conjugate* to one another in regard to a quadric when the polar plane of either point contains the other point. Two planes are said to be *conjugate* to one another in regard to a quadric when the pole of either plane lies on the other plane.

Now, supposing the quadric not to be a cone, consider any two planes, α, β, whose line of intersection meets the quadric in the points F and G; let the polar line of FG, which is the line joining the poles, say A and B, respectively, of the planes α, β, meet the quadric in F' and G'. We have seen that the lines, lying on the quadric, which intersect in F, pass through F' and G', as do the lines, lying on the quadric, which intersect in G. If the planes α, β be conjugate to one another, the point A, the pole of the plane α, lies on the plane β, and similarly B on α; so that the planes α, β meet the line $F'G'$, which is the line AB, in points which are harmonic conjugates of one another in regard to F' and G'. The tangent lines at F, of the conics in which the quadric is met by the planes α, β, meet the line $F'G'$, however, in the same two points as do the planes α, β; for these tangent lines lie in the tangent plane of the quadric at F, which, we have seen, contains the generators of the quadric at F, and contains the line $F'G'$. Thus we have the result: *if two conic sections of a quadric be made by planes which are conjugate to one another in regard to the quadric, the tangent lines of these sections, at either one of the two points of the quadric at which the conics intersect, are harmonic conjugates of one another in regard to the two generators of the quadric at this point.*

The fact which corresponds dually to this may be remarked; it will be seen that, incidentally, it has been established by the argument given. To the points of the quadric which lie in a plane, correspond dually the tangent planes of the quadric which pass through a point; as the points lie on a conic, so these planes are the tangent planes of a quadric cone. As the conics in which two planes meet the quadric have two points in common, these being the points of intersection, with the quadric, of the line of intersection of the two planes, so the tangent cones drawn to the quadric, from two points, have two tangent planes in common, these being the tangent planes drawn to the quadric from the line joining the vertices of the two cones. To the tangent line of a plane conic section of the quadric, at any point of this, corresponds a generator of a tangent cone. To the tangent lines of two plane sections, at a common point, which both lie in the tangent plane of the quadric at this point, correspond, then, two generators of two cones of contact of the quadric, lying in a common tangent plane of these cones, and intersecting in the point where this plane touches the quadric. If A and B be the vertices of these cones, and F be one of the two points of the quadric for which the tangent plane contains the line AB, this plane is one of the tangent planes of the quadric which pass through A, and, therefore, a tangent plane of the cone of contact whose vertex is A; and is, similarly, a tangent plane of the cone of vertex B. The lines AF, BF are the two generators of the cones which correspond to the tangents at a common point of the plane sections. By what we have seen above, if A and B be conjugate points in regard to the quadric, so that they are harmonic conjugates of one another in regard to the two points, F' and G' say, in which the line AB meets the quadric, then, the lines FA, FB are harmonic conjugates in regard to the two generators of the quadric which meet in F.

Ex. 4. We can, in an infinite number of ways, find four points, A, B, C, D, such that each is the pole of the plane containing the other three. Let D be an arbitrary point, and A an arbitrary point lying in the polar plane of D. The polar plane of A will, then, pass through D; and it will meet the polar plane of D in a line. On this line take an arbitrary point, B; the polar plane of B will then pass through A and D, and meet this line in a point C; as C is in the polar plane of D and A and B, its polar plane contains D, A, B. The four points A, B, C, D are, therefore, four such points as were desired.

When referred to A, B, C, D as points of reference, the equation of the quadric must reduce to

$$ax^2 + by^2 + cz^2 + dt^2 = 0,$$

containing only squares of the coordinates. For, if the point D be joined to any point, $xA + yB + zC$, of the plane ABC, the joining line meets the quadric in two points which are harmonic conjugates in regard to D and the point, $xA + yB + zC$, where this line meets the plane ABC. Two such points will be of symbols

$$xA + yB + zC + tD \text{ and } xA + yB + zC - tD;$$

namely, to any point (x, y, z, t) of the quadric is associated a point $(x, y, z, -t)$. Thus the equation of the quadric, $f(x, y, z, t) = 0$, must be such as to be the same as $f(x, y, z, -t) = 0$, for all values of x, y, z. It is similarly unaffected by change of the sign of x only, or y only, or z only; and thus has the form in question. Or the same result may be obtained by remarking that the polar plane of D, of which the equation is $ux + vy + wz + dt = 0$, must reduce to $t = 0$, so that $u = v = w = 0$; and similarly for A, B, C.

It follows that if $a_rx + b_ry + c_rz + d_rt = 0$ be the equations, for $r = 1, 2, 3, 4$, of four planes such that any three intersect in the pole of the remaining one, the equation of the quadric is capable of the form

$$\sum_{r=1}^{4} A_r (a_rx + b_ry + c_rz + d_rt)^2 = 0.$$

Further, the tangential equation of the quadric

$$ax^2 + by^2 + cz^2 + dt^2 = 0 \text{ is } a^{-1}l^2 + b^{-1}m^2 + c^{-1}n^2 + d^{-1}p^2 = 0.$$

And, if (x_r, y_r, z_r, t_r) be four points, for $r = 1, 2, 3, 4$, such that any one is the pole of the plane containing the other three, the tangential equation of the quadric is capable of the form

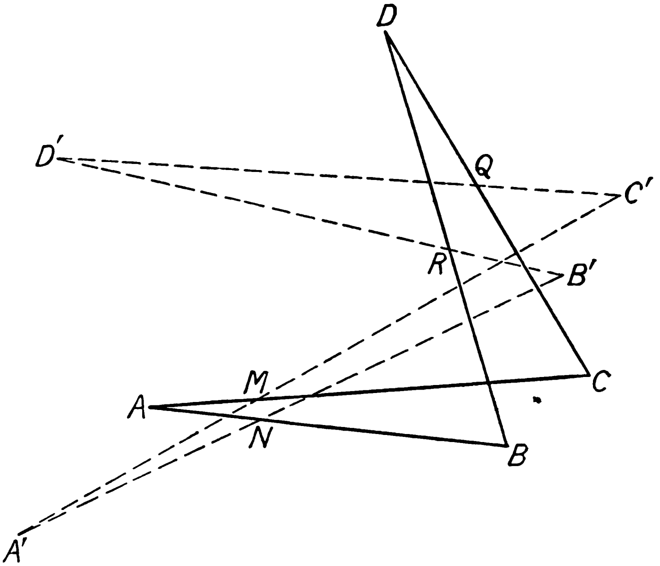
$$\sum_{r=1}^{4} (lx_r + my_r + nz_r + pt_r)^2 = 0.$$

For a cone, one of the four points, say D, must be at the vertex; the equation then is $ax^2 + by^2 + cz^2 = 0$; the polar plane of *any* point of the line DA is then the plane DBC, and so for DB and DC; the plane $t = 0$ is then arbitrary.

Four points A, B, C, D, of which each is the pole of the plane containing the other three, are said to form a self-polar tetrad in regard to the quadric.

Ex. 5. When two lines, l, m, are such that the polar line, l', of l, in regard to a quadric, intersects the other line m, it can be shewn that the polar line, m', of m intersects l. For, if M be the point where l' meets m, the polar plane of M, a point of l', contains the line l; but as M is on m, this polar plane also contains the line m'. Thus l and m' meet. Two such lines, l, m, are said to be *conjugate* in regard to the quadric.

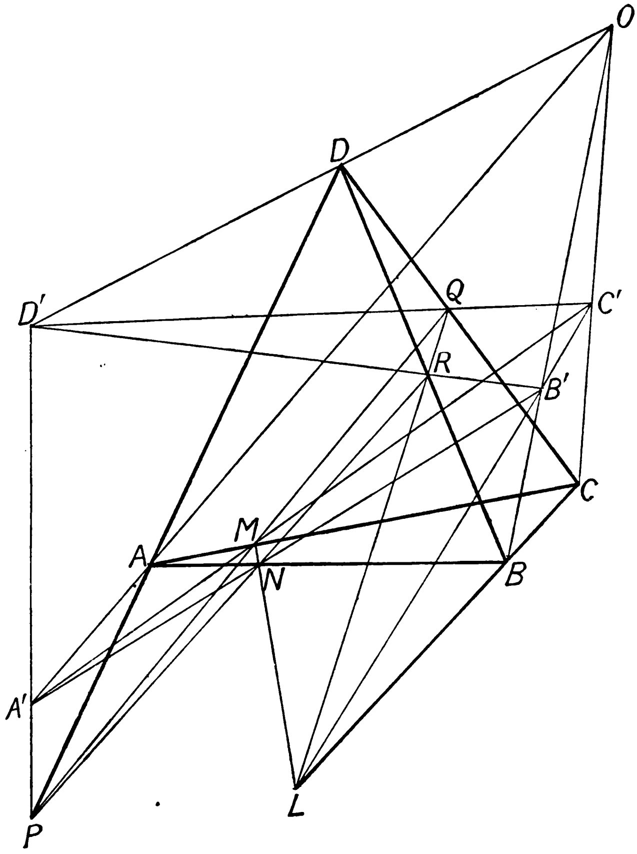
It can be shewn that, if A, B, C, D be four points such that two pairs of opposite joining lines of these be conjugate, say CA conjugate to BD and AB conjugate to CD, then also the third pair of joins, BC and AD, are conjugate lines. Let D', A', B', C' be the poles, respectively, of the planes ABC, BCD, CAD, ABD, so



that $C'D'$, which is the polar line of AB, intersects CD, say in Q; and similarly, $B'D'$, $C'A'$, $B'A'$, the respective polar lines of AC, BD, CD, meet the joins BD, AC, AB, say in the points R, M, N.

Considering the section, of the quadric and of the planes DBC, DCA, DAB, by the plane $A'B'C'$, we obtain a conic, and three lines, which, in virtue of the definition of a polar plane by a harmonic range, are the polar lines, with respect to this conic, respectively of the points A', B', C'. The three intersections of these lines respectively with $B'C'$, $C'A'$, $A'B'$ are thus in line (Vol. II, p. 31). Thus, the line $B'C'$ intersects the line MN in a point lying on the plane DBC, which is, therefore, on the line BC. By considering the section by the plane $D'B'C'$, we similarly prove that QR and $B'C'$ intersect on BC. Let this point of intersection of $B'C'$, QR, MN, BC be called L. A similar argument will shew that QM, RN, AD and $A'D'$ meet in a point, say P. It is then easy to prove that any two of the four lines AA', BB', CC', DD' meet one another, so that these four lines all meet in one point, say O, and the two tetrads A, B, C, D and A', B', C', D' are in perspective from O, corresponding lines and planes, belonging to these, intersecting the plane $QMNR$ in the same point or line. The consequence that BC, being met by $B'C'$, is conjugate to AD, is contained in this statement.

The complete figure so obtained is one in which there are fifteen points, twenty lines and fifteen planes, each point lying on four of these lines and on six of these planes, each line containing three of the points and lying in three of the planes, each plane containing six of the points, and four of the lines. So that the structure of the figure may be indicated by the scheme



$$15\,(\,\cdot\,, 4, 6)\ 20\,(3, \cdot\,, 3)\ 15\,(6, 4, \cdot\,)\,;$$

the figure is in fact the same as that which is used in the proof of Desargues' theorem for two triangles lying in the same plane (of which a diagram is given as frontispiece to Vol. I).

In this figure the points A', B', C', D' are introduced as the poles of certain planes, and these are in perspective with A, B, C, D

from the point O. It can however be shewn that the points of the figure are symmetrical in their relations to the quadric. For first, the line AA', joining the poles of the planes $B'C'D'$, BCD, is the polar line of the intersection of these, that is, of the line QR; and, similarly, the lines BB', CC', DD' are, respectively, the polars of the lines MQ, NR, MN, so that the point O is the pole of the plane, $MQRN$, containing these four lines; and also, the point P, where MQ, NR intersect, is the pole of the plane OBC, containing the lines BB', CC'; and, similarly, L, M, N, Q, R are, respectively, the poles of the planes OAD, OBD, OCD, OAB, OCA. Thus the six points in which the polar plane of O meets the joins of A, B, C, D (which are also the points in which this plane meets the joins of A', B', C', D'), are the poles of the planes joining O to the respectively opposite joins of A, B, C, D (which are the same as the planes joining O to the corresponding joins of A', B', C', D'). Every one of the fifteen points of the figure is, however, the centre of perspective of two tetrads of points, of which the respective sets of six joins occur in the figure; the remaining six points of the figure are then on the polar plane of the centre of perspective, and are the poles of the six planes of the figure passing through this centre. For each of the two tetrads, a join, l', is conjugate to the opposite join, l, the polar line of l being a join in the second tetrad, meeting l'. Either tetrad consists of the poles of the planes containing three points of the other tetrad. The point O and A, B, C, D together form a pentad of points, with the property that the pole of the plane containing any three of these lies on the line joining the other two; the same statement is true of the pentad O, A', B', C', D'. For each of the fifteen points of the figure a similar pair of pentads exists. Such a pentad may be called a *self-conjugate pentad*. It may be regarded as a generalisation of a self-polar tetrad, just as, in a plane, a set of four points, of which every pair of opposite joins consists of conjugate lines in regard to a conic (Vol. II, p. 33), may be regarded as a generalisation of a self-polar triad.

Conversely given two tetrads of points which are in perspective, there is a definite quadric in regard to which the completed figure has the reciprocal properties which have been announced.

The whole may be discussed with the symbols; and it appears that the quadric can be expressed, in point coordinates, or in tangential coordinates, by an equation involving five squares, for each of the ways in which the figure may be regarded. Let A, B, C, D be four points such that the lines AC, BD are conjugate, and, also, the lines AB, CD are conjugate, in regard to a certain quadric surface. Let the equation of this surface, in point coordinates referred to A, B, C, D, be $(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0$, and A', B', C', D' be, as before, the poles of the planes BCD, CAD, ABD, ABC; then the planes $A'B'D'$, $B'C'D'$, which are the polar planes of C and A respectively, have the respective equations

$$gx + fy + cz + wt = 0, \quad ax + hy + gz + ut = 0;$$

in order that $B'D'$, the line of intersection of these planes, should intersect BD, the points, given by $fy + wt = 0$, $hy + ut = 0$, where these planes meet BD, must be the same, so that $fu = hw$. In order that $C'D'$ may intersect CD, the results of putting $x = 0$, $y = 0$ in the equations $ax + hy + gz + ut = 0$, $hx + by + fz + vt = 0$, namely $gz + ut = 0$, $fz + vt = 0$, must be the same, so that $fu = gv$. From $gv = hw$ it then follows at once that the joins BC, AD are conjugate. It follows also, writing σ in place of any one of the equal expressions vw/f, wu/g, uv/h, that the equation of the quadric is capable of the form

$$(a - \sigma^{-1}u^2)x^2 + (b - \sigma^{-1}v^2)y^2 + (c - \sigma^{-1}w^2)z^2 + (d - \sigma)t^2 \\ + \sigma^{-1}(ux + vy + wz + \sigma t)^2 = 0;$$

here $x = 0$, $y = 0$, $z = 0$, $t = 0$ are the equations of the planes BCD, ..., ABC; and it can be shewn that $ux + vy + wz + \sigma t = 0$ is the equation of the plane containing the six points of intersection of corresponding joins of the two tetrads A, B, C, D and A', B', C', D'. This plane is the polar of the point, O, in which the lines AA', BB', CC', DD' meet. It will appear that, if the cofactors of a, b, c, ... in the determinant Δ, formed by the coefficients in the equation of the quadric, be, as above (p. 24), denoted by A, B, C, ..., then there exist also the relations $FU = GV = HW$, and that, if λ denote any one of the equal expressions VW/F, WU/G, UV/H, then the coordinates of O are (U, V, W, λ). That the polar plane of this point should have the equation above given involves identities which are reducible to those given.

We may investigate the theorem with the tangential equation of the quadric, supposed of the form

$$(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0.$$

The condition that the line $B'D'$ should meet the line BD is that a plane with equation of the form $x = mz$ should contain the points B', D', whose coordinates are, respectively, (H, B, F, V) and (U, V, W, D); this requires that $HW = FU$. The condition that the line $C'D'$ should meet the line CD, similarly, requires that the points (G, F, C, W), (U, V, W, D) should lie in a plane with equation of the form $x = ny$; and this involves that $GV = FU$. The consequence $GV = HW$, of these equations, is easily seen to secure that the joins BC, AD are conjugate in regard to the quadric. Denoting the equal quantities VWF^{-1}, etc., as just said, by λ, these identities allow the tangential equation of the quadric to be put into the form

$$(A-\lambda^{-1}U)\,l^2+(B-\lambda^{-1}V)\,m^2+(C-\lambda^{-1}W)\,n^2+(D-\lambda)\,p^2$$
$$+\lambda^{-1}(Ul+Vm+Wn+\lambda p)^2=0,$$

where $l=0$, $m=0$, $n=0$, $p=0$, $Ul+Vm+Wn+\lambda p=0$, are, respectively, the tangential equations of the points A, B, C, D, O.

In general, we may determine the tangential equation of a quadric whose point equation is

$$Ex^2+Qy^2+Rz^2+Kt^2+(ex+qy+rz+kt)^2=0.$$

For this, we have to express that, for a point (x', y', z', t') satisfying this equation, and the equation $lx'+my'+nz'+pt'=0$, the equation

$$Exx'+Qyy'+Rzz'+Ktt'+(ex'+qy'+rz'+kt')\,(ex+qy+rz+kt)=0$$

is the same as $lx+my+nz+pt=0$; namely that, with M' written for $ex'+qy'+rz'+kt'$, we have

$$\frac{Ex'+M'e}{l}=\frac{Qy'+M'q}{m}=\frac{Rz'+M'r}{n}=\frac{Kt'+M'k}{p}.$$

These equations, if $lx'+my'+nz'+pt'=0$, involve that (x', y', z', t') lies on the quadric. Denoting these equal fractions by θ, we have

$$Ex'+M'e-\theta l=0,\quad \ldots,\quad Kt'+M'k-\theta p=0,$$

from which we infer

$$M'+M'\,(e^2E^{-1}+\ldots+k^2K^{-1})-\theta\,(elE^{-1}+\ldots+kpK^{-1})=0\,;$$

if $1+e^2E^{-1}+\ldots+k^2K^{-1}$ be denoted by ω, and $elE^{-1}+\ldots+kpK^{-1}$ by L, this equation is $M'=\omega^{-1}\theta L$. We, therefore, have

$$Ex'+\theta\,(e\omega^{-1}L-l)=0,\ \ldots,\ Kt'+\theta\,(k\omega^{-1}L-p)=0,$$

so that, by $lx'+\ldots+pt'=0$, we deduce

$$E^{-1}l^2+\ldots+K^{-1}p^2-\omega^{-1}L^2=0\,;$$

this is the tangential equation sought. More explicitly it is

$$\left(1+\frac{e^2}{E}+\ldots+\frac{k^2}{K}\right)\left(\frac{l^2}{E}+\ldots+\frac{p^2}{K}\right)-\left(\frac{e}{E}\,l+\ldots+\frac{k}{K}\,p\right)^2=0.$$

It is easily seen that a special form of self-conjugate pentad is constituted by a self-polar tetrad and an arbitrary point.

A particular case of a tetrad, of which each pair of opposite joins are conjugate lines in regard to a quadric, arises when the quadric, regarded as the aggregate of its tangent planes, reduces to all the planes passing through the tangent lines of a conic. This degeneration of a quadric is that which is the dual of the case when the quadric, regarded as the locus of its points, reduces to all the points on the generating lines of a quadric cone. In the latter case, the polar plane of any point passes through the vertex of the cone. In the former case, which we consider here, the pole of any plane is a point in the plane of the conic, being the pole, in regard to the conic, of the line in which the plane meets the plane of the conic; the polar line of any line, which does not lie in the plane of the conic, is the line, in this plane, which is the polar line, in regard to the conic, of the point in which the former line meets this plane. Two lines, of which neither lies in the plane of the conic, are then conjugate lines if they meet in this plane in two points which are conjugate points in regard to this conic. This being understood, we still have the result that, if the joins AC, BD, of four points A, B, C, D, be conjugate lines in regard to this degenerate quadric, and also the joins AB, CD be conjugate lines, then the remaining pair of joins BC, AD are conjugate lines. And we have the further result that the lines joining A, B, C, D, respectively to the poles A', B', C', D', of the planes BCD, CAD, ABD, ABC, all of which are in the plane of the conic, are four lines which meet in a point. In this case the points L, M, N, P, Q, R, are the points where the joins of A, B, C, D meet the plane of the conic.

Ex. 6. The unsymmetrical algebra of Ex. 5 may be replaced by symmetrical algebra, which suggests itself naturally if we regard the figure as arising by projection of a figure in four dimensions. The theorem so suggested is one for which there is an analogue in space of any number of dimensions; for instance we may have, in a plane, a set of four points such that the join of any two is conjugate to the join of the other two, with respect to a conic in that plane. This is analogous to the self-conjugate pentad. By taking the poles of these joins we obtain six other points, lying by threes on four lines. Thereby we have a figure of ten points and ten lines, of which three points lie on each line, and three lines pass through each point. This is analogous to the complete figure obtained above, and is self-polar in regard to the conic; as the figure obtained in Ex. 5 may be regarded as consisting of two tetrads in perspective from a point, together with the six points of intersection of the corresponding joins of the two tetrads, so the figure of which we speak, in a plane, may be regarded as consisting, in ten ways, of two triads in perspective from a point, together with the three points of intersection of corresponding joins of the two triads, these lying in line. The centre of perspective is the pole of this line. The conic in regard to which the polarity has place may have its equation expressed as a sum of four squares, in terms of the three joins of one of the triads which are in perspective, and of the axis of perspective corresponding, and this mode of expression arises in as many ways as the relation in question. But the analytical expression is suggested most naturally by regarding the figure as arising from a figure in three dimensions, that namely of the complete intersection of five planes. This figure is also self-polar, in a particular way, in regard to a quadric cone.

The existence of this polarity may be regarded as a generalisation of the theorem that if we take four arbitrary lines in a plane, and an arbitrary point O of the plane, the pair of lines joining O to two opposite pairs of intersection of the four lines, are harmonic conjugates in regard to a certain pair of lines drawn through O. This is the dual of the Greek theorem that the joins of four points of a plane cut an arbitrary line in pairs of points in involution.

The general theorem to which reference is here made will be understood from the Examples on p. 218 of Vol. II.

Ex. 7. Consider now a tetrad of points in three dimensions, and a fixed quadric, Σ, such that no one of the three pairs of opposite joins of these points consists of lines which are conjugate in regard to Σ. We prove that, if A', B', C', D' be the poles of the planes BCD, CAD, ABD, ABC, respectively, then the four lines, AA', BB', CC', DD', are such that they have an infinite number of transversals, and that they may be regarded as lines belonging to one system of lines of a quadric surface.

The condition that no two opposite joins, such as AB, CD, are conjugate lines, involves that no two of the lines AA', BB', CC', DD' intersect. For, if AA' meets BB', then $A'B'$ meets AB; but $A'B'$ is the polar line of CD, and only meets AB when AB, CD are conjugate.

Now consider the plane $A'B'C'$; let it meet the lines DA, DB, DC respectively in A_1, B_1, C_1. Then, in virtue of the definition of the polar plane of a point in regard to a quadric, by means of ranges of four harmonic points, it follows that the lines B_1C_1, C_1A_1, A_1B_1 are the polar lines, respectively of A', B', C', in regard to the conic in which the plane $A'B'C'$ meets the quadric Σ. Thus, by a theorem for two triads which arise by taking polars in regard to a conic (Vol. II, p. 31), the triads A', B', C' and A_1, B_1, C_1 are in perspective, and the lines $A'A_1$, $B'B_1$, $C'C_1$ meet in a point. Thus the planes DAA', DBB', DCC' meet in a line; and this line meets AA', BB' and CC'; it evidently also meets DD', at D. By a similar argument there is a line through each of the points A, B, C which meets all the lines AA', BB', CC', DD'. The three lines AA', BB', CC', of which, as we have seen, no two intersect, determine a quadric surface, containing all the points of all the transversals of these three lines; the line DD' contains four points of this quadric surface, namely the points where it is met by the four transversals of AA', BB', CC', DD' which are found above. Thus DD' lies entirely on this quadric surface.

Let the tangential equation of the given quadric surface, Σ, referred to the points A, B, C, D, be, as before,

$$(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0.$$

Then, as the poles of the planes ABC, BCD, etc., are (U, V, W, D), (A, H, G, U), etc., the planes DAA', DBB', DCC' are at once seen to intersect in the line, through the point D, whose equations are $Fx = Gy = Hz$; and the planes ABB', ACC', ADD' similarly intersect in the line whose equations are $V^{-1}y = W^{-1}z = FV^{-1}W^{-1}t$. These two lines are at once verified to lie entirely upon the quadric surface whose equation is

$$(GV - HW)(Fxt + Uyz) + (HW - FU)(Gyt + Vzx) \\ + (FU - GV)(Hzt + Wxy) = 0,$$

and, as this equation is symmetrical, it is that containing the four lines AA', BB', CC', DD'. This equation is independent of the coefficients A, B, C, D.

Let this quadric be denoted by $\Phi(A, B, C, D)$, and let E be any point lying thereon. We may similarly form, with poles of the planes ABC, BCE, CAE, ABE, taken in regard to the same quadric Σ, a quadric $\Phi(A, B, C, E)$. One common point of these quadrics is the point D', which is the pole of the plane ABC in regard to the quadric Σ, and the lines $D'D$, $D'E$ lie, by construction, respectively on the quadrics $\Phi(A, B, C, D)$ and $\Phi(A, B, C, E)$; through D' there passes another generator of each of these quadrics. This other generator is, in fact, the same for both. For, if P, Q, R be the poles, in regard to Σ, respectively of the planes BCE, CAE, ABE, the three points D', A', P, being poles of planes which intersect in the line BC, are on a line, the polar line of BC in regard to Σ; therefore the plane $D'AA'$, containing $D'A'$, equally contains $D'P$. The generator of $\Phi(A, B, C, D)$ of which we are speaking is a line through D' meeting the lines AA', BB', CC'; thus the planes $D'AA'$, $D'BB'$, $D'CC'$ meet in a line, and, of these, the plane $D'AA'$ contains $D'P$, as we have seen; this common line, then, equally meets the lines AP, BQ, CR, and is the generator spoken of, drawn through D', of the quadric $\Phi(A, B, C, E)$.

It can be further shewn that the quadric $\Phi(A, B, C, E)$ passes through the point D. But this result follows from the less particular result given in the following example (8).

Ex. 8. Let A, B, C and P, Q, R be any six points, in three dimensions; suppose that the polar lines of the lines QR, RP, PQ, in regard to any quadric, Σ, which will be lines passing through the pole, say O, of the plane PQR in regard to the quadric Σ, meet the plane ABC in points A', B', C', respectively, such that the lines AA', BB', CC' meet in a point. Then, if the polar lines, in regard to Σ, of the lines BC, CA, AB, which will be lines passing through the pole, say D, of the plane ABC, meet the plane PQR, respectively, in points P', Q', R', the lines PP', QQ', RR' also meet in a point.

For, by the hypothesis made, the planes $AA'O$, $BB'O$, $CC'O$ meet in a line. Now A is the intersection of the lines AB, AC, and is thus the pole of the plane containing the polar lines of these, DR', DQ', respectively, namely, A is the pole of the plane $DQ'R'$; and $A'O$ is the polar line of QR. Thus, the plane joining A to the line $A'O$ is the polar of the point in which the plane $DQ'R'$ is met by the line QR; in other words, the plane $AA'O$ is the polar of the point of intersection of the lines $Q'R'$ and QR. Similarly the planes $BB'O$ and $CC'O$ are, respectively, the polar planes of the points $(R'P', RP)$ and $(P'Q', PQ)$. As these three planes meet in a line, it follows that the points $(Q'R', QR)$, $(R'P', RP)$, $(P'Q', PQ)$ are in line, so that the triads P', Q', R' and P, Q, R are in perspective, as was to be shewn.

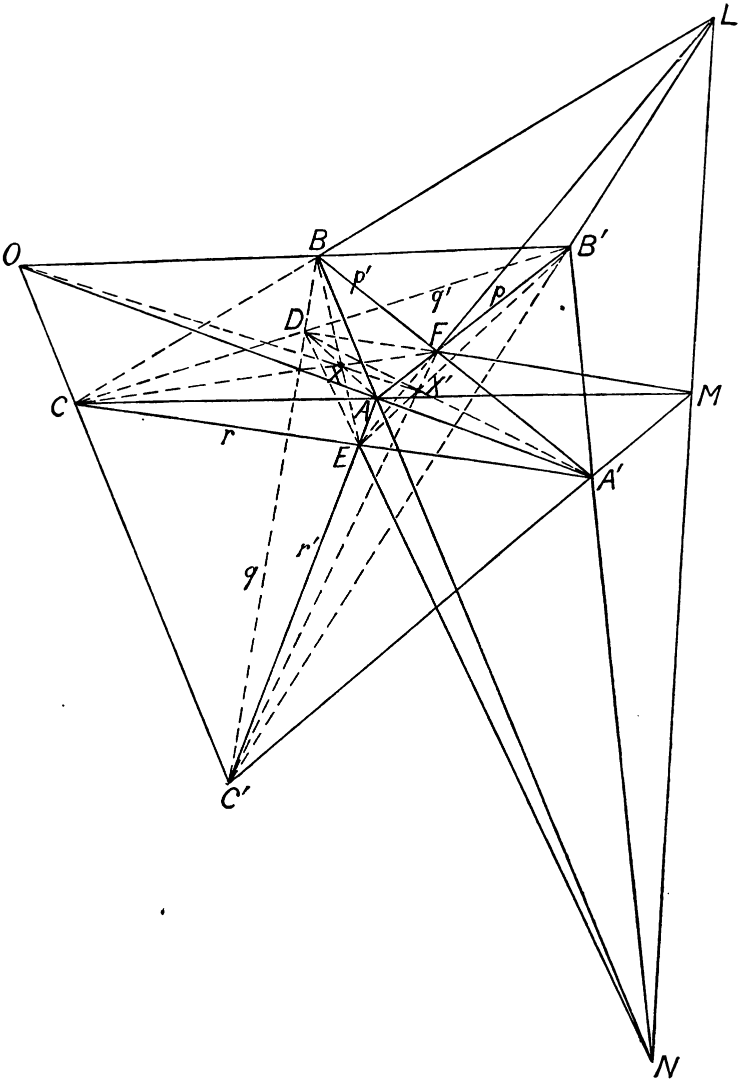
The proposition dual to the above may also be enunciated; but first it may be well to remark that three planes, α, β, γ, are said to be in perspective with three planes, α', β', γ', whose point of intersection is the same as the point of intersection of α, β, γ, when the plane containing the lines $\beta\gamma$, $\beta'\gamma'$, the plane containing the lines $\gamma\alpha$, $\gamma'\alpha'$, and the plane containing the lines $\alpha\beta$, $\alpha'\beta'$, are three planes which meet in a line. When this is so, the line of intersection of the planes α, α', the line β, β' and the line γ, γ', are three lines which lie in a plane. This being clear, the dual of the result above given is this: let α, β, γ be three planes, meeting in a point (α, β, γ), and (λ, μ, ν) be three other planes, meeting in a point (λ, μ, ν). Consider the polar lines of the lines (μ, ν), (ν, λ) and (λ, μ); these polars are lines lying in the polar plane of the point (λ, μ, ν). Suppose that the planes joining the point (α, β, γ) to these polar lines are in perspective with the planes α, β, γ. Then, also, the polar lines of the lines (β, γ), (γ, α), (α, β)—which polars are lines lying in the polar plane of the point (α, β, γ)—when joined by planes to the point (λ, μ, ν), give three planes in perspective with the planes λ, μ, ν.

A particular case of this is the result referred to in the preceding example. Let DBC, DCA, DAB, or λ, μ, ν, be three planes, whose respective poles are A', B', C', with reference to a quadric Σ. Let E be a point from which can be drawn a line to meet the lines AA', BB', CC', so that the planes EAA', EBB', ECC' meet in a line. From this it follows that if the planes EBC, ECA, EAB be called, respectively, α, β, γ, the planes joining E to the polar lines, respectively $B'C'$, $C'A'$, $A'B'$, of λ, μ, ν, are in perspective with the planes α, β, γ. Now let P, Q, R, respectively, be the poles of the planes EBC, ECA, EAB, so that the planes joining D to the polars of the lines EA, EB, EC are, respectively, the planes DQR, DRP, DPQ. By what we have seen above, these planes are in perspective with the planes λ, μ, ν, respectively. Therefore the planes DAP, DBQ, DCR meet in a line. Hence, a line can be drawn from D to meet the lines AP, BQ, CR. That is, as stated in the last example, when E is a point of the quadric $\Phi(A, B, C, D)$, then D is a point of the quadric $\Phi(A, B, C, E)$.

Ex. 9. It has been remarked, in Ex. 5, that when two tetrads of points, in three dimensions, are in perspective, there is a definite quadric, in regard to which each point of either tetrad is the pole of the plane containing three points of the other tetrad. A particular case is that when three of the points of either tetrad lie upon their corresponding planes, which are thus tangent planes of the quadric. This case arises when we consider a skew hexagon, formed with three generators of a quadric of one system, and three generators of the other system (Dandelin, *Gergonne's Annales de Math.* xv (1824, 5), p. 387; Hesse, *Crelle*, xxiv (1842), p. 40); and the figure leads to a proof of the concurrence, in a Steiner point, of three of the Pascal lines of six points of a conic, and to the theorem that two of these Steiner points are conjugate points in regard to the conic (Vol. II, Note II, p. 231).

It will be simplest, first, to describe the figure which is obtained: let O, A, B, C be four points of which no three are in line; let A', B', C' be any points lying respectively on OA, OB, OC, the lines BC', $B'C$ meeting in D, the lines CA', $C'A$ meeting in E, and the lines AB', $A'B$ meeting in F. The lines BC, EF then meet on $B'C'$ at the point, L, where the planes $OB'C'$, $A'BC$, AEF meet; so CA, FD, $C'A'$ meet, say in M, and the lines AB, DE, $A'B'$ meet, say in N; the points L, M, N are in the line in which the planes ABC, $A'B'C'$, DEF meet. Further, the lines AD, BE, CF meet in a point, say X; also the lines $A'D$, $B'E$, $C'F$ meet in a point, say X'; and the points O, X, X' are in line. The two tetrads A, B, C, X and A', B', C', X' are, thus, in perspective from O, the plane of perspective being D, E, F, L, M, N; but A' lies on the plane BCX, as A lies on the plane $B'C'X'$, and, similarly, B, B', C, C' lie each on their corresponding planes; the figure is thus a particular case of one often considered, of which the scheme of the indices of structure is $15(\cdot, 4, 6)\ 20(3, \cdot, 3)\ 15(6, 4, \cdot)$. If we take, for the symbols of A, B, C, in regard to A', B', C', O, $A = O + A'$, etc., we have $D = O + B' + C'$, etc.; $X = 2O + A' + B' + C'$; $X' = O + A' + B' + C'$.

We may initiate the figure by taking six generators, AB', $B'C$, CA', $A'B$, BC', $C'A$, which we shall call, respectively, p, q', r, p', q, r', of a quadric surface. The point A, where the generators AB', AC' intersect, is then the pole of the plane $B'C'X'$, and so on; the line



AD is the polar line of the line $B'C'$, etc., and X is the pole of the plane $A'B'C'$; so the line $A'D$ is the polar line of the line BC, etc., and X' is the pole of the plane ABC; the line AA' is the polar line of the line EF, etc.; hence the line AXX' is the polar line of the line LMN.

It can be proved that the equation of the quadric, referred to A, B, C, X, is capable of the form

$$x^2+y^2+z^2-t^2-(x+y+z-t)^2=0,$$

this being the particular case of the equation

$$(a+b+c+d+1)(ax^2+by^2+cz^2+dt^2)-(ax+by+cz+dt)^2=0,$$

which arises when $a=b=c=-1$, $d=1$; this last is that of the quadric in regard to which the two tetrads,

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),$$

and

$$(a^{-1}+1, 1, 1, 1), (1, b^{-1}+1, 1, 1), (1, 1, c^{-1}+1, 1), (1, 1, 1, d^{-1}+1),$$

are obtained from one another by reciprocation.

Now consider the section of the quadric by an arbitrary plane. The generators p, q', ... will meet this plane in six points, say P, Q', ..., lying on a conic; the plane of two consecutive generators, p, q', will meet this plane in the line PQ'; the intersection of the opposite joins, PQ', $P'Q$, of the six points, P, Q', etc., of the plane, will be the point in which the plane is met by the line of intersection of the opposite planes, p, q' and p', q, of the skew hexagon formed by the generators. Taking the generators in a definite order, p, q', r, p', q, r', the three lines of intersection of the three pairs of opposite planes, each plane defined by two consecutive generators, namely of the planes pq', $p'q$; qr', $q'r$; rp', $r'p$, are in one plane. This plane meets an arbitrary plane in a line. This is then the Pascal line of the six points, P, Q', R, P', Q, R', in the plane. By taking the generators in a different order we obtain other Pascal lines in this plane, corresponding to different orders of the same six points P, Q', ... We consider in particular the six orders obtained by interchanging the alternate generators p', q', r' among themselves, leaving p, q, r unaltered. We find that the three Pascal lines corresponding to the three cyclically different orders p', q', r'; q', r', p'; r', p', q', meet in a point, one of the Steiner points, the three planes in the space figure which give these Pascal lines, by their intersection with the arbitrary plane, being planes which meet in a line, say l; and, then, that the three planes corresponding to the other three orders r', q', p'; q', p', r'; p', r', q' meet in another line, say l', which is the polar line of l in regard to the quadric. Wherefore these latter give rise to a Steiner point, for the conic, which is conjugate to the former Steiner point in regard thereto. The details may be tabulated thus:

the orders of the generators

$$p, p', r, r', q, q';\quad p, r', r, q', q, p';\quad p, q', r, p', q, r';$$
$$p, q', r, r', q, p';\quad p, p', r, q', q, r';\quad p, r', r, p', q, q',$$

correspond respectively to the skew hexagons

$$B', F, A', E, C', D;\quad F, A, E, C, D, B;\quad A, B', C, A', B, C';$$
$$F, B', C, E, C', B;\quad A, F, A', C, D, C';\quad B', A, E, A', B, D;$$

the pairs of opposite planes of these, each plane containing two consecutive sides of the skew hexagon, meet on three lines which are the joins, respectively, of the three points

$$A, B, C;\quad A', B', C';\quad D, E, F;\quad A, A', D;\quad B, B', E;\quad C, C', F;$$

the planes determined by the first three triads of points meet in the line LMN; the planes determined by the second three triads of points meet in the line OXX'. We have seen that this line is the polar line of the line LMN.

This proves the statement made.

Ex. 10. We have considered above, in Ex. 5, a set of five points which are such that the pole of the plane containing any three of them, taken in regard to a quadric, lies on the join of the other two, calling it a self-conjugate pentad. We consider now a set of six points which are such that the pole of the plane containing any three of them, taken in regard to a quadric, lies on the plane containing the other three. The pentad was such that the tetrad, constituted by any four of its points, had each of its three pairs of opposite joins as conjugate lines, in regard to the quadric. For a hexad, we take any four arbitrary points A, B, C, D, and consider the quadric, $\Phi(A, B, C, D)$, of which four generators are the lines joining A, B, C, D, respectively, to the poles, A', B', C', D', of the planes BCD, CAD, ABD, ABC, taken in regard to the fixed quadric, Σ. If E and F form, with A, B, C, D, such a hexad as is desired, the plane EFD must contain the pole, D, of the plane ABC, so that the line EF must meet DD'; and it must similarly meet AA', BB', CC'. Thus EF must be a generator of $\Phi(A, B, C, D)$, of the opposite system from the lines AA', etc. Again, the plane ADF must contain the pole, P, of the plane BCE, or DF must meet AP; similarly, DF must meet BQ and CR, where Q and R are, respectively, the poles of the planes CAE and ABE. Thus DF must be a generator of the quadric $\Phi(A, B, C, E)$, of the opposite system from the lines AP, etc. We therefore proceed as follows: we take E to be any point of $\Phi(A, B, C, D)$, and on the generator of this through E, of the opposite system from the lines AA', etc., we take the point F, in which the quadric $\Phi(A, B, C, E)$, which passes through E, meets this generator again. Then as EF meets DD', the line ED', which lies on $\Phi(A, B, C, E)$, meets DF; we have shewn (above, Ex. 8) that $\Phi(A, B, C, E)$ contains D; thus the line DF contains three points of $\Phi(A, B, C, E)$, and is a generator of this, through D, meeting ED', and, thus, of the opposite system from the lines AP, etc. We can now shew that the hexad A, B, C, D, E, F, so constructed, is such a self-conjugate hexad as is desired, in regard to the quadric Σ.

For first, as EF meets DD', the plane DFF' contains the pole, D', of the plane ABC; for a similar reason, the planes AEF, BEF, CEF, are, respectively, conjugate to the opposite planes of the hexad, BCD, CAD, ABD. Again, as DF is a generator of $\Phi(A, B, C, E)$, of the opposite system from the lines AP, etc., it follows that AP meets DF, and the plane ADF contains the pole, P, of the opposite plane, BCE, of the hexad; similarly, the planes, BDF, CDF are conjugate to the respectively opposite planes, CAE, ABE, of the hexad. Finally, we proved (Ex. 8, above), that if E be on $\Phi(A,B,C,D)$, then D is on $\Phi(A, B, C, E)$; by a similar argument, since F is on $\Phi(A, B, C, E)$, it follows that E is on $\Phi(A, B, C, F)$, and the point E may be constructed from F, just as was F from E. Wherefore it follows that DE is a generator of $\Phi(A, B, C, F)$, and that the planes ADE, BDE, CDE are conjugate to the respectively opposite planes, BCF, CAF, ABF, of the hexad. The proof of the property is thus complete, for each of the ten pairs of opposite planes of the hexad. In the particular case when E is so taken that EF contains the pole, D', of the plane ABC, it may be shewn that A, B, C, E, F constitute a self-conjugate pentad.

It has been remarked that the equation of the quadric $\Phi(A,B,C,D)$, referred to A, B, C, D, if the tangential equation of the fixed quadric, Σ, be $(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0$, is

$$(GV - HW)(Fxt + Uyz) + (HW - FU)(Gyt + Vzx) + (FU - GV)(Hzt + Wxy) = 0.$$

If the point E be (ξ, η, ζ, τ), it can be shewn that the point F, or $(\xi', \eta', \zeta', \tau')$, is given by

$$\frac{\xi'}{\tau'} = \frac{G\eta - H\zeta}{\eta W - \zeta V}, \quad \frac{\eta'}{\tau'} = \frac{H\zeta - F\xi}{\zeta U - \xi W}, \quad \frac{\zeta'}{\tau'} = \frac{F\xi - G\eta}{\xi V - \eta U}.$$

Further, it can be shewn that these values secure that all the determinants, of three rows and columns, of the array

$$\left\| \begin{array}{cccccc} F, & G, & H, & U, & V, & W \\ \eta\zeta, & \zeta\xi, & \xi\eta, & \xi\tau, & \eta\tau, & \zeta\tau \\ \eta'\zeta', & \zeta'\xi', & \xi'\eta', & \xi'\tau', & \eta'\tau', & \zeta'\tau' \end{array} \right\|,$$

are zero. It follows thence that the quadric Σ is capable of being written, tangentially, in the form

$$Pl^2 + Qm^2 + Rn^2 + Kp^2 + (l\xi + m\eta + n\zeta + p\tau)^2 + (l\xi' + m\eta' + n\zeta' + p\tau')^2 = 0;$$

this form shews at once that the plane containing any three of the points A, B, C, D, E, F is conjugate, in regard to this quadric, to the plane containing the other three points.

To find the point equation of the quadric represented by this equation, say $\Psi = 0$, we have to find the locus of a point (x, y, z, t), where x is $\partial\Psi/\partial l$, etc., and $lx + my + nz + pt = 0$; that is, writing L for $l\xi + \ldots + p\tau$, and L' for $l\xi' + \ldots + p\tau'$, we have to eliminate l, m, n, p, and ρ, from the equations

$$Pl + \xi L + \xi' L' + \rho x = 0, \text{ etc.,} \quad lx + my + nz + pt = 0.$$

These equations give, if we write

$$U = \frac{x\xi}{P} + \ldots + \frac{t\tau}{K}, \quad U' = \frac{x\xi'}{P} + \ldots + \frac{t\tau'}{K}, \quad S = \frac{x^2}{P} + \ldots + \frac{t^2}{K},$$

$$\omega = 1 + \frac{\xi^2}{P} + \ldots + \frac{\tau^2}{K}, \quad \phi = \frac{\xi\xi'}{P} + \ldots + \frac{\tau\tau'}{K}, \quad \omega' = 1 + \frac{\xi'^2}{P} + \ldots + \frac{\tau'^2}{K},$$

the following equations

$$\begin{aligned} UL + U'L' + \rho S &= 0, \\ \omega L + \phi L' + \rho U &= 0, \\ \phi L + \omega' L' + \rho U' &= 0, \end{aligned}$$

and hence

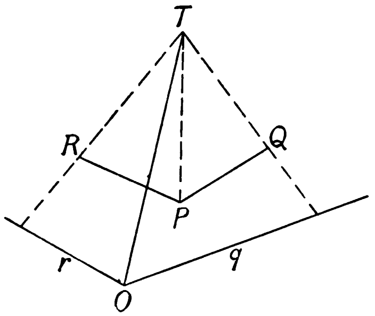
$$\begin{vmatrix} U, & U', & S \\ \omega, & \phi, & U \\ \phi, & \omega', & U' \end{vmatrix} = 0,$$

or
$$S(\omega\omega' - \phi^2) - U^2\omega' + 2UU'\phi - U'^2\omega = 0.$$

The expression of the equation of a quadric, referred to a self-polar tetrad, by four squares, has been considered above. The expression of a quadric by five, or more, squares, is considered by Paul Serret, *Géométrie de direction*, Paris, 1869, pp. 265, etc. The theory of the self-conjugate pentads and hexads has been much developed by Reye; see *Crelle's Journal*, Vols. LXXVII (1874), LXXXII (1877), etc.

Ex. 11. Given two arbitrary quadrics, S and Σ, it is possible to find three points of S of which every two are conjugate in regard to Σ, in an infinite number of ways; or, four points of S, forming a tetrad of which every pair of opposite joins consists of conjugate lines in regard to Σ, so that (Ex. 5, above) a fifth point can be found, not necessarily lying on S, such that the five form a self-conjugate pentad, in an infinite number of ways; or, lastly, five points of S, forming five points of a self-conjugate hexad, also in an infinite number of ways. We consider these cases in turn.

In regard to the first case, take an arbitrary point P, on S; then a point Q of S, lying anywhere on the conic in which S is met by the polar plane of P taken in regard to Σ; the polar plane of Q in regard to Σ will pass through P, and will meet the conic spoken of in two points. If R be one of these, then P, Q, R are three points of S of which every two are conjugate to one another, in regard to the quadric Σ. The pole of the plane PQR, say T, which does not necessarily lie on S, will then form, with P, Q, R, a tetrad which is self-polar in regard to Σ.



For the self-conjugate pentad, take three arbitrary points P, Q, R lying on S, and the polar lines, in regard to Σ, of the joins QR, RP, PQ; these lines, respectively p, q, r, will then meet in the point, O, which is the pole, in regard to Σ, of the plane PQR. The two planes Qq, Rr will meet in a line, passing through O; and this line will meet the quadric S in two points; let T be one of these. The four points P, Q, R, T, of the quadric S, are then such that the join TQ intersects the polar line, q, in regard to Σ, of the opposite join RP, so that TQ, RP are conjugate lines in regard to Σ. The opposite joins TR, PQ are equally conjugate lines in regard to Σ. Thus (by Ex. 5, above), the remaining pair of opposite joins, TP, RQ, are also conjugate. Wherefore, as has been shewn, the tetrad of points formed by the poles, in regard to Σ, of the planes PQR, TQR, TRP, TPQ, is in perspective with T, P, Q, R, and the centre of perspective, say U, forms, with T, P, Q, R, a self-conjugate pentad in regard to Σ. But U does not necessarily lie upon the quadric S.

For the self-conjugate hexad, take four arbitrary points, P, Q, R, T, of S: then form the quadric $\Phi\,(P, Q, R, T)$ in regard to Σ, as above explained (Ex. 7); on the curve of intersection, of this quadric Φ, with S, we can then take an arbitrary point E, and from this determine, as above explained, a point F, such that P, Q, R, T, E, F form a self-conjugate hexad, of which the points P, Q, R, T, E, but not necessarily F, are upon S.

Ex. 12. It is very easy now to prove with help of the equation to a quadric, that if upon a quadric S there be, either a self-polar tetrad, or a self-conjugate pentad, or a self-conjugate hexad, in regard to another quadric Σ, then an infinite number, of either of these figures, can be found whose points lie upon S. And though it is desirable, also, to establish this result without help of the symbols, the symbolical proof retains its interest. It depends on the facts we have remarked, that in regard to a self-polar tetrad, or a self-conjugate pentad, or a self-conjugate hexad, the tangential equation of Σ is capable of expression as a sum of squares, respectively four, five, or six in these respective cases.

For brevity let us call a set of h points, (x_r, y_r, z_r, t_r), for $r=1, \ldots, h$, which are so related to the quadric Σ, that the tangential equation of this is capable of a form

$$\sum_{r=1}^{h} A_r (lx_r + my_r + nz_r + pt_r)^2 = 0,$$

involving the squares of the left sides of the equations of these points, a self-conjugate h-ad in regard to Σ. Suppose, then, that a quadric S contains the points of a self-conjugate h-ad in regard to Σ. Suppose further that it has been shewn that $k-1$ points can be taken on S, so related to Σ that they can be completed by another point, not necessarily lying on S, to form a self-conjugate k-ad in regard to Σ; this has been proved, in the preceding example 11, for the cases of $k=4$, 5, 6. It can then be shewn that this remaining point of the k-ad equally lies on S. For if $(\xi_s, \eta_s, \zeta_s, \tau_s)$ be the points of the k-ad, for $s=1, \ldots, k$, we have, by identifying the two forms of which the tangential equation of Σ is capable, an identity, in regard to l, m, n, p, of the form

$$\sum_{r=1}^{h} A_r (lx_r + my_r + nz_r + pt_r)^2 + \sum_{s=1}^{k} P_s (l\xi_s + m\eta_s + n\zeta_s + p\tau_s)^2 = 0;$$

namely, we have the equations of forms such as

$$\sum_r A_r x_r^2 + \sum_s P_s \xi_s^2 = 0, \quad \sum_r A_r y_r z_r + \sum_s P_s \eta_s \zeta_s = 0.$$

If however the point equation of the quadric S be $f(x)=0$, where $f(x) = (a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2$, these equations lead to

$$\sum_r A_r f(x_r) + \sum_s P_s f(\xi_s) = 0,$$

and we are given that the $h+k-1$ equations, $f(x_r)=0, f(\xi_\sigma)=0$, hold, where $\sigma = 1, \ldots, k-1$. It follows then, supposing that P_k is not zero, that is, supposing that the $k-1$ points of the k-ad which have been taken on S do not form a $(k-1)$-ad, that we also have $f(\xi_k)=0$. So that the quadric S contains the point completing the k-ad, as was to be shewn.

But we see that the condition that there exists upon S a h-ad in regard to Σ may be replaced by another, in order to prove that all the points of the k-ad lie on S. For suppose that the tangential equation of Σ is $(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0$, and, as before, that $k-1$ points of a k-ad in regard to Σ have been taken on S. By identifying the two forms of which the tangential equation of Σ is capable, we have again ten equations of forms such as

$$-A + \sum_s P_s \xi_s^2 = 0, \quad -F + \sum_s P_s \eta_s \zeta_s = 0,$$

and, therefore,

$$-\Theta + \sum_s P_s f(\xi_s) = 0,$$

where

$$\Theta = aA + bB + cC + dD + 2fF + 2gG + 2hH + 2uU + 2vV + 2wW.$$

If then S be so conditioned, relatively to Σ, that $\Theta = 0$, it follows, as before, that all the points of the k-ad lie on S. But, conversely, this same equation shews that, if it be possible to find a k-ad of which all the points lie on the quadric S, then $\Theta = 0$.

When $\Theta = 0$, the quadric S is said to be *outpolar* to Σ.

Ex. 13. If P, P' be two points which are conjugate to one another in regard to a quadric, Σ, and Q, Q' two other such points, and if l, m be the respective polar lines of the lines PP', QQ', in regard to Σ, then a quadric S exists containing the lines l, m and the four points P, P', Q, Q'.

To see this, recall, first, that if planes be drawn through a line, p, the poles of these planes lie on the line, p', which is the polar of p; and the planes through a second line, q, drawn, respectively, conjugate to the former planes, are those which contain these poles; these planes, therefore, form an axial pencil related to that formed by the planes through p.

With the figure we have assumed, the polar plane of the point P' contains both the line l and the point P, namely P' is the pole of the plane lP; thus the planes mP' and lP are conjugate. In this way we see that the planes lP, lP', lQ, lQ' are conjugate, respectively, to the planes mP', mP, mQ', mQ; so that the axial pencil of planes, $l(P, P', Q, Q')$, is related to the axial pencil $m(P', P, Q', Q)$, and, therefore, also to the pencil $m(P, P', Q, Q')$. There is thus (above, p. 4), a quadric surface containing the lines l, m, and the points P, P', Q, Q', of which the transversals, drawn from these four points to meet both l and m, are all generators.

A corollary from this theorem is, that, if P, P', A, A' and Q, Q', B, B' are two tetrads of points, both self-polar in regard to a quadric Σ, then the quadric surface containing the points P, P', A, A', Q, B, B', and the lines AA', BB', passes through the point Q'.

From this can be deduced the theorem proved, with the help of the symbols, in Ex. 12, that every quadric surface containing the seven points P, P', A, A', Q, B, B', contains also the eighth point Q'. But we do not enter into this at present.

Ex. 14. It has been seen (Ex. 7, above) that the lines joining four points A, B, C, D, respectively to the poles of the planes BCD, CAD, ABD, ABC, taken in regard to a quadric with respect to which A, B, C, D have general positions, are generators, of the same system, of another quadric.

Conversely, if four lines, p, q, r, s, of which no two intersect, passing, respectively, through A, B, C, D, be generators of the same system of a quadric, and A', B', C', D' be arbitrary points respectively on p, q, r, s, then there exists a quadric in regard to which the points A', B', C', D' are the poles, respectively, of the planes BCD, CAD, ABD, ABC. In particular, if A', B', C', D' be the points in which these planes are met by p, q, r, s, there exists a quadric touching these planes at these points.

For if the lines p, q, r, s be such generators, a line, s', can be drawn through D to meet all of p, q, r, so that the planes (DA, p), (DB, q) and (DC, r) meet in a line. The condition for this to be so is, that the lines p, q, r should meet the planes BCD, CAD, ABD, respectively, in points whose coordinates, relatively to A, B, C, D, are of the forms $(0, H, G, U)$, $(H, 0, F, V)$, $(G, F, 0, W)$. Considering, similarly, the three lines q, r, s, we see that the line s meets the plane ABC in a point whose coordinates are, with this notation, $(U, V, W, 0)$; the planes (DA, p), (DB, q), (DC, r) meet in a line which meets the plane ABC in the point of coordinates $(F^{-1}, G^{-1}, H^{-1}, 0)$. The quadric whose tangential equation is

$$Fmn + Gnl + Hlm + Ulp + Vmp + Wnp = 0,$$

then touches the planes BCD, etc., in the points where these are met by p, q, r, s; and, if (A, H, G, U), (H, B, F, V), etc., be the coordinates of any points A', B', etc., lying, respectively, on the lines p, q, etc., these points are the poles of the planes BCD, etc., in regard to the quadric whose equation is

$$(A, B, C, D, F, G, H, U, V, W \between l, m, n, p)^2 = 0.$$

We have excluded the consideration of cases in which the lines p, q, etc., lie in the planes ABC, ABD, ACD, BCD.

Dually, if lines l, m, n, k, lying respectively in the planes, BCD, CAD, ABD, ABC, are generators of a quadric, of the same system, and arbitrary planes be drawn through these lines, there is a quadric for which these are the polar planes of A, B, C, D, respectively.

Ex. 15. Consider the twelve points in which the six joins of four points, A, B, C, D, are met by an arbitrary quadric. Three of these, lying on the joins of one of these points, D, to A, B, C, define a plane. We may thus, in thirty-two ways, specify four planes each of which contains three of the points, no two of these planes intersecting in one of the twelve points. If we consider the line in which the plane ABC is met by the plane containing the three points chosen on *DA*, *DB*, *DC*, and the three other lines in which the planes *BCD*, *CAD*, *ABD* are met by the corresponding planes, the four lines so obtained are generators of a quadric, of the same system. (Chasles, *Aperçu historique*, 1837, Note XXXII, p. 400.)

Dually, if we draw the pairs of tangent planes to a quadric from the lines *BC*, *CA*, *AB*, *DA*, *DB*, *DC*, and denote by D' the point of intersection of three planes, one chosen from each of the three pairs of planes drawn from *BC*, *CA*, *AB*, and by A' the point of intersection of three planes drawn from *BC*, *DB*, *DC*, not including the one already taken which is drawn from *BC*, and so on, so as to obtain four points D', A', etc., each the intersection of three of the twelve tangent planes, then the four lines DD', AA', ... are generators of the same system of a quadric.

For, let A' be the intersection of planes β, γ', δ, drawn, respectively, through *DB*, *DC*, *BC*; let B' be the intersection of planes γ, α', ϵ, drawn, respectively, through *DC*, *DA*, *CA*; and let C' be the intersection of planes α, β', ζ, drawn, respectively, through *DA*, *DB*, *AB*. The planes β, γ', γ, α', α, β', being tangent planes of the quadric, are tangent planes of the quadric cone which is drawn from *D* to the quadric. Therefore, by Brianchon's theorem (Vol. II, p. 25), the three planes (DA', DA), (DB', DB), (DC', DC) meet in a line, say k; for DA' is the intersection of the planes β, γ', and *DA* the intersection of the opposite planes α', α; and so on. The lines AA', BB', CC' are, therefore, all met by the line k, which equally meets DD'. These four lines AA', etc., are similarly met by a line through each of the points *A*, *B*, *C*. They are thus such generators as was stated.

Ex. 16. If every intersection of a quadric with a join, such as *AD*, of four points *A*, *B*, *C*, *D*, be joined by a plane to the opposite edge, *BC*, the twelve planes so obtained, touch another quadric. If the point equation of the first quadric be

$$(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0,$$

the tangential equation of the second quadric is

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{-f}{bc}, \frac{-g}{ca}, \frac{-h}{ab}, \frac{-u}{ad}, \frac{-v}{bd}, \frac{-w}{cd} \between l, m, n, p\right)^2 = 0.$$

Ex. 17. The transversal is drawn from a point, *O*, to each pair of opposite joins of four points *A*, *B*, *C*, *D*, so giving rise to six points. Six other points, one on each join, are similarly obtained from another point, O'. Shew that the twelve points lie on a quadric. If the points O, O' be, respectively, (ξ, η, ζ, τ), $(\xi', \eta', \zeta', \tau')$, referred to *A*, *B*, *C*, *D*, the quadric has a point equation obtained from

$$\left(\frac{x}{\xi} + \frac{y}{\eta} + \frac{z}{\zeta} + \frac{t}{\tau}\right)\left(\frac{x}{\xi'} + \frac{y}{\eta'} + \frac{z}{\zeta'} + \frac{t}{\tau'}\right) = 0,$$

by changing, in this, the signs of the six terms in yz, zx, ..., yt, zt. The relation itself is a particular case of that of Ex. 16, when the second quadric there referred to becomes a point-pair; but it will be found to be of importance.

Ex. 18. The point equation of the quadric, touching the planes BCD, CAD, ABD, ABC, whose tangential equation is

$$Fmn + \ldots + Ulp = 0,$$

is, by the rule proved above (p. 24),

$$\begin{vmatrix} 0, & H, & G, & U, & x \\ H, & 0, & F, & V, & y \\ G, & F, & 0, & W, & z \\ U, & V, & W, & 0, & t \\ x, & y, & z, & t, & 0 \end{vmatrix}.$$

Prove that, if we put

$$\xi = x(FVW)^{\frac{1}{2}}, \quad \eta = y(GWU)^{\frac{1}{2}}, \quad \zeta = z(HUV)^{\frac{1}{2}}, \quad \tau = t(FGH)^{\frac{1}{2}},$$

this is the same as

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 - 2(\eta\zeta + \xi\tau)\cos\alpha - 2(\zeta\xi + \eta\tau)\cos\beta - 2(\xi\eta + \zeta\tau)\cos\gamma = 0,$$

where $\alpha + \beta + \gamma = \pi$. In fact

$$\cos\alpha = \tfrac{1}{2}(GV + HW - FU)(GVHW)^{-\frac{1}{2}}, \text{ etc.}$$

Ex. 19. Prove that if (x, y, z, t) be any point of the quadric

$$fyz + gzx + hxy + uxt + vyt + wzt = 0,$$

so also is the point $(Ax^{-1}, By^{-1}, Cz^{-1}, Dt^{-1})$, where $A, = 2fvw$; $B, = 2gwu$; $C, = 2huv$; $D, = 2fgh$, are the minors of the diagonal elements, in the usual determinant of four rows and columns formed with the coefficients of the quadric.

Let P be any point of the quadric, and l, l' be the two generators through P; let the quadric cone be constructed, with vertex at any one, D, of the four points of reference for the coordinates, to contain P, and also the joins, DA, DB, DC, of D to the other three points of reference, and also to contain the generator at D which is of the same system as l; shew that these four cones all meet l again in the same point Q. Making a similar construction for the other system of generators, we obtain a point, Q', on l'. Shew that we can pass from Q to Q' by the above transformation.

Prove that, if the equation of the quadric be put into the form

$$(b - c)(axt + yz) + (c - a)(byt + zx) + (a - b)(czt + xy) = 0,$$

and P be (ξ, η, ζ, τ), one of the lines l, l' is given by

$$x(\eta - \zeta) + y(\zeta - \xi) + z(\xi - \eta) = 0,$$

$$y(\zeta - b\tau) - z(\eta - c\tau) + t(b\eta - c\zeta) = 0,$$

the point Q being given by

$$xt^{-1} = (b\eta - c\zeta)(\eta - \zeta)^{-1}, \quad yt^{-1} = (c\zeta - a\xi)(\zeta - \xi)^{-1},$$
$$zt^{-1} = (a\xi - b\eta)(\xi - \eta)^{-1},$$

and that the other line of l, l' is given by

$$ax(b\eta - c\zeta) + by(c\zeta - a\xi) + cz(a\xi - b\eta) = 0,$$
$$y(\zeta - a\tau) - z(\eta - a\tau) + at(\eta - \zeta) = 0,$$

the point Q' being

$$xt^{-1} = bc(\eta - \zeta)(b\eta - c\zeta)^{-1}, \quad yt^{-1} = ca(\zeta - \xi)(c\zeta - a\xi)^{-1},$$
$$zt^{-1} = ab(\xi - \eta)(a\xi - b\eta)^{-1}.$$

It may also be shewn that, if

$$U = a(b-c)x + b(c-a)y + c(a-b)z,$$
$$V = (a-b)y + (c-a)z + a(b-c)t,$$

the equation of the quadric is capable of the form

$$UV = (by - cz)(y - z)(c - a)(a - b).$$

The coordinates, and equations, of a line, in three dimensions. Elements of the theory of a linear complex. Some elementary results, in regard to lines in space, largely algebraical in character, which it is sometimes convenient to be able to refer to, may be collected here.

Referred to four given points, A, B, C, D, a line may evidently be given in general by the two points in which it meets two, definitely chosen, of the planes BCD, CAD, ABD, ABC. As the position of a point in a plane depends on the two ratios of its three coordinates, we see thus that a line depends upon four parameters, or, in order to avoid the occurrence of infinite parameters, upon the ratios of five parameters. It is desirable to choose these so that the representation shall be valid for all positions of the line; the representation above suggested would fail if the line were entirely contained in one of the two chosen planes. We owe to Cayley (*Coll. Papers*, IV, p. 447, 1860; and VII, p. 66) the discovery that a convenient representation is obtained only by taking the ratios of *six* parameters, which are connected by an equation, reducing them effectively to five.

Let (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) be the coordinates of any two points of a line. We consider the six quantities

$$l = t_1x_2 - t_2x_1, \quad m = t_1y_2 - t_2y_1, \quad n = t_1z_2 - t_2z_1,$$
$$l' = y_1z_2 - y_2z_1, \quad m' = z_1x_2 - z_2x_1, \quad n' = x_1y_2 - x_2y_1;$$

if, in place of one of the two points, we take any other point of the line, putting, for example, in place of x_2, y_2, z_2, t_2, respectively $\theta x_1 + \phi x_2$, $\theta y_1 + \phi y_2$, etc., it is at once seen that the ratios of l, m, n, l', m', n' remain unaltered. These ratios are therefore given without ambiguity when the line is given. They are evidently connected by the equation

$$ll' + mm' + nn' = 0.$$

Further, it is at once seen that the coordinates, x, y, z, t, of any point of the line, satisfy the four equations $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\tau = 0$, where

$$\xi = l't + mz - ny, \quad \eta = m't + nx - lz,$$
$$\zeta = n't + ly - mx, \quad \tau = -(l'x + m'y + n'z),$$

which, when $ll' + mm' + nn' = 0$, are equivalent only to two equations, since we have

$$l\xi + m\eta + n\zeta = 0, \quad l\tau + m'\zeta - n'\eta = 0.$$

The equation $\xi = 0$ is the equation of the plane which joins the line to the reference point A, and similarly for η, ζ, τ, the equation $\tau = 0$ being that of the plane joining the line to the reference point D. As the line can evidently be determined in all cases by two of these four planes, we see, conversely, that the ratios of any six quantities, l, m, n, l', m', n', which are connected by the equation $ll' + mm' + nn' = 0$, suffice to determine a line. It is these quantities which are called the *coordinates of a line* in threefold space.

These coordinates may also be expressed in terms of the coordinates of any two planes which contain the line. If the equations of two such planes be

$$a_1x + b_1y + c_1z + d_1t = 0, \quad a_2x + b_2y + c_2z + d_2t = 0,$$

the line coordinates are given, as regards their ratios, by

$$l = b_1c_2 - b_2c_1, \quad m = c_1a_2 - c_2a_1, \quad n = a_1b_2 - a_2b_1,$$
$$l' = d_1a_2 - d_2a_1, \quad m' = d_1b_2 - d_2b_1, \quad n' = d_1c_2 - d_2c_1.$$

This is seen at once by obtaining, from the equations of the two planes, the equations of the planes joining the line to the four points of reference, A, B, C, D, and comparing these with the forms above expressed in terms of the line coordinates.

It is easily verified that the line coordinates of the six joins of the points of reference are, for DA, $(1, 0, 0; 0, 0, 0)$, and, for BC, $(0, 0, 0; 1, 0, 0)$, and so on. We may, therefore, symbolically regard any line as represented in terms of these six joins, in the form

$$l(DA) + m(DB) + n(DC) + l'(BC) + m'(CA) + n'(AB).$$

Ex. 1. The conditions that the plane $ax + by + cz + dt = 0$ should contain the line (l, m, n, l', m', n') are two of the following four:

$$ld + m'c - n'b = 0, \quad md + n'a - l'c = 0, \quad nd + l'b - m'a = 0, \quad la + mb + nc = 0.$$

Ex. 2. The plane joining the line (l, m, n, l', m', n') to the point (x, y, z, t) is that whose coordinates are ξ, η, ζ, τ, where, as above, $\xi = l't + mz - ny$, $\tau = -(l'x + m'y + n'z)$, etc. The point of intersection of the line with the plane whose coordinates are (a, b, c, d) is that whose coordinates are

$$ld + m'c - n'b,\ md + n'a - l'c,\ nd + l'b - m'a, \text{ and } -(la + mb + nc).$$

Ex. 3. Expressing the condition that four points, (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), $(\xi_1, \eta_1, \zeta_1, \tau_1)$ and $(\xi_2, \eta_2, \zeta_2, \tau_2)$, should lie in a plane by the vanishing of a determinant in which the first row consists of x_1, y_1, z_1, t_1, and the third row consists of $(\xi_1, \eta_1, \zeta_1, \tau_1)$, etc., and expanding this determinant as a sum of products of binary determinants formed from the first two rows, and from the third and fourth rows, we see that the condition that the two lines

$$(l_1, m_1, n_1, l_1', m_1', n_1'),\ (l_2, m_2, n_2, l_2', m_2', n_2'),$$

or, say, λ_1 and λ_2, should intersect one another, is $\varpi_{12} = 0$, where

$$\varpi_{12} = l_1 l_2' + l_2 l_1' + m_1 m_2' + m_2 m_1' + n_1 n_2' + n_2 n_1'.$$

When this condition is satisfied, the six quantities such as $\xi l_1 + \eta l_2$, $\xi l_1' + \eta l_2'$, whatever ξ and η may be, satisfy the necessary condition, $\Sigma(\xi l_1 + \eta l_2)(\xi l_1' + \eta l_2') = 0$, that they should be the coordinates of a line. Symbolically, we may say that $\xi\lambda_1 + \eta\lambda_2$ then represents a line. It is easy to see that this is a line passing through the intersection of λ_1 and λ_2, and lying in the plane of these. Similarly, if $\lambda_1, \lambda_2, \lambda_3$ be three lines which meet in a point, not lying in a plane, the line $\xi\lambda_1 + \eta\lambda_2 + \zeta\lambda_3$ is, for proper choice of the ratios $\xi : \eta : \zeta$, any line through this point of intersection.

Ex. 4. If $\lambda_1, \lambda_2, \lambda_3$ be three lines of which no two intersect, the condition that $\xi\lambda_1 + \eta\lambda_2 + \zeta\lambda_3$ should be a line is that ξ, η, ζ should satisfy the condition

$$\varpi_{23}\xi^{-1} + \varpi_{31}\eta^{-1} + \varpi_{12}\zeta^{-1} = 0.$$

There are ∞^1 sets of values of $\xi : \eta : \zeta$ satisfying this condition; they are expressible rationally in terms of a single variable parameter, for instance in the form $\xi = \varpi_{23}$, $\eta = -\varpi_{31}\theta^{-1}$, $\zeta = \varpi_{12}(\theta - 1)^{-1}$. As the condition that two lines should intersect is of the first degree in the coordinates of either line, it follows that any line which intersects $\lambda_1, \lambda_2, \lambda_3$ also intersects $\xi\lambda_1 + \eta\lambda_2 + \zeta\lambda_3$; thus the lines expressed by this last formula are those, of the same system of generators as $\lambda_1, \lambda_2, \lambda_3$, of the quadric determined by these three.

Ex. 5. From Ex. 2, shew that the two tangent planes which can be drawn, from the line (l, m, n, l', m', n'), to touch the quadric whose tangential equation is $f(u, v, w, p) = 0$, where u, v, w, p are the coordinates of a plane, are given by the equation

$$f[l't + mz - ny,\ m't + nx - lz,\ n't + ly - mx,\ -(l'x + m'y + n'z)] = 0.$$

Ex. 6. The polar line of the line (l, m, n, l', m', n'), in regard to the quadric $ax^2 + by^2 + cz^2 + dt^2 = 0$, has the coordinates

$$(bcl',\ cam',\ abn',\ adl,\ bdm,\ cdn).$$

Thus the condition that the polar line of the line λ_1, in regard to the quadric, should meet the line λ_2, is

$$\Sigma\,(adl_1l_2 + bcl_1'l_2') = 0,$$

which is symmetrical in regard to the two lines. Two such lines are said to be *conjugate* in regard to the quadric.

Ex. 7. The condition that the line (l, m, n, l', m', n') should touch the quadric $ax^2 + by^2 + cz^2 + dt^2 = 0$ is

$$\Sigma\,(adl^2 + bcl'^2) = 0, \text{ or } \Sigma\,[l\,(ad)^{\frac{1}{2}} + l'\,(bc)^{\frac{1}{2}}]^2 = 0.$$

This is also the condition that the line should intersect its polar line in regard to the quadric.

Ex. 8. If instead of reference points A, B, C, D, we take other points of reference, X, Y, Z, T, the line coordinates, with reference to X, Y, Z, T, of a line whose coordinates relative to A, B, C, D are (l, m, n, l', m', n'), are certain linear functions of l, m, n, l', m', n'. The coefficients in these linear functions are, in fact, the line coordinates, relative to the original points of reference, A, B, C, D, of the six lines which are the joins of the new points of reference, X, Y, Z, T. We may forecast this result if we remark that, by Ex. 4, the condition that a line (l, m, n, l', m', n') should meet the line DA is $l' = 0$, and the condition that the line should meet BC is $l = 0$; if then (L, M, N, L', M', N') be the coordinates of the line referred to X, Y, Z, T, the condition $L = 0$ will be the condition that the line (l, m, n, l', m', n') should meet the line YZ; denoting the coordinates of YZ by $(\lambda_{23}, \ldots, \lambda_{23}', \ldots)$ we may thence expect that L is proportional to $l\lambda_{23}' + l'\lambda_{23} + \ldots$; and similarly for the others.

In fact, in terms of the symbols of X, Y, Z, T, let the symbols of A, B, C, D be, respectively,

$$A = a_1X + a_2Y + a_3Z + a_4T, \ldots, \; D = d_1X + d_2Y + d_3Z + d_4T,$$

so that, for all values of x, y, z, t,

$$xA + yB + zC + tD = \xi X + \eta Y + \zeta Z + \tau T,$$

where

$$\xi = a_1x + b_1y + c_1z + d_1t, \ldots, \; \tau = a_4x + b_4y + c_4z + d_4t.$$

Then (ξ, η, ζ, τ) are the coordinates, referred to X, Y, Z, T, of a point whose coordinates referred to A, B, C, D, are (x, y, z, t), and the planes $YZT, \ldots, XYZ$, referred to A, B, C, D, have the respective equations $\xi = 0, \ldots, \tau = 0$. The line coordinates of the line YZ, the intersection of $\xi = 0$, $\tau = 0$, are then given by

$$\lambda_{23} = b_4c_1 - b_1c_4, \ldots, \; \lambda_{23}' = d_4a_1 - d_1a_4, \ldots.$$

But, if (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) be two points of the line (l, m, n, l', m', n'), we have

$$L = \tau_1\xi_2 - \tau_2\xi_1 = \begin{vmatrix} a_4x_1 + b_4y_1 + c_4z_1 + d_4t_1, & a_4x_2 + b_4y_2 + c_4z_2 + d_4t_2 \\ a_1x_1 + b_1y_1 + c_1z_1 + d_1t_1, & a_1x_2 + b_1y_2 + c_1z_2 + d_1t_2 \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} x_1, & t_1 \\ x_2, & t_2 \end{vmatrix} \begin{vmatrix} a_4, & d_4 \\ a_1, & d_1 \end{vmatrix}$$

$$= \Sigma\,(l\lambda_{23}' + l'\lambda_{23}),$$

which is the formula expected. The value of L' is obtained by similar work in which $(a_2, b_2, \ldots)$, $(a_3, b_3, \ldots)$, respectively, replace $(a_4, b_4, \ldots)$ and $(a_1, b_1, \ldots)$. If then the line coordinates of the line TX be denoted by $\lambda_{41}, = b_2c_3 - b_3c_2$, $\lambda_{41}', = d_2a_3 - d_3a_2$, etc., we find

$$L' = \Sigma\,(l\lambda_{41}' + l'\lambda_{41})\,;$$

and so for the others.

Ex. 9. For the quadric surface whose equation is $x^2 + y^2 - z^2 = t^2$, prove that any generating line of one system has coordinates of the form $(p, q, 1, p, q, -1)$, where $p^2 + q^2 = 1$; and any generating line of the other system has coordinates of the form $(p', q', 1, -p', -q', 1)$, where $p'^2 + q'^2 = 1$. The coordinates of a line of the former system thus satisfy the three conditions $l - l' = 0$, $m - m' = 0$, $n + n' = 0$, whereby, from the ∞^4 existing lines of space there are selected the ∞^1 lines forming the system of generators; for the other system the corresponding conditions are $l + l' = 0$, $m + m' = 0$, $n - n' = 0$.

Ex. 10. *The focal system, and linear complex.* In the course of this chapter we have considered a correspondence between a point and a plane, the polar plane of the point in regard to a quadric, in which to the plane also corresponds the point, this correspondence having the property: (*a*), that, if the polar plane, ϖ, of any point, P, contain the point Q, then the polar plane of Q also contains P. From this it follows that the pole of any plane through P must lie on ϖ; for if Q' be this pole, since its polar contains P, the polar, ϖ, of P must contain Q'. It also follows that the polars of three points which are in line, must be planes which intersect in a line; for if P' be any point on the line of intersection of the polar planes of two points P and Q, the polar plane of P' must contain both P and Q, and therefore, also, any other point, R, of the line PQ, from which it follows that the polar plane of R contains P'. From this again it follows that the polar points of three planes which meet in a line are three points in line. The correspondence also had the property: (*b*), that the plane ϖ corresponding to P does not in general contain P, this being so only when the point P is on the quadric of reference. In this relation the coordinates of the plane corresponding to a point, of coordinates (x, y, z, t), are given by formulae of the form

$$ax + hy + gz + ut, \quad hx + by + fz + vt,$$
$$gx + fy + cz + wt, \quad ux + vy + wz + dt,$$

of which the coefficients form a symmetrical system.

We may, however, have a correspondence in which to a point corresponds a plane, and to this plane this point, in which the property (*a*) continues to hold but (*b*) fails, every point lying in the plane which corresponds to it.

Such a correspondence is effected by making correspond to a point (x, y, z, t) the plane whose coordinates are, with any constants a, b, c, a', b', c', the following, respectively,

$$cy - bz - a't, \quad -cx + az - b't, \quad bx - ay - c't, \quad a'x + b'y + c'z,$$

wherein the coefficients form a skew-symmetrical system. The equation of this plane, when X, Y, Z, T are current coordinates, is then

$$X(a't + bz - cy) + Y(b't + cx - az) \\ + Z(c't + ay - bx) - T(a'x + b'y + c'z) = 0,$$

or $$a'(tX - Tx) + b'(tY - Ty) + c'(tZ - Tz) \\ + a(yZ - zY) + b(zX - xZ) + c(xY - yX) = 0,$$

and the correspondence clearly has the properties referred to. It is supposed that, generally, the coefficients $a, b, \dots$ do not satisfy the condition $aa' + bb' + cc' = 0$. If they do, there is a fixed line of coordinates (a, b, c, a', b', c'), and the correspondence is that by which we make correspond, to any point, the plane joining this point to the fixed line; this is a degenerate relation in that, conversely, the point corresponding to a plane, through the fixed line, is not a definite, but is any, point of this plane.

We may agree to speak of the plane corresponding to any point as the polar plane of that point, and of the point as the pole of the plane. The polar planes of the points of any line are planes which intersect in another line, as we have remarked above; and, dually, the poles of the planes through any line lie on another line. We may agree to speak of either of these lines as the polar line of the other. If P be any point on the polar plane of a point O, the polar plane of P passes through O, and through P; thus the line OP is its own polar line. This is then true of every line, through any point O, lying in the polar plane of O; and is therefore also true of every line, in any plane ϖ, which passes through the pole of ϖ. The line coordinates $(l, m, \dots)$ of any such self-polar line evidently satisfy the equation

$$a'l + b'm + c'n + al' + bm' + cn' = 0.$$

The aggregate of such self-polar lines is called a *linear complex*; it consists of ∞^3 lines. The aggregate of points with their polar planes, and of lines of the linear complex, may be called a *focal system*.

The representation of the relation takes a very simple form if the points of reference for the coordinates be taken so that two of these points of reference are on the polar line of the line joining the other two points of reference. Suppose these points, $A, B, C, D,$

are such that D and C have for polar planes, respectively, the planes DAB and CAB; the coordinates, of the polar plane of (x, y, z, t), namely

$$a't + bz - cy, \quad b't + cx - az, \quad c't + ay - bx, \quad -(a'x + b'y + c'z),$$

must then reduce, respectively, to $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, when for (x, y, z, t) we have $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$. From these two facts in turn we deduce $a' = 0 = b'$, and $a = 0 = b$. The plane corresponding to (x, y, z, t) has then the equation

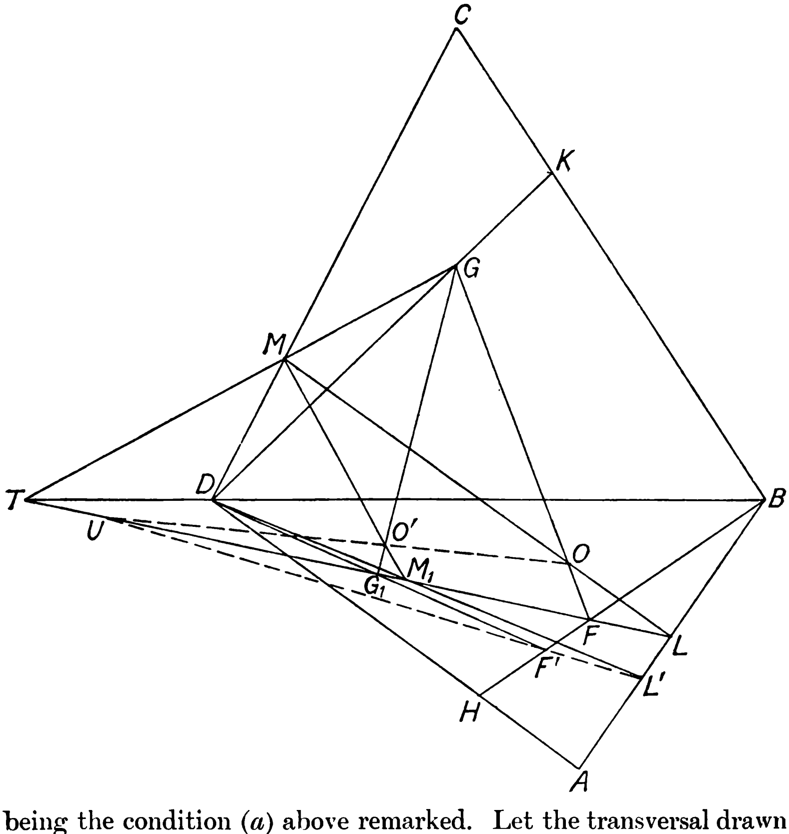
$$c'(tZ - Tz) + c(xY - yX) = 0,$$

and the lines of the linear complex are those for which $c'n + cn' = 0$.

It is clear that the polar plane of any point of a line of the linear complex contains this line. Also that two polar lines do not intersect; for, if they did, the polar plane of any point of one of the lines, as passing through this point and the other line, would contain the first line, which would then be its own polar line and coincide with the other line. Any line which meets two polar lines is evidently a line of the linear complex, since it lies in the polar planes of the two points in which it meets the polar lines.

Now consider any two lines of the linear complex, say λ and λ': the polar plane of any point, P, of λ, is a plane through λ, meeting λ', say, in a point P'. It is clear, or easily verifiable from the equations, that as P describes λ, the range (P'), on λ', described by P', is related to the range (P), on λ, described by (P). Let Q' be the position of P', on λ', corresponding to the position, Q, of P, on λ. Then the polar plane of P contains P', by hypothesis, and contains Q, on the line λ; the polar plane of Q' similarly contains Q and P'. Thus the cross lines PQ' and $P'Q$ are polar lines, one of the other; and therefore, as remarked above, every line meeting PQ' and $P'Q$ is a line of the linear complex. We thus infer that if $A, P, Q, \ldots, D$ and $C, P', Q', \ldots, B$, be related ranges on any two non-intersecting lines in space, the transversal drawn from any point, O, to the lines PQ', $P'Q$, that drawn from O to PR', $P'R$, that drawn from O to QR', $Q'R$, and so on, are in one plane, and are lines of a linear complex in which AB is the polar line of CD. Dually if we join the points in which an arbitrary plane meets PQ' and $P'Q$, all such joins meet in a point of this plane, the pole of the plane in the focal system.

We may thus initiate the theory of the linear complex, as has been remarked in Vol. I, p. 59, without the use of coordinates. Let A, H, D be three points in line, and C, K, B be three points in another line. These are sufficient to determine on the two lines, respectively, two related ranges, $A, H, P, Q, \ldots, D$, and $C, K, P', Q', \ldots, B$; and it is, as there explained, an immediate consequence of the theory of related ranges that the transversals drawn from an arbitrary point, O, to the pairs of cross joins such as (AB, CD), (BH, DK), $(PQ', P'Q)$, lie in a plane through O. This plane is determined by the first two pairs of cross joins, and, therefore, by the points A, H, D and C, K, B alone. In order to establish that the correspondence is that of a focal system, it is necessary to see that if another point, O', be taken in the plane so corresponding to O, then the plane similarly corresponding to O' passes through O, this



being the condition (a) above remarked. Let the transversal drawn from O to AB and CD meet these, respectively, in L and M; and the transversal drawn from O to BH and DK meet these in F and G respectively. Let the plane of these two transversals meet BD in T. Let O' be any other point of this plane. Let MO' meet FL in M_1, and GO' meet FL in G_1; let DM_1 meet AB in L', and DG_1 meet BH in F'. The point L' is in the plane $O'CD$, and is, therefore, the point in which the transversal from O' to AB, CD meets AB. The point F' is in the plane $O'DK$, and is, therefore, the point in which the transversal from O' to BH, DK meets BH. Thus the plane corresponding to O' is the plane $O'F'L'$, and it is desired to shew that this plane contains O. For this it is necessary and sufficient that the lines OO' and $L'F'$ should meet; their point of meeting will then be on FL, which is the line common to the planes LTM and BAD. What we shew, then, is that OO' and $L'F'$ meet the line FL in the same point. Considering, first, the intersections of FL with the pairs of joins of the four points M, O', O, G, we see that OO' meets FL in the point which is conjugate to T in the involution determined by the pairs of points F, M_1 and L, G_1. Considering, next, the intersections with FL of the pairs of joins of the four points F', L', B, D, we see that $F'L'$ meets FL in the point which is the conjugate of T in the involution determined by the same pairs of points F, M_1 and L, G_1. This proves the result required. (Cf. the diagram of p. 61, Vol. I. The tetrads O, M, G, B and D, F', L', O' are Moebius tetrads.) It can easily be seen that, if we take A, B, C, D for points of reference for coordinates, choosing these, as we may, so that H is $(1, 0, 0, 1)$ and K is $(0, 1, 1, 0)$, then the plane, $LFMG$, corresponding to O, has the equation $Tz - Zt = Xy - Yx$, where (x, y, z, t) are the coordinates of O, and (X, Y, Z, T) are current coordinates. This verifies that the linear complex obtained in this way agrees with that originally defined. In the construction used here, the lines of the linear complex passing through O are the transversals of the pairs of cross joins such as PQ', $P'Q$; to obtain these lines it is evidently sufficient to take P and P' always the same, corresponding to one another in the two related ranges, and to vary only the corresponding points Q and Q'. Thus we come back to a construction given by Sylvester, in 1861. (*Math. Papers*, II, p. 237.) In particular when P, P' are taken, respectively, as D and B, we have the result, that, given two related pencils of lines $D(C, K, Q', \ldots, B)$, and $B(A, H, Q, \ldots, D)$, in different planes, with a common corresponding ray, DB, the transversals drawn from any point O to the pairs of corresponding rays DQ', BQ, lie on a plane, which is the polar plane of O in a general focal system.

The line which is the polar of a line (l, m, n, l', m', n'), for the focal system considered, is the intersection of the two planes whose equations are

$$x(a't_1 + bz_1 - cy_1) + \ldots - t(a'x_1 + b'y_1 + c'z_1) = 0,$$
$$x(a't_2 + bz_2 - cy_2) + \ldots - t(a'x_2 + b'y_2 + c'z_2) = 0,$$

where $l = t_1x_2 - t_2x_1$, $l' = y_1z_2 - y_2z_1$, etc. Its line coordinates are thence easily computed to be

$$\Delta l - Ma, \ldots, \quad \Delta l' - Ma', \ldots$$

where $\Delta = aa' + bb' + cc'$, $M = a'l + \ldots + al' + \ldots$.

The condition for the polar lines to intersect would then be $M^2 = 0$, or the original line would be a line of the linear complex, and coincide with its polar. When this is not so, if we write for the computed line coordinates of the polar line, respectively, $-\Delta\lambda$, $-\Delta\lambda'$, etc., we may say that $a = M^{-1}(l+\lambda)$, $a' = M^{-1}(l'+\lambda')$, etc.

In particular when the equation of the linear complex is $n'-kn=0$, the polar line is of coordinates $(l, m, k^{-1}n', l', m', kn)$. Also, the condition, in general, that the polar line of one line, λ_1, should intersect another line, λ_2, is of the form $\Delta\varpi_{12} - M_1M_2 = 0$, which is symmetrical in regard to the lines.

Consider now two linear complexes. It can be shewn that there are in general two lines which are polars of one another in both the corresponding focal systems. Any transversal of these two polar lines is then a line of both linear complexes. The ∞^2 lines so obtained, which are said to form a *linear congruence*, are then the lines common to the two given linear complexes. If the equations for these two complexes be

$$a'l + \ldots + al' + \ldots = 0, \quad A'l + \ldots + Al' + \ldots = 0,$$

any line common to both satisfies, for all values of σ, the equation

$$(A' + \sigma a')\, l + \ldots + (A + \sigma a)\, l' + \ldots = 0\,;$$

herein we may choose σ so that the six quantities given by $\lambda = A + \sigma a$, $\lambda' = A' + \sigma a'$, etc., are the coordinates of a line; this requires only

$$AA' + BB' + CC' + \sigma\,(Aa' + A'a + \ldots) + \sigma^2\,(aa' + bb' + cc') = 0,$$

of which there are two different roots in general. This shews that all the lines common to the two complexes meet both of two definite lines, which are then a pair of common polar lines for these.

Incidentally we infer that the ratios of the functions of the coefficients, in the equations of the two linear complexes, which occur in this quadratic equation for σ, are unaltered by change of the reference points for the coordinates, and that these functions must have a geometrical significance. The first and last of these functions are those which vanish when the linear complexes, respectively, reduce to lines all meeting a given line, as we have seen. The condition that the remaining function should vanish,

$$Aa' + A'a + Bb' + B'b + Cc' + C'c = 0,$$

is that either linear complex should consist of pairs of lines which are polars of one another in regard to the other complex; or also, is that, if we take the polar plane of an arbitrary point, in the first focal system, and then the pole of this plane in the other focal system, we reach the same point independently of the order in which the linear complexes are considered. This is easily verified from the general equations. If the complexes be referred to their common pair of polar lines, however, and take the forms $n' - kn = 0$, $n' - Kn = 0$, the polar line $(l, m, K^{-1}n', l', m', Kn)$, of (l, m, n, l', m', n'), in regard to the second focal system, belongs to the first if $Kn = kK^{-1}n'$, which, when $n' = kn$, becomes $K^2 = k^2$, and leads to $K + k = 0$, the complexes not being identical. This however is the form of the condition in question for these particular equations of the complexes. Again, if (ξ, η, ζ, τ) be the pole in $n' - Kn = 0$ of the polar plane of (x, y, z, t) in $n' - kn = 0$, the planes

$$Xy - Yx - k(Tz - Zt) = 0, \quad X\eta - Y\xi - K(T\zeta - Z\tau) = 0,$$

are identical, so that $\xi = x$, $\eta = y$, $\zeta = kK^{-1}z$, $\tau = kK^{-1}t$; for this to be independent of the order in which the complexes are considered, we again require $k^2 = K^2$.

Two linear complexes for which the function $Aa' + A'a + \ldots$ vanishes are said to be *apolar* to one another.

Ex. 11. A focal system is by definition a self dual construct. In particular the pole of the plane $ux + vy + wz + pt = 0$, corresponding to the linear complex $a'l + al' + \ldots = 0$, has the coordinates

$$ap + b'w - c'v, \quad bp + c'u - a'w, \quad cp + a'v - b'u, \quad -(au + bv + cw).$$

Ex. 12. If $P, Q, R, \ldots, P', Q', R', \ldots$, be related ranges on any two skew lines, any line meeting one pair of cross joins, such as PQ' and $P'Q$, meets an infinite number of other pairs of cross joins, such as RS', $R'S$, one of the four points R, S, R', S' being arbitrary.

Ex. 13. Let A, B, C be three points in a line, A', B', C' three points in another line not intersecting the former, and P, Q, R three points in a third line not meeting either of the other two lines. From P let the transversal be drawn to the lines AB', $A'B$, and also the transversal to the lines AC', $A'C$, and the plane of these two transversals be considered. Similarly the plane containing the transversals from Q, to AB', $A'B$, and to AC', $A'C$; and the plane containing the transversals from R, to AB', $A'B$, and to AC', $A'C$. Prove that these three planes meet in a line.

Ex. 14. Take five arbitrary lines in space of which no two intersect. By omitting each in turn we obtain five sets each of four lines; and these four lines have a common pair of transversals. From an arbitrary point a single line can be drawn to meet the two lines of any such pair of transversals. Prove that the five single lines so obtained lie in a plane (Sylvester, *Math. Papers*, II, p. 237).

In fact, the equation of a linear complex being linear in the line coordinates, there is a definite linear complex containing the five given lines. If a line meet two lines of a linear complex, its polar line, in the associated focal system, also meets these two lines; thus the two common transversals of four skew lines of a linear complex are polar lines one of another, and any line meeting both these transversals is also a line of the linear complex.

Ex. 15. If λ be a transversal of three lines, p, q, r, belonging to a linear complex, of which no two intersect, and λ' be the polar line of λ, then λ' is also a transversal of p, q, r, and every line meeting λ and λ' is a line of the linear complex. Conversely, if t be any line of the complex, the two common transversals of p, q, r and t are polar lines of one another. It is easy to prove that the pairs of polar lines which meet p, q, r determine an involution of pairs of points on any one of these lines. Thus a linear complex appears as an aggregate of lines meeting a quadric surface each on two generators, of one system, which are pairs in an involution of these generators. The involution is determined by the three generators p, q, r of the other system taken in turn with two other generators, say t, u, of that system, and the linear complex by the five lines p, q, r, t, u. (Chasles, *Liouville's J.*, IV (1839), p. 348.)

Ex. 16. By expressing the planes which join an arbitrary line to the four points of reference for the coordinates, in terms of two of these planes, and the points in which the line meets the planes containing three of these points of reference in terms of two of these points, shew that the axial pencil of the planes is related to the range of the points (cf. Vol. I, p. 30).

Ex. 17. Prove that the polar plane, in any focal system, of any one, say, D, of the four reference points for coordinates A, B, C, D, contains the poles in this system, of the three planes DBC, DCA, DAB. So that A, B, C, D, and the poles of the four planes such as these, form two Moebius tetrads (cf. Vol. I, pp. 61, 91). Conversely, any two Moebius tetrads define a focal system.

Ex. 18. For the linear complex $al' + a'l +$ etc. $= 0$, the two tetrads of Ex. 17 are both self-polar in regard to the quadric

$$a'bcx^2 + b'cay^2 + c'abz^2 + a'b'c't^2 = 0.$$

Ex. 19. If any two generators of the same system of a quadric be taken, and from each of the points A, B, C, D, forming a self-polar tetrad in regard to the quadric, there be drawn the transversal to these two generators, meeting the planes BCD, ..., ABC respectively in A', ..., D', prove that A, B, C, D and A', B', C', D' are two Moebius tetrads.

Ex. 20. If, in any focal system, A', B', C', D' are the respective poles of the planes BCD, ..., ABC, and U, U', on AA', are harmonic conjugates in regard to A and A', etc., prove that, if U, V, W are in line, the plane $U'V'W'$ contains DD'; but, if the line UVW meets DD', then U', V', W' are also in a line meeting DD', the lines UVW, $U'V'W'$ then meeting DD' in points harmonically conjugate in regard to D and D'. (Cf. p. 144, below.)

Ex. 21. If A, A', B, B', C, C' be any six points, and U, U', on AA', are harmonic conjugates in regard to A and A', while $BVB'V'$, $CWC'W'$ are, similarly, harmonic ranges, prove that, U, V, W being in line, there is no position of this line for which U', V', W' are in line, unless the quadric containing the lines BC, BC', $B'C$, $B'C'$, and the point A, also contains A'.

Ex. 22. Shew that the equations of any four non-intersecting lines can be put into the forms

$$y = px,\ t = qz; \qquad y = -px,\ t = -qz;$$
$$y = p^{-1}x,\ t = q^{-1}z; \quad y = -p^{-1}x,\ t = -q^{-1}z.$$

Taking, then, the four points of respective coordinates $(1, p, m, mq)$, $(1, -p, m, -mq)$, $(p, 1, mq, m)$, $(-p, 1, -mq, m)$, where m is arbitrary, say P, Q, R, S, and also the points P', Q', R', S', whose coordinates are obtained from those of P, Q, R, S, respectively, by changing the sign of m, prove that these form two Moebius tetrads, both self-polar in regard to the quadric $yz - xt = 0$. Further, that the polar plane of any point of one tetrad, in the focal system given by $yz' - y'z + tx' - t'x = 0$, contains three points of the other tetrad, the polar plane of P containing Q', R', S', etc.

Ex. 23. Prove that two of the lines of any given linear complex are generators of the same system of any given quadric surface.

Ex. 24. If we be given a fixed conic, we may take, for any two lines which do not lie in the plane of the conic, the pole, in regard to the conic, of the line joining the points where the lines meet the plane of the conic; and draw from this pole the transversal to the two lines. Now take three lines, p, q, r, not lying in the plane of the conic; let p', q', r' be the transversals, determined as described, respectively of the pairs (q, r), (r, p) and (p, q); then let p'', q'', r'' be the transversals, similarly determined, respectively of the pairs (p, p'), (q, q') and (r, r'). Shew that the transversals so determined, respectively of the pairs (q'', r''), (r'', p''), (p'', q''), coincide in one line. The proof depends on two triads, of points and lines, which are polar reciprocals of one another in regard to the conic (cf. Vol. II, p. 27).

Ex. 25. Shew that, of points of which the coordinates, in terms of a varying parameter θ, are $(\theta^3, \theta^2, \theta, 1)$, there are three lying on an arbitrary plane. These points are therefore said to lie on a cubic curve. Shew that any plane containing the line which is given by the two equations $x - 2y\theta + z\theta^2 = 0$, $y - 2z\theta + t\theta^2 = 0$, meets the curve in two coincident points at the point θ; this line is called the tangent of the curve at this point. Shew that all the tangents of the curve belong to the linear complex expressed by $l + 3l' = 0$.

Ex. 26. Given five points in three dimensions joined in order by five lines, shew that there is a focal system in which each of the points is the pole of the plane containing the two lines which meet in this point, each of the five joins being a line of the associated linear complex.

Ex. 27. Prove that, in regard to a quadric surface expressed in the general form

$$(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0,$$

the line coordinates, $(\lambda, \mu, \nu, \lambda', \mu', \nu')$, of the polar line of the line (l, m, n, l', m', n'), are such that

$$\begin{vmatrix} 0, & -\nu, & \mu, & \lambda' \\ \nu, & 0, & -\lambda, & \mu' \\ -\mu, & \lambda, & 0, & \nu' \\ -\lambda', & -\mu', & -\nu', & 0 \end{vmatrix} = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} \begin{vmatrix} 0, & -n', & m', & l \\ n', & 0, & -l', & m \\ -m', & l', & 0, & n \\ -l, & -m, & -n, & 0 \end{vmatrix} \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix},$$

where the matrices are to be multiplied by the rule explained in Vol. I, p. 67, the rows of a multiplying matrix being combined with the columns of that multiplied.

Ex. 28. Prove that the tangent lines of the curve of intersection of two quadrics which meet an arbitrary line are eight in number; and that, if the line be a generator of one of the two quadrics, the eight tangents are also generators of this.

The focal system was first investigated by Moebius, in connexion with the reduction of a system of forces acting on a single rigid body, and called by him a *Null* system, as consisting of lines about which the sum of the moments of the forces of the system vanishes (*Lehrbuch der Statik*, 1837). The relations were also studied by Chasles in connexion with the kinematics of a rigid body (1843; *Aperçu historique*, Sec. Edit., 1875, p. 614). On account of these applications the system is often referred to as a *Screw*. The kinematical relations will be considered at a later stage.

CHAPTER II

RELATIONS WITH A FIXED CONIC

SPHERES, CONFOCAL SURFACES; QUADRICS THROUGH THE INTERSECTION OF TWO GENERAL QUADRICS

In the plane geometry of Vol. II we have seen that the ordinary metrical relations are particular cases of the relation of the figure to two arbitrary Absolute points, or, more generally, to an arbitrary Absolute Conic; the recognition of this adds greatly to the breadth of view obtained, without increasing the difficulty of proof. A similar gain is found in the geometry of three dimensions. Here we consider an arbitrary Absolute Conic. In a later Volume we shall consider an arbitrary Absolute Quadric.

When we use coordinates we shall most generally suppose the points of reference, A, B, C, D, so taken that A, B, C form a self-polar triangle in regard to this Absolute Conic, and suppose, therefore, that the equations of this conic have the forms $t=0$, $x^2+y^2+z^2=0$. This is immaterial; but it will enable the reader easily to make comparison with metrical formulae which may be familiar.

Parallel lines and planes. Middle point. Lines and planes at right angles. Two lines which meet the plane of the absolute conic in the same point, not themselves lying in this plane, may be said to be *parallel*, relatively to this plane; two planes which meet this absolute plane in the same line may similarly be said to be parallel; a line and a plane may be said to be parallel if they have a common point lying on the absolute plane. Thus, for instance, a plane can be drawn through any line, which shall be parallel to any other given line not meeting the former line.

If the line joining two points A, B, which do not lie in the absolute plane, meet this plane in C, and O be the harmonic conjugate of C in regard to A and B, then O may be spoken of as the *middle point* of A and B, in the relative sense employed.

If two lines, not lying in the absolute plane, meet this plane in two points which are conjugate in regard to the absolute conic, then the lines may be said to be *at right angles*, or *perpendicular* to one another. If a plane and a line meet the absolute plane in a line and a point, respectively, of which the point is the pole of the line in regard to the absolute conic, then the plane and the original line may be said to be at right angles. Similarly, two planes which meet the absolute plane in lines which are conjugate in regard to the absolute conic may be said to be at right angles. Immediate consequences are, that, if two lines are at right angles, a plane can be drawn through either line which shall be at right angles to the other; or, that, if a line and a plane are at right angles, any plane drawn through the line is at right angles to the given plane; or again, that a line can be drawn which shall be at right angles to both of two given lines which do not meet one another; this being the common transversal of the lines, drawn from the point, in the absolute plane, which is the pole of the line joining the points in which the given lines meet the absolute plane.

Ex. 1. Prove that coordinates can be chosen so that the equations of the absolute conic have the forms $t=0$, $x^2+y^2+z^2=0$, and at the same time the equations of any two lines, which do not meet one another, and do not lie in the absolute plane, have the forms

$$lx-my=0,\quad z-ct=0,\quad \text{and}\quad lx+my=0,\quad z+ct=0.$$

Ex. 2. Shew that the equations of any three lines, of which no two intersect, can be supposed to be of the respective forms

$$(y-bt=0,\ z+ct=0),\ (z-ct=0,\ x+at=0),\ (x-at=0,\ y+bt=0);$$

and that the quadric generated by the transversals of these three lines has the equation

$$ayz+bzx+cxy+abct^2=0.$$

Ex. 3. If between nine quantities (l_1, m_1, n_1), $(l_2, \ldots)$, $(l_3, \ldots)$, there hold the six relations

$$l_1^2+m_1^2+n_1^2=1,\qquad l_2^2+m_2^2+n_2^2=1,\qquad l_3^2+m_3^2+n_3^2=1,$$
$$l_2l_3+m_2m_3+n_2n_3=0,\quad l_3l_1+m_3m_1+n_3n_1=0,\quad l_1l_2+m_1m_2+n_1n_2=0,$$

then there also hold the six following relations

$$l_1^2+l_2^2+l_3^2=1,\qquad m_1^2+m_2^2+m_3^2=1,\qquad n_1^2+n_2^2+n_3^2=1,$$
$$m_1n_1+m_2n_2+m_3n_3=0,\quad n_1l_1+n_2l_2+n_3l_3=0,\quad l_1m_1+l_2m_2+l_3m_3=0,$$

while the determinant, of three rows and columns, (l_1, m_2, n_3), has its square equal to 1.

This result, which is often utilised, may be obtained at once by elementary algebra; or, if we use the notation of matrices (explained Vol. I, p. 67), putting Δ for the matrix of three rows whose first row consists of the elements l_1, m_1, n_1, the second and third rows consisting, respectively, of the elements l_2, m_2, n_2 and l_3, m_3, n_3, and $\bar{\Delta}$ for the transposed matrix, obtained from Δ by change of rows into columns, the given equations are equivalent to $\Delta\bar{\Delta}=1$; this leads to $\bar{\Delta}=\Delta^{-1}$, and hence to $\bar{\Delta}\Delta=1$, which is equivalent to the deduced relations. Geometrically, the given relations shew that the conic $x^2+y^2+z^2=0$ is the same as

$$(l_1x+l_2y+l_3z)^2+(m_1x+m_2y+m_3z)^2+(n_1x+n_2y+n_3z)^2=0,$$

so that the lines $l_1x+l_2y+l_3z=0$, etc., determine a self-polar triad for this conic. This proves at once that $l_1m_1+l_2m_2+l_3m_3=0$, etc.

Spheres and Circles. A quadric which meets the absolute plane in the absolute conic is called a sphere, relatively to this conic. A degenerate case is when the quadric consists of the absolute plane and any other plane. Another degenerate case is that in which the sphere is the cone formed by the lines which join an arbitrary point to the points of the absolute conic. The pole of the absolute plane in regard to a sphere is called its *centre*. Let this centre be O, and let any line through O meet the sphere in P and P', and meet the absolute plane in Q. Then O and Q are harmonic conjugates in regard to P and P', or O is the middle point of P, P'. Further, the polar planes, in regard to the sphere, of all points of the line PP', meet in a line, which is, therefore, on the absolute plane, and is the polar line of the point Q in regard to the absolute conic. Thus, the tangent planes of the sphere at P and P' are both at right angles to the line PP'; or, the tangent plane of the sphere, at any point not lying on the absolute conic, is at right angles to the line joining this point to the centre. At a point of the sphere which lies on the absolute conic the tangent plane passes through the pole of the absolute plane, that is, through the centre of the sphere, and all such planes are tangent planes of the tangent cone or cone of contact drawn to the sphere from its centre. (Cf. Chap. I, p. 26.) It may be proved easily that, if P, P' be any two points of a sphere, the plane through the middle point of P, P', at right angles to the line PP', passes through the centre of the sphere.

In the second Volume we have defined a circle, in a plane, as a conic containing two particular points chosen in that plane as absolute points. Let us now agree that for any plane of space the two absolute points shall be the two points in which the plane meets the absolute conic. Then a *circle* will be a conic whose two points of intersection with the absolute plane lie upon the absolute conic. This being understood, consider any plane section of a sphere. Any two plane sections of a quadric are conics with two points in common, the two points where the line of intersection, of the planes of the conics, intersects the quadric. Thus any plane section of a sphere is a circle, relatively to the absolute conic. And, incidentally, it appears, as a quadric can (Chap. I, p. 7) be drawn through two conics which have two points in common, to pass through an arbitrary point, that a sphere can be drawn to contain an arbitrary circle and an arbitrary point. And, hence, that a sphere can be drawn through four arbitrary points in general position.

As in the case of a quadric in general, there pass through any point, P, of a sphere two lines, lying entirely on the sphere, contained in the tangent plane of the sphere at P. These lines meet the absolute plane in two points which, as they lie on the sphere, are on the absolute conic. Any two lines through P, in the tangent plane at P, which are harmonic in regard to these two generating lines of the sphere at P, evidently meet the absolute plane in two points which are conjugate in regard to the absolute conic, and are therefore at right angles to one another. Any plane section of the sphere also meets the two generating lines at P; if this section be projected, from P, on to any plane, ϖ, it becomes, evidently, a conic passing through the two points in which the generating lines of the sphere at P meet the plane ϖ; it becomes then a circle, in the sense above adopted, if the plane ϖ be so taken as to meet the absolute conic in points lying on the tangent plane of the sphere at P. In other words, a plane section of the sphere projects, from any point, P, of the sphere, into a circle, if the plane, upon which the projection is made, is parallel to the tangent plane of the sphere at P.

Ex. 1. Prove that the polar plane of any point in regard to a sphere is at right angles to the line joining that point to the centre of the sphere. This joining line is in fact the polar line of the line in which the absolute plane is met by the polar plane.

Ex. 2. Prove that when the section of the sphere by a plane, ϖ, is projected from a point, H, of the sphere, into a circle on a plane, the centre of this circle is the projection of the pole, in regard to the sphere, of the plane ϖ. The line of intersection of the plane ϖ with the tangent plane of the sphere at H is, in fact, the polar line of the line joining H to the pole of the plane ϖ.

Ex. 3. The line joining any point, Q, of the section of a sphere by a plane, ϖ, to the pole, P, of the plane ϖ, is at right angles to the tangent line of the section at Q. For, a line drawn from P, in the tangent plane of the sphere at Q, meets the sphere in two points lying on the generators of the sphere at Q, and meets the plane ϖ, say in P', upon the tangent plane of the sphere at Q; so that P' is on the tangent line at Q of the section ϖ. But P and P' are harmonic in regard to the points in which the line PP' meets the sphere.

Ex. 4. At either common point of the sections of a sphere by two planes which are conjugate to one another, each plane containing the pole of the other in regard to the sphere, the tangent lines of the sections are at right angles.

Ex. 5. When the two sections of the preceding example are projected, from a point H of the sphere, on to a plane parallel to the tangent plane at H, they become two circles which cut at right angles.

Relations of two spheres. Coaxial spheres. Now consider two spheres. In the first place, they have two points of contact, lying on the absolute conic; that is, there are two points, common to both spheres, on the absolute conic, at each of which the tangent planes of the two spheres are the same. For, let O and O' be the centres of the two spheres, let the line OO' meet the absolute plane in T, and let TA, TB be the tangent lines drawn from T to the absolute conic, touching this in A and B. Then the plane AOT, containing the common tangent line, AT, of the spheres at the common point A, is a tangent plane of both the spheres at this point A; and BOT is, similarly, a common tangent plane at the common point B. This plane equally contains the other centre, O'. The points of contact of the spheres are thus the two points in which either sphere is touched by the two planes which can be drawn to touch it, from the line of centres, OO'. In the next place, the two spheres have in common, beside the absolute conic, a further plane section, which is, therefore, a circle. For, an arbitrary plane cuts the two spheres in circles which, beside their common points lying on the absolute conic, have two further common points; thus the spheres have, beside the absolute conic, an infinite number of points common; let P, Q, R be three of these; then, the plane PQR cuts the spheres in two conics which, beside P, Q, R, have two points in common, those namely where the plane meets the absolute conic; but two conics with five points in common coincide with one another. Thus the two spheres have in common the circle in which the plane PQR meets either; and, therefore, no other common points beside those on the absolute conic. At either of the two points common to this circle and the absolute conic, the tangent plane of either sphere contains the tangent lines of both these conics, and is hence the tangent plane of both spheres. The two points of contact of the spheres are thus the points where their common circular section meets the absolute conic. The plane of this common circular section is called the *radical plane* of the spheres. Through this common circular section can be drawn, as we have seen, an infinite number of further spheres, one through any arbitrary point. These form what is known as a *coaxial system of spheres*. There are, in particular (Chap. I, p. 10), two cones which both contain the absolute conic and the common circle of the two spheres. The vertices of these cones are called the *limiting points* of the system of coaxial spheres. The limiting points lie on the line joining the centres, O and O', of the two original spheres, which also contains the centres of all the spheres of the coaxial system. Further, any sphere drawn through the two limiting points has its centre on the radical plane, and cuts at right angles every sphere of the coaxial system; that is, the tangent plane of such a sphere, at any point common to it and one of the coaxial spheres, is at right angles to the tangent plane of the latter sphere at this point. These facts may be proved from what has been said. We proceed however to utilise the algebraic representation.

Taking reference points A, B, C, D, of which D is arbitrary, but not on the absolute plane, while A, B, C form a self-polar triad in regard to the absolute conic, so that the equations of this conic may be taken to be $t=0$, $x^2+y^2+z^2=0$, the equation of a sphere is necessarily of the form

$$a(x^2+y^2+z^2)+2uxt+2vyt+2wzt+dt^2=0.$$

When the coefficient a vanishes this becomes the aggregate of the absolute plane $t=0$ and another plane; in general we suppose that a is not zero. This equation is the same as

$$(ax+ut)^2+(ay+vt)^2+(az+wt)^2=(u^2+v^2+w^2-ad)\,t^2,$$

and the point $(u, v, w, -a)$, having for polar the plane $t=0$, is the centre. The points common to this sphere and any other, of equation

$$a'(x^2+y^2+z^2)+2u'xt+2v'yt+2w'zt+d't^2=0,$$

satisfy the condition

$$[2(au'-a'u)\,x+2(av'-a'v)\,y+2(aw'-a'w)\,z+(ad'-a'd)\,t]\,t=0;$$

thus, beside the points lying on $t=0$, there are common points on the radical plane, whose equation is obtained by equating to zero the first factor of the left side. This plane meets $t=0$ on the line given by $t=0$ and

$$(au'-a'u)\,x+(av'-a'v)\,y+(aw'-a'w)\,z=0,$$

which is the polar line, in regard to the absolute conic, of the point $(au'-a'u,\ av'-a'v,\ aw'-a'w,\ 0)$, in which the line joining the centres of the spheres meets the absolute plane; so that the radical plane is at right angles to this joining line. We may thus take the point of intersection of this plane and the line of centres for the reference point D, and the point A to be on the line of centres, so that the radical plane becomes $x=0$. This choice requires that $v=0$, $w=0$, $v'=0$, $w'=0$, $ad'-a'd=0$. Supposing a and a' not to be zero, we may put $a=1$, $a'=1$; then $d=d'$ and the equations of the spheres become

$$x^2+y^2+z^2+2uxt+dt^2=0, \quad x^2+y^2+z^2+2u'xt+dt^2=0.$$

Any coaxial sphere, being required to meet $x=0$ on the circle $x=0$, $y^2+z^2+dt^2=0$, will then have an equation

$$x^2+y^2+z^2+2Uxt+dt^2=0;$$

in particular when $U^2 = d$ this represents a cone whose vertex is $(-U, 0, 0, 1)$. The two limiting points are then $(d^{\frac{1}{2}}, 0, 0, 1)$ and $(-d^{\frac{1}{2}}, 0, 0, 1)$. If, for a moment, we put $X = x + d^{\frac{1}{2}}t$, $T = x - d^{\frac{1}{2}}t$, $A = \frac{1}{2}(1 + ud^{-\frac{1}{2}})$, $A' = \frac{1}{2}(1 + u'd^{-\frac{1}{2}})$, $B = 1 - A$, $B' = 1 - A'$, the equations of the two spheres become

$$AX^2 + BT^2 + y^2 + z^2 = 0, \quad A'X^2 + B'T^2 + y^2 + z^2 = 0,$$

which shew that the four points of intersection of the four planes $X = 0$, $T = 0$, $y = 0$, $z = 0$, form a self-polar tetrad for both the spheres, and, therefore, for all the spheres of the coaxial system (Chap. I, p. 33). Herein, however, the planes $y = 0$, $z = 0$ are not unique, being any two planes through the line of centres which are at right angles to one another. The equation of any sphere through the two limiting points is at once seen to be of the form

$$x^2 + y^2 + z^2 + 2Vyt + 2Wzt - dt^2 = 0,$$

having its centre on the radical plane $x = 0$. We proceed to verify that this meets all the spheres of the coaxial system at right angles; and for this purpose we prove that the condition that two spheres,

$$a(x^2 + y^2 + z^2) + 2uxt + 2vyt + 2wzt + dt^2 = 0, \quad a'(x^2 + y^2 + z^2) + \text{etc.} = 0,$$

should cut at right angles, is

$$2(uu' + vv' + ww') - ad' - a'd = 0.$$

The condition that two planes, of equations $lx + my + nz + pt = 0$, $l'x + \text{etc.} = 0$, should be at right angles, is that the lines

$$lx + my + nz = 0, \quad l'x + \text{etc.} = 0,$$

in which these planes meet $t = 0$, should be conjugate in regard to the absolute conic; that is, $ll' + mm' + nn' = 0$. If now (ξ, η, ζ, τ) be a common point of the two spheres, the respective tangent planes, of such equations as

$$x(a\xi + u\tau) + y(a\eta + v\tau) + z(a\zeta + w\tau) + t(u\xi + v\eta + w\zeta + d\tau) = 0,$$

will be at right angles if

$$(a\xi + u\tau)(a'\xi + u'\tau) + (a\eta + v\tau)(a'\eta + v'\tau) + (a\zeta + w\tau)(a'\zeta + w'\tau) = 0;$$

this, however, is the same as

$$a'S + aS' + [2(uu' + vv' + ww') - ad' - a'd]\,\tau^2 = 0,$$

where $S = 0$, $S' = 0$ are the equations of the two spheres, with (ξ, η, ζ, τ) written for (x, y, z, t). As the point (ξ, η, ζ, τ) satisfies both equations, and is supposed not to lie on the absolute plane, we see that if the spheres cut at right angles at one point, the specified condition is satisfied, and the spheres then cut at right angles at every point; and conversely. The condition is evidently satisfied for any sphere through the limiting points, taken with any one of the coaxial spheres.

Ex. 1. If two circles, not lying in the same plane, have two points in common, a sphere can be described to contain both of them. For these circles together with the absolute conic form three conics of which every two have two points in common; and we have seen (p. 10) that three such conics lie upon a quadric. In the present case this quadric, as containing the absolute conic, is a sphere.

Ex. 2. Prove that the three radical planes, for each pair of three given spheres, meet in a line.

Ex. 3. Given any three circles, no two in the same plane, prove that there is another circle meeting each of them in two points.

If $S_r = 0$ be the equation of a sphere, where

$$S_r = a_r(x^2 + y^2 + z^2) + 2u_r xt + 2v_r yt + 2w_r zt + d_r t^2,$$

it is at once clear that for any six spheres there exists an identity $\lambda_1 S_1 + \lambda_2 S_2 + \ldots + \lambda_6 S_6 = 0$, where $\lambda_1, \ldots, \lambda_6$ are functions of a_r, u_r, v_r, w_r, d_r $(r = 1, \ldots, 6)$. Now let $S_1 = 0$, $S_2 = 0$ be any two chosen spheres through the first of the three given circles, $S_3 = 0$, $S_4 = 0$ be any two chosen spheres through the second of the three given circles, and $S_5 = 0$, $S_6 = 0$ be any two chosen spheres through the third of the three given circles. The equation $\lambda_1 S_1 + \lambda_2 S_2 = 0$ then represents a sphere through the first circle, and the equation $\lambda_3 S_3 + \lambda_4 S_4 = 0$ represents a sphere through the second circle. In virtue of the identity remarked, any point common to these two spheres equally lies on the sphere $\lambda_5 S_5 + \lambda_6 S_6 = 0$, which contains the third circle. Thus the circle common to the first two spheres equally lies on the third. This circle intersects each of the three given circles in two points, since any two circles on the same sphere meet in the two points in which the line of intersection of their planes cuts the sphere.

It may be remarked at once that a similar argument shews that any three lines in space of four dimensions are met by another line. We shall see later how to pass from a circle in three dimensions to a line in four dimensions, or conversely.

Ex. 4. For any two spheres,

$$a(x^2 + y^2 + z^2) + 2uxt + 2vyt + 2wzt + dt^2 = 0,$$
$$a'(x^2 + y^2 + z^2) + 2u'xt + \text{etc.} = 0,$$

let

$$D = u^2 + v^2 + w^2 - ad, \quad E = 2uu' + 2vv' + 2ww' - ad' - a'd,$$
$$D' = u'^2 + v'^2 + w'^2 - a'd';$$

we have shewn that $E = 0$ is the condition for the two spheres to cut at right angles; prove that the first sphere becomes a cone passing through the absolute conic if $D = 0$; also, that the condition that the two spheres should touch, that is, have the same tangent plane, at a point not lying on the absolute plane $t = 0$, is $E^2 = 4DD'$. Further, that the condition that the plane $lx + my + nz + pt = 0$ should touch the first sphere is

$$D(l^2 + m^2 + n^2) - (ul + vm + wn - ap)^2 = 0.$$

Thus, the common tangent planes of the two spheres satisfy the condition

$$D^{-1}(ul + vm + wn - ap)^2 = D'^{-1}(u'l + v'm + w'n - a'p)^2;$$

these common tangent planes are thus the planes drawn to touch either sphere from either of the points whose coordinates are

$$(\lambda u + \lambda' u', \lambda v + \lambda' v', \lambda w + \lambda' w', -\lambda a - \lambda' a'),$$

where λ, λ' are such that $\lambda^2 = 1/D$, $\lambda'^2 = 1/D'$. These two points are called the *centres of similitude* of the two spheres.

Prove further that any sphere, drawn through the vertex of a quadric cone which contains the absolute conic, satisfies the condition $E = 0$, formed for itself and the cone regarded as a degenerate sphere; and may, therefore, be said to cut the cone at right angles.

Shew, too, that, if $l = ua' - u'a$, $l' = vw' - v'w$, etc., the equation of the two common tangent planes of the spheres at the points on the absolute conic at which they touch, is

$$(l't + mz - ny)^2 + (m't + nx - lz)^2 + (n't + ly - mx)^2 = 0.$$

Ex. 5. If two plane sections of a quadric cone be taken, prove that a quadric can be described to contain these, passing through an arbitrary point not lying on either of the planes. Shew also that a sphere can be constructed having a given centre, to pass through a further arbitrary point.

Ex. 6. If O, D be given points, and a variable plane be drawn through D, and a point P taken so that the line OP is at right angles to the plane, and the middle point of O and P lies on the plane, shew that the locus of P is the sphere with centre D which passes through O.

Ex. 7. Four lines in the absolute plane meet in six points, the opposite intersections being conjugate in regard to the absolute conic; any four planes, one through each of these lines, meet, in threes, in the points A, B, C, D. These four points are then such that any two opposite joins of them are two lines at right angles. Prove that, if the line be drawn through each of these points which is at right angles to the plane containing the other three points, these four lines meet in a point, say P. Prove, further, that the six middle points of the joins of the pairs of A, B, C, D lie on a sphere; let its centre be G. Prove that G lies on the line joining P to the centre of the sphere which contains A, B, C, D. A line can be drawn to be at right angles to both of any two opposite joins of

A, B, C, D; taking the intersections of this line with these joins, and the four other points similarly arising for the other two pairs of opposite joins, prove that these six points lie on the sphere containing the middle points of the six joins.

Relations of a quadric in general with the absolute conic. The circular sections. Suppose that the quadric considered does not touch the absolute plane, but meets it in a conic which has four distinct intersections with the absolute conic. These two conics have then a common self-polar triad of points, which we may take for points of reference for coordinates, A, B, C (Vol. II, p. 23). Incidentally also we may remark that if the given quadric be expressed, in regard to any reference points, by an equation wherein the coefficients are real, and the absolute conic, referred to these, be imaginary, then the points A, B, C are real (Vol. II, p. 165). If D be the pole of the absolute plane in regard to the quadric, the four points A, B, C, D will form a self-polar tetrad in regard to the quadric. Referred to A, B, C, D the equation of the quadric will then be of the form

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=t^2,$$

unless the quadric be a cone, consisting of points lying on lines all passing through the point D; in this latter case the equation will be of the form $x^2/a+y^2/b+z^2/c=0$. We may, if we please, include both these equations in the single form $x^2/a+y^2/b+z^2/c=mt^2$.

The point D, the pole of the absolute plane, is called the *centre* of the quadric, or, in the case of the cone, its vertex. The lines DA, DB, DC are called the axes of the quadric; every two of them are at right angles, relatively to the absolute conic. The planes DBC, DCA, DAB are called the *principal planes*. It can now be seen, dealing first with the case when the quadric is not a cone, that through each axis there pass two planes which meet the quadric in sections which are circles, and that all sections in planes parallel to one of these circular sections are also circular sections. For, if P, Q, R, S be the four points common to the quadric and the absolute conic, the section of the quadric by any plane through two of these points is a conic meeting the absolute conic in two points, and is therefore a circle, by the definition. The join of two of these points, P, Q, R, S, contains one of the points A, B, C (Vol. II, p. 23), and a definite plane can be drawn through this join to contain, also, the point D. As two of these joins pass through each of A, B, C, what we have said in regard to circular sections is clear. Of any one of the six sets of parallel planes each meeting the quadric in a circle, there will be two which touch the quadric, these being the tangent planes to the quadric from the line, say, for example, PQ, through which the planes of this set all pass. We thus obtain twelve points of contact, of such tangent planes; these are called *umbilici*. Taking either one of the two generating lines of the quadric at the point P, it is met by one of the two generating lines of the quadric at the point Q, say in U, which will be an umbilicus, while the other generating line at P will be met by the other generating line at Q in another umbilicus, say U'. If B be the one of the three points A, B, C through which PQ passes, the line UU', which is the polar line of PQ, lies in the polar plane of B, that is, in the principal plane DAC, and passes through the centre D, this being the pole of the plane ABC. The twelve umbilici thus lie in fours on the three principal planes, and in pairs on six lines through D. Moreover, as we may consider the intersection of a generating line of the quadric at P with a generating line at Q, or R, or S, we see that the twelve umbilici lie in threes upon the eight generators of the quadric at P, Q, R, S, there being two of these generators through each umbilicus.

If the equation of the quadric be taken in the form remarked above, $\Sigma x^2/a = t^2$, the equations of the absolute conic being $t = 0$, $\Sigma x^2 = 0$, and p, q, r be defined by

$$(a-b)(a-c)p = a, \quad (b-c)(b-a)q = b, \quad (c-a)(c-b)r = c,$$

the point whose coordinates are

$$(a+\lambda)p^{\frac{1}{2}}, \quad (b+\lambda)q^{\frac{1}{2}}, \quad (c+\lambda)r^{\frac{1}{2}}, \quad 1,$$

describes a line, as λ varies. The point of this arising for $\lambda = \infty$, namely $(p^{\frac{1}{2}}, q^{\frac{1}{2}}, r^{\frac{1}{2}}, 0)$, satisfies both the equations $\Sigma x^2/a = 0$, $\Sigma x^2 = 0$, and is thus one of the four points P, Q, R, S; it is at once seen that $\Sigma\,(a+\lambda)^2/(a-b)(a-c) = 1$, so that the line lies entirely upon the quadric. By taking all the possibilities of sign for $p^{\frac{1}{2}}$, $q^{\frac{1}{2}}$, $r^{\frac{1}{2}}$, we thus obtain the eight generators of the quadric which contain the twelve umbilici. In particular, the umbilici are obtained by the values $-a$, $-b$, $-c$, of λ.

These eight generators are a set of lines which may be derived from any one of them by a process of harmonic inversion which often occurs, and may be described: if we are given a point O, and a plane ϖ not containing O, we may, from any point P, obtain another point, P', as that point on the line OP which is the harmonic conjugate of P, in regard to O and the point where OP meets the plane ϖ. Then as P describes any line, P' also describes a line, meeting the former on the plane ϖ; and as P describes any plane, P' also describes a plane, meeting the former plane on the plane ϖ. Suppose now we have a tetrad of points, A, B, C, D; we can employ this process of harmonic inversion taking, for O, any one of these four points and, for the associated plane ϖ, the plane containing the other three. Thereby we have four inversions, and we may subject a point to one of these, or to a combination of two or more of them, taken in succession. It is at once seen that thereby only seven new points are obtained; if the original point be of coordinates (x, y, z, t) relative to A, B, C, D, the aggregate of eight points are those of coordinates $(\pm x, \pm y, \pm z, t)$. Similarly from a line, by repeated inversions with this tetrad, eight lines in all are obtained. It is clear from the above expressions that the eight generators containing the umbilici are such a set of eight lines.

Case of a cone. When the quadric reduces to a cone whose vertex is not on the absolute plane, so that its equation is capable of the form $x^2/a + y^2/b + z^2/c = 0$, the equations of the absolute conic being $t = 0$, $x^2 + y^2 + z^2 = 0$, there will still be, in the general case, four intersections P, Q, R, S, of the cone with the absolute conic, and therefore, as before, three principal planes, through the vertex of the cone, the planes $x = 0$, $y = 0$, $z = 0$, intersecting in pairs in the axes of the cone. It is still the case that a plane containing two of the points P, Q, R, S meets the cone in a circle, so that there are six sets of parallel planes each meeting the cone in a circle. The circular sections containing the axes of the cone degenerate however each into two generators of the cone; and the umbilici all coincide with the vertex. A modification of statement is clearly necessary when two or more of the points P, Q, R, S coincide. The case most frequently considered is that in which the cone touches the absolute conic in two distinct points, say, when Q coincides with P and S with R. There are then still triads which are self-polar both in regard to the absolute conic and in regard to the section of the cone by the absolute plane, one point of all of these being the common pole of PR in regard to the two conics, the other two being any pair of points on PR which are harmonic conjugates in regard to P and R. The cone, in this case, has one unique axis, and all sections by planes at right angles to this are circles, so that the cone is said to be a *cone of revolution*, or a *right circular cone*. Any two lines at right angles to one another, through the vertex of the cone, in the plane at right angles to this unique axis, may be regarded as the other two axes. Besides the circular sections spoken of there are no others which are not degenerate. Finally, if the cone has its vertex, C, on the absolute plane, in which case it is called a *cylinder*, there will be two generators of the cone lying in the absolute plane. If these meet the absolute conic in distinct points, respectively, P and R, and Q and S, so that P, C, R are in line, as are Q, C, S, then any plane through either of the four lines PQ, QR, RS, SP, meets the cylinder in a circle. But when P coincides with Q and R coincides with S, or the cylinder touches the absolute plane, no undegenerate circular sections lie in planes through PR. And it is also possible that the vertex of the cone should be on the absolute conic. The necessary modifications may easily be stated. When the cone meets the absolute plane in two distinct lines, its vertex, ***C***, not being on the absolute conic, we may take two points, ***A*** and ***B***, on the polar line of ***C*** in regard to the absolute conic, which shall be conjugate to one another both in regard to the absolute conic, and in regard to the cone, and take for ***D*** a point on the polar line of the absolute plane in regard to the cone; then, with ***A***, ***B***, ***C***, ***D*** as point of reference, the equations of the absolute conic may be supposed to be $t=0$, $x^2+y^2+z^2=0$, and the equation of the cone to be of the form $ax^2+by^2=t^2$. This construction fails if the cone touches the absolute plane, its vertex still not being on the absolute conic. If then ***C*** be the vertex of the cone, and ***CA*** the generator of contact with the absolute plane, the point ***A*** being on the polar line of ***C*** in regard to the absolute conic, and if ***B*** be the pole of ***CA*** in regard to the absolute conic, and ***D*** be any point not on the absolute plane, the triad ***A***, ***B***, ***C*** is self-polar in regard to the absolute conic, whose equations may therefore be supposed to be of the form $t=0$, $x^2+y^2+z^2=0$, while the equation of the cone is of the form $y^2=(Ax+By+Ct)\,t$. This is simplified if we take for ***CD*** the other generator of the cone, beside ***CA***, along which the tangent plane contains ***CB***; the equation then reduces to the form $y^2=mxt$. The case when the vertex of the cone lies on the absolute conic may be similarly dealt with.

Ex. Prove that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

represents a right circular cone (touching the absolute conic in two points), (1) if $f=g=h=0$ and two of a, b, c be equal, (2) if $g=h=0$ and $(a-b)(a-c)=f^2$, (3) when none of f, g, h is zero, if $a-f^{-1}gh=b-g^{-1}hf=c-h^{-1}fg$. Remark also that in general the equation of the cone is capable of the form

$$(a-f^{-1}gh)\,x^2+(b-g^{-1}hf)\,y^2+(c-h^{-1}fg)\,z^2$$
$$+fgh\,(f^{-1}x+g^{-1}y+h^{-1}z)^2=0.$$

Quadric touching the absolute plane. When the quadric touches the absolute plane at a point, *C*, not lying on the absolute conic, there will be two distinct lines of the surface, passing through *C*, lying in the absolute plane, say these are ***PCR*** and ***QCS***, where ***P***, ***Q***, ***R***, ***S*** are on the absolute conic. It is then possible to take two points, *A* and *B*, on the polar line of *C* in regard to the absolute conic, which shall be conjugate in regard to this conic, and also harmonic conjugates of one another in regard to the two lines ***PCR***, ***QCS***. From the line *AB* it is possible to draw another tangent plane to the quadric beside the absolute plane. Let this touch the quadric in the point D. Then, referred to A, B, C, D, the equation of the quadric assumes the form $ax^2+by^2+2wzt=0$. For, consider the most general form possible

$$ax^2+by^2+cz^2+dt^2+2fyz+2gzx+2hxy+2uxt+2vyt+2wzt=0;$$

then, the intersection with the absolute plane is to reduce to two lines, passing through the point C, harmonic in regard to CA, CB, and two such lines are represented by equations of the form $t=0$, $Ax^2+By^2=0$; we infer, therefore, that $c=f=g=h=0$; further, the intersection with $z=0$ is to reduce to two lines given by equations of the form $z=0$, $lx^2+2mxy+ny^2=0$; we infer, therefore, that $d=u=v=0$. This establishes the form in question. The form of the equations of the absolute conic may still be supposed to be $t=0$, $x^2+y^2+z^2=0$, the triad A, B, C being self-polar in regard to this. Such a quadric may be called a *paraboloid*. There are four sets, each of parallel planes, meeting the surface in circles, namely those containing any one of the lines PQ, QR, RS, SP. The points of contact of tangent planes, other than the absolute plane, drawn to the quadric through these lines, give four umbilici; these lie in pairs on four generators of the surface passing each through one of the points P, Q, R, S. The case when the quadric touches the absolute plane at a point of the absolute conic may be similarly dealt with.

Ex. 1. For the surface $ax^2+by^2+2wzt=0$, if $c=a-b$, the umbilici are given by

$$[0,\ (-ac)^{\frac{1}{2}},\ \tfrac{1}{2}c,\ ab/w] \quad\text{and}\quad [(bc)^{\frac{1}{2}},\ 0,\ \tfrac{1}{2}c,\ ab/w].$$

Ex. 2. Any two circular sections of a quadric, whose planes are not parallel, lie on a sphere. For two plane sections of a quadric meet in two points, and three conics, of which every two have two points in common, lie on a quadric, as has been remarked.

Ex. 3. The condition for a given plane to be a circular section of a given quadric is that the plane should meet the absolute conic in points lying on the quadric. If the absolute conic be $t=0$, $x^2+y^2+z^2=0$, the plane be $lx+my+nz+pt=0$, and the quadric be in general form $ax^2+\ldots+2wzt=0$, prove that the conditions are the two

$$\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2}.$$

Ex. 4. Prove that the axes and asymptotes of a plane section of a quadric are parallel to those of the section by any other parallel plane, and that the centres of these sections, by parallel planes, lie in line. Prove that the axes of the section of the quadric $ax^2+by^2+cz^2=t^2$ by the plane $lx+my+nz=0$, being the lines through the centre, in this plane, which are conjugate to one another both in regard to the quadric, and in regard to the absolute conic ($t = 0$, $x^2 + y^2 + z^2 = 0$), are the intersections of the plane with the cone

$$x^{-1}l(b-c) + y^{-1}m(c-a) + z^{-1}n(a-b) = 0.$$

Ex. 5. If the general equation $ax^2 + \ldots + 2wzt = 0$ break up into two linear factors, and so represent two planes, the condition that these planes should be at right angles is that the lines in which they meet the absolute plane should form a conic outpolar to the absolute conic (Vol. II, pp. 36, 144). With the usual form for the equations of the absolute conic, the condition for this is $a + b + c = 0$.

The pair of tangent planes which can be drawn from the line whose coordinates are (l, m, n, l', m', n'), to the quadric

$$x^2/a + y^2/b + z^2/c = t^2,$$

are at right angles if

$$(b+c)l^2 + (c+a)m^2 + (a+b)n^2 = l'^2 + m'^2 + n'^2.$$

Ex. 6. Prove that the tangent planes of a quadric at the points of a generator form an axial pencil which is related to the range formed by their points of contact; and that the normals of the surface at the points of the generator, that is the lines through these points at right angles to the tangent planes, meet this generator, and a certain line in the absolute plane, in two related ranges. These normals are therefore generators of a quadric which touches the absolute plane.

If the equation of the given quadric be taken in the form $xU - yV = 0$, where U, V are linear in x, y, z, t, the equations of the absolute conic being as before, shew that the normals describe the surface $xV_0 + yU_0 = 0$, where U_0, V_0 are obtained from U, V, respectively, by putting $x = 0$ and $y = 0$.

The line of striction for one system of generators of a quadric. If two lines which do not intersect one another meet the absolute plane in P and Q, respectively, and R be the pole of the line PQ in regard to the absolute conic, the transversal drawn from R to the given lines is a line at right angles to both, and the only such, as has been remarked. Now suppose the two lines coincide in one generator of a given quadric; the line PQ will then be replaced by the tangent line, at P, of the section of the quadric by the absolute plane. The transversal from R is then replaced by a line from R meeting the quadric in two coincident points; the point of coincidence will therefore be the intersection of the generator with the polar plane of R in regard to the quadric. This construction determines a single point on the generator; the locus of these points, for all the generators of one system, is called the *line of striction* corresponding to the system.

Ex. 1. If the equation of the quadric be $x^2/a^2+y^2/b^2-z^2/c^2=t^2$, the absolute conic being as before, the coordinates of a point of the quadric can be represented in terms of two parameters, λ, μ, in the forms $x/a=1-\lambda\mu$, $y/b=\lambda+\mu$, $z/c=1+\lambda\mu$, $t=\lambda-\mu$. Then λ is constant for all points of a generator of one system, given by the equations $x/a+z/c=\lambda^{-1}(t+y/b)$, $x/a-z/c=\lambda(t-y/b)$; and μ is constant on the generator of the other system given by

$$x/a+z/c=-\mu^{-1}(t-y/b), \quad x/a-z/c=-\mu(t+y/b).$$

The parameters of the generators of the two systems which meet in a point of the section by $t=0$ are equal, their point of meeting being of coordinates $a(1-\lambda^2)$, $2b\lambda$, $c(1+\lambda^2)$, 0.

Carrying out then the construction above explained it may be shewn that the line of striction, for the generators (λ), meets the generator λ in the point for which μ is given by

$$\mu\lambda(\lambda^2+A)+A\lambda^2+1=0, \quad \text{where } A=(2b^{-2}+c^{-2}-a^{-2})/(a^{-2}+c^{-2}).$$

From this it may be seen that the line of striction is a curve meeting an arbitrary plane in four points, and meeting any generator, μ, of the other system, in three points.

Ex. 2. By the same construction, for the quadric represented by the equation $a^{-2}x^2-b^{-2}y^2=2zt$, the tangent line at the point P being however replaced in this case by a generator lying in the absolute plane, shew that the lines of striction are the sections of the surface by the planes $a^{-3}x \pm b^{-3}y=0$. These sections both touch the absolute plane.

Ex. 3. With the notation above, a generator, λ, of the quadric meeting the absolute plane in P, and R being the pole, in regard to the absolute conic, of the tangent line, at P, of the section of the quadric by the absolute plane, prove that, if RP meet this section again in T, and S be the point of the line of striction which lies on the generator considered, then ST is a generator of the other system.

Ex. 4. Hence shew that if a pair of tangent planes of the quadric, at right angles to one another, be drawn through the given generator, their points of contact being the points H, K of this generator, then the pair H, K belong to an involution of pairs of points on this generator, in which the point S, on the line of striction, is the point conjugate to P.

Ex. 5. Shew that the quartic curve in Ex. 1 lies on a cubic cone whose vertex is $(0, b, 0, 1)$, which intersects the quadric surface also, doubly, in the μ-generator for which $\mu=0$, namely the line given by $x/a=z/c$, $t=y/b$, this being a double line of the cone.

Shew further that a cubic surface can be drawn through the curve and any two μ-generators of the quadric surface; and that no other quadric surface than the original contains the curve.

In the construction given, as the point P varies, the point R describes a conic in the absolute plane, the polar reciprocal, in regard to the absolute conic, of the conic described by P. The polar planes of the points R envelope a quadric cone, whose vertex is the centre of the quadric. The tangent planes of this cone are thus related, in the sense employed in Vol. I, to the generators of the quadric, through the points P; and the curve is the locus of the intersection of a plane of the cone with the corresponding generator. More generally, if $P=0$, $Q=0$, $R=0$, $S=0$, $U=0$, $V=0$, $W=0$ be arbitrary planes, we may consider the locus of the intersection of the plane $U\theta^2+2V\theta+W=0$, which envelopes the cone $UW=V^2$, with the line $P=\theta R$, $Q=\theta^{-1}S$, which is a corresponding line of the quadric $PQ=RS$. The locus is a curve of this quadric lying on the cubic surface $UP^2+2VPR+WR^2=0$, which also intersects the quadric in the generator $P=0$, $R=0$ counted twice. The curve lies also on the cubic surface

$$U(P-\phi S)(P-\psi S)+V[(P-\phi S)(R-\psi Q)+(P-\psi S)(R-\phi Q)] \\ +W(R-\phi Q)(R-\psi Q)=0$$

which contains two arbitrary generators, ϕ, ψ, of the other system, given by $P=\phi S$, $R=\phi Q$, etc.

The construction for the line of striction is evidently not limited to the case when the generators lie on a quadric surface.

Ex. 6. Notice that the two generators of the surface

$$a^{-2}x^2+b^{-2}y^2-c^{-2}z^2=t^2,$$

at the point (x_0, y_0, z_0, t_0), are the intersections of the tangent plane with the two planes $(xy_0-x_0y)/ab=\pm(zt_0-z_0t)/c$.

The director sphere of a quadric. If three tangent planes of a quadric, which does not touch the absolute plane, be such that every two are at right angles, it can be shewn that the point of intersection of the three planes lies on a definite sphere, whose centre is the centre of the quadric.

We can suppose the equation of any plane put into the form $lx+my+nz=pt$, with l, m, n so chosen that $l^2+m^2+n^2=1$. Further the condition that two planes,

$$l_1x+m_1y+n_1z=p_1t, \quad l_2x+m_2y+n_2z=p_2t,$$

should be at right angles, when the equations of the absolute conic are $t=0$, $x^2+y^2+z^2=0$, is that $l_1l_2+m_1m_2+n_1n_2=0$. If then we have three planes of which every two are at right angles, with equations $l_rx+m_ry+n_rz=p_rt$, we may suppose the six equations $l_r^2+m_r^2+n_r^2=1$, $l_rl_s+m_rm_s+n_rn_s=0$ to be satisfied. Therefore, as remarked above (Ex. 3, p. 71), the six equations such as $l_1^2+l_2^2+l_3^2=1$, $l_1m_1+l_2m_2+l_3m_3=0$, are also satisfied. Wherefore, if (x, y, z, t) be the point of intersection of the planes, from $\Sigma (l_r x + m_r y + n_r z)^2 = t^2 \Sigma p_r^2$, we infer $x^2 + y^2 + z^2 = t^2 \Sigma p_r^2$.

Now suppose the planes all touch the quadric whose equation is $x^2/a + y^2/b + z^2/c = t^2$; the conditions for this are the three of the form $al_r^2 + bm_r^2 + cn_r^2 = p_r^2$; these however lead to $\Sigma p_r^2 = a + b + c$. Thus the point of intersection of the planes lies on the sphere whose equation is $x^2 + y^2 + z^2 = t^2 (a + b + c)$.

And it is clear that if the planes touch, not all the same quadric, but, respectively, the three quadrics whose equations are

$$x^2 (a + \lambda_1)^{-1} + y^2 (b + \lambda_1)^{-1} + z^2 (c + \lambda_1)^{-1} = t^2, \text{ etc.},$$

so that we have $(a + \lambda_r) l_r^2 + (b + \lambda_r) m_r^2 + (c + \lambda_r) n_r^2 = p_r^2$, or $al_r^2 + bm_r^2 + cn_r^2 = p_r^2 - \lambda_r$, then $\Sigma p_r^2 = \Sigma \lambda_r + a + b + c$. In this case the point of intersection of the planes lies on the sphere whose equation is $x^2 + y^2 + z^2 = a + b + c + \Sigma \lambda_r$.

When the quadric touches the absolute plane, and has an equation $ax^2 + by^2 + 2wzt = 0$, the condition that a plane

$$lx + my + nz + pt = 0,$$

should touch the quadric is $a^{-1} l^2 + b^{-1} m^2 + 2w^{-1} np = 0$. If then three planes touch this quadric and intersect in the point (x, y, z, t), we have three equations such as

$$t (a^{-1} l_r^2 + b^{-1} m_r^2) - 2w^{-1} n_r (l_r x + m_r y + n_r z) = 0.$$

If every two of the planes be at right angles, we can thus infer, as above, by adding these equations, that $t (a^{-1} + b^{-1}) - 2zw^{-1} = 0$. The locus of the point of intersection of the three tangent planes thus degenerates into a plane, the absolute plane being, in a certain sense, also part of this. The plane in question passes through the polar line, on the absolute plane, in regard to the absolute conic, of the point of contact of the quadric with the absolute plane.

If we take the general tangential equation of a quadric, for a plane, $lx + my + nz + pt = 0$, touching this, which contains a point (x, y, z, t), the equation

$$t^2 (Al^2 + Bm^2 + Cn^2) + D (lx + my + nz)^2 + 2Ft^2 mn + \dots$$
$$- 2Utl (lx + my + nz) - \dots = 0$$

holds. If we have three such planes, mutually at right angles, satisfying this equation, we infer, as above, by addition of the three equations, the equations of the absolute conic being as before, that

$$t^2 (A + B + C) + D (x^2 + y^2 + z^2) - 2Utx - 2Vty - 2Wtz = 0.$$

This is then the equation of the locus of the point of intersection of the planes; it represents a sphere whose centre is (U, V, W, D), which is the centre of the quadric, the pole of the plane $t = 0$. The locus degenerates into the absolute plane $t = 0$, together with another plane, when $D=0$; this is the condition that the quadric should touch the plane $t=0$. In general, the sphere is called the *director sphere* of the quadric.

We may consider the matter from a less algebraical point of view, if we recall what was said in Chap. I (p. 52) in regard to the condition that one quadric be outpolar to another. The condition that a quadric S', given by a point equation $a'x^2+\ldots+2h'xy=0$, should be outpolar to another quadric Σ given by a tangential equation $Al^2+\ldots+2Hlm=0$, is $a'A+\ldots+2h'H=0$. This condition may be proved, as was the corresponding condition for conics (Vol. II, p. 142), to be independent of the choice of the points of reference for coordinates. Evidently if $S_1'=0, S_2'=0$ be two quadrics both outpolar to Σ, so is any quadric whose equation is $S_1'+\lambda S_2'=0$; or, if $\Sigma_1=0$, $\Sigma_2=0$ be two quadrics both inpolar to S', so is any quadric whose equation is $\Sigma_1+\mu\Sigma_2$. The condition does not apply directly to the case when the inpolar quadric Σ is a cone; there is no unique tangential equation to a cone, just as there is no unique point equation to a conic, regarded as existing in three dimensions. We may however speak of a cone S as being outpolar to a cone Σ of the same vertex as S, when the conic sections of these cones by any plane are in the corresponding relation to one another. The condition for this is easily expressed when the common vertex of the cones is taken as one of the points of reference for coordinates. This being understood, a cone, of vertex P, which is outpolar to the enveloping cone drawn from P to a quadric Σ, is outpolar to Σ; in particular a pair of planes which are conjugate to one another in regard to a quadric Σ, form, together, a degenerate quadric outpolar to Σ. Assuming these results, which may easily be verified, consider again a point, P, from which pass three tangent planes of a quadric, Σ, which are mutually at right angles in regard to the absolute conic, ω. These planes touch the enveloping cone from P to Σ; this cone thus meets the absolute plane in a conic, ϵ, which is inpolar to ω, being touched by three lines in the absolute plane which form a self-polar triad in regard to ω. Thus the conic ω is outpolar to ϵ (Vol. II, p. 33), and the cone $P\omega$, joining P to ω, is outpolar to the enveloping cone $P\epsilon$, and, therefore, by what has been said, is outpolar to the quadric Σ. The director sphere thus appears as the locus of a point P such that the conic $P\omega$, joining P to the absolute conic ω, is outpolar to the quadric. This result is analogous to the fact that the director circle of a conic is the locus of a point which is such that the lines joining it to the two absolute points of the plane of the conic form together a degenerate conic outpolar to the conic, in regard to which they are conjugate lines. If we adopt this definition we may prove that the locus in question is a sphere by remarking that if P and P' be two points, there is a system of coaxial spheres of which P and P' are the limiting points, the cones $P\omega$, $P'\omega$ intersecting in two conics, one, ω, lying on the absolute plane, the other lying in a plane which is at right angles to PP' and passes through the middle point of P, P'. This second plane contains the centre of the quadric; for this statement is the same as that the two planes containing the complete intersection of the two cones, $P\omega$, $P\omega'$, are conjugate to one another in regard to the quadric Σ; now it has been shewn above (p. 75) that if the equations of the two cones be, respectively, $\Gamma = 0$, $\Gamma' = 0$, the equation of the two planes is of the form $\Gamma - \lambda\Gamma' = 0$; as Γ and Γ' are both outpolar to Σ, so then is the aggregate of the two planes. This involves that they are conjugate to one another in regard to the quadric Σ. It follows now easily (cf. Ex. 6, p. 78, above) that P' lies on the sphere constructed to pass through P which has its centre at the centre of the quadric. More generally, if we consider any two spheres which are outpolar to the quadric Σ, these define a system of coaxial spheres. If P and P' be the limiting points of this system, the cones $P\omega$, $P'\omega$, whose equations, as has been seen (p. 75, above) are obtained linearly from the equations of these spheres, are equally outpolar to Σ. Thus P and P' lie on the director sphere. Therefore this sphere cuts both the given spheres at right angles, as we have proved. Thus, all spheres outpolar to Σ cut the director sphere at right angles. This is the analogue in three dimensions of Gaskin's theorem for conics (Vol. II, p. 48).

When a conic ϵ is inpolar to a conic ω, lying in the same plane, if any tangent line of ϵ be taken, and the pole of this in regard to ω, the two tangents of ϵ drawn from this pole are conjugate to one another, as well as to the chosen tangent, in regard to ω. Wherefore, if any tangent plane be drawn to the quadric Σ from a point, P, of the director sphere, and the line at right angles to this be drawn through P, the two tangent planes of the quadric drawn through this line are at right angles to one another. In particular, if we consider a point P common to the director sphere and the quadric Σ, and take for the tangent plane drawn through P that which touches the quadric at P, since a tangent plane of the quadric can be drawn through either generating line of the quadric at P to contain also any other line through P, we infer that the two generating lines of the quadric at P are at right angles to one another. Conversely, it can be shewn that the points of the quadric at which the generators are at right angles lie upon a sphere; this would furnish another definition of the director sphere.

Ex. 1. Let the plane of a conic, ω, meet a quadric, Σ, in the conic σ; let a generator, l, of Σ meet σ in P; let the polar of P in regard to ω meet σ in Q and R. Through each of Q and R there is a generator of Σ which intersects l; let these be m_1 and m_2. The two tangent planes of Σ drawn through QR are those containing, respectively, m_1 and m_2; for, for instance, m_1 is met by the generator through R which is of the same system as l. Any pair of planes through QR which are conjugate to one another in regard to Σ are then harmonic conjugates in regard to the planes (QR, m_1) and (QR, m_2). We can infer that there are two generators m_1, m_2, of the quadric Σ, which intersect a given generator l, and are at right angles to this, and, if these meet l in P_1 and P_2, that the plane at right angles to l which contains the middle point of P_1, P_2, passes through the centre of the quadric. From this it follows that the sphere, whose centre is the centre of the quadric, which contains the point P_1, also contains the point P_2. The generator m_2 is, similarly, met at right angles by another generator beside l, in a point similarly lying on the same sphere; and so on continually.

Ex. 2. When the quadric is that representable, as above (p. 85), in terms of parameters λ, μ, by the equations

$$x/a = 1 - \lambda\mu, \quad y/b = \lambda + \mu, \quad z/c = 1 + \lambda\mu, \quad t = \lambda - \mu,$$

prove that the condition that two generators, (λ), (μ), of different systems, should be at right angles, is

$$a^2(1 - \lambda^2)(1 - \mu^2) + 4b^2\lambda\mu + c^2(1 + \lambda^2)(1 + \mu^2) = 0,$$

shewing also that this arises from

$$x^2 + y^2 + z^2 = (a^2 + b^2 - c^2)t^2.$$

Ex. 3. Supposing the equation of the quadric to be

$$x^2/a + y^2/b + z^2/c = t^2,$$

we have shewn above (p. 80) that a generator of the quadric at one of its intersections with the absolute conic is given, varying λ, by the points expressed by

$$\frac{x}{t} = (a + \lambda)p^{\frac{1}{2}}, \quad \frac{y}{t} = (b + \lambda)q^{\frac{1}{2}}, \quad \frac{z}{t} = (c + \lambda)r^{\frac{1}{2}},$$

where $p = a/(a - b)(a - c)$, etc. The point of this line on the absolute plane, $(p^{\frac{1}{2}}, q^{\frac{1}{2}}, r^{\frac{1}{2}}, 1)$, arises by $\lambda = \infty$. This line, meeting the director sphere once at this point, will meet this sphere in one other point. Shew that this is given by $\lambda = 0$, being $(ap^{\frac{1}{2}}, bq^{\frac{1}{2}}, cr^{\frac{1}{2}}, 1)$. Shew further that the cubic equation in θ expressed by

$$a^2p\,(a + \theta)^{-1} + b^2q\,(b + \theta)^{-1} + c^2r\,(c + \theta)^{-1} = 1$$

possesses only the root $\theta = 0$.

Ex. 4. The director sphere of a quadric is the locus of a point such that the cone joining this to the absolute conic is outpolar to the enveloping cone from this to the quadric. More generally, prove that the locus of a point such that the enveloping cone drawn from this to the quadric $x^2+y^2+z^2=mt^2$ is outpolar to the enveloping cone of the quadric $x^2/a+y^2/b+z^2/c=t^2$, is represented by

$$x^2(b+c)+y^2(c+a)+z^2(a+b)+m\left[x^2+y^2+z^2-(a+b+c)\,t^2\right]=0.$$

Dually, prove that a plane whose section with the quadric expressed by $ax^2+by^2+cz^2=t^2$ is outpolar to the section with

$$a_1^{-1}x^2+b_1^{-1}y^2+c_1^{-1}z^2=t^2,$$

touches the quadric expressed by $A^{-1}x^2+B^{-1}y^2+C^{-1}z^2=t^2$, where $\quad A=a_1(bb_1+cc_1+1)\,(aa_1+bb_1+cc_1)^{-1}$, etc.

Ex. 5. If from a point, T, three lines, TP, TQ, TR, can be drawn, to touch a quadric surface E, of which every two are at right angles, the enveloping cone from T to the surface E is outpolar to the cone $T\omega$, joining T to the absolute conic. If the equation of E be $a^{-1}x^2+b^{-1}y^2+c^{-1}z^2=t^2$, the equations of the absolute conic being as before, prove that the plane PQR is a tangent plane of the surface $(a+\lambda)^{-1}x^2+\text{etc.}=t^2$, where $\lambda^{-1}+a^{-1}+b^{-1}+c^{-1}=0$, say, of the surface F. The locus of T is then that of the pole, in regard to E, of the tangent plane of F, namely is the polar reciprocal of F in regard to E, whose equation is $a^{-2}(a+\lambda)\,x^2+\text{etc.}=t^2$; this locus intersects E on the curve where the common tangent planes of E and F touch E. (Cf. Vol. II, p. 133, and Ex. 2, p. 93, below.)

Confocal quadrics, defined algebraically. The quadrics represented, with different values of λ, by the equation

$$(a+\lambda)^{-1}x^2+(b+\lambda)^{-1}y^2+(c+\lambda)^{-1}z^2=t^2,$$

are said to be confocal. It is perhaps easier to deal with the theory of such surfaces, in the first instance, with the help of the algebraic symbols; a more synthetic treatment is suggested below.

If we replace (x, y, z, t) by the coordinates of a particular point, (x_1, y_1, z_1, t_1), we obtain a cubic equation for the parameter λ; there are thus, in general, three confocals passing through a given point. It is easy to prove that, if a, b, c be real, and (x_1, y_1, z_1, t_1) be real, the three values of λ are real; for definiteness suppose $a>b>c$; it can then be shewn that the three roots λ_1, λ_2, λ_3 are such that

$$-a<\lambda_3<-b<\lambda_2<-c<\lambda_1;$$

it is only necessary to assume that, if the cubic polynomial

$$(\lambda+a)(\lambda+b)(\lambda+c)\,t_1^2-(\lambda+b)(\lambda+c)\,x_1^2-\ldots-\ldots$$

assumes values of different sign upon the substitution of two real values of λ, then it vanishes for one (or three) intermediate values of λ. Of the three confocal surfaces thus obtained to pass through the point (x_1, y_1, z_1, t_1), every two cut at right angles. For the condition that the tangent planes,

$$(a+\lambda)^{-1}xx_1+(b+\lambda)^{-1}yy_1+(c+\lambda)^{-1}zz_1=tt_1,$$

and

$$(a+\mu)^{-1}xx_1+(b+\mu)^{-1}yy_1+(c+\mu)^{-1}zz_1=tt_1,$$

should cut at right angles, namely $(a+\lambda)^{-1}(a+\mu)^{-1}x_1^2 + \text{etc.} = 0$, is obtained at once by subtraction of the two equations

$$(a+\lambda)^{-1}x_1^2 + \text{etc.} = t_1^2, \quad (a+\mu)^{-1}x_1^2 + \text{etc.} = t_1^2.$$

Thus any two different confocals cut at right angles at every common point.

For every one of the confocal surfaces the tetrad of points, $(1, 0, 0, 0)$, etc., is self-polar. In each of the four planes of this tetrad there is a conic associated with the confocal system; for instance, in the plane $x=0$, the conic $(b-a)^{-1}y^2 + (c-a)^{-1}z^2 = t^2$, obtainable from the general equation of the confocals by putting $x=0$ and $\lambda = -a$; and so on; the conic in the plane $t=0$ is $x^2+y^2+z^2=0$, obtainable by putting $t=0$ and $\lambda = \infty$. These four conics are called the *focal conics* of the system; each may be regarded as lying on two degenerate surfaces of the confocal system.

When (x_1, y_1, z_1, t_1) are not all real, the three confocals through this point need not be distinct. We have, in particular, remarked above (p. 90) that, when this point is $(ap^{\frac{1}{2}}, bq^{\frac{1}{2}}, cr^{\frac{1}{2}}, 1)$, in which $p = a/(a-b)(a-c)$, etc., the three roots λ of the cubic equation are all $\lambda = 0$, the only confocal of the system through this point being $a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = t^2$. More generally, the confocals through any point, P, of coordinates $[(a+\theta)p^{\frac{1}{2}}, (b+\theta)q^{\frac{1}{2}}, (c+\theta)r^{\frac{1}{2}}, 1]$, are given by $\lambda = 0$, and $\lambda = \theta$, of which the root θ is a double root, as may easily be verified; for every value of θ this point P lies on a generator, of the surface $a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = t^2$, passing through one of the intersections of this quadric with the conic $t=0$, $x^2+y^2+z^2=0$; this generator, it may easily be seen, touches the surface

$$(a+\theta)^{-1}x^2 + \text{etc.} = t^2$$

at the point P. Thus, this generator touches all the confocals (other than that on which it lies); through any point of it there pass only two distinct confocals, the one on which it lies, and that which it touches there; through one point of it, lying on the director sphere of the surface on which it lies, when this director sphere is defined with the focal conic of the plane $t=0$ as absolute conic (cf. p. 90, above), there passes the surface only, of the confocal system. Further, it may easily be verified that through any point, (x_1, y_1, z_1, t_1), where the quadric $a^{-1}x^2 + \text{etc.} = t^2$ is met by the cone $a^{-2}x^2 + \text{etc.} = 0$, there pass, of the confocal system, only this quadric, counting twice, and the surface of parameter $\lambda = (x_1^2 + y_1^2 + z_1^2)\,t_1^{-2} - (a+b+c)$; at these points the tangent plane of the surface $a^{-1}x^2 + \text{etc.} = t^2$ is such as also to touch the conic $t=0$, $x^2+y^2+z^2=0$.

Ex. 1. Prove that the eight generators of the surface

$$a^{-1}x^2 + \text{etc.} = t^2$$

at the points where this meets the conic $t = 0$, $x^2 + y^2 + z^2 = 0$, lie on the surface expressed by the equation

$$[x^2 + y^2 + z^2 - t^2 (a + b + c)]^2 - 4abc\,(a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2)\,t^2 = 0.$$

Ex. 2. Prove that the polar reciprocal of the quadric

$$(a + \lambda)^{-1}x^2 + \text{etc.} = t^2,$$

in regard to the quadric $a^{-1}x^2 + \text{etc.} = t^2$, that is, the locus of the poles, in regard to the second quadric, of the tangent planes of the first, is the quadric given by

$$a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 - t^2 + \lambda\,(a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2) = 0.$$

Consider now the locus of the poles of an arbitrary plane, say of equation $lx + my + nz = pt$, in regard to all the surfaces of the confocal system. It is evidently the line of which the general point has the coordinates $[(a + \theta)\,l,\ (b + \theta)\,m,\ (c + \theta)\,n,\ p]$, for varying values of θ. This line meets the plane $t = 0$ in the point $(l, m, n, 0)$, and is at right angles to the given plane, when the absolute conic is the focal conic of the plane $t = 0$. The line meets the plane itself in the point for which θ is given by $(a + \theta)\,l^2 + \text{etc.} = p^2$; thus there is one surface of the confocal system touching the plane, that namely for which the parameter λ has this particular value of θ. The line may be called the *axis* of the plane.

It may happen that the axis of a plane lies in the plane itself, namely this is so when the condition $(a + \theta)\,l^2 + \text{etc.} = p^2$ is satisfied for all values of θ. Then the plane, satisfying

$$l^2 + \text{etc.} = 0, \quad al^2 + \text{etc.} = p^2,$$

touches the focal conic in the plane $t = 0$, and touches the surface $a^{-1}x^2 + \text{etc.} = t^2$. In this case, however, we also have, for any value of λ, the equation $(a + \lambda)\,l^2 + \text{etc.} = p^2$, and the plane touches every surface of the confocal system; it touches, then, also the other three focal conics. The points of contact all lie on the line which is the axis of the plane; this line meets all the focal conics. When the plane has the equation $xp^{\frac{1}{2}} + yq^{\frac{1}{2}} + zr^{\frac{1}{2}} = t$, where, now,

$$p = a/(a - b)\,(a - c), \text{ etc.},$$

being then the tangent plane of the quadric $a^{-1}x^2 + \text{etc.} = t^2$ at the point $(ap^{\frac{1}{2}}, bq^{\frac{1}{2}}, cr^{\frac{1}{2}}, 1)$, considered above, the axis is a generator of this quadric, as we have seen. We can thus make the statements: (*a*) Confocal surfaces have the property that a plane which touches two of them, touches all, the points of contact being in line. Such a plane touches all the focal conics, in points lying on this line. Conversely, a plane touching two of these conics, or touching one of these and one of the surfaces, touches all; (*b*) Through any point four planes of this character can be drawn, these being the common tangent planes of the enveloping cones drawn from this point to two of the surfaces. Exception arises when the point is on one of the focal conics; then two such planes can be drawn through this point, both passing through the tangent line of this conic at this point.

Alternative ways of initiating the system of confocal quadrics. If we take any two quadrics which have a common self-polar tetrad of points, that is, as will be seen, any two quadrics which are not in particular relation, their equations may be supposed to be $X^2+Y^2+Z^2=T^2$ and $A^{-1}X^2+B^{-1}Y^2+C^{-1}Z^2=T^2$. Then, a plane, of equation $lX+mY+nZ=pT$, which touches both these surfaces satisfies the two equations $l^2+m^2+n^2=p^2$, $Al^2+Bm^2+Cn^2=p^2$, and, therefore, satisfies, also, the equation

$$(A+\sigma)\,l^2+(B+\sigma)\,m^2+(C+\sigma)\,n^2=(1+\sigma)p^2,$$

whatever σ may be. This plane thus touches the surface

$$P^{-1}X^2+Q^{-1}Y^2+R^{-1}Z^2=T^2, \text{ where } P=(A+\sigma)/(1+\sigma).$$

If we put $x=(A-1)^{-\frac{1}{2}}X$, etc., $t=T$, with $a=(A-1)^{-1}$, etc., and $\lambda=(1+\sigma)^{-1}$, this last equation becomes

$$(a+\lambda)^{-1}x^2+(b+\lambda)^{-1}y^2+(c+\lambda)^{-1}z^2=t^2,$$

which is of the form we have taken to define the system of confocal quadrics.

Or again, suppose we take, in different planes, any two conics, which have no common point; on the line of intersection of the planes of the conics, let A, B be the points which are conjugate to one another in regard to both the conics; let C and D be the poles of this line in regard to the two conics, respectively. Then, referred to A, B, C, D, the equations of the two conics may be supposed to be, respectively,

$$x^2+y^2+z^2=0,\ t=0, \quad \text{and} \quad Ax^2+By^2=t^2,\ z=0;$$

taking c arbitrary we can then take a and b so that $A=(a-c)^{-1}$, $B=(b-c)^{-1}$. Now consider a plane, $lx+my+nz=pt$, which touches both these conics. We then have

$$l^2+m^2+n^2=0, \quad \text{and} \quad (a-c)\,l^2+(b-c)\,m^2=p^2.$$

From these, whatever λ may be, there follows

$$(a+\lambda)\,l^2+(b+\lambda)\,m^2+(c+\lambda)\,n^2=p^2.$$

The plane thus touches any surface whose equation is of the form $(a+\lambda)^{-1}x^2+\text{etc.}=t^2$. Equally, it touches a definite conic in the plane $x=0$, and a definite conic in the plane $y=0$.

Ex. Our original definition of confocal surfaces was for surfaces not touching the absolute plane. If however, as here, we regard as confocal to a given quadric any quadric touching all the tangent planes of this which also touch the absolute conic, we may define quadrics confocal with the surface of equation $a^{-1}x^2 + b^{-1}y^2 = 2zt$, which touches the absolute plane. For the condition that the plane $lx + my + nz = p$ should touch this is $al^2 + bm^2 + 2pn = 0$. The general confocal is then that given tangentially by

$$(a + \kappa)\, l^2 + (b + \kappa)\, m^2 + \kappa n^2 + 2pn = 0\,;$$

it is easily seen that the point equation of this is

$$(a + \kappa)^{-1}x^2 + (b + \kappa)^{-1}y^2 = 2zt + \kappa t^2.$$

The relations of the four focal conics. Begin, as just explained, with two arbitrary conics, having no point in common, lying in different planes; define thereby two other planes, and in each of these a further conic, touched by the common tangent planes of the two given conics; take the four planes as $t = 0, z = 0, x = 0, y = 0$, and denote the conics lying therein by δ, γ, α, β, respectively. It appears then, from what has been said, and is immediately verifiable, that the poles of an arbitrary plane in regard to these conics, that is, the poles, in regard to these conics, of the lines in which the plane intersects the planes of the four conics, respectively, lie in line. Now take the tangent line, say p, at any point, P, of one of the conics α, β, γ, say β; let this line meet the plane of the conic δ in the point T; from T let the tangent lines, TH, TK, be drawn to the conic δ, touching this in H and K. The points P and H are the poles of the plane PTH, respectively, in regard to β and δ; this plane, therefore, touches α and γ at points lying on the line PH. Consider, now, the conics, α', γ', in the plane of the conic δ, which are obtained by projecting the conics α, γ from the point P. Since the plane PHT touches α on PH, the conic α' touches δ at H, as does the conic γ', for the same reason. Similarly, α' and γ' touch δ at K. If we regard δ as the absolute conic, the plane PHK is at right angles to the line PT, that is, to the tangent of the conic β at the point P. The cones joining P to the conics α, γ have thus, both, contact along two generators with the cone joining P to the absolute conic, δ. They are, therefore, what we have above described as right circular cones, or cones of revolution. But, further, as the plane PHT touches all the confocal surfaces derivable from the focal conics, at points lying on the line PH, it similarly follows that the enveloping cone drawn from P to any one of these surfaces is a right circular cone, likewise touching the cone $P\delta$ along the generators PH and PK. Therefore, if we consider the conic in which the plane PHK cuts one of these confocal surfaces, this is touched by the lines PH, PK. If, then, we take, as the two absolute points of any plane, the points in which this plane meets the absolute conic, we have the result that a plane drawn through a point P of one of the focal conics α, β, γ, at right angles to the tangent line of this conic at this point, meets any of the confocal surfaces in a conic of which this point is one focus. When P is at one of the points where the conic, say β, meets the plane of α, (or of γ), the plane PHK, at right angles to the tangent line p, is the plane of this new conic, α. We thus infer that the four points where the plane of one conic, α, of the three α, β, γ, is met by the other two conics, β and γ, are the foci of this conic α.

Ex. 1. Prove that the equation of the enveloping cone drawn from the point, $(\xi, 0, \zeta, 1)$, of the conic β, whose equations are $y = 0$, $(a-b)^{-1}x^2 + (c-b)^{-1}z^2 = t^2$, to the confocal surface of equation $(a+\lambda)^{-1}x^2 + \text{etc.} = t^2$, may be put in the form

$$H\,[(x-t\xi)^2+y^2+(z-t\zeta)^2] - [(a-b)\,x\xi^{-1} + (b-c)\,z\zeta^{-1} - (a-c)\,t]^2 = 0,$$

where $$H = (a+\lambda)(a-b)\,\xi^{-2} - (c+\lambda)(b-c)\,\zeta^{-2}.$$

Ex. 2. Let the line joining the point $(\xi, 0, \zeta, 1)$, or P, of the conic β, to the point $(x, y, 0, 1)$, or Q, of the conic γ, meet the surface $a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = t^2$ in the points L, M. We consider the symbol $(P, Q; L, M)$, explained Vol. II, p. 166. This is the value of λ_1/λ_2, where λ_1, λ_2 are the roots of the quadratic equation obtained by substituting $(x+\lambda\xi, y, \lambda\zeta, 1+\lambda)$ for the coordinates in the equation of the quadric. It is found that this symbol has a value $F(P)/\Phi(Q)$, where $F(P)$ depends only on P, and $\Phi(Q)$ depends only on Q. In fact, with $m^2 = (a-c)(a-b)^{-1}$,

$$F(P) = (a^{-\frac{1}{2}}\xi + m^{-1})/(a^{-\frac{1}{2}}\xi - m^{-1}), \quad \Phi(Q) = (a^{-\frac{1}{2}}x + m)/(a^{-\frac{1}{2}}x - m).$$

If, then, we take two points Q, say Q_1, Q_2, on the conic γ, the ratio of $(P, Q_1; L_1, M_1)$ to $(P, Q_2; L_2, M_2)$ does not vary with the position of P upon the conic β. This result includes that given in Ex. 4, p. 209, of Vol. II.

Verify that if P_1, P_2 be two points of the conic β, and Q_1, Q_2 two points of the conic γ, there are two enveloping cones, of any surface of the confocal system associated with the conics, which touch the four lines P_1Q_1, Q_1P_2, P_2Q_2, Q_2P_1.

Ex. 3. If the absolute points of any plane of space be the intersections of the plane with a single conic, taken as Absolute, prove that through any point there can be drawn six planes whose sections with a given quadric shall have that point as a focus.

If two quadrics touch one another at all the points of a plane section, prove that the tangent plane at an umbilicus of one of these meets the other in a conic of which the umbilicus is one focus.

Ex. 4. The plane at right angles to a focal conic at a point P meets the polar plane of P, in regard to any chosen surface of the confocal system, in a line l. Prove that if any plane through l meet this surface in the conic ζ, the cone joining P to the points of ζ is a right circular cone.

The axes of a confocal system of quadrics. We have seen that the poles of a plane in regard to all the confocal surfaces lie on a line. The lines so determined are of great importance; the necessity, for dynamical purposes, of considering the properties of these lines, is in fact responsible for much of the theory of confocal quadrics. The fundamental property, just stated, depends on the circumstance that the tangential equation of the general confocal surface is of the form $U+\lambda V=0$, where λ is variable; the dual theorem, that the polar planes of a point, in regard to all quadrics whose point equation is of the form $S+\lambda T=0$, pass through a line, will arise below.

By the pole of a plane in regard to one of the focal conics is meant the pole, in regard to this conic, of the line in which the plane meets the plane of the conic. Thus, by what we have seen, the poles of a plane in regard to the surfaces of the confocal system lie on the line joining the poles of this plane in regard to two of the focal conics. This line is then such as to meet the planes of these conics in points, say H and Q, whose polar lines in regard to these conics are two lines which intersect one another (on the line of intersection of the planes of the conics), say in T; and this is a sufficient condition for such a line. The polar lines of T in regard to the conics contain then, respectively, the points H and Q; but, as T is on the line of intersection of the planes of the conics, these polar lines also contain, respectively, the poles, in regard to the conics, of the line of intersection of their planes; let these two poles, which are fixed points, be called B and D. The general construction for a line which is the locus of the poles of a plane in regard to the confocal surfaces is then as follows: Take the pencil of lines through B, in the plane of one of the conics, obtained as the polars, in regard to this conic, of the points, T, of the line of intersection of the planes of the conics; take the pencil of lines through D similarly obtained with the other conic; to any line of the former pencil corresponds a line of the latter, both having T for pole; take any point H of the former line and any point Q of the latter line. The line HQ is such a line as is desired. Such a line depends then on three parameters, one determining the position of T, two others determining the positions of H and Q on definite lines; we say then that there are ∞^3 such lines, the totality of all possible lines being ∞^4, as we have seen in Chap. I. There is in fact one such line arising from any plane.

Ex. A line being given, as in Chap. I, by the equations

$$l'x+m'y+n'z=0,\quad l't+mz-ny=0,$$

etc., the polar lines of the points, $(l, m, n, 0)$, $(0, -n', m', l)$, where the line meets the planes $t=0$, $x=0$, respectively, in regard to the conics $t=0$, $x^2+y^2+z^2=0$ and $x=0$, $(b-a)^{-1}y^2+(c-a)^{-1}z^2=t^2$, intersect on the line $t=0$, $x=0$, provided $all'+bmm'+cnn'=0$. This is then the condition for such lines.

Let such a line be called an *axis of the confocal system*; the plane, of which the line is the locus of poles for the surfaces of the confocal system, if it is not one of the planes touching all the surfaces of the confocal system, touches one of the confocals, and the line passes through the point of contact, and is at right angles to the plane, with respect to any one of the focal conics regarded as absolute conic. The point of contact may be called the *focus* of the plane. If the plane is tangent to all the quadrics of the confocal system, the axis contains all the points of contact, and, in particular, meets each of the focal conics, being a generator of the surface of the confocal system which passes through the point of meeting, as we have seen above. Thus, eight of these axes lie on any quadric of the system, and there are ∞^1 such lines.

The axes of the confocal system which lie in any general plane, ϖ, touch a conic, which touches each of the principal planes $x=0$, $y=0$, $z=0$, $t=0$. This is clear from the construction for such an axis given above. For, let the lines in which the plane ϖ meets the two principal planes there considered, respectively, be the lines m and n; to any point, U, of m there corresponds a line joining it to the point B of the first plane; to this there corresponds a line through D of the related pencil in the second plane, and this meets the line n in a point, V. The ranges (U), (V), respectively on the intersecting lines m and n, being thus related, the line UV, which is one of the axes of the plane ϖ, envelopes a conic touching the lines m and n. On the line of intersection of the two principal planes considered, there are two points, say A and C, such that BA and DA are the polars of C in regard to the conics, respectively, and BAD is a principal plane. If the plane ϖ meet BA and DA respectively in H and Q, the line HQ is tangent to the conic. Similarly the conic touches the principal plane BCD. In fact any line in one of the four principal planes, or through one of the four principal points, is an axis of the confocal system.

The range of four points, in which an axis of the confocal system, lying in the plane ϖ, meets the four principal planes, is thus related to the range in which any other axis in this plane meets these four planes. This range is, however, related to that in which an axis lying in another plane, ϖ', meets the principal planes. For, to any tangent line, t, of the conic in the plane ϖ can be found two tangents of the conic in the plane ϖ' which intersect the former, say t_1' and t_2'; then the lines t and t_1', being axes of the confocal system in the plane (t, t_1'), meet the four principal planes in related ranges. Thus we reach the conclusion that the axes of the confocal system are the lines upon which the intersections with the principal planes are all related ranges, each of four points. The planes joining an axis to the four principal points, A, B, C, D, are then, also, all related axial pencils, of four planes (Vol. I, p. 30); from this it follows that the axes of the confocal system which pass through an arbitrary point lie on a quadric cone containing the points A, B, C, D.

Ex. If the points $(0, -n', m', l)$, $(n', 0, -l', m)$, $(-m', l', 0, n)$, $(l, m, n, 0)$, in which a line whose coordinates are (l, m, n, l', m', n') meets the principal planes, be denoted, respectively, by the symbols P, Q, R, S, we have $lQ = n'S + mP$, $lR = -m'S + nP$. Two such lines are therefore met by the principal planes in related ranges if mm'/nn' be the same for both (Vol. I, p. 154, Vol. II, p. 166). We saw above that the condition for an axis is $all' + bmm' + cnn' = 0$; this, however, in virtue of $ll' + mm' + nn' = 0$, gives $mm'/nn' = (c-a)/(a-b)$.

The lines in space of which the coordinates satisfy a single relation, as here, are said to form a *complex*. We have already briefly considered the linear complex (above, Chap. I). The particular quadratic complex here arising is called the *tetrahedral* complex. It is a general property of quadratic complexes, that the lines of the complex lying in an arbitrary general plane touch a conic, and the lines of the complex passing through an arbitrary general point form a quadric cone. The latter statement appears, if the point be (x_0, y_0, z_0, t_0), by substituting in the quadratic relation connecting l, m, n, l', m', n' which defines the complex, for l, l', etc., respectively, $tx_0 - t_0x$, $yz_0 - y_0z$, etc.; the result, in virtue of $l't_0 = -mz_0 + ny_0$, $m't_0 = -nx_0 + lz_0$, $n't_0 = -ly_0 + mx_0$, is a quadratic relation connecting $tx_0 - t_0x$, $ty_0 - t_0y$, $tz_0 - t_0z$, which is the equation of the cone. The former statement follows in the same way if the line coordinates be defined by the coordinates of two planes, of which, in the proposition considered, the given plane may be taken as one.

For the tetrahedral complex, the cone is given by

$$(b-c)\,x_0l^{-1} + (c-a)\,y_0m^{-1} + (a-b)\,z_0n^{-1} = 0,$$

where $l = tx_0 - t_0x$, etc. The conic, touched by the lines of the complex which lie in the plane $Ax + By + Cz + Dt = 0$, lies on the cone whose equation is

$$[A\,(b-c)\,x]^{\frac{1}{2}} + [B\,(c-a)\,y]^{\frac{1}{2}} + [C\,(a-b)\,z]^{\frac{1}{2}} = 0.$$

It lies also on the cone

$$[(b-c)\,Dt]^{\frac{1}{2}} + [C\,(c-a)\,z]^{\frac{1}{2}} + [B\,(a-b)\,y]^{\frac{1}{2}} = 0.$$

A line is an axis of the system of confocals only if it contains the poles of some plane, taken in regard to two of the surfaces; in this case it contains the poles of this plane in regard to all the confocals. But, if the pole of a plane, in regard to a particular quadric, lie on a line, the polar line of this line, in regard to this quadric, lies in the plane. Thus a line is an axis of the confocals if its polar lines in regard to two of the confocals lie in one plane, in which case its polar lines in regard to all the confocals lie in this plane. This is the same as saying that a line which is an axis of the confocal system is the intersection of the polar planes, in regard to any two of the confocals, of a suitably chosen point. By taking the particular case when one of the two selected confocals is replaced by a focal conic, we reach the conclusion that a line is an axis of the system of confocals if it is at right angles, in regard to one of the focal conics, taken as absolute conic, to its polar line in regard to one of the confocals.

Ex. 1. The polar line of (l, m, n, l', m', n') in regard to the surface $a^{-1}x^2 + \text{etc.} = t^2$ is the line whose coordinates are

$$(al', \ldots, -bcl, \ldots).$$

The condition that these should be at right angles in regard to the absolute conic $t = 0$, $x^2 + y^2 + z^2 = 0$, is $all' + bmm' + cnn' = 0$; and this is also the condition that they should be at right angles in regard to $x = 0$, $(b-a)^{-1}y^2 + (c-a)^{-1}z^2 = t^2$. The condition that the second line should intersect the line

$$[(a+\theta)\,l', \ldots, -(b+\theta)(c+\theta)\,l, \ldots]$$

is also the same.

Ex. 2. If p be the line containing the poles of a plane, ϖ, in regard to the confocals, say the axis of ϖ, meeting ϖ in P, prove that the axes of planes ϵ, drawn through p, lie in the plane ϖ, and meet any two of the principal planes in related ranges, so that they touch a conic touching these planes; further that two of these axes meet in P, and are at right angles to one another in regard to any one of the focal conics, taken as absolute conic; also that any one of these axes is the polar line of p in regard to a properly chosen one of the confocals.

We may consider the locus of the point, in the plane ϖ, which is the intersection of any plane ϵ, drawn through p, with the axis of this plane. If we speak of the conic, in the plane ϖ, which is the envelope of the axes of the confocals lying in this plane, as a parabola, the absolute points of this plane being the intersections of this plane with the focal conic $t = 0$, $x^2 + y^2 + z^2 = 0$, then the point, P, being the intersection of two tangents of the parabola which are at right angles in regard to these absolute points, is on the directrix; and the locus is that of the intersection of a tangent of the parabola with the line drawn at right angles to this tangent from a particular point, P, of the directrix. It may be shewn that this is a curve meeting an arbitrary line of the plane in three points, which passes twice through P.

Ex. 3. If four planes meet in threes in four points, A, B, C, D, which form a self-polar tetrad in regard to a quadric, the lines in which a plane, ϖ, meets these planes intersect in six points, lying on the joins of A, B, C, D, and these six points, it is easily seen, consist of three pairs of points conjugate to one another in regard to the conic section of the quadric by the plane ϖ. This conic is therefore outpolar to any conic in the plane ϖ which touches the four lines spoken of (Vol. II, p. 33).

Thus, in particular, the conic in the plane ϖ, touched by the axes of the confocal quadrics which lie in this plane, is inpolar to the section of any one of the confocals by this plane. Dually, the quadric cone of axes of the confocals which pass through an arbitrary point is outpolar to the enveloping cone drawn from this point to any one of the confocals.

Ex. 4. Shew that if two points, P, Q, of one of the confocals, be such that the lines through these points at right angles respectively to the tangent planes of the quadric at these points, with respect to any one of the focal conics, (called the *normals* of the quadric at these points), meet one another, then PQ is an axis of the confocal system, and conversely.

Ex. 5. For a conic in space, the centre is the pole, in regard to the conic, of the line in which the plane of the conic meets the absolute plane; and the axes of the conic are the lines through its centre at right angles to one another in regard to the absolute conic and also conjugate to one another in regard to the conic itself. Consider now a conic which is the section of one of the confocals by a plane; take, for instance, the principal plane $t=0$ for absolute plane, and the focal conic of this plane for absolute conic; let D be the centre of the quadric, so determined, O the centre of the conic, and OU, OV its axes, the points U, V being on the line, l, in which the plane meets $t=0$. The line DO is then the polar line of l, in regard to the quadric. We desire to see that the axis OU is at right angles to its polar line in regard to the quadric, and, in fact, that the polar line of OU passes through V. For this, as the polar plane of O contains the line UV, or l, it is only necessary to see that the polar plane of U contains V; this is clear because U and V are conjugate in regard to the quadric. Thus a line, which is an axis of the conic section of a quadric of the confocal system by some plane through the line, is an axis of the confocal system. This condition is necessary that a plane may be possible through the line giving a conic section of which the line is an axis.

If we draw through l one of the tangent planes of the quadric, touching it in P, on the line DO, then the polar line of PU is the line PV. These lines are the normals at P of the two confocals, to the given quadric, which pass through P; and are parallel to the axes of any conic section of the given quadric by a plane parallel to the tangent plane at P (that is, intersecting this tangent plane on the absolute plane).

Ex. 6. Taking one of the confocals, an arbitrary plane, and the axis of this plane, the polar line of this axis, in regard to this confocal, lies in the plane, and meets the section of the surface by the plane in two points. Prove that the tangent planes of the surface at these points are at right angles to the plane, and pass through its axis.

Ex. 7. If (l, m, n, l', m', n') satisfy the equation

$$all' + bmm' + cnn' = 0,$$

it is the axis, in regard to the confocal system, of the plane

$$lx + my + nz = pt,$$

where p has the value $mn(b-c)/l'$, or the equal values, $nl(c-a)/m'$, $lm(a-b)/n'$. The normals of the surface $a^{-1}x^2 + \text{etc.} = t^2$, at the points, (x_1), (x_2), where the line meets it, intersect in $(\xi, \eta, \zeta, 1)$, where $a\xi l = px_1x_2$, etc. The plane through the line whose conic section with this surface has the line for an axis has the equation

$$al'(l't + mz - ny) + bm'(m't + nx - lz) + cn'(n't + ly - mx) = 0.$$

Find the equation of the locus of the centres of the conic sections of all the confocals by an arbitrary plane.

Ex. 8. If the normal of a surface, S, of the confocal system, at P, meet the surface S again in P', and meet another confocal, Σ, of the system, in Q and Q', prove that the normals of Σ at Q and Q' intersect in a point lying on the tangent plane of S at P'. Further, that, if Σ vary, the locus of the point of intersection is a conic, in the tangent plane at P'.

Ex. 9. We have seen that the axes of the confocals which lie in a plane touch a conic, and those which pass through a point are generators of a cone. Prove that the points of the conics lying in planes passing through a line (l, m, n, l', m', n') lie on a surface, which is touched by the tangent planes of the cones whose vertices are on this line. Shew that the points of this surface are expressible by two parameters, θ, ϕ, in the forms

$$\beta\gamma x/t = (\theta + a)^2 (l'\phi - p)^{-1}, \quad \gamma\alpha y/t = (\theta + b)^2 (m'\phi - q)^{-1}, \text{ etc.},$$

where $\alpha = b - c$, $\beta = c - a$, etc., and $p = (mn' - m'n)(l'^2 + m'^2 + n'^2)^{-1}$, etc. See Plücker's *Neue Geometrie des Raumes*, zw. Abth.; Leipzig, 1868, pp. 193 ff.

The normals of a quadric, and of confocal quadrics, which pass through a point. In the case of confocal conics, we considered, starting from an arbitrary point, the aggregate of lines formed by the polars of this point in regard to all the confocals. The lines were tangents of a parabola, touching the two axes of the confocals (Vol. II, p. 120). For the system of confocal surfaces we may similarly consider the aggregate of the polar planes of an arbitrarily taken fixed point, (x_0, y_0, z_0, t_0), in regard to all the surfaces of the system. The equation of the general plane of this aggregate is $(a+\lambda)^{-1}xx_0+(b+\lambda)^{-1}yy_0+(c+\lambda)^{-1}zz_0=tt_0$; particular planes of the aggregate are the principal planes $x=0$, $y=0$, $z=0$, $t=0$, arising respectively for $\lambda=-a$, $\lambda=-b$, $\lambda=-c$, $\lambda=\infty$. Through an arbitrary point there pass three of these planes, which, when the point is (x_0, y_0, z_0, t_0), are mutually conjugate in regard to all the focal conics. Such an aggregate of planes, each determined by a chosen value of a parameter, λ, is called a developable; and, in this case, as three planes of the aggregate pass through an arbitrary point, it is called a cubic developable. If we take a particular quadric of the confocal system, say $a^{-1}x^2+\text{etc.}=t^2$, and then consider the poles, in regard to this confocal, of all the planes of the developable, the points obtained lie upon a curve, of which three points lie in an arbitrary plane, so that the curve is said to be of the third order. It may properly be regarded as the polar reciprocal of the developable in regard to the quadric chosen. The coordinates of the points of this curve are given by

$$[ax_0(a+\lambda)^{-1},\ by_0(b+\lambda)^{-1},\ cz_0(c+\lambda)^{-1},\ t_0],$$

for varying values of λ; it contains the principal points forming the common self-polar tetrad for the confocal surfaces (respectively for $\lambda=-a, -b, -c, \infty$), and also the point (x_0, y_0, z_0, t_0), for $\lambda=0$. The theory of cubic curves in space will be considered below; such a curve can, in an infinite number of ways, be regarded as consisting of all the points common to two quadric surfaces which have also a generator in common, other than the points of this generator, of which only two lie on the curve. The line which joins two points of the curve being called a chord of the curve, the chords which pass through an arbitrary point of the cubic curve in space are the generators of a quadric cone, and the curve may be defined as consisting of the intersection of two of these cones, other than the generator joining their vertices, which is common to both. For the particular cubic curve now under consideration, every chord, being the polar line, in regard to the surface $a^{-1}x^2+\text{etc.}=t^2$, of the intersection of the polar planes of (x_0, y_0, z_0, t_0) in regard to two confocals of the system, is, like this line, an axis of the confocal system. The chords of the curve passing through an arbitrary point of the curve, (x_1, y_1, z_1, t_1), are then the axes of the confocal system through this point, lying on the cone whose equation is

$$(b-c)\,x_1(xt_1-x_1t)^{-1}+(c-a)\,y_1(yt_1-y_1t)^{-1}+\text{etc.}=0.$$

Consider now a point common to the surface, $a^{-1}x^2 + \text{etc.} = t^2$, and the cubic curve. The tangent plane of the surface at this point will be the polar plane of (x_0, y_0, z_0, t_0) in regard to one of the confocals; thus (x_0, y_0, z_0, t_0) will lie on the axis of this plane. This axis is at right angles to the plane, with respect to every one of the focal conics taken as absolute conic, and is the normal of the surface $a^{-1}x^2 + \text{etc.} = t^2$ at the point of contact of the plane. The normals of this surface which can be drawn through the point (x_0, y_0, z_0, t_0) are then the lines joining this point to the intersections of this surface with the cubic curve. The number of such intersections, of a cubic curve with a quadric surface, is in fact six. In the present case they may be determined from the roots of the sextic equation

$$\frac{ax_0^2}{(a+\lambda)^2} + \frac{by_0^2}{(b+\lambda)^2} + \frac{cz_0^2}{(c+\lambda)^2} = t_0^2.$$

Ex. 1. If a line be drawn from the point (x_0, y_0, z_0, t_0), at right angles, in regard to the focal conic $t = 0$, $x^2 + y^2 + z^2 = 0$, to any plane, ϖ, drawn through the centre of the surface $a^{-1}x^2 + \text{etc.} = t^2$, such that this line meets the diameter of this plane (namely the polar line of the line in which the plane ϖ meets the plane $t = 0$), prove that the locus of the point of meeting is the cubic curve just described.

Ex. 2. Shew that the two lines of Ex. 1, that through (x_0, y_0, z_0, t_0), and the diameter, are such as to meet any plane in two related plane systems (in the sense of Vol. I, p. 148). In general, if two corresponding points, P_1 and P_2, of two related plane systems, which may be in different planes, be joined, respectively, to two fixed points O_1, O_2, of arbitrary position, shew that, for positions of P_1 for which O_1P_1 and O_2P_2 meet, the locus of the point of meeting is a cubic curve.

Ex. 3. Shew that the plane containing the three points (λ_1), (λ_2), (λ_3), of the cubic curve just described, has the equation

$$\frac{x}{ax_0}\frac{(a+\lambda_1)(a+\lambda_2)(a+\lambda_3)}{(a-b)(a-c)} + \frac{y}{by_0}\frac{(b+\lambda_1)(b+\lambda_2)(b+\lambda_3)}{(b-c)(b-a)} + \frac{z}{cz_0}\frac{(c+\lambda_1)(c+\lambda_2)(c+\lambda_3)}{(c-a)(c-b)} = \frac{t}{t_0}.$$

Ex. 4. Shew that the six roots of the sextic equation which gives the six points of the quadric $a^{-1}x^2 + \text{etc.} = t^2$ at which the normals pass through (x_0, y_0, z_0, t_0), are such that

$$\prod_{r=1}^{6} (a + \lambda_r) = -a(a-b)^2(a-c)^2 x_0^2 t_0^{-2},$$

with similar values for the products of the quantities $b + \lambda_r$, and of the quantities $c + \lambda_r$; hence, denoting these six points by

$N_1, N_2, \ldots, N_6$, prove that, if the equation of the plane $N_1N_2N_3$ be written in the form $a^{-\frac{1}{2}}lx + \text{etc.} = t$, the equation of the plane $N_4N_5N_6$ will be of the form $a^{-\frac{1}{2}}l^{-1}x + \text{etc.} = -t$.

Ex. 5. Through every point of the cubic curve there is a line, called the tangent of the curve, such that any plane through this line meets the curve in two points coincident at this point, and in only one other point. Shew that the tangent at the point (x_0, y_0, z_0, t_0) is the line drawn through this point at right angles to the polar plane of this point in regard to the surface $a^{-1}x^2 + \text{etc.} = t^2$. In general only one cubic curve can be drawn to contain six given points. The present cubic curve is thus determined by the facts that it passes through the principal points, the common self-polar tetrad for the confocals, passes through (x_0, y_0, z_0, t_0), and has the line described, as the tangent line at (x_0, y_0, z_0, t_0).

Ex. 6. The chords $N_1N_2, N_1N_3, \ldots, N_1N_6$ of the cubic curve, being axes of the confocal system, lie upon the quadric cone, of vertex N_1, containing the axes of the confocals which pass through N_1. This cone is independent of the point, O, of the normal at N_1, from which the other normals ON_2, etc., are drawn. Thus, as O varies on the normal at N_1, the points N_2, etc., vary on the curve of intersection of the surface $a^{-1}x^2 + \text{etc.} = t^2$ with the cone in question.

Ex. 7. Taking an arbitrary plane section of the surface

$$a^{-1}x^2 + \text{etc.} = t^2,$$

and a fixed point N_1 of this section, there are two other points of this section which are such that the normals of the surface at these points meet the normal of the surface at N_1. For, if the normal at N_2 meet the normal at N_1, the line N_1N_2 is an axis of the confocals; and the axes which lie in this plane touch a conic lying in this plane; two of them, therefore, pass through N_1, say N_1N_2 and N_1N_3; the normals of the surface at the points N_2 and N_3 of the section intersect the normal at N_1, generally in points which do not coincide. If, however, these points coincide, the normals at N_2N_3 intersect one another, and the line N_2N_3 is an axis of the confocals. When this is so, the conic, Σ, of the plane, enveloped by all the axes of the confocals which lie in this plane, is triangularly inscribed in the conic section of the quadric by the plane. This conic, Σ, we have seen, is always inpolar to the other. When both facts are true, the two conics are in the particular relation discussed in Vol. II, p. 149, each being both inpolar and outpolar to the other. It may be proved that, if the equation of the plane be $Ax + By + Cz = Dt$, the condition for this particular relation is

$$\Sigma\,(b-c)^2\,(bcB^2C^2 - aA^2D^2) = 0\,;$$

and that, then, the locus of the intersection of the existing triads of concurrent normals is a line.

The normals of the confocals which are at right angles to a given line, and the polars of this line in regard to the confocals. Consider a line l, which we suppose not to be an axis of the confocal system; in particular, it does not lie in one of the four principal planes nor pass through one of the four principal points. These possibilities are considered below. Every plane, ϖ, drawn through l determines a further line, λ, the axis of the plane ϖ; every confocal, S', also determines a further line, l', the polar line of l in regard to S'. The pole, in regard to S', of any plane through l, in particular of the plane ϖ, lies on l'; the poles of the plane ϖ, in regard to all the surfaces of the confocal system, in particular in regard to the surface S', lie on the axis λ. Thus the lines l' and λ have a point in common. If we take different surfaces S', the corresponding lines l' do not intersect, or l would be one of the axes of the confocals. By considering in the first instance three surfaces S', and the corresponding lines l', we see that the axes, λ, of all the planes, ϖ, drawn through l, are the generators, of one system, of a quadric surface; and, then, that the polar lines l', for all the confocal surfaces, are the generators of the other system of this surface. The system of polar lines l' will have a line in each of the principal planes, the polars, namely, in regard to the focal conics, respectively, of the point in which the line l meets these planes. The quadric surface in question thus touches each of the principal planes.

Considering any particular two of the polar lines l', the ranges in which these are met by all the lines λ will be related to one another, and these both related to the axial pencil of planes ϖ from which the lines λ are determined. For, let X, T be the intersections of the line l with two of the principal planes, respectively; and x, t the polar lines of these points, respectively, with regard to the focal conics in these planes. Any plane, ϖ, through l, will meet these planes, respectively, in two lines, say p and s, passing through X and T, and the poles, say P and S, of these lines, in regard to the appropriate conics, will be points of the lines x and t, respectively. As ϖ varies, the lines p, s will describe related pencils in their respective planes, and the points P, S will, therefore, describe related ranges. The line λ, corresponding to a plane ϖ, is the join PS; this line, therefore, meets x and t in ranges related to the axial pencil of planes ϖ; and it is a general property of quadric surfaces that the line λ meets all the generators l', of which x and t are two particular ones, in related ranges. The line λ, or PS, may evidently be regarded as the intersection of corresponding planes of two related axial pencils of planes, one having x, the other having t, for axis. The point where the line λ meets ϖ, which is the point of contact with one surface of the confocal system of a tangent plane drawn thereto from the line l, may, therefore, be regarded as the intersection of corresponding planes from three related axial pencils of planes, having, respectively, the lines x, t and l for axes. The line of intersection of corresponding planes of two of these axial pencils, as follows from what was said in Chapter I, describes a quadric surface of which both the axes are generators. We thus have, here, three quadric surfaces, say (x, t), (x, l), (t, l), of which any two have a generator in common. It follows that the point where the plane ϖ, through l, meets its corresponding axis, λ, describes a curve which is that common to two quadric surfaces having a common generator. Such a curve evidently meets an arbitrary plane in three points; it is in fact identical with such a cubic curve as that considered above. The theory shews that the curve has two points on each of the lines l, x, t; by similar reasoning it has two points on every polar line, l', of the line l, in regard to the confocal surfaces. And, by definition, it has a simple point on every axis, λ, of a plane ϖ passing through l.

Ex. 1. Putting $b-c=\alpha$, $c-a=\beta$, $a-b=\gamma$, prove that the polar lines of the line whose coordinates are $(l, m, n, \alpha u, \beta v, \gamma w)$, in regard to surfaces confocal with $a^{-1}x^2+\text{etc.}=t^2$, lie on the quadric surface whose equation is

$$uvw\,(u^{-1}x+v^{-1}y+w^{-1}z)\,(lx+my+nz)-plxt-qmyt-rnzt+lmnt^2=0,$$

where $$p=mv+nw, \quad q=nw+lu, \quad r=lu+mv.$$

Ex. 2. The cubic curve which is the locus of points of contact of tangent planes from this line to the confocals has, for the coordinates of any point,

$$L\,(M^2\gamma-N^2\beta+1), \qquad M\,(N^2\alpha-L^2\gamma+1),$$
$$N(L^2\beta-M^2\alpha+1), \qquad L^2+M^2+N^2,$$

where L, M, N are linear functions of a variable parameter, θ, of which L is given by

$$L=\alpha u\theta+(\gamma mw-\beta nv)\,(\alpha^2u^2+\beta^2v^2+\gamma^2w^2)^{-1},$$

and M, N have similar forms.

When the line l is one of the axes of the confocal system the preceding argument fails. We have already (above, Ex. 2, p. 100) considered the case when the line is an axis in general position. The polars of the line then lie in a plane, and the points of contact of tangent planes to the confocals, drawn through the line, lie on a plane curve having the property of meeting any line of its plane in three points.

When the line l lies in one of the principal planes, its polar lines also lie in a plane, through the corresponding principal point; and the points of contact of the tangent planes, through the line l, lie on a conic in this plane, which meets the principal plane considered in two points conjugate to one another in regard to the focal conic of this plane, the tangents of the locus at these two points meeting in the corresponding principal point. In this case the polar lines of l, in regard to the confocals, all pass through a principal point, that corresponding to the plane in which l lies; by an argument already given, they all meet the axis, λ, in regard to the confocals, of any particular plane, ϖ, drawn through l. Thus these polar lines lie in a plane, σ. The line in which this plane σ meets the principal plane considered is, in fact, the locus of the poles of l in regard to the sections of the confocals by this principal plane. If the plane σ meet l in Q, and O be the pole of l in regard to the focal conic of the principal plane, and P the point of contact of a tangent plane from l to one of the confocals, it is at once seen that the pencils $Q\,(P)$ and $O\,(P)$ are related. This proves the statement made.

Lastly, when the line l passes through one of the principal points, the polar lines of l, in regard to the confocals, lie in the corresponding principal plane, as do the axes of the planes drawn through l, all these lines touching a conic, lying in this plane. The circumstances are then sufficiently dealt with by the remarks made above for the case when l is any axis of the confocals.

Ex. When the line l is the line given by $t = 0$, $l'x + m'y + n'z = 0$, the polar lines of this line in regard to the confocals lie in the plane $\alpha x/l' + \beta y/m' + \gamma z/n' = 0$, where $\alpha = b - c$, etc. The points of contact of tangent planes to the confocals drawn from this line lie on the conic expressed by $x = l'(\theta^2 + \gamma m'^2 - \beta n'^2)$, etc., $t = \theta\,(l'^2 + m'^2 + n'^2)$, where θ is a variable parameter.

The enveloping cones drawn from a point to the surfaces of the confocal system, and the tangent planes drawn from a line. We have seen that any plane which touches two surfaces of the confocal system touches every one of these surfaces. Thus, if the two enveloping cones be drawn from an arbitrary point O to two of the surfaces, the four common tangent planes of these two cones, which touch both these surfaces, are equally tangent planes of the cone drawn from O enveloping any other of the surfaces. The enveloping cones from O are thus a system with four common tangent planes. They will meet any one of the principal planes in a system of conics having four common tangent lines, which, in particular, will be tangents of the focal conic of that plane. We may denote the point-pairs in this plane through which these four tangents pass respectively by S_1, H_1; S_2, H_2; S_3, H_3; the three lines S_1H_1, S_2H_2, S_3H_3 will then be a self-conjugate triad in regard to every one of the conics, in particular in regard to the focal conic of the plane. There are thus, through O, three planes forming a self-polar triad for every one of the enveloping cones; and, in regard to any one of the focal conics taken as an absolute conic, these planes are mutually at right angles. The line of intersection of any two of these planes thus contains the pole of the third plane in regard to every one of the focal conics, and, therefore, contains the pole of this plane in regard to all the surfaces of the confocal system. From this it is clear that the three planes are the tangent planes at O of the three confocals of the system which pass through O. Further, a point-pair such as S_1, H_1 is a degenerate form of a conic of the system touching the four lines in the principal plane considered, the tangents of this conic being all lines passing through S_1 or H_1. A degenerate enveloping cone from O consists therefore of all planes passing through one of the lines OS_1, OH_1. Thus these lines are the generating lines at O of one of the confocals which pass through O. The generating lines of the other confocals are, respectively, OS_2, OH_2 and OS_3, OH_3.

The three planes are common *principal* planes of the cones; the six lines OS_1, OH_1, etc., are common *focal* lines of the cones, with reference to the focal conic, taken as absolute conic, in the principal plane considered. As is easy to see, any one of these lines has the property that a plane drawn at right angles to it, in regard to the absolute conic, meets the cone in a conic of which one focus is the intersection of the line with the plane.

Consider now the pairs of tangent planes which can be drawn to the surfaces of the confocal system from an arbitrary line. Taking any point, O, of this line, the pairs of tangent planes from the line to the surfaces are the pairs of tangent planes, from the line, drawn to the enveloping cones of the surfaces which have O for vertex. These cones, however, we have seen, have four common tangent planes; thus, as in the theory of conics in one plane which have four common tangent lines, the pairs of tangent planes to the confocals (two to each surface) are an axial pencil of pairs of planes in involution. There are thus two planes through the line, the double elements of the involution, each of which consists of two coincident tangent planes drawn from the line to one of the surfaces; in other words there are two surfaces of the confocal system which touch the line. And the tangent planes, through the line, of these two confocals, are harmonic conjugates in regard to the pair of tangent planes drawn from the line to any other one of the confocal surfaces. In particular, considering one of the focal conics, and taking it as absolute conic, the tangent planes through the line, of the two confocal surfaces which touch the line, are at right angles to one another in regard to this focal conic. When the line is a generator of one of the confocals, either plane through it touching another confocal, touches all the confocals, and the proof, and theorem, fail.

Ex. 1. Let the equations of the three surfaces of the confocal system, which pass through any point $(x_0, y_0, z_0, 1)$, be

$$(a+\lambda_r)^{-1}x^2 + (b+\lambda_r)^{-1}y^2 + (c+\lambda_r)^{-1}z^2 = t^2,$$

for $r = 1, 2, 3$. Also, denote $a+\lambda_r$, $b+\lambda_r$, $c+\lambda_r$, respectively, by a_r, b_r, c_r, and put $p_1^2 = a_1b_1c_1(a_1-a_2)^{-1}(a_1-a_3)^{-1}$, etc. It may easily be verified that $x_0^2(a-b)(a-c) = a_1a_2a_3$, $y_0^2(b-c)(b-a) = b_1b_2b_3$, $z_0^2(c-a)(c-b) = c_1c_2c_3$, and that $p_1^{-2} = a_1^{-2}x_0^2 + b_1^{-2}y_0^2 + c_1^{-2}z_0^2$, etc. Take for points of reference for coordinates the point $(x_0, y_0, z_0, 1)$, and the points on the plane $t = 0$ which lie on the normals, at $(x_0, y_0, z_0, 1)$, of the three confocals through this point; let such coordinates be denoted by $(\xi, \eta, \zeta, 1)$. Prove, then, that the enveloping cone drawn from $(x_0, y_0, z_0, 1)$ to the confocal surface $(a+\lambda)^{-1}x^2 + \text{etc.} = t^2$ has for equation

$$(\lambda_1-\lambda)^{-1}\xi^2 + (\lambda_2-\lambda)^{-1}\eta^2 + (\lambda_3-\lambda)^{-1}\zeta^2 = 0.$$

Also, that the generators, at $(x_0, y_0, z_0, 1)$, of the three confocals which pass through this point, lie in threes upon four planes, touching all the confocal surfaces, whose equations are

$$(\lambda_2-\lambda_3)^{\frac{1}{2}}\xi \pm (\lambda_3-\lambda_1)^{\frac{1}{2}}\eta \pm (\lambda_1-\lambda_2)^{\frac{1}{2}}\zeta = 0.$$

Also, that the polar plane of $(x_0, y_0, z_0, 1)$ in regard to the surface $(a+\lambda)^{-1}x^2 + \text{etc.} = t^2$ has for equation

$$(\lambda_1-\lambda)^{-1}p_1\xi + (\lambda_2-\lambda)^{-1}p_2\eta + (\lambda_3-\lambda)^{-1}p_3\zeta - 1 = 0.$$

Denoting this by $U = 0$, prove that the equation of the surface $(a+\lambda)^{-1}x^2 + \text{etc.} = t^2$ is

$$(\lambda_1-\lambda)^{-1}\xi^2 + (\lambda_2-\lambda)^{-1}\eta^2 + (\lambda_3-\lambda)^{-1}\zeta^2 - MU^2 = 0,$$

where $M = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)/(a+\lambda)(b+\lambda)(c+\lambda)$. This fails when λ is one of $\lambda_1, \lambda_2, \lambda_3$; prove that the equation of the surface $(a+\lambda_1)^{-1}x^2 + \text{etc.} = t^2$ is capable of either of the two forms

$$(\lambda_1-\lambda_2)^{-1}(p_1\eta - p_2\xi)^2 + (\lambda_1-\lambda_3)^{-1}(p_1\zeta - p_3\xi)^2 + \xi^2 - 2p_1\xi = 0,$$

$$p_1\left[N_1\xi^2 + (\lambda_1-\lambda_2)^{-1}\eta^2 + (\lambda_1-\lambda_3)^{-1}\zeta^2\right]$$
$$- 2\xi\left[(\lambda_1-\lambda_2)^{-1}\eta p_2 + (\lambda_1-\lambda_3)^{-1}\zeta p_3 + 1\right] = 0,$$

where $N_1 = a_1^{-1} + b_1^{-1} + c_1^{-1} - (a_1-a_2)^{-1} - (a_1-a_3)^{-1}, = p_1^2\Sigma a_1^{-3}x_0^2$.

Ex. 2. If we take for points of reference for coordinates the common centre of the confocals and the three points where the normals of the confocals through $(x_0, y_0, z_0, 1)$ meet the plane $t = 0$, denoting the coordinates by $(X, Y, Z, 1)$, prove that the sections of the three confocals through $(x_0, y_0, z_0, 1)$ by the planes $X = 0$, $Y = 0$, $Z = 0$ (the planes through the centre parallel, in regard to $t=0$, to the tangent planes at $(x_0, y_0, z_0, 1)$), have the respective equations

$$X=0,\quad (a_1-a_2)^{-1}Y^2+(a_1-a_3)^{-1}Z^2=1\,;$$
$$Y=0,\quad (a_2-a_1)^{-1}X^2+(a_2-a_3)^{-1}Z^2=1,\ \text{etc.}$$

These three conics are therefore such that any plane touching two of them touches the third, and touches the focal conic of the confocals which lies in $t=0$.

Shew that three confocal surfaces can be constructed to have their common centre at $(x_0, y_0, z_0, 1)$, their axes being the normals at this point of the original confocals, to pass through the common centre of the original confocals, and have for normals at this point the axes of the original confocals (cf. Chasles, *Aperçu historique*, 1837, Note xxv, p. 359). Prove, also, that if $(x_0, y_0, z_0, 1)$ lie on a plane drawn through the original centre touching all the original confocals, then this plane equally touches all the new confocals.

Ex. 3. Considering four points P, Q, P', Q' of a line, of which the symbols of P', Q' are expressible by those of P, Q by $P'=P+mQ$, $Q'=P+nQ$, we have previously considered the symbol denoted by $(P', Q'\,;\ P, Q)$, which is equal to $m^{-1}n$ (Vol. II, p. 166). We may call this the cross-ratio of P', Q' in regard to P, Q, or of P, Q in regard to P', Q'. Now consider the involution of pairs of points, on the line, of which P, Q are one pair and P', Q' are another pair. Any pair of this involution is given by an equation $\phi+\lambda\psi=0$, where $\phi=0, \psi=0$ are quadratic equations giving two arbitrarily chosen pairs of the involution (Vol. II, p. 111). Any pair of points of the involution is thus given by one parameter, λ. Let λ, λ' be the respective values of this parameter for the two pairs P, Q and P', Q'. Also let w, w' be the respective values of this parameter for the two double pairs of the involution. We may consider the quantity E defined by $E=(\lambda'-w)(\lambda'-w')^{-1}/(\lambda-w)(\lambda-w')^{-1}$, which we may here speak of as the cross-ratio of the parameters λ, λ' in regard to the parameters w, w'. There is a definite relation connecting the cross-ratio $(P', Q'\,;\ P, Q)$ with the cross-ratio E. This may be expressed, putting the former equal to $e^{2i\alpha}$, and the latter equal to $e^{2i\beta}$, by $\sin\alpha\sin\beta=\pm 1$. This result can be immediately proved by interpreting the involution on a conic (cf. Vol. II, p. 210). Similar remarks hold of course for involutions of pairs of lines of a plane pencil of lines, or for involutions of pairs of planes of an axial pencil.

We apply this now to the involution of pairs of tangent planes which can be drawn from a line to the surfaces of a confocal system, remarking that the parameter which determines one of these pairs of the involution may be taken identical with that which determines the particular surface to which the planes are tangent. For the pair of tangent planes, drawn from the line of coordinates (l, m, n, l', m', n') to the surface $(a+\lambda)^{-1}x^2 + \text{etc.} = t^2$, is given by $\phi + \lambda\psi = 0$, where $\phi = a\xi^2 + b\eta^2 + c\zeta^2 - \tau^2$, $\psi = \xi^2 + \eta^2 + \zeta^2$, in which $\xi = l't + mz - ny$, etc., $\tau = -(l'x + m'y + n'z)$.

We thus have the result that if, from a line which touches the two confocals of parameters respectively $\lambda = p$, $\lambda = q$, there be drawn the pair of tangent planes to the surface of parameter μ, and also the pair of tangent planes to the surface of parameter ν, then the cross-ratio of the former pair of tangent planes in regard to the latter pair, say $e^{2i\alpha}$, depends upon the cross-ratio of the parameters μ, ν in regard to the parameters p, q, say $e^{2i\beta}$, these being connected by a relation $\sin\alpha \sin\beta = \pm 1$. The former cross-ratio is thus the same as for any other line also touching the two surfaces p, q, proper regard being paid to the order in which the pair of tangent planes to a confocal are taken.

Through any point of the surface of parameter p there can be drawn two lines, lying in the tangent plane of this surface at this point, which also touch the surface of parameter q, these being generators of the enveloping cone from this point to the surface (q). From what has been shewn above in regard to the enveloping cones drawn to the confocals from any point, it is clear that these two lines are a pair of an involution of lines through the point, in the tangent plane, obtained by varying q, of which the double rays are the normals of this point of the other two confocals which pass through the point. It can be shewn that there exist curves on the surface (p) with the property that, the tangent line of the curve, at any point, is a tangent line also of the surface (q); all these curves, which are called *geodesics*, touch the curve in which the surface (p) is met by the surface (q), any tangent line of this latter curve being a common tangent line of the surfaces (p) and (q).

Ex. 4. The two geodesics which can be drawn, as in Ex. 3, from any point of the confocal (p), to touch the curve of intersection, say (p, q), of the surfaces (p) and (q), may suggest the two tangent lines which can be drawn in a plane, to a conic, from an arbitrary point. It can in fact be shewn that, if the various curves, (p, q), of the surface (p), obtained by its intersection with various confocal surfaces (q), be projected from one of the principal points, say $(0, 0, 1, 0)$, each giving rise thereby to a quadric cone, then these cones have four common tangent planes. If we consider on the principal plane $t = 0$, the focal conic on the plane, and the section of the surface (p), there will be two common chords of these conics which intersect in the centre of projection $(0, 0, 1, 0)$; each of these two chords is the intersection of two of the four common tangent planes spoken of. Taking then the focal conic of $t = 0$ as absolute conic, and the two intersections of any plane with this conic as the absolute points of this plane, we reach the conclusion that the curves (p, q'), for various confocals (q'), project from $(0, 0, 1, 0)$, upon any plane of circular section of the surface (p) which does not contain this point, into a system of confocal conics, $[p, q']$. For definiteness consider one of the planes of circular section which contains the common centre of all the confocals, and also the point $(0, 1, 0, 0)$. Two foci of the confocal conics are then projections of umbilici in the plane $y = 0$.

Now consider a particular point of the surface (p), say P, and the two tangent lines through P which also touch the surface (q). These are the intersections with the tangent plane at P of the two tangent planes drawn to the surface (q) from the normal, at P, of the surface (p). The cross-ratio of these planes in regard to the two tangent planes drawn from this normal to the focal conic of the plane $t = 0$ (which touch this conic at points lying on the tangent plane at P) depends, we have seen, upon the cross-ratio of the two parameters, q and ∞, giving these confocals, in regard to the two parameters, q_1 and q_2, of the two confocals, other than (p), which pass through P, namely depends on $(q - q_1)/(q - q_2)$. Again, the point $(0, 0, 1, 0)$ being C, as the pairs of tangent planes from CP, to the cones projecting from C the curves (p, q'), are in involution, the cross-ratio of the two tangent lines from $[P]$, the projection of P, to the conic $[p, q]$, in regard to the two lines joining $[P]$ to the absolute points of the plane of projection, depends in the same way upon the same quantity $(q - q_1)/(q - q_2)$. Thus, defining the measure of angle as in Vol. II, pp. 167, 192, we may say that the two tangents from P to the surface (q) intersect at the same angle as the two tangents from $[P]$ to the conic $[p, q]$. The two former lines however do not project into the two latter.

Ex. 5. Consider the curve of intersection of two confocals $(a + \mu)^{-1}x^2 + \text{etc.} = t^2$, $(a + \nu)^{-1}x^2 + \text{etc.} = t^2$, say the surfaces (μ), (ν). Through any point, P, of this curve there pass two generators of the third confocal, say (λ), which can be drawn through P. Consider one of these two generators, say p. Through any other point P', of this generator, there pass, beside the surface (λ), two other confocals, (μ') and (ν'), and the curve of intersection of these, say the curve (μ', ν'). Consider all these curves (μ', ν'), for varying positions of P' on this chosen generator p. They are all intersected by one of the two generators of the third confocal which can be drawn through any other point, Q, of the first curve (μ, ν), say the generator q. The generators, q, so obtained meet any one of the principal planes in the points of a conic. Also, if the generator q meet the curve (μ', ν') in the point Q', the range of points Q' on q, as the curve (μ', ν') varies, but so as always to meet the generator p, in P', is related to the range of points P' on the generator p. A particular case is that the two points, say Q_1 and Q_2, in which any particular confocal (M), other than (λ), is met by the generator q, each describes, on this confocal (M), as Q varies on the curve (μ, ν), a curve, (M, N), which is the intersection of the surface (M) with another confocal (N). Thus, if the surface (M) be met by p in P_1 and P_2, and Q', Q'' be two positions of Q corresponding to the positions P', P'', of P, the cross-ratio $(Q', Q''; Q_1, Q_2)$ is equal to $(P', P''; P_1, P_2)$. As in the case of a plane, where we may measure the interval between two points with reference to an absolute conic, we may thence speak of the interval (Q', Q''), with reference to the surface (M), as being equal to the interval (P', P''). But, also as in the case of a plane, we may replace (M) by one of the focal conics, and so obtain a Euclidean measure of distance (cf. Vol. II, p. 184). In this sense we may speak of the two generators p, q as being divided into segments of equal length by the curves (μ', ν'). The measurement of distance, and the theory of movements, will be considered in a later volume.

The analytical proof of what has been stated is very easy. It is easy to verify that one of the eight points of intersection of the surfaces (λ), (μ), (ν), say, the point P, has coordinates

$$\left[\frac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}\right]^{\frac{1}{2}}, \quad \left[\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-c)(b-a)}\right]^{\frac{1}{2}},$$

$$\left[\frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}\right]^{\frac{1}{2}}, \qquad 1,$$

or say $[(a+\lambda)^{\frac{1}{2}}U, \ (b+\lambda)^{\frac{1}{2}}V, \ (c+\lambda)^{\frac{1}{2}}W, \ 1]$. Consider now the point in the plane $t=0$ whose coordinates are

$$[(a+\lambda)^{\frac{1}{2}}u, \quad (b+\lambda)^{\frac{1}{2}}v, \quad (c+\lambda)^{\frac{1}{2}}w, \quad 0],$$

where

$$u=[(a+\lambda')/(a-b)(a-c)]^{\frac{1}{2}}, \quad v=[(b+\lambda')/(b-c)(b-a)]^{\frac{1}{2}}, \quad \text{etc.},$$

say the point T. Here λ' is to be so taken that

$$(b-c)[(a+\mu)(a+\nu)(a+\lambda')]^{\frac{1}{2}}+(c-a)[(b+\mu)(b+\nu)(b+\lambda')]^{\frac{1}{2}} \\ +(a-b)[(c+\mu)(c+\nu)(c+\lambda')]^{\frac{1}{2}}=0,$$

a condition which, for given values of μ and ν, is satisfied by two values of λ'. Thence, in addition to the identity, $u^2+v^2+w^2=0$, we have the relation $Uu+Vv+Ww=0$. We also have the identity $U^2+V^2+W^2=1$. Now consider, for any value of h, the point of coordinates

$$(a+\lambda)^{\frac{1}{2}}[U+hu], \quad (b+\lambda)^{\frac{1}{2}}[V+hv], \quad (c+\lambda)^{\frac{1}{2}}[W+hw], \quad 1;$$

for all values of h this point lies on the surface $(a+\lambda)^{-1}x^2+\text{etc.}=t^2$; for fixed values of λ, μ, ν, and one of the two corresponding fixed values of λ', this point describes a line as h varies. The point is, therefore, the general point of one of the two generators above explained, at P, of the surface (λ). It meets the plane $t=0$ in the point T, which, as λ varies, μ, ν and therefore λ' remaining fixed, describes the conic $(a+\lambda')^{-1}x^2+\text{etc.}=0$, $t=0$. The line similarly meets the plane $x=0$ in the point of coordinates $(0, -w, v, iu)$, for we easily see that $vW-wV=iu$, $wU-uW=iv$, etc. As λ varies this point describes the conic $(b+\sigma)^{-1}y^2+(c+\sigma)^{-1}z^2=t^2$, $x=0$, where $\sigma=(bv^2+cw^2)u^{-2}$.

To find the curve (μ', ν') which passes through any point of the generator p, we have to solve for μ', ν' the three equations such as

$$(a+\lambda)^{\frac{1}{2}}(U+hu)=(a+\lambda)^{\frac{1}{2}}U',$$

wherein U' denotes $[(a+\mu')(a+\nu')/(a-b)(a-c)]^{\frac{1}{2}}$. Herein λ, μ, ν, h are given, and λ' is that, of the two possible values, which belongs to the generator p, through P, the other value of λ' belonging to the other generator of the surface (λ) through this point. The three equations are equivalent to two, as we see from $U'^2+V'^2+W'^2=1$, and lead to definite values of μ', ν'. These values are however independent of λ; thus as λ, and the surface (λ), vary, the curves (μ', ν') are all met by generators, q, meeting the curve (μ, ν), each lying on one of the surfaces (λ). That the ranges on the generators q, by the various curves (μ', ν'), are related, is clear from the fact that the value of h which belongs to a point P' on p is the same as that belonging to the point Q' of q which lies on the same curve (μ', ν'). Incidentally this argument shews that the lines q meet any one of the confocals, say (M), upon two curves (μ', ν'), say (M, N_1) and (M, N_2). In particular, to find the locus of the other intersection, beside the curve (μ, ν), of the lines q with the surface (μ), we shall be required to solve, for ν_1, the equations

$$(a+\lambda)^{\frac{1}{2}}(U+hu)=(a+\lambda)^{\frac{1}{2}}U_1,$$

where

$$U_1=[(a+\mu)(a+\nu_1)/(a-b)(a-c)]^{\frac{1}{2}}, \text{ etc.}$$

Beside $\Sigma U_1^2=1$, these equations require $\Sigma(a+\mu)^{-1}U_1^2=0$, which determines h; then ν_1 is to be found from

$$\Sigma U_1 u=0, \quad \text{or} \quad \Sigma(b-c)[(a+\mu)(a+\lambda')(a+\nu_1)]^{\frac{1}{2}}=0;$$

of the two roots of this equation for ν_1, one is ν itself. The remainder of the statement made is now clear.

The equation above, by which λ' is determined from μ and ν, is capable of interpretation by elliptic functions, and will be considered in a later section. Geometrically, it expresses that one of the intersections of the three confocals (μ), (ν), (λ') lies on one of the four planes drawn from the common centre of the confocals to touch all these surfaces, the equation of such a plane being $\Sigma x(b-c)^{\frac{1}{2}}=0$.

Thus, if λ_1', λ_2' denote the two values of λ' when μ and ν are given, we infer that each of these four planes contains two of the intersections of (μ), (ν), (λ_1'); and two of the intersections of (μ), (ν), (λ_2').

If the equation connecting μ, ν and λ' be rationalised it will be found to be symmetrical in regard to the two sets μ, ν, λ' and a, b, c. Another irrational form of the equation is therefore

$$(\mu - \nu) L' + (\nu - \lambda') M + (\lambda' - \mu) N = 0,$$

where

$$L' = (a + \lambda')^{\frac{1}{2}} (b + \lambda')^{\frac{1}{2}} (c + \lambda')^{\frac{1}{2}}, \text{ etc.}$$

Further, denoting the equation by $(\mu, \nu, \lambda') = 0$, it can be shewn that λ, μ, ν can be eliminated from the three equations $(\mu, \nu, \lambda') = 0$, $(\nu, \lambda, \mu') = 0$, $(\lambda, \mu, \nu') = 0$, the result being $(\lambda', \mu', \nu') = 0$.

Ex. 6. From the preceding results we have an immediate proof of the porism of Poncelet[1] in regard to conics (Vol. II, p. 55), so far as it relates to triangles. Here it may be stated in the form: If the join of a point, M, lying on the conic $(a + \mu)^{-1} x^2 + \text{etc.} = 0$, to a point, N, lying on the conic $(a + \nu)^{-1} x^2 + \text{etc.} = 0$, be a tangent of the conic $x^2 + y^2 + z^2 = 0$, the locus of the point of intersection, T, of the other tangents drawn to this latter conic from the points M, N is a third conic, $(a + \lambda')^{-1} x^2 + \text{etc.} = 0$. The four conics involved all have four common tangents. To prove this, consider, in the first instance, any three confocals of the system of confocal quadrics, (λ), (μ), (ν), intersecting in a point O. Let the generators at O, of the surface (λ), meet the plane $t = 0$ in the points T, T_1, lying on the conic $(a + \lambda)^{-1} x^2 + \text{etc.} = 0$, $t = 0$; the generators at O, of the surface (μ), meet the plane $t = 0$ in the points M, M_1, lying on the conic $(a + \mu)^{-1} x^2 + \text{etc.} = 0$; and the generators at O, of the surface (ν), meet the plane $t = 0$ in the points N, N_1, lying on the conic $(a + \nu)^{-1} x^2 + \text{etc.} = 0$. By what we have seen (p. 110), the six points T, T_1, M, M_1, N, N_1 lie in threes on four lines, in which the plane $t = 0$ is met by the four planes drawn from O to touch all the confocals; we suppose the notation so chosen that these lines are MNT_1, TMN_1, TNM_1, $T_1M_1N_1$; these lines all touch the conic $x^2 + y^2 + z^2 = 0$, $t = 0$. If the line joining O to the common centre, D, of the confocals, meet the plane $t = 0$ in H, the polar line of DH, in regard to any of the surfaces (λ), (μ), (ν), lies in the plane $t = 0$, and, therefore, contains the points of contact of the tangents drawn from H to the section of that confocal by $t = 0$; thus these points of contact lie on the tangent plane at O of the corresponding confocal. In other words, the six tangents of the conics at T, T_1, M, M_1, N, N_1 meet in H. Now, retaining the surfaces (μ), (ν) the same, allow the surface (λ) to vary, the point O varying on the curve (μ, ν). Then the points M, N will vary on the conics (μ), (ν), the line MN continuing to touch the focal conic, and, by what is proved above, the point T, the intersection with $t=0$ of one of the generators at O of the surface (λ), will describe a conic (λ'), where λ' is connected with μ, ν by the relation $(\mu, \nu, \lambda')=0$. The other value of λ' which satisfies this equation gives the conic (λ_1'), described by the point T_1, where the other generator at O of the surface (λ) meets $t=0$.

---

[1] See frontispiece of this Volume.

Conversely, if, as above, M, N be two points of the plane $t=0$, lying respectively on the conics (μ), (ν), the line MN touching the focal conic, and the other tangents to this conic from M and N intersecting in T, we can construct the figure above used, and the locus of T, as follows: By hypothesis the conics (μ), (ν) on which M and N lie, and the focal conic touched by MN, have four common tangents. We can therefore refer them to a common self-polar triad, and suppose their equations to be as we have taken them above, and then, further, taking an arbitrary point D, not in the plane $t=0$, suppose these conics to arise from a confocal system of quadrics, as above, with D as centre. Then, through each of the lines TM, MN, NT there pass two planes which touch all the surfaces of this confocal system, a pair of such planes having equations of the form

$$lx+my+nz\pm pt=0, \quad \text{where} \quad p^2=al^2+bm^2+cn^2.$$

The three pairs of planes give rise to eight points through each of which passes a plane of each pair, these eight points lying in couples on four lines through D. Taking one of these eight points for the point O, such a figure as that above described can be constructed; but, the conics (μ), (ν) being given, and, therefore, the point, H, of intersection of the tangents of these conics at M and N, it is necessary to take for O one of the two points such that DO passes through H. These two points O give rise to the same surface (λ), and the same conic (λ'), the point T being the intersection of two generators of the surface (λ).

When M, N vary, respectively on the conics (μ), (ν), the conic (λ) varies, and T remains on (λ'), while H describes a conic, the projection from D of the curve (μ, ν). The conic (λ'), and the conic (λ), have, at T, tangents which are conjugate in regard to the focal conic of the plane $t=0$. We may similarly consider the conics, (μ'), through M, and (ν'), through N, whose tangents at these points are conjugate, respectively, to those of (μ) and (ν). Then, when M and N vary on (μ'), (ν'), respectively, T also varies on (λ'). Calling for brevity the line conjugate, in regard to the focal conic, to a tangent of one of the conics, the *normal*, we may thus say that when a triad of points, one on each of the conics, is such that their joins touch the focal conic, not only the tangents of two of these conics and the normal of the other, meet in a point, but also the normals of these conics meet in a point, and these points of concurrence describe conics.

Ex. 7. If the line joining two points P_0, P, of respective coordinates $(x_0, y_0, z_0, 1)$, $(x, y, z, 1)$, meet the confocal surface $(a+\theta)^{-1}x^2 + \text{etc.} = t^2$ in the points M, N, and the cross-ratio $(P_0, P; M, N)$ be written e^{2iw}, prove that

$$\cosh w = \frac{(\theta+a)(\theta+b)(\theta+c) - \Sigma x x_0 (\theta+b)(\theta+c)}{[\Pi(\theta-\lambda)(\theta-\lambda_0)]^{\frac{1}{2}}},$$

where λ_0, μ_0, ν_0 are the parameters of the confocals through P_0, and λ, μ, ν the parameters of the confocals through P, so that we may replace x_0 by $[(a+\lambda_0)(a+\mu_0)(a+\nu_0)/(a-b)(a-c)]^{\frac{1}{2}}$, and x by the same function of λ, μ, ν. This expression is unaffected by the interchange of λ and λ_0, whereby P_0 is replaced by a point P_0' on the same curve (μ_0, ν_0), and P is replaced by a point P' on the same curve (μ, ν), provided the signs of xx_0, yy_0, zz_0 are unaltered.

The value of $\cosh^2 w$ is unity, not only for $\theta = -a, -b, -c, \infty$, but also for two other values of θ; these give the two confocals touched by the line P_0P. Thus the line $P_0'P'$ also touches these two confocals. And the cross-ratio $(P_0, P; M, N)$ is equal to that obtained, for the same surface θ, for the line $P_0'P'$.

By supposing θ to become infinitely great, and regarding the focal conic of the plane $t=0$ as the absolute conic, it can hence be inferred, as will be seen later, that the *distances* P_0P and $P_0'P'$, which arise as the limits of $\theta^{\frac{1}{2}}w$, are equal. This result is known as Ivory's theorem, and is of importance in the theory of the Newtonian attraction of a material ellipsoid. The geometrical theory is capable of great development. See a paper by A. L. Dixon, *Messenger of Mathematics*, XXXII, 1903, p. 177, and a note by the author, *Proc. Camb. Phil. Soc.* XX, 1920, p. 129.

Ex. 8. The planes touching all the confocals of the system are those represented by the equation $Px + Qy + Rz = t$, where

$$P = (a+\lambda)^{\frac{1}{2}}(a-b)^{-\frac{1}{2}}(a-c)^{-\frac{1}{2}}, \text{ etc.}$$

These planes, depending on a single parameter, determine a developable. They meet an arbitrary plane in the tangent lines of a curve. Upon each plane, of the developable, there is a line, of which the points are given by $x = t(a+\phi)P$, etc., for varying values of ϕ; the intersections of these lines with the arbitrary plane are the points of the curve spoken of. The planes of the developable meet, each in three coincident points, a curve of which the general point is given by $x = t(a+\lambda)^{\frac{3}{2}}(a-b)^{-\frac{1}{2}}(a-c)^{-\frac{1}{2}}$, etc. The lines in question are called the generating lines of the developable; each of them touches all the confocals, other than one of these, of which it is a generator; through this line there is one plane of the developable touching all the confocals at points of this line. More generally, the line of intersection of any two planes of the developable is a generator of one of the confocal surfaces; for the surface drawn to touch an arbitrary plane through this line of intersection will be touched by three planes through the line; it will thus contain the line entirely. If we call the line of intersection of two planes of the developable an *axis* of the developable, (it corresponds dually to a chord of a curve), we thus see that the aggregate of all the generators of the confocals is the same as the aggregate of the axes and generating lines of the developable.

The line given by $x = t(a + \phi)P$, etc., is a generator of the surface $(a + \lambda)^{-1}x^2 + \text{etc.} = t^2$, touching the surface $(a + \theta)^{-1}x^2 + \text{etc.} = t^2$ at the point given by $\phi = \theta$; this point is one of the eight intersections of the surfaces (λ), (θ) with the quadric which is the polar reciprocal of the surface (λ), on which the line is a generator, with respect to the surface (θ). (Cf. pp. 80, 92.)

Dual method of developing the theory of confocal surfaces. If any two quadric surfaces be given, the curve in which they intersect is such as to have four points upon an arbitrary plane, these being the intersections of the conics in which the plane severally meets the quadrics. Through this curve, and an arbitrary point, can be drawn another quadric; for if $S = 0$, $S' = 0$ be the equations of the given quadrics, the equation $S + \lambda S' = 0$ represents a quadric passing through the common points of the two given quadrics; and the value of λ can be taken so that this passes through the arbitrary point. No other quadric can be drawn through this curve and this point, since, of such a quadric, five points are determined lying on an arbitrary plane drawn through this point. Thus the equation $S + \lambda S' = 0$ is that of the general quadric through the curve $S = 0$, $S' = 0$. We have seen that the tangential equation of the general quadric of the confocal system is of the form $\Sigma + \lambda\Sigma' = 0$; we may, then, clearly develop, in dual form, the theory of a confocal system, as that of the quadrics $S + \lambda S' = 0$. Without entering into a detailed account, it is interesting to consider what corresponds to some of the leading theorems already obtained for confocals. We suppose that the surfaces $S = 0$, $S' = 0$ have a common self-polar tetrad, so that their equations can be supposed to be, respectively,

$$Ax^2 + By^2 + Cz^2 + Dt^2 = 0, \quad A'x^2 + \text{etc.} = 0;$$

their curve of intersection will then not contain a line, or a conic, as part, as is possible for two quadrics in general. Through the curve of intersection there pass four quadric cones; in fact $S + \lambda S' = 0$ represents a cone for $\lambda = -A/A'$, etc. These cones, each enveloped by an aggregate of planes passing through one of the four principal points, of which two planes can be drawn to contain an arbitrary line through this point, correspond dually to the four focal conics of the confocal system, each the locus of points in a principal plane, of which two points lie on an arbitrary line of the plane. The coordinates can be chosen so that the equation of the system of quadrics now considered is

$$(a+\mu)x^2+(b+\mu)y^2+(c+\mu)z^2=t^2,$$

where $\mu=\infty$ gives rise to one of the cones, $x^2+y^2+z^2=0$. The other three cones then arise for $\mu=-a$, $\mu=-b$, $\mu=-c$. It is at once clear that the polar planes of an arbitrary point, in regard to the system of quadrics now being considered, all pass through a line. These polar planes have an equation $P+\mu P_0=0$, where P, P_0 are the polar planes of the point in regard to the fixed quadrics $ax^2+\text{etc.}=t^2$, and $x^2+y^2+z^2=0$. The line is the intersection of four planes, arising for $\mu=\infty, -a, -b, -c$, each containing one of the principal points, and the axial pencil formed by these planes is related to that similarly arising for any other such line. These are the lines corresponding to what have been called the axes of the confocal system; as in that case they constitute a tetrahedral complex; every line in a principal plane, or through a principal point, belongs to this complex. Considering, next, an arbitrary line, this is intersected, by the quadrics considered, in pairs of points belonging to an involution, the two points of intersection with any quadric being determined by one of them. There are thus two quadrics of the system touching the line. The proof and theorem fail if the line contains two points of the common curve of intersection of the quadrics, in which case the line is a generator of one of the quadrics; when the line is a tangent of this curve, it is a generator of one of the quadrics, and touches all the other quadrics at this point. In the general case, the tangent planes of the quadrics, at the points where the line meets them, form an aggregate of which three pass through an arbitrary point; also the polar lines of the given line in regard to the quadrics are generators of a quadric, which passes through the four principal points. When the line belongs to the tetrahedral complex, the tangent planes of the quadrics, at the point where the line meets them, all pass through a point. In particular, when the line passes through one of the principal points, these tangent planes touch a quadric cone, whose vertex is in the corresponding plane. Lastly, considering an arbitrary plane, there are three quadrics of the system which touch this plane. The quadrics of the system meet this plane in conics having four points in common; the generators, in this plane, of the three quadrics which touch this plane, are the three line-pairs which contain these four points. If we consider the aggregate of planes touching two particular quadrics of the system, and, in any such plane, the generators of the third quadric of the system other than these two, which touches this plane, then these generators are projected from one of the principal points by tangent planes of two particular quadric cones.

Ex. 1. The polar lines of the line (l, m, n, l', m', n'), in regard to the quadrics of equation $(a+\lambda)x^2 + \text{etc.} = t^2$, lie on the quadric expressed by

$$l'\alpha yz + m'\beta zx + n'\gamma xy - lxt - myt - nzt = 0,$$

where $\alpha = b - c$, $\beta = c - a$, $\gamma = a - b$. When the line belongs to the tetrahedral complex expressed by $all' + bmm' + cnn' = 0$, this quadric is a cone of vertex $(l', m', n', -\sigma)$, where σ is $\alpha m'n'/l, = \beta n'l'/m$, etc. When the line passes through one of the principal points the quadric breaks up into two planes; for instance, when $l' = m' = n' = 0$, it is $t(lx + my + nz) = 0$.

Ex. 2. The tangent planes of the quadrics $(a+\mu)x^2 + \text{etc.} = t^2$, at the points where these are met by the line $x/l = y/m = z/n$, touch the cone

$$(xP + yQ + zR)(xp + yq + zr) = t^2,$$

where $(l^2 + m^2 + n^2)P = l(\gamma m^2 - \beta n^2)$, $(l^2 + m^2 + n^2)p = l$, etc., so that $Pp + Qq + Rr = 0$.

CHAPTER III

CUBIC CURVES IN SPACE. THE INTERSECTION OF TWO OR MORE QUADRICS

The curve of intersection of two quadrics. A conic is a curve of which the points are in unique correspondence with the points of a line; it is defined by the intersection of associated rays of two related flat pencils of lines in a plane. In three dimensions there exists a curve, in many ways simpler than a conic, of which the points are also in unique correspondence with the points of a line. Three points of this curve lie on an arbitrary plane, so that the curve is called a cubic. The curve may be defined, analogously to a conic in a plane, by the intersection of associated planes of three related axial pencils of planes. The curve may also be defined as the partial intersection of two quadric surfaces. We have, however, in the theory of confocal quadrics, been led to consider a curve which is the intersection of two quadrics, and is not a cubic. It will therefore add to clearness if, before considering the cubic curve in detail, we enumerate all the possibilities in regard to the intersection of two quadrics; the proof that the list is exhaustive will be given in a later Volume. Since an arbitrary plane meets each of the quadrics in a conic, and two conics in the same plane intersect in four points, the complete intersection of two quadrics must meet an arbitrary plane in four points. This intersection may however consist of several irreducible pieces (curves), of which two may coincide.

(1) The curve of intersection of two quadrics may consist of one piece, that is of one curve, which will then be a quartic, as meeting an arbitrary plane in four points. In the most general case the points of this curve are not in unique correspondence with the points of a line. If we take two points of the curve, and the line, or *chord* of the curve, which joins them, and consider the planes through this chord, (which are in correspondence with the points of a line), each of these planes gives rise to two points of the curve. The coordinates of the points of the curve are in fact rational functions of two parameters, ξ, η, themselves connected by a relation $\eta^2 = f(\xi)$, wherein $f(\xi)$ is a polynomial in ξ of the fourth order; conversely, both ξ and η are rational functions of the ratios of the coordinates of a point of the quartic curve.

There is however a particular case of this in which the curve of intersection still consists of a single quartic curve; namely when the curve crosses itself, or has a double point. Then any plane drawn through this point and another fixed point of the curve has but one further intersection with the curve; the points of the curve are in this case in unique correspondence with the points of a line. The cases in which the curve crosses itself twice occur below; the curve does not then consist of one piece.

(2) The curve may consist of a line and a curve consisting of one piece. This last is then the cubic curve above referred to; it meets the line in two points.

(3) The curve may consist of two conics, which then have two points of intersection. We have met with this possibility in the last chapter, in the case of two spheres, one of the common conics being then the absolute conic. The two conics may touch one another.

(4) In particular one of the conics may consist of two straight lines, each of which meets the other conic. When the point of intersection of these lines lies on the other conic, the plane of the lines contains the tangent line of this conic at this point.

(5) Or both conics may consist of two lines. In this case the two quadrics have in common two generators of each of the two systems of either quadric.

(6) Two of the four lines may coincide, in which case the intersection consists of a line at every point of which the two quadrics have the same tangent plane, together with two lines intersecting this.

(7) More particularly still, the four lines may consist of two, each occurring twice; the lines will intersect, and the tangent planes of the quadrics will be the same at every point of both.

(8) Finally, the intersection may consist of a single conic at every point of which the quadrics have the same tangent plane. This conic thus counts doubly in the intersection.

In the cases (6), (7), (8) we say that the common curve is multiple, or contains a multiple portion.

Ex. 1. The quadrics whose equations are $x^2+y^2-z^2-t^2=0$, $ax^2+by^2+cz^2+dt^2=0$, intersect in a curve of which the points are expressed by $x=\theta+\phi$, $y=1-\theta\phi$, $z=1+\theta\phi$, $t=\theta-\phi$, provided θ, ϕ are connected by the equation

$$A(1+\theta^2\phi^2)+B(\theta^2+\phi^2)+2C\theta\phi=0,$$

where $A=b+c$, $B=a+d$, $C=a-b+c-d$.

The quadrics whose equations are

$$2xy+z^2-t^2=0,\quad 2xy+y^2+(1-m^2)z^2-(1-n^2)t^2=0$$

intersect in a curve expressed by

$$x=PQ,\quad y=2m^2n^2,\quad z=mn^2(\theta+\theta^{-1}),\quad t=m^2n(\theta-\theta^{-1}),$$

where

$$2P = (\theta - \theta^{-1})\,m + (\theta + \theta^{-1})\,n, \quad 2Q = (\theta - \theta^{-1})\,m - (\theta + \theta^{-1})\,n,$$

θ being variable. This curve crosses itself at (0, 1, 0, 0), either of the two planes, $mz + nt - \theta y = 0$, $mz - nt - \theta^{-1}y = 0$, meeting the curve in only one further point.

The quadrics whose equations are

$$2xy + z^2 + t^2 = 0, \quad 2xy + 2yz + z^2 + (1 - n^2)\,t^2 = 0$$

intersect in a curve expressed by

$$x = -(\theta^2 + n^4), \quad y = \tfrac{1}{2}\theta^4, \quad z = n^2\theta^2, \quad t = \theta^3,$$

where θ varies. This curve has a cusp at (0, 1, 0, 0), any plane $2y = \theta t$ meeting the curve in only one further point.

Ex. 2. The quadrics whose equations are

$$xy + zt = 0, \quad 2xy + 2czt + y^2 + t^2 = 0$$

intersect in a curve expressed by

$$x = -(1 + \theta^2), \quad y = 2(1 - c)\,\theta^2, \quad z = \theta(1 + \theta^2), \quad t = 2(1 - c)\,\theta,$$

as well as in the line $y = 0$, $t = 0$. The curve meets an arbitrary plane in three points, and meets the line in (1, 0, 0, 0) and (0, 0, 1, 0).

The quadrics whose equations are

$$xy + zt = 0, \quad 2xy + 2zt + 2yt + z^2 = 0$$

intersect in a curve expressed by $x = 1$, $y = -2\theta^3$, $z = 2\theta^2$, $t = \theta$, as well as in the line $y = 0$, $z = 0$. This line meets the curve in two coincident points, any plane $y + \theta z = 0$ meeting the curve in only one further point.

Ex. 3. The quadrics $x^2 + y^2 + z^2 + t^2 = 0$, $x^2 + y^2 + cz^2 + dt^2 = 0$ meet in two conics having two points in common. The quadrics $2xy + z^2 + t^2 = 0$, $2xy + y^2 + z^2 + dt^2 = 0$ meet in two conics touching one another on the line $y = 0$, $t = 0$.

Ex. 4. The quadrics $2xy + z^2 - t^2 = 0$, $2xy + y^2 + c(z^2 - t^2) = 0$ intersect in two lines $y = 0$, $z = \pm t$, and in a conic, lying in the plane $y + 2x(1 - c) = 0$, meeting both these lines. The quadrics $2xy + z^2 - t^2 = 0$, $2xy + 2yz + z^2 - t^2 = 0$ intersect in two lines $y = 0$, $z = \pm t$, and in a conic, lying in the plane $z = 0$, which contains the intersection of the two lines and is touched, at this point, by the plane of these.

Ex. 5. The quadrics $x^2 + y^2 + z^2 + t^2 = 0$, $x^2 + y^2 + c(z^2 + t^2) = 0$ have four lines in common.

Ex. 6. The quadrics $xy + zt = 0$, $xy + zt + yt = 0$ have in common the lines $x = 0 = t$, $y = 0 = z$, and $y = 0 = t$, of which the last counts doubly, the tangent planes of the quadrics at any point of this being the same.

Ex. 7. The quadrics $xy+zt=0$, $2xy+2zt+y^2=0$ have in common the lines $y=0=t$ and $y=0=z$, each doubly, the tangent planes of the quadrics at any point of either being the same.

Ex. 8. The quadrics $x^2+y^2+z^2+t^2=0$, $x^2+y^2+z^2+dt^2=0$ touch one another at every point of the conic $t=0$, $x^2+y^2+z^2=0$, and have no other common point.

Ex. 9. We have seen that the theory dual to that of the theory of confocal surfaces is that of the quadrics passing through the common curve of two quadrics of sufficient generality. In particular, the dual of a focal conic, of which every tangent line lies in (two of the) planes touching all the confocal surfaces, is a quadric cone of which every generating line passes through a point of the common curve of the two quadrics. It is thus of interest to determine, in each of the thirteen cases just enumerated, the quadric cones containing the complete curve of intersection of the two quadrics. Moreover, the tangent planes touching the confocal surfaces are the common tangent planes of any two of the focal conics. It is thus of interest to consider the aggregate of tangent planes common to two conics which are not in so general relation to one another as are two of the focal conics; in particular, of two conics in different planes which both touch the line of intersection of their planes. It is this problem, essentially, which we now proceed to consider, in its dual form.

The cubic curve in three dimensions. Consider first two quadric cones, of vertices O and Q, so related that the line OQ is a generator of both, the tangent planes of the cones along this generator being different. These cones meet an arbitrary plane in two conics, among the four intersections of which will be the point in which the line OQ meets the plane. There are then three points of the plane, not on this line, which belong to both cones. The curve of points common to the two cones, other than their common generator, thus meets an arbitrary plane in three points, and is, for this reason, called a *cubic* curve. Any plane through OQ meets each cone in a further generator; the point of intersection of these two lines is a point of the curve; the other two points of the curve lying in this plane are the points O and Q. If R be any point of the curve, any plane drawn through R meets the curve in two other points; if then we consider the aggregate of lines joining R to all other points of the curve, there are two of these in any plane through R. We infer, therefore, that these lines are the generators of a quadric cone. This may be deduced also by remarking that the axial pencils of planes which join two fixed generators of a quadric cone to a variable generator are related pencils, and conversely. For let P be a variable point of the curve; as OP is a variable generator of the quadric cone of vertex O, of which OQ and OR

are two fixed generators, the axial pencils $OQ(P)$ and $OR(P)$ are related; also, as QP is a variable generator of the quadric cone of vertex Q, of which QO and QR are two fixed generators, the axial pencils $QO(P)$ and $QR(P)$ are related. As then the axial pencils $RO(P)$ and $RQ(P)$ are related, it follows that RP is the variable generator of a quadric cone of vertex R, of which RO and RQ are two fixed generators. The cubic curve may then be defined as the locus of the common points of the two quadric cones with vertices at O and R, other than their common generator OR. It follows too that the axial pencil of planes, joining the chord OQ to the points P of the curve, is related to the axial pencil joining the chord QR to the same points P, and, therefore, to the axial pencil joining any other chord, RS, to the same points P of the curve. In view of what has been said previously in regard to related ranges, this is sufficiently clear also from the fact that an arbitrary plane through OQ, by determining a definite further point, P, of the curve, determines a definite plane RSP, and conversely. The curve is thus like a conic in being the seat of related ranges, of points lying upon it, any four points, P_1, P_2, P_3, P_4, of the curve, being related to four other points, P_1', P_2', P_3', P_4', if the planes joining the first four points, to an arbitrary chord of the curve, be related to the planes joining the second four points to another, also arbitrary, chord. In other words, the curve, like a conic, may be regarded as having its points in (1, 1) correspondence with the points of a line. The curve is, in fact, as appears from what has been said, the locus of the intersection of three corresponding planes; one from each of three related axial pencils of planes, of which the axes are any three chords of the curve.

Consider any two points D, A, of the curve, and the quadric cones which project the curve from these two points, respectively, say (D) and (A). Let the tangent plane of the cone (D) along DA meet the cone (A) also in the line AB, and the tangent plane of the cone (A) along AD meet the cone (D) also in the line DC. Let B be determined as the intersection of the tangent plane of (D), along DC, with AB; and C be determined as the intersection of the tangent plane of (A), along AB, with DC. Then, with A, B, C, D as fundamental points for coordinates, the cones (D), (A) may be supposed to have the respective equations $xz = y^2$ and $yt = z^2$. The former is generated by the intersection of planes of equations $x = y\theta$, $y = z\theta$; the latter by the intersection of planes of equations $y = z\theta$, $z = t\theta$, where θ is variable. The coordinates of any point of the curve are then of the form $(\theta^3, \theta^2, \theta, 1)$. The three points in which any plane, of equation $lx + my + nz + pt = 0$, meets the curve, are then given by the roots of the cubic equation $l\theta^3 + m\theta^2 + n\theta + p = 0$. In particular, the plane $x = 0$, or BCD, meets the curve in three coincident points at the point D, for which $\theta = 0$; and the plane $t = 0$, or BCA, meets the curve in three coincident points at the point A, for which $\theta^{-1} = 0$. The line DC meets the curve in two coincident points at the point D, any plane, $x = y\theta$, through this line, meeting the curve in only one other point; so the line AB meets the curve in two coincident points at the point A, any plane, $z = t\theta$, through this line, meeting the curve in only one other point. It appears thus that there is through any point of the curve a line, called the *tangent*, meeting the curve in two coincident points at this point; and a plane through the tangent, called the *osculating plane*, which meets the curve in three coincident points at this point. The two arbitrary points A, D of the curve being taken, the points B and C are the intersections of the tangents at these points, respectively, with the line in which the osculating planes of the curve, at these points, intersect.

More generally, given any three lines, the equations of three corresponding planes of axial pencils, which are related to one another, having these lines as axes, can be supposed to be of the forms $P - \theta Q = 0$, $P_1 - \theta Q_1 = 0$, $P_2 - \theta Q_2 = 0$, where each of P, Q, etc. is a linear function of coordinates x, y, z, t, with any fundamental points of reference. Supposing that the axes, and the manner of relation of the axial pencils, are so general that these three planes do not meet in a line, the point of intersection of these planes has coordinates, found by solving the three linear equations, each proportional to an expression of the form $a\theta^3 + b\theta^2 + c\theta + d$. We may then suppose x, y, z, t respectively equal to $f_1(\theta), f_2(\theta), f_3(\theta), f_4(\theta)$, where each of these functions is such an expression. There can be no identical relation, $\Sigma A_r f_r(\theta) = 0$, where the four coefficients A_r are independent of θ, or the locus of (x, y, z, t) would lie in a plane, which we suppose not to be the case. Wherefore, by reverting the equations, we can express each of θ^3, θ^2, θ, 1 as a linear function of x, y, z, t, say $X = \theta^3$, $Y = \theta^2$, $Z = \theta$, $T = 1$. We then have the expression of the curve found above, with X, Y, Z, T as coordinates.

Taking any two chords, l, m, of the cubic, a line, p, can be drawn, from any point, P, of the curve to meet both l and m. This line lies in both the planes $l(P)$, $m(P)$. When P varies on the curve, these planes, by what we have seen, describe related axial pencils, so that the line p describes a quadric surface containing both l and m. Thus a quadric surface can be drawn containing the curve and any two of its chords. The generators of this surface which are of the same system as l and m all meet the generator, p, through any particular point of the curve; the plane containing p and any one of these generators, say n, meets the curve in three points; as the curve lies on the quadric surface, and the plane meets this only in the lines p, n, these three points must lie on these lines; by the construction,

p meets the curve in only one point, P. Wherefore the generators, n, of the same system as l and m, which meet p, are all chords of the curve. We have seen that we may consider related ranges of points upon the curve; we may thus consider involutions of pairs of points upon the curve. In fact the chords, n, meeting the line p, intersect the curve in pairs of points of such an involution, that, namely, determined by the two pairs of points of the curve which are on the chords l and m, respectively. This is clear by considering the cone which projects the curve from P; the two generators of this cone which join P to the two points where one of the chords n meets the curve, lie on a plane passing through the fixed line p; as for conics, this is the necessary and sufficient condition for an involution of pairs of generators of the cone. We thus see that, if we consider any involution of pairs of points of the cubic curve, the chords of the curve, each containing a pair of the involution, are the generators, of one system, of a quadric surface containing the curve. Also, if an arbitrary line, p, be drawn through a point, P, of the curve, not meeting the curve again, any variable plane through p meets the curve in two points, each determining the other, which are a pair of an involution upon the curve. Of the chords joining the pairs of this involution there is one through any point, say O, of the line p. Thus, through any point O of space can be drawn a chord of the curve; for take any point, P, of the curve, and consider, in what has just been said, the line p as being the line OP. Two such chords are not possible through O, since the plane containing these would meet the curve in four points. It appears thus that through any line, p, which meets the curve in one point, and the curve, a quadric surface can be drawn; and further that if a series of such lines p be taken, all containing a point O, these quadric surfaces have all a common generator. More generally it appears that a quadric surface can be drawn to contain the curve and two arbitrary points not lying on the curve; for, taking the chords of the curve passing, respectively, through these points, this is the surface containing the curve, and these two chords.

This last result is in accordance with the fact, which we can reach without difficulty, that the equation of the general quadric surface containing the curve is of the form $\lambda U + \mu V + \nu W = 0$, where U, V, W are definite quadric functions, and λ, μ, ν are arbitrary constants. This is the equation of a general quadric conditioned to pass through seven given points of independent position; for the unconditioned equation of a quadric contains ten terms. In fact, a cubic curve which does not lie wholly upon a quadric surface meets this in six points, as we see by substituting in the equation of the quadric respectively θ^3, θ^2, θ, 1 for the coordinates. Wherefore, a quadric surface prescribed to pass through seven general points of a cubic curve contains the curve entirely.

We may remark at this point: (*a*), that a cubic curve can be made to pass through six arbitrary points of space, say A, B, C, D, O, Q; for the five points A, B, C, D, O lie on a definite quadric cone of vertex Q, and the five points A, B, C, D, Q lie on a definite quadric cone of vertex O. The curve is the intersection, other than the common generator OQ, of these two cones; (*b*), that through five general points, say A, B, C, D, E, of a quadric surface, two cubic curves can be drawn which lie entirely upon the surface. For, if l, m be the generators of the surface at E, a definite quadric cone can be constructed, with vertex at E, to contain one of these generators, say l, and also the four points A, B, C, D; the points common to this cone and the quadric, other than the points of the line l, constitute a cubic curve, passing through E. For, let P be any point of the curve of intersection, and p the line drawn from P to meet l and to meet any other fixed generator, k, of the quadric, of the same system as l, so that p is also a generator of the quadric; the quadric is the locus of the points of the lines p, as P varies, and p is the intersection of corresponding planes of two related axial pencils whose axes are l and k, two of these planes being $l(P)$ and $k(P)$. The quadric cone drawn with vertex at E is however the locus of the lines of intersection of corresponding planes of two related axial pencils whose axes are l and (for example) the line EA, two of these planes being $l(P)$ and EAP. By taking k to pass through A, the point P, which is the intersection of the corresponding planes $l(P)$, $k(P)$, EAP, appears as lying on another quadric cone, of vertex A, which is the locus of the intersection of the corresponding planes, $k(P)$, AEP, of two related axial pencils. Another cubic curve lying on the given quadric surface, and passing through the points E, A, B, C, D, is that lying on the cone of vertex E described to contain A, B, C, D and the second generator, m, at the point, E. As we see by considering the intersection of the cone of vertex E, and of the quadric, with an arbitrary plane drawn through l, the former cubic curve has l for a chord; the latter has m for a chord. We see, further: (*c*), that every cubic curve on a quadric surface has all the generators of one system as chords, meeting every generator of the other system each in one point. For the plane of two intersecting generators of the quadric meets the quadric only on these lines, and meets, therefore, a cubic curve lying on the surface only on these lines. The curve cannot meet either of these lines in three points, since else an arbitrary plane through this line would have no further intersection with the curve, which, therefore, would degenerate into the line itself (taken three times). If the curve meets one of these lines in two points it must meet every generator of the surface, which meets this, in one point; if it meets one of these lines in one point, it must meet every generator of the surface, which meets this, in two points. Thus also, a cubic curve, in (*b*), through E, A, B, C, D, must have either l or m for a chord; there is then no other cubic through these points than those two found in (*b*).

We have defined a cubic curve as the locus of the points common to two quadric cones which have a common generator, and have shewn that quadric surfaces can be drawn through the curve; two of these with a chord of the curve as common generator intersect then further in the curve. Conversely, if we take any two quadric surfaces with a common generator, their remaining intersection is a curve meeting an arbitrary plane in three points. That this is not a more general cubic curve than that above found, may be shewn in detail by proving, as in the above argument under (*b*), that it can be regarded as the intersection of two quadric cones with a common generator.

The construct which is dual to a cubic curve will be that of the aggregate of planes touching two quadric surfaces which have a common generator, other than those passing through this line; or, as follows by the dual of the reasoning above given, of the aggregate of planes touching two conics in different planes which both touch the line of intersection of these planes; or, also, of the aggregate of planes meeting three lines in the corresponding points of three related ranges upon these lines, the ranges being such that the three corresponding points do not always lie in line. There is, evidently, an ∞^1 of such planes, that is, the planes are each determined by a value of a single varying parameter; they are therefore said to constitute a *developable*. And, of such planes, three pass through an arbitrary point, as in the dual case. The aggregate is thus called a cubic developable. Consider this developable as defined by the common tangent planes of two conics in different planes, touching the line of intersection of their planes respectively in A and B. Let P, Q be points, of these two conics respectively, which are such that the tangent lines of these conics at P and Q intersect one another, in the point C, of the line AB. From any point, O, of the line PQ, the conics are projected into two quadric cones; these touch one another along the line OPQ. These cones have four common tangent planes, of which OAB is one, and $OPQC$ is another; this last is to be regarded as two coincident planes; and there is another plane tangent to both cones passing through O; let this plane touch the conics, respectively, in P' and Q', and meet the line AB in C', so that $C'P'$ and $C'Q'$ touch the conics respectively at P' and Q'; let $C'P'$, $C'Q'$ meet CP, CQ, respectively, in M and N, so that MN passes through O, as being on the plane $C'P'Q'$ and intersecting PQ. Then, as O varies on the line PQ, the point C' will vary on AB, and M, N will vary on CP, CQ, respectively. But, as $C'P'M$ is a tangent of one conic, of which AB and

CP are fixed tangents, the ranges (C'), (M) will be related; likewise the ranges (C'), (N) are related. Thus MN envelopes a conic in the plane CPQ, of which CP, CQ are tangents, as also is PQ, since when C' is at C, the points P' and Q' are respectively at P and Q. Let T be the point of contact of this conic with PQ. Then, when O is at T, this third plane of the developable drawn from O, which in general is $OC'P'Q'$, coincides with the former one $OCPQ$. Thus, the developable is such that in any plane of it, (CPQ), there is a line, (PQ), such that, of the three planes of the developable drawn from any point of the line, two coincide with this plane; and on this line there is a point, (T), with the property that the three planes of the developable drawn from it all coincide with this plane. The line is called a *generator* of the developable; it is the dual of the tangent line of the cubic curve above explained. The point T is similarly the dual of the osculating plane of the cubic curve.

It can be proved, however, that, like the conic, the cubic curve is a self-dual construct, the osculating planes forming a cubic developable; and, similarly, the cubic developable is self-dual, the points T above obtained describing a cubic curve, the osculating plane of which, at the point T, is the plane of the developable in which T has been constructed. Thus, not only does the osculating plane of the cubic curve, at a point P, have all its three intersections with the curve coincident at P, but the three osculating planes which can be drawn of the cubic curve through a general point, all coincide, when this point is at P, with the osculating plane at P. The tangent lines of the cubic curve are the generators of the cubic developable which is the aggregate of its osculating planes, and, dually, the tangent line of the locus of the points T is the generating line PQ. The locus of the points T, regarded as arising from the developable, is often called the *cuspidal edge* of the developable.

The proof that the cubic curve is a self-dual construct in the sense explained is very easy with the symbols. The curve being regarded as the locus of the point $(\theta^3, \theta^2, \theta, 1)$, the plane whose equation is $x-py+qz-rt=0$, where $p=\alpha+\beta+\gamma$, $q=\beta\gamma+\gamma\alpha+\alpha\beta$, $r=\alpha\beta\gamma$, evidently contains the three points of the curve for which $\theta=\alpha$, $\theta=\beta$, $\theta=\gamma$. Thus the plane whose equation is

$$x-3\theta y+3\theta^2 z-\theta^3 t=0$$

is the osculating plane at the point θ. Evidently three such planes pass through an arbitrary point. Further, these planes meet the plane $t=0$, the osculating plane at $\theta^{-1}=0$, in the lines

$$x-3\theta y+3\theta^2 z=0,$$

which are tangent lines of the conic $t=0$, $3y^2=4xz$, the point of contact of one such line with this conic being $(3\theta^2, 2\theta, 1, 0)$; and these planes meet the plane $x=0$, the osculating plane at $\theta=0$, in the lines $3y - 3\theta z + \theta^2 t = 0$, which are tangent lines of the conic $x = 0$, $3z^2 = 4yt$, the point of contact of one such line with this conic being $(0, \theta^2, 2\theta, 3)$. The generator of the developable formed by these planes, being the line joining the points of contact of the variable plane with these conics, is the line whose general point is expressed by $(3\theta^3, 2\theta^2 + \lambda\theta^2, \theta + 2\lambda\theta, 3\lambda)$; this is immediately proved to be the tangent of the original curve at the point θ. The general point of a tangent line of the cubic, of which, as has been remarked, the coordinates are $3\theta^3, (\lambda + 2)\theta^2, (2\lambda + 1)\theta, 3\lambda$, is easily shewn to describe a surface of which the equation is $V^2 = 4UW$, where $U = xz - y^2$, $V = xt - yz$, $W = yt - z^2$. Thus four tangent lines of the cubic curve meet an arbitrary line; or, through an arbitrary line can be drawn four planes each of which contains a tangent line of the cubic curve. Dually, the generators of the cubic developable meet an arbitrary plane in the points of a curve of which four points lie on an arbitrary line of this plane. This plane meets the cuspidal edge, which is a cubic curve, in three points. Each of these three points is a *cusp* of this plane quartic curve; that is, it is a double point of this curve, or a point such that every line, in the plane of the curve, drawn through this point, meets the curve in two coincident points at this point (and, therefore, meets the quartic curve in two other points); and it is a double point at which the two tangents of the curve coincide. It is for this reason that the name cuspidal edge is given.

As the notion of a developable, and its cuspidal edge, is fundamental for the understanding of the dual of a curve, it may be desirable to refer to a concrete representation. Let two pieces of paper, laid one over the other, be simultaneously cut by a pair of scissors along an open portion of an arbitrary curve, say an arc of a circle, which thus becomes a portion of the boundary of each piece of paper. Let the two pieces of paper be now connected along this arc by means of tissue paper gummed to both pieces of paper, to the upper side of the upper piece of paper and to the lower side of the lower piece. Now let tangent lines be drawn to this arc on the upper side of the upper piece of paper, and on the lower side of the lower piece; let the portion of a tangent line on the upper piece which lies on one side of the point of contact with the arc be coloured red, and the continuation of this tangent which lies on the lower piece be also coloured red; let the other portion of this tangent line on the upper piece be coloured blue, and the continuation of this tangent which lies on the lower piece be also coloured blue; let this be done, in a continuous way, for a succession of tangent lines. If now the two pieces of paper be lifted by a corner of the upper piece, not lying on the arc, the paper being stiff enough for the red tangent lines, which are partly on one sheet and partly on the other, to remain straight, the blue lines being allowed to curve, and the arc not remaining in a plane, these red lines will represent generators of a developable surface of which the arc is a portion of the cuspidal edge; at each point of this edge there will be a red tangent line. (Thomson and Tait, *Natural Philosophy*, Part I (1879), p. 114, § 149.)

We may usefully consider the properties of a cubic developable in more detail, taking the steps dual to those followed for the cubic curve. As a line joining two points of the curve is called a chord, let the line of intersection of two planes of the developable be called an *axis*. As for the cubic curve we took two quadric cones with a common generator, so now we take two conics, say Σ and Σ', in different planes, say ϖ and ϖ', both conics touching the line of intersection of their planes. A plane of the developable will be any plane touching both these conics; it will meet the two planes in tangents of the two conics, of which each will determine the other by the fact that they intersect on the line of intersection of the two planes. In particular each of the two given planes will be a plane of the developable. In the case of the cubic curve two chords determine a quadric surface containing the curve, and from any point of the cubic there passes a line, intersecting these two chords and all the other chords which are generators of this quadric, of the same system; these chords determine an involution of pairs of points of the cubic curve, and project from any point of the curve into chords of a conic meeting in a point. Conversely, if an arbitrary line be drawn from any point of the cubic curve, and the chord be drawn to the cubic from any point of this line, these chords are generators of a quadric surface. In the dual figure, if any point, P, be taken in the plane ϖ, two tangents can be drawn from P to the conic Σ, meeting the line of intersection of the planes ϖ, ϖ' in two points; from each of these can be drawn, beside this line of intersection, a tangent line to the conic Σ', and these meet in a point, P', of the plane ϖ'. The line PP' is an axis of the developable, and the construction establishes a $(1, 1)$ correspondence between the points of the planes ϖ, ϖ', which is in fact of the kind considered (Vol. I, p. 148). In the original figure there is similarly a correspondence between the planes drawn through the vertex of one of the two cones with the planes drawn through the vertex of the other cone. Consider now various points P lying on an arbitrary line of the plane ϖ. These will be the dual of various planes passing through the vertex of one of the two cones of the original figure, all having in common a line through this vertex. The axes PP' will then be all generators of a quadric surface, touching all the planes of the developable; the pairs of these planes through the various axes PP' will be pairs of an involution of planes of the developable; the locus of the points P' will be a line, which, like the line on which P is taken, is a generator of the quadric surface generated by the lines PP'. So long as this quadric surface does not degenerate, these two corresponding lines of the planes ϖ, ϖ' do not intersect. This happens, however, when the line on which P varies is a tangent line of the conic Σ; then the locus of P' is a tangent of the conic Σ', and the plane of these intersecting lines is a plane of the developable. In the original figure the corresponding statement is: if from two arbitrary points as centres lines be drawn generating two related central systems, so that the points in which these lines meet a plane, not passing through the centres, are two related plane systems, then corresponding lines, one from each centre, do not in general intersect; when, however, they do, their point of intersection lies on a cubic curve, and the intersecting lines are generators of two quadric cones which intersect in this curve. This gives a new definition of the curve. The theorem relates to the most general way of relating two central systems of lines. What particular cases can arise will be seen from the detailed consideration of two related plane systems given later.

Examples in regard to the cubic curve in space. *Ex.* 1. The tangent line of the curve at any point is a generating line of the quadric cone projecting the curve from this point; the osculating plane of the curve at this point is the tangent plane of this cone along this generator.

Ex. 2. The osculating planes at any three points of the curve meet in a point lying in the plane containing these points. If the point of meeting be (x_0, y_0, z_0, t_0), the plane has the equation $xt_0 - 3yz_0 + 3zy_0 - tx_0 = 0$. This is the focal plane of the point (x_0, y_0, z_0, t_0) in regard to the linear complex expressed by $l + 3l' = 0$; in particular when (x_0, y_0, z_0, t_0) is on the curve, it is the osculating plane of the curve at this point, all the tangent lines of the curve belonging to this linear complex. Further, if any line be taken, and the four tangents of the cubic curve which meet this line, the second of the two transversals of these four tangents is the polar line of the first in regard to this linear complex.

Ex. 3. Since the four planes which join a chord of the cubic curve to four arbitrarily chosen fixed points, A, B, C, D, of the curve form a pencil related to that of the planes joining these points to any other chord, it follows that all the chords of the curve belong to a quadratic tetrahedral complex depending upon A, B, C, D (see Chap. II, p. 99 above). If the values of the parameter θ which belong to the points A, B, C, D be, respectively, a, b, c, d, the coordinates, relative to A, B, C, D, of any point of the cubic, can be taken to be $(\theta - a)^{-1}$, $(\theta - b)^{-1}$, $(\theta - c)^{-1}$, $(\theta - d)^{-1}$, and the line coordinates l, l', etc., of any chord are then such that

$$ll'/(b-c)(a-d) = mm'/(c-a)(b-d) = \text{etc.}$$

Ex. 4. Shew that the general equation of a quadric surface containing the cubic curve, of which the general point is $(\theta^3, \theta^2, \theta, 1)$, is $A\xi + B\eta + C\zeta = 0$, where $\xi = xz - y^2$, $\eta = xt - yz$, $\zeta = yt - z^2$, one system of generators of this meeting the curve in the pairs of points (θ), (ϕ), of the involution given by $A\theta\phi + B(\theta + \phi) + C = 0$. Further, that the line coordinates of the chord of the curve passing through any point (x, y, z, t) are, respectively, $\eta^2-\xi\zeta$, $\eta\zeta$, ζ^2, $-\xi\zeta$, $\xi\eta$, $-\xi^2$.

Ex. 5. We have seen that a cubic curve lying on a quadric surface meets every generator of the surface of one system in one point, and every generator of the other system in two points. Thus, taking the intersections, with a fixed arbitrary plane section of the quadric, of the generators through the points of the curve, there is determined on this section a (1, 2) correspondence (see Vol. II, p. 134). Conversely, such a correspondence upon any plane section of the quadric determines two cubic curves lying on the surface.

Ex. 6. If A, B, C, A', B', C' be any six points of the cubic curve, the three quadric surfaces containing the curve, of which the first contains the two chords BC', $B'C$, the second contains the two chords CA', $C'A$, and the third the two chords AB', $A'B$, have all in common, as generator, another chord of the curve (cf. Vol. II, p. 16). Further, any chord of the curve is met in pairs of points of an involution by the quadrics through the six points.

Ex. 7. A cubic developable is formed, we have seen, by the planes joining corresponding points of three related ranges of points, on three lines, of which no three corresponding points are in line. If we have five lines l, m, l', m', p, in space, of which no two intersect, these lines not having a common transversal, and, from a variable point of p, the transversal be drawn to l and m, and also the transversal to l' and m', the plane of these two transversals describes a cubic developable, of which the five given lines are axes (cf. Ex. 22, below). If P, Q describe related ranges upon a conic, and R describe a range related to both, on a line not in the plane of the conic, prove that, in general, the planes PQR form a cubic developable. Also, if P, Q, R describe three related ranges of points on a cubic curve, the planes PQR form a cubic developable. Further, if the points P, Q of a cubic curve are in (2, 2) correspondence, so that to any position of P corresponds two positions, Q and Q', of Q (and conversely), then the planes PQQ' form a cubic developable, and the point of this plane which is the intersection of the osculating planes at P, Q, Q', of the original cubic, describes another cubic curve.

Ex. 8. From the variable point P, of a fixed straight line, the chord PQR is drawn to a cubic curve, and thereon is taken the point, T, which is the harmonic conjugate of P in regard to R and Q. Thus T is conjugate to P in regard to all quadrics containing the curve. Prove that the locus of T is a cubic curve, meeting the given cubic in four points. Further, if $P_0Q_0R_0T_0$ be a particular position of the line $PQRT$, and O be a fixed point of the cubic, and the transversal from O to $PQRT$ and $P_0Q_0R_0T_0$ meet the latter in

P', prove that the ranges (P), (P') are related, the planes joining O to the chords $PQRT$ being tangents of a quadric cone. This cone also touches the fixed line, on which P is taken. Also, for a fixed position of P, the polar planes of P in regard to the cones which project the cubic from the points of the curve, envelope a quadric cone whose vertex is T.

Ex. 9. If six lines, of which no two intersect, be all met by another line, shew that a variable plane, whose points of intersection with the seven lines lie on a conic, describes a cubic developable, of which the first six lines are axes. (Cf. Ex. 7, p. 195, below.)

Ex. 10. It is a familiar fact that if $f, = ax^2 + 2hxy + by^2$, and $f_1, = a_1x^2 +$ etc., be two quadratic forms in x, y, and we take the two values of x/y for which, for proper values of λ/λ_1, the form $\lambda f + \lambda_1 f_1$ is a perfect square, the quadratic $F, = Ax^2 +$ etc., which vanishes for these values of x/y, is such that both the expressions $Ab - 2Hh + Ba$, $Ab_1 - 2Hh_1 + Ba_1$ vanish (Vol. II, p. 8). More generally, if $f_r, = a_{r0}x^n + na_{r1}x^{n-1}y + \ldots + a_{rn}y^n$, for $r = 1, 2, \ldots, n$, be n forms each homogeneous of the nth order in x, y, there are n sets of values of the ratios of $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that the form $\lambda_1 f_1 + \ldots + \lambda_n f_n$ is the nth power of a linear form, $px + qy$. The form F, of the nth order, which vanishes for the n values of x/y for which these linear forms vanish, is expressed by a determinant of $(n+1)$ rows and columns of which the $(s+1)$th row consists of the elements $a_{1s}, a_{2s}, \ldots, a_{ns}, y^{n-s}(-x)^s$. Denoting this form by $A_0x^n + nA_1x^{n-1}y +$ etc., each of the n expressions

$$A_0\, a_{r,n} - nA_1 a_{r,n-1} + \binom{n}{2} A_2\, a_{r,n-2} - \text{etc.}$$

is zero, and F is the form which is *apolar* to all the forms $f_1, \ldots, f_n$ (cf. Vol. II, p. 114, etc.).

Ex. 11. Any plane drawn through an arbitrary point, O, has an equation of the form $\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 = 0$, where

$$U_r = a_r x + 3b_r y + 3c_r z + d_r t \quad (r = 1, 2, 3),$$

and $U_r = 0$ are three planes of general position through this point. If f_r denote what U_r becomes by putting θ^3, θ^2, θ, 1, respectively, for x, y, z, t, it is in accordance with Ex. 10 that three osculating planes of the cubic curve can be drawn through the point O. By that example, the three points of the cubic curve, say A, B, C, where these osculate the curve, correspond to a cubic form which is apolar to that corresponding to the three intersections of the curve with any plane drawn through O. We call O the pole of the plane ABC. Thus, in particular, any plane drawn through the chord of the cubic which passes through O, meets the curve in three points giving rise to a cubic which is apolar with the cubic associated with the points

A, *B*, *C*. Thus we infer that the Hessian of this last cubic (*A*, *B*, *C*), corresponds to the ends of the chord of the curve drawn through *O* (cf. Vol. II, p. 124). Further, it follows, as on p. 136 of Vol. II, that an involution of sets of three points upon the cubic curve is given by the intersections of the curve with planes through an arbitrary fixed line; and, then, as here, that if one of these sets of points, *A*, *B*, *C*, be such that the pole, *O*, of the plane *ABC*, lies upon this line, then this is the case for every one of the sets, every two of which are then apolar. In general, the poles of planes passing through a line (l, m, n, l', m', n') lie on the line $(-3l', m, n, -\frac{1}{3}l, m', n')$.

Ex. 12. We have seen in Vol. II (p. 124) that the Hessian points of three points, *A*, *B*, *C*, of a conic, may be obtained by taking the three points such as the intersection of the line *BC* with the tangent of the conic at *A*, and then the points where the line containing these three points meets the conic. When the cubic curve is projected to a conic, on an arbitrary plane, from any point, *D*, of itself, any point of this conic may evidently be represented by the parameter, θ, appropriate to the point of the cubic of which it is the projection. From this we have a construction for the Hessian pair of the three points *A*, *B*, *C* of the cubic, and for the chord of this curve drawn through the pole, *O*, of the plane *ABC*: namely, if we take the line in which the plane *DBC* is met by the plane joining *D* to the tangent line of the curve at *A*, the three such lines lie on a plane through *D*, whose intersections, other than *D*, with the curve, are the Hessian points of *A*, *B*, *C*. The chord joining these points meets the plane *ABC* in its pole, *O*. Or, if the tangent lines of the curve at *A*, *B*, *C*, respectively, meet the planes *DBC*, *DCA*, *DAB* in *A′*, *B′*, *C′*, the plane *A′B′C′* contains *D*, and contains the Hessian chord of *A*, *B*, *C*.

Ex. 13. Now let, again, *A*, *B*, *C*, *D* be any four points of the cubic curve. As the osculating planes of the curve at *A*, *B*, *C* meet in a point, *O*, of the plane *ABC*, and similarly for the osculating planes of any three of these four points, the four points such as *O* form a Moebius tetrad both in- and circumscribed to *A*, *B*, *C*, *D* (Vol. I, p. 61). But, also, if *A′*, *B′*, *C′*, *D′* be the points where the tangent lines of the curve at the points *A*, *B*, *C*, *D*, respectively, meet the planes *DBC*, etc., it follows from the preceding example: (i), that the points *A′*, *B′*, *C′*, *D′* form a tetrad in- and circumscribed to *A*, *B*, *C*, *D*, the plane *A′B′C′* containing *D*, etc., and, (ii), that the points *A′*, *B′*, *C′*, *D′* are a tetrad which is also in- and circumscribed to the tetrad of the four poles such as *O*, the plane *A′B′C′* containing *O*, while *D′* lies in the osculating plane at *D*. Thus, if *L* be the pole of the plane *DBC*, namely the intersection of the osculating planes at *D*, *B*, *C* and, similarly, *M*, *N* be the poles of the planes *DCA*, *DAB*, respectively, the three tetrads *A*, *B*, *C*, *D*; *L*, *M*, *N*, *O*;

A', B', C', D' are such that every two are a pair of Moebius tetrads. It can be shewn to follow from this that the four planes $AA'L$, $BB'M$, $CC'N$, $DD'O$ meet, in threes, in a tetrad in- and circumscribed to each of the three tetrads referred to. For this it is sufficient to shew that the three planes $AA'L$, $BB'M$, $CC'N$ meet in a point lying on the planes ABC, $A'B'C'$ and in the osculating plane at D. The theorem, however, is a case of that enunciated in the succeeding example.

Ex. 14. The general tetrad of points, A_1, B_1, C_1, D_1, forming a Moebius tetrad with four points A, B, C, D, are a set whose coordinates, relative to A, B, C, D, are capable of the forms

$$(0, -n_1, m_1, -l_1), (n_1, 0, -l_1, -m_1), (-m_1, l_1, 0, -n_1), (l_1, m_1, n_1, 0),$$

depending on the ratios of the arbitrary quantities l_1, m_1, n_1. Another such tetrad, depending on l_2, m_2, n_2, form a Moebius tetrad with A_1, B_1, C_1, D_1 if $l_1l_2 + m_1m_2 + n_1n_2 = 0$. Thus a set of four tetrads of which every two are Moebius tetrads arises if $(l_1, m_1, n_1, 0)$, $(l_2, m_2, n_2, 0)$, $(l_3, m_3, n_3, 0)$ are a self-polar triad of points in regard to the conic $t = 0$, $x^2 + y^2 + z^2 = 0$.

Two tetrads, A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2, each both inscribed and circumscribed to A, B, C, D, can also be obtained by taking AA_1, BB_1, CC_1, DD_1 as the transversals to two arbitrary generators, of the same system, of an arbitrary quadric, in regard to which A, B, C, D are self-polar, and AA_2, BB_2, CC_2, DD_2, as the transversals to two other generators of the same system. When the two latter generators are harmonic conjugates in regard to the two former, then the two tetrads, A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2, are also mutually inscribed. In this case the four planes such as DD_1D_2, AA_1A_2 give rise, by their intersections, to a fourth tetrad inscribed and circumscribed to the previous three tetrads. (See pp. 67, 68, above.)

Ex. 15. Four such tetrads are also representable, for any values of a, b, c, d, by the rows (or by the columns) of the scheme

$$\begin{array}{cccc}
iPi & iPj & iPk & iP \\
jPi & jPj & jPk & jP \\
kPi & kPj & kPk & kP \\
Pi & Pj & Pk & P
\end{array}$$

where $P = ai + bj + ck + d$, the symbols i, j, k are such that

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k,$$

and $xi + yj + zk + t$ represents the point (x, y, z, t).

Ex. 16. We may consider a general involution of sets of four points upon the cubic curve, given (Vol. II, p. 136) by the condition $U + \lambda V = 0$, for varying values of λ, where U, V are each polynomials of the fourth order in the parameter θ which gives a point of the curve. It can be shewn: (a), that the tetrads of points of the cubic so obtained are all self-polar in regard to a certain quadric; (*b*), that the planes joining three points of the tetrad are those of a cubic developable. In fact, if θ_1, θ_2 be two roots of the quartic equation, so that $U_1+\lambda V_1=0$, $U_2+\lambda V_2=0$, the identity $(U_1V_2-U_2V_1)/(\theta_1-\theta_2)=0$, which does not depend on λ, is linear in regard to both the two sets (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), where $x_1=\theta_1^3$, $y_1=\theta_1^2$, $z_1=\theta_1$, $t_1=1$, $x_2=\theta_2^3$, etc., and is symmetrical in regard to the two points with these coordinates. Thus it expresses that these points are conjugate in regard to a definite quadric. Further, denoting this relation by $(\theta_1, \theta_2)=0$, we also have $(\theta_1, \theta_3)=0$, $(\theta_1, \theta_4)=0$. Thus the coordinates of the plane joining the points $(\theta_2, \theta_3, \theta_4)$, which are 1, $-(\theta_2+\theta_3+\theta_4)$, $\theta_2\theta_3+$ etc., $-\theta_2\theta_3\theta_4$, appear as proportional to four cubic polynomials in θ_1. Therefore, as θ_1 varies, the plane $(\theta_2, \theta_3, \theta_4)$ describes a cubic developable. This may be spoken of as the polar reciprocal of the given cubic curve in regard to the quadric surface obtained. The theorems, however, arise incidentally below (Ex. 28).

Ex. 17. If a cubic curve in space, and a conic, have three points in common, a quadric surface exists containing both curves. Also, if two cubic curves in space have five points in common, a quadric surface exists containing both curves; this is not necessarily the case for two cubic curves with only four common points. Prove, further, that two cubic curves lying on a quadric surface have four common points, or five common points; the former is the case when the curves are such as to have the lines of the same system of generators of the quadric as chords, the latter when this is not so. Shew that in the latter case the cubic curves have no common chord other than a join of two of the five common points.

Ex. 18. It has been shewn that a cubic curve can be described to contain six arbitrary points. Shew that a cubic curve can be constructed to pass through five arbitrary points and have a given line as chord. In fact, two independent quadrics can be constructed to pass through the five points and have the line as a generator. Or, if coordinates be taken with four of the five given points as points of reference, a transformation of the form

$$x'=Ax^{-1},\ y'=By^{-1},\ z'=Cz^{-1},\ t'=Dt^{-1},$$

changes the given line into a cubic curve through the four points of reference; from the point which arises, by this transformation, from the fifth given point, a single chord can be drawn to this curve. The curve which is sought is that obtained from this chord by applying the inverse of the transformation referred to.

Ex. 19. Now consider the problem of describing a cubic curve through four given points to have two given lines as chords. Through such a curve would pass a quadric surface containing the lines as generators; in general, however, a quadric surface cannot be drawn through four given points to contain also two given lines. The construction of the cubic curve is, therefore, impossible in general. Suppose however the given lines are two generators, of the same system, of a quadric surface containing the four given points. Then we have seen that a cubic curve can be drawn to satisfy the given conditions, lying on the quadric surface and passing through a fifth arbitrary point of this surface (p. 129, above). Consider the case when the given lines intersect.

Ex. 20. Next suppose we are given three points and three lines of general position. We can clearly describe a cubic curve to have these lines as chords, containing the three points. We have only to consider related axial pencils of planes having these lines as axes, these pencils being related by the fact that three corresponding planes meet in each of the three given points.

Ex. 21. If we are given two points and four lines, of which no two intersect, a cubic curve can be constructed passing through the two given points to have each of the four given lines as a chord. For, let A, B be the given points and l, m, l', m' the given lines. Consider the transversal, a, drawn from A to meet l and m, and the transversal, a', drawn also from A, to meet l' and m', and the plane of these two transversals. Consider also the transversal, b, drawn from B to meet l and m, and the transversal, b', also drawn from B, to meet l' and m', and the plane of these two transversals. These two planes meet in a line, n, which is the only line other than AB which meets the four lines a, a', b, b'. If a cubic curve exists through A, B with l, m, l', m' as chords, it lies on a quadric containing l and m, of which therefore a, b are also generators; for each of a, b contains, beside, the point A or B, two other points of the quadric, lying, respectively, on l and m. The curve equally lies on a quadric containing l' and m', of which therefore a', b' are also generators. These two quadrics, having this cubic in common, must further intersect in a line, which then is a chord of the curve, and of the same system of generators of the first quadric as are l, m; it will therefore meet a and b. Similarly it will meet a', b'. The common generator is, therefore, the line n. Conversely, let a quadric, which is unique, be constructed to contain the lines l, m, n; this quadric then contains a, b and, therefore, A and B. Similarly, the quadric containing the lines l', m', n contains a', b' and, therefore, A and B. These two quadrics, having n in common, determine the cubic curve required, as their remaining intersection.

Ex. 22. If we are given one point, and five lines of which no two intersect, these lines not having a common transversal, a cubic curve exists passing through the given point of which the five given lines are chords. Denote the point by A, and the lines by l, m, l', m', p. Remark first that, by means of the two given pairs of lines, l, m and l', m', a definite plane is determined through any point, P, of space, namely the plane of the transversals drawn from P to l, m and from P to l', m'. Denote this plane by (P). As P varies on the line, p, this plane, describing related ranges on l, m, l', m', p, describes a cubic developable, of which these lines are axes. Of this developable there is then one definite axis on the plane (A), say the line n; this is the intersection of two planes of the developable, passing through positions of P, say B and C, upon the line p; the three planes of the developable which pass through B consist of the plane (B), and the two planes intersecting in the axis p. Evidently, now, A is the intersection of a line meeting l, m, n and a line meeting l', m', n; as, similarly, are B and C. The two quadric surfaces, with n as common generator, determined, respectively, by l, m, n and l', m', n, thus meet in a cubic curve having l, m, l', m' as chords, passing through A, B, C; this curve, therefore, has also p as a chord, and is the curve desired.

It may be remarked, in anticipation, that when the five given lines have a common transversal (in which case there is a position of P on p for which the plane (P) is indeterminate) there are cubic curves in infinite number having these lines as chords; but these all lie on a surface (a cubic surface; *v.* p. 175); one such cubic curve can be drawn passing through two points of this surface.

Ex. 23. If we have two cubic curves in space, of general position, there are ten lines each of which is a chord of both curves. To prove this, first notice that through any point, O, of one cubic, ρ, can be drawn six chords of this curve each of which meets the other curve, σ. For the points of the curve, into which the curve σ is projected from O on to an arbitrary plane, have coordinates proportional to cubic polynomials of the parameter which gives the position of a point of the curve σ; if these be substituted in the equation of the conic into which the curve ρ is projected from O, the result is an equation of order six. Now take any variable point, P, of the curve ρ, and draw from it the chord to the curve σ, meeting it in Q and Q'. From Q draw the chord to the curve ρ, and so from Q'. The point P of the curve ρ thus gives rise to four other points of this curve; let P' be one of these. To pass back from P' to P, we must first draw from P' a chord of ρ which meets σ, say in Q, which is possible in six ways; and then from Q we must draw a chord of σ which meets ρ, in the point P, also possible in six ways. The point P' thus arises from thirty-six different possible positions of P. Thus, if θ, θ' be the parameters giving the positions of P, P', respectively, upon the curve ρ, there is a relation $(\theta, \theta')=0$, an integral polynomial in both θ and θ', of order 36 in

θ and of order 4 in θ'. Or, as it is usual to say, there is a correspondence (36, 4) between P and P' (cf. Vol. II, p. 134). The values of θ and θ' are therefore the same for $36+4$ or 40 values of θ. When this is the case the chord PQQ', drawn from the point P, of ρ, to meet σ in Q and Q', meets ρ again, say in the point H. This common chord however arises in four ways in the construction given: not only when P coincides with a corresponding position P' arising with Q as intermediary, but also when P coincides with a corresponding position arising with Q' as intermediary; and not only when P coincides with P', but also when H coincides with a corresponding position, arising through Q or Q'. The number of common chords of the two curves is therefore one quarter of 40, namely is ten.

Ex. 24. We can hence shew that, if we be given six lines, of general position, there are six cubic curves all having these six lines as chords. Let these lines be l, m, l', m', p, q. We have shewn in Ex. 22 that a cubic curve having the five lines l, m, l', m', p as chords has, as another chord, an axis of the cubic developable described by the planes (P), there explained, as P moves on p. Similarly this cubic, if it have q also as a chord, has as another chord an axis of the cubic developable described by the planes (Q), as Q moves on q. This other chord is the same in both cases, being that meeting the curve in the common pair of the two involutions determined thereon by l, m and by l', m'. These two cubic developables have, however, by the dual of Ex. 23, ten common axes. Of these, four are the lines l, m, l', m'. There remain then six others. Taking any one of these, say n, the cubic required is the locus other than n, of the common points of the quadrics (l, m, n), (l', m', n).

The theorem that there are six cubic curves having six given chords is subject to modification when these chords are not lines of general position. For instance, when the six lines have a common transversal, it may be shewn that there is only one proper, or undegenerate, cubic curve having these as chords. Or again, when every five of the six given lines have a common transversal, there is an infinite number of such curves. Or, as another case, when the six lines, say a, b, c, l, m, n, are such that each of the three sets of five, b, c, l, m, n; c, a, l, m, n; a, b, l, m, n, have a common transversal, there is no proper cubic curve with the six lines as chords; in this case six degenerate curves, each consisting of three lines of which one meets the other two, can be found.

Ex. 25. The theory of the last preceding examples is essentially due to Cremona (*Crelle's J.*, Vol. LX, 1862, p. 191. See also Cayley, *Papers*, Vol. VII, p. 170, 1870). Another method of proof employs a method of transformation which it is desirable to explain. We have already above employed a reversible cubic transformation $x' = Ayzt$, $y' = Bzxt$, $z' = Cxyt$, $t' = Dxyz$, which changes a line into a cubic curve having four fixed points. A reversible cubic transformation of much generality is obtained by taking three fixed quadric surfaces, and making correspond to any point, (x, y, z, t), the point of intersection, (x', y', z', t'), of the polar planes of (x, y, z, t) in regard to these surfaces. Then, as (x, y, z, t) describes any line, the three polar planes describe axial pencils of planes, all related to the range of points (x, y, z, t), whose axes are the polar lines of the line in regard to the three quadric surfaces. Thus (x', y', z', t') describes a cubic curve, having these lines as chords. We can however choose the three quadric surfaces so that the cubic curve so obtained has four fixed lines as chords, which are independent of the line of which the cubic curve is the transformation, and may be taken quite arbitrarily; conversely, in this case, any cubic curve having these four lines as chords is transformed into a line. By the transformation so obtained, the theorem, for example, that a cubic curve can be described to pass through two given points and have four given lines as chords, becomes the theorem that a line can be drawn through two points. Again, the theorem that a cubic curve can be drawn through a given point to have five given lines as chords, when these lines have no common transversal, becomes the theorem that a line can be drawn through an arbitrary point to be a chord of a given cubic curve. Lastly, the cubics having six given lines, of general position, as chords become, by the transformation, the common chords, other than four given ones, of two given cubics.

We may obtain the particular cubic transformation in question, in dual form, in a simple manner, by proceeding as follows: Taking the Moebius tetrads, A, B, C, D and A', B', C', D', the latter points being of equations relative to the former, respectively $U=0$, $V=0$, $W=0$, $P=0$, where $U = c'v - b'w - ap$, $V = -c'u + a'w - bp$, $W = b'u - a'v - cp$, $P = ua + vb + wc$, the point-pairs (A, A'), (B, B'), (C, C'), (D, D') are not independent, their equations being connected by the identity $uU + vV + wW + pP = 0$. Thus the poles of an arbitrary plane in regard to these four point-pairs lie in one plane, as do the poles of an arbitrary plane in regard to three independent quadrics. The pole of a plane in regard to a point-pair, such as A, A', is the point of the line AA' which is the harmonic conjugate, in regard to A and A', of the point where the plane meets the line AA'. The new plane may be taken to correspond to the original plane, which, conversely, corresponds to the new plane by the same rule. It is this correspondence which is effected by the transformation in question. Consider, now, all the planes which correspond, by this rule, to the planes, say α, which can be drawn through an arbitrary fixed line, say l. If a plane α meet the lines AA', BB', CC', respectively, in U, V, W, the ranges (U), (V), (W), being sections of an axial pencil of planes, are related; therefore, the ranges (U'), (V'), (W'), where U', V', W' are the points lying respectively on AA', BB', CC' for which the ranges A, U, A', U'; B, V, B', V'; C, W, C', W' are harmonic, are also related. Whence, the plane $U'V'W'$ describes a cubic developable. Of this developable each of the lines AA', BB', CC', DD' is an axis. For instance, there are two planes α, passing through l, of which the corresponding planes both pass through DD', those, namely, containing l and the two common transversals of the four lines l, AA', BB', CC'; it is, in fact, a property of two Moebius tetrads that, when U, V, W are in line, the plane $U'V'W'$ contains DD' (Chap. I, p. 67, Ex. 20). The two planes α may also be defined as the two tangent planes through l of the quadric described to have the lines AA', BB', CC' as generators.

It is now to be shewn, conversely, that given any four lines, of which no two intersect, it is possible to find two Moebius tetrads, A, B, C, D and A', B', C', D', of which A, A' are on the first line, B and B' on the second, and so on. This is always possible; formulae applicable for this purpose are given in Ex. 22, p. 68; it is only necessary to express that the four pairs of points A, A'; B, B', etc., each harmonic conjugates in regard to the points in which their join is met by the two common transversals of the four joins, are not independent.

Dually, the polar planes of a point, P, in regard to the plane pairs, $x\xi = 0$, $y\eta = 0$, $z\zeta = 0$, $t\tau = 0$, where

$$\xi = a't + bz - cy,\ \ldots,\quad \tau = -(a'x + b'y + c'z),$$

meet in a point P', which, as P describes a line, l, of general position, describes a cubic curve having as chords the four lines, k, such as $x = 0 = \xi$. The points of intersection of the cubic with one of these lines, k, correspond to the points where the original line, l, meets the quadric containing the other three of the lines k.

An incidental consequence of the last remark may be referred to in passing: In order that it may be possible to describe a cubic curve touching each of the four given lines, it is necessary and sufficient that the four quadrics each containing three of these lines should all be touched by a line. It can be shewn that they are then touched by an infinite number of lines, all generators of a quadric, of the same system. There is then an infinite number of cubic curves touching the four lines. This is also clear by considering that if we take four tangents of a cubic curve there is an infinite number of linear complexes to which these four tangents belong; the poles, in regard to any one of these complexes, of the osculating planes of the given cubic curve, describe another cubic curve also touching the four given tangents. The problem of finding a common tangent line of the four quadrics which each contain three of four non-intersecting lines is the same, in fact, as of finding a plane section of a quadric surface, which contains four points A, B, C, D, which shall touch the sections of this surface by the four planes BCD, CAD, ABD, ABC. For the surface

$$fyz + gzx + hxy + t\,(ux + vy + wz) = 0,$$

the condition that this shall be possible can be shewn to be $l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0$, where $l = \sigma\,(fu)^{\frac{1}{2}}$, $m = \epsilon\,(gv)^{\frac{1}{2}}$, $n = \sigma\epsilon\,(hw)^{\frac{1}{2}}$, in which $\sigma^2 = 1$, $\epsilon^2 = 1$. If $(l_1, m_1, n_1, l_1', m_1', n_1')$, etc., be the line coordinates of four tangent lines of a cubic curve, and

$$\varpi_{12} = l_1 l_2' + l_2 l_1' + \text{etc.},$$

it can be proved that $\alpha^{\frac{1}{4}} + \beta^{\frac{1}{4}} + \gamma^{\frac{1}{4}} = 0$, where $\alpha = \varpi_{23}\varpi_{14}$, etc.

Ex. 26. There is another problem of construction for cubic curves which may be referred to: Given four lines of general position, and upon each of them an involution of pairs of points, there are two cubic curves having these four lines as chords whose points of meeting with each of these lines are a pair of the involution lying thereon. This may be deduced from a theorem, given by Chasles (*Liouville's J.*, t. IV, 1839, p. 348), that the lines of space which meet the generators of one system of a quadric surface in points lying on pairs of a given involution, of pairs of these generators, are lines of a linear complex. (Above, Ex. 15, p. 67.)

Ex. 27. Consider a cubic curve, θ, and a quadric surface, Σ. Let l, m be any two chords of the curve. To any plane, α, drawn through l, meeting the curve again in a point P, can be drawn a plane, β, through m, conjugate to the former in regard to Σ. As α varies, describing an axial pencil about l, the plane β will describe a related axial pencil about m, and if β meet the curve again in Q, the ranges (P), (Q) will be related. The pairs of points P, Q will then belong to an involution on the curve if, for one pair, not only Q corresponds to P but P also corresponds to Q (Vol. II, p. 2); that is, if not only the planes lP, mQ be conjugate in regard to Σ, but also the planes lQ, mP be conjugate.

Now suppose that upon the curve θ there are four points, A, B, C, D, forming a self-polar tetrad in regard to Σ. Take any point, T, of the curve, and let its polar plane in regard to Σ meet the curve in X, Y, Z. We can then shew that T, X, Y, Z are a self-polar tetrad in regard to Σ. For consider pairs of planes, conjugate to one another in regard to Σ, drawn respectively through the chords BC and YZ; the plane joining YZ to D, which is the pole of the plane BCA, is conjugate to this plane, joining BC to A; similarly the plane joining YZ to A is conjugate to the plane joining BC to D. Wherefore, these pairs of planes determine an involution on the curve. But the plane joining BC to T is conjugate to the plane joining YZ to X, since T is the pole of the plane XYZ; wherefore the plane BCX contains the pole of the plane YZT. By similar reasoning the pole of the plane YZT lies on each of the six planes joining X to a pair of the points A, B, C, D. This pole is therefore at X. Similarly, Y, Z are the poles of the planes ZXT, XYT, respectively, and X, Y, Z, T are a self-polar tetrad.

A cubic curve which contains four points forming a self-polar tetrad in regard to a quadric Σ will be spoken of as *outpolar* to Σ. It contains not only other self-polar tetrads, as has been proved, but is such that it is met by any quadric, S, which is outpolar to Σ, and does not contain the curve, in six points, say H, forming what has been called a self-conjugate hexad in regard to Σ (Chap. I, p. 47), that is, six points such that the plane containing any three of them is conjugate, in regard to Σ, to the plane containing the other three. A descriptive proof of this may be given. But we may argue as follows: Since there are three independent quadric surfaces containing the curve, all of which contain the points of any self-polar tetrad of points of the curve, we have, through the six points H, four quadrics all of which are outpolar to Σ, namely the three quadrics containing the curve and the quadric S. The equations of these four quadrics are linearly independent, since S does not contain the curve; and, as there are ten terms in the equation of a quadric, the number of quadrics through six points, with equations which are linearly independent, is just four. The number could be more than four only if every quadric described through five, or fewer, of these points, necessarily passed through the others (as a plane described through three of four coplanar points necessarily passes through the other). The six points H have not this character, since, else, a degenerate quadric consisting of a plane through three of the points and a plane through two others would necessarily pass through the sixth. Thus we can infer that every quadric passing through these points H is outpolar to the quadric Σ; in particular a pair of planes containing these six points form together such an outpolar quadric, that is they are conjugate in regard to Σ. Conversely, of such a self-conjugate hexad of points upon the cubic curve, five points may be taken arbitrarily. In particular, if we take five points upon the curve, X, Y, Z, T, U, of which T, U are the extremities of the chord of the curve drawn through the pole of the plane XYZ, in regard to the quadric Σ, then every other point of the curve forms with these five a self-conjugate hexad;

thence it follows that the pole of the plane containing any three of the points X, Y, Z, T, U lies on the line joining the other two. These points constitute then what is called a self-conjugate pentad.

We have defined the cubic as outpolar to Σ when it contains four points forming a self-polar tetrad of Σ. It can, however, be shewn that if the cubic contains the points of a self-conjugate hexad (or of a pentad), it contains also the points of a self-polar tetrad. This is obvious from what was shewn in Chap. I (p. 51); for, thence, the four linearly independent quadrics containing the six points will all be outpolar to Σ; in particular, the three quadrics passing through the curve. Also, it may be remarked, in the notation of p. 47, that if A, B, C, D, E, F be a self-conjugate hexad of points of the curve, the quadric $\Phi(A, B, C, D)$ is that quadric containing the curve which contains the chord EF and is outpolar to Σ.

Ex. 28. A cubic curve, θ, consists of the points common to three quadrics of which every two have a common generator, and is outpolar to a quadric Σ when this last is inpolar to the three former quadrics containing the curve. Similarly a cubic developable, Θ, consists of the planes common to three quadrics regarded as aggregates of their tangent planes, of which every two have a common generator, and the cubic developable may be spoken of as inpolar to a quadric S when this last is outpolar to the three tangential quadrics defining the developable. When Σ consists of the tangent planes of S, we may speak of the curve θ as outpolar to the developable Θ. This case has already arisen in a preceding example (Ex. 16); if an involution of sets of four points be taken on θ, each of these sets forms a self-polar tetrad in regard to a certain quadric, and the planes containing threes of the four points are those of a developable Θ.

Ex. 29. If A, B, C, D and A', B', C', D' be two tetrads of points both self-polar in regard to a quadric, the cubic curve, described to contain A, B, C, D and C', D', has $A'B'$ for a chord, meeting this in two points, P, Q, which are harmonic conjugates in regard to A' and B'. For, by what we have seen, the polar plane of D', in regard to the quadric, which passes through C', meets the curve again in points, P and Q, such that D', C', P, Q are a self-polar tetrad. Thus P, Q lie on the join of A', B' and are harmonic in regard to these.

The equation of the general quadric through A, B, C, D, C', D' is of the form $\lambda U + \mu V + \nu W + \rho\Phi = 0$, where $U=0$, $V=0$, $W=0$ may be taken to be independent quadrics containing the cubic curve through these six points. As these three quadrics contain P and Q, we see that the quadrics through the six points meet the line PQ in pairs of points of an involution, of which P, Q are one pair. One such quadric, therefore, contains both A' and B'. Thus, there are three independent quadrics, containing the eight points A, B, C, D, A', B', C', D', any quadric containing A, B, C, D, B', C', D' necessarily passing also through A'.

Any seven general points may be regarded as being seven points from two tetrads which are both self-polar in regard to a properly chosen quadric. For a quadric can be found having four points as a self-polar tetrad, in regard to which every two of three other arbitrary points are conjugate to one another; the pole of the plane of these three, in regard to this quadric, then forms a second self-polar tetrad.

Thus we infer that if two tetrads, of points of general position, be both self-polar in regard to a quadric, any two tetrads formed from these eight points are also two self-polar tetrads in regard to a properly chosen quadric. Also, that all quadrics passing through seven general points pass also through an eighth point. This eighth point may be found by drawing the cubic curve through any six of the seven given points, and then the chord to this curve from the seventh point, and taking thereon the harmonic conjugate of the seventh point in regard to the two points where the chord meets the curve. Or, if the seven given points be A, B, C, D, B', C', D', the eighth point is A', constructed as above, as the pole of the plane $B'C'D'$ in regard to the quadric for which A, B, C, D is self-polar and every two of B', C', D' are conjugate.

In another phraseology, if we have two self-polar tetrads in regard to a quadric, the cubic curve containing any six of the points is outpolar to the point-pair constituted by the remaining two, regarded as a degenerate tangential quadric. The squares of the left sides of the equations of the eight points are in fact connected by a linear relation (cf. Chap. I, p. 51).

Eight points which are such that all quadrics through seven of them necessarily pass through the remaining one, are said to be eight *associated* points. Some particular properties of such a set are given below (p. 154).

Ex. 30. Prove that the squares of the left sides of the equations of eight points of a cubic curve are connected by a linear relation of the form $\Sigma c_r P_r^2 = 0\ (r = 1 \dots 8)$. See, for instance, *Proc. Lond. Math. Soc.* IX, 1910, p. 174. For the theory of outpolar cubics, see also Wakeford, *ibid.*, XXI, 1922, p. 101, and *ibid.* p. 114. For the descriptive theory, see Telling, *Proc. Camb. Phil. Soc.*, XXI, 1922, p. 249.

Ex. 31. Consider a focal system (p. 60), and a quadric, such that two particular lines which are polars in the focal system are also polars of one another in regard to the quadric. Let P_1, P_2 be any two points of which the focal planes, respectively ϖ_1, ϖ_2, intersect in the line P_1P_2. Denote by δ the interval P_1P_2 measured with respect to the quadric, in the manner employed for a conic in Vol. II, p. 168, and by θ the angular interval between the planes ϖ_1, ϖ_2, similarly measured with respect to the tangent planes drawn from the line P_1P_2 to the quadric. Consider $\sin\theta/\sin\delta$, expressing this when the focal system and quadric are given by

$$n + kn' = 0, \quad Ax^2 + By^2 + Cz^2 + Dt^2 = 0.$$

Shew that the quadric can be so taken that $\sin\theta/\sin\delta$ is independent of P_1 and P_2 (cf. Lie-Scheffers, *Berührungstransformationen*, 1896, p. 233. The corresponding theorem given by Lie for a non-linear complex, *ibid.*, p. 308, is not true).

The rational curve of order n in space of n dimensions. A conic in a plane, and the cubic curve in three dimensions, are evidently both particular cases of a curve in space of n dimensions, of which the coordinates, $x_0, x_1, \ldots, x_n$, of any point of the curve, are respectively proportional to $(n+1)$ integral polynomials in a variable parameter, say θ. We do not enter into the general theory here; but it seems well to remark that any curve in space of r dimensions, in which $r < n$, of which the coordinates of any point are expressible as polynomials of order n in a variable parameter, θ, may be regarded as arising, by projection, from this curve of order n in space of n dimensions; which is thus of fundamental importance. For instance, to illustrate a general proof of this theorem, consider a curve in a plane of which the coordinates, x, y, z, of any point, are expressible as cubic polynomials in θ. Take a point (X, Y, Z, T), in space of three dimensions, of which X, Y, Z are precisely the same functions of θ as are x, y, z, while T is an arbitrary cubic polynomial. This point (X, Y, Z, T) then describes a cubic curve in three dimensions. But to any point of this curve there follows by the equations $x/X = y/Y = z/Z$, that is by projection from the point $(0, 0, 0, 1)$, a corresponding point of the given plane curve. This plane curve has, therefore, a double point, lying on the chord of the space cubic which can be drawn to it from $(0, 0, 0, 1)$; but the space cubic has no double point. For, if it had, a plane, drawn through this double point and two arbitrary points of the curve, would meet the curve in four points, and, therefore, contain it entirely. By a similar argument the curve in space of n dimensions is without double points.

It is supposed in general, in speaking of a curve which is representable rationally by a parameter, that not only is one point of the curve given by each value of the parameter, but, conversely, only one value of the parameter corresponds to a general point of the curve. The parameter can therefore be defined as a rational function of the coordinates of the point of the curve. It can be shewn that if a parameter has been chosen for which this converse expression is not possible, another parameter, a rational function of the former, can be substituted for which this is possible (cf. Vol. II, p. 136. The several values of the unsuitable parameter which correspond all to the same point of the curve are sets of a general involution).

It is clear that the number of points of the general rational curve of the nth order of which the coordinates satisfy a homogeneous equation of the rth order in the $(n+1)$ coordinates, is nr. It is characteristic of the curve that these can be arbitrarily taken. For instance, there are $2r$ points of a conic lying upon a general curve in the plane whose equation is given by a homogeneous equation of order r in the coordinates; but, conversely, such a curve of order r can be drawn through any $2r$ points taken upon the conic, as may be readily verified. Or, again, a surface of equation of order r can be drawn through any $3r$ points taken arbitrarily upon the cubic curve in space of three dimensions, not entirely containing the curve. For instance, a quadric surface can be drawn through any six points of the curve, not entirely containing the curve; or a cubic surface, not entirely containing the curve, through any nine points.

Ex. 1. Find the most general relation, homogeneously of the third order in x, y, z, t, which is identically satisfied by $x=\theta^3$, $y=\theta^2$, $z=\theta$, $t=1$.

Ex. 2. Shew that if a general cubic surface, expressed by the vanishing of a homogeneous relation of the third order in x, y, z, t, be put through two non-intersecting generators of a quadric surface, the remaining intersection is a curve meeting an arbitrary plane in four points, of which the coordinates of a point are expressible rationally by a parameter. Shew, also, how to define, by intersections of manifolds in space of four dimensions, a rational curve of order four which projects into this curve.

Ex. 3. Shew that the equations, for x, y, z, t in terms of a parameter θ,

$$(p\theta+a)x=\theta x_2+ax_1,\ (p\theta+b)y=\theta y_2+by_1,\ \ldots,\ (p\theta+d)t=\theta t_2+dt_1$$

represent a quartic curve containing the points of reference and the two points (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2); and that it can be made to pass through two further arbitrary points by suitable choice of the constants a, b, c, d, p. In how many ways is this possible? Find a quadric surface, and a cubic surface containing this curve.

Ex. 4. Shew that a conic, regarded as a curve in three dimensions, depends upon eight parameters. In particular, given seven lines of which no two intersect, which have a common transversal, prove that there exists a single conic which meets every one of the seven lines and also meets the transversal. Analytically, we may regard the conic as the intersection of two quadric cones whose vertices are on the transversal, which touch along this line; we are thus led to seven linear homogeneous equations in eight unknown coefficients. (See Ex. 7, p. 195, below.)

It may be proved that there are ninety-two conics meeting eight lines of general position. See J. Lüroth, *Crelle*, LXVIII (1868), and C. Hierholzer, *Math. Annal.* II (1870).

The curve of intersection of two general quadric surfaces. If two quadric surfaces have a common self-polar tetrad of points, their curve of intersection is essentially different from the cubic curve which has been considered. The coordinates of its points cannot be expressed rationally in terms of a parameter; they can be expressed rationally in terms of *two* parameters, of which one is the square root of a quartic, or cubic, polynomial in the other, or, what is the same thing, if x, y, z, t be the coordinates of a point of the curve, a cubic curve can be found in a plane, in which the coordinates are ξ, η, ζ, this curve being expressed by the vanishing of a homogeneous cubic polynomial in ξ, η, ζ, such that the ratios of x, y, z, t are rational functions of the ratios of ξ, η, ζ, and the ratios of ξ, η, ζ are rational functions of the ratios of x, y, z, t. Such a cubic curve is obtained by projecting the space curve on to a plane from any point of itself. Further, this space curve has the property that, of its intersections with an arbitrary surface, not all can be arbitrarily taken, one of them being always determined when the others are assigned. For instance, the curve meets an arbitrary plane in four points; but, of these, one is determined by the plane containing the other three. Or again, it can be shewn that the curve meets a general quadric surface in eight points (see below); and that the quadric surfaces through seven of these all pass through the remaining one, which, therefore, cannot be taken arbitrarily upon the curve when these seven are given.

Ex. 1. If two quadric surfaces be expressed by the respective equations

$$x^2 + y^2 - z^2 = t^2, \quad ax^2 + by^2 - cz^2 = t^2,$$

any point upon the former is given in terms of θ and ϕ by the expressions $x = \theta + \phi$, $y = 1 - \theta\phi$, $z = 1 + \theta\phi$, $t = \theta - \phi$; if this point lie also on the latter there is a (2, 2) relation connecting θ and ϕ, by means of which either is expressible rationally in terms of the other, and the square root of a quartic polynomial in this other. The systematic consideration of this relation is best undertaken in close connexion with the theory of elliptic functions, and we do not enter upon this here.

Ex. 2. Prove that the quartic curve in space now being considered meets in two points every generator of any quadric surface which contains the curve. Further, that if a variable plane be drawn through two points of the curve, the join of its two other intersections with the curve describes a quadric surface containing the curve. Further, that such a curve can be drawn through eight general points of space. Also, that two chords of the curve can be drawn through an arbitrary point, not lying thereon; and that no line exists meeting the curve in three points.

Ex. 3. Shew that the algebraic determination of the common self-polar tetrad of two general quadrics can be carried out as for the common self-polar triad of two conics given p. 141 of Vol. II; and that the theorem for the reality of the roots of the determining quartic equation, given for the cubic on p. 165 of Vol. II, remains also true.

The determination of a quadric by given conditions. The equation of a quadric contains ten terms; that the quadric should pass through a point of assigned coordinates imposes one linear condition for the coefficients in its equation. That the quadric should pass through several, say through r, given points, requires r such linear conditions for the ten coefficients. If these conditions are independent there are $10-r$ of the coefficients in terms of which the remaining ones can be linearly expressed; then the equation of the general quadric through the r assigned points is of the form $\Sigma\lambda_s U_s = 0$, with $s = 1, \ldots, 10-r$, wherein the $\lambda_1, \lambda_2, \ldots$ are arbitrary, but $U_1=0$, $U_2=0$, ... represent definite quadrics passing through the assigned points. When this is so the assigned points are called *independent*. It may happen however that, of the r linear conditions for the ten coefficients, one or more is obtainable as a linear aggregate of the others; a very simple example would be that in which four of the assigned points were in line; then a quadric containing any three of these would contain the line, and the condition that it contained the fourth point would be a consequence of the conditions that it contained these three. Algebraically, as the equations are linear, the equation expressing the fourth condition would be a linear function of the equations expressing the three others. These considerations have their application chiefly when the assigned points are already conditioned in some way, as by lying on a curve, or a surface.

For instance, upon the curve of intersection of two quadric surfaces there must be sets of eight points furnishing independent conditions for quadrics required to pass through them; there cannot be nine such independent points; for, by definition, such nine points entirely determine a quadric, whereas, in the present case, if $U=0$, $V=0$ be the given quadrics, the quadric $U+\lambda V=0$ contains all points of their common curve, in particular the nine points, and can yet, by choice of λ, be made to pass through a further arbitrary point. But there must be as many as eight independent points upon the curve; for if, for example, there were but seven, this would mean that every quadric through these seven necessarily contained all points of the common curve; it would then be possible to describe a quadric through this curve and two other arbitrary points. We can, however, prove that only one quadric can be passed through this curve and a further single point, O; namely by considering the section of the quadrics with a general plane drawn through O. In this argument we have spoken of a quadric passing through all the points common to two given quadrics, as if this were a description quite free of ambiguity; when the given quadrics have multiple points of intersection, as for example in the case when they touch at all points of a plane conic, the description needs interpretation. This is furnished by first agreeing as to what is meant by a conic passing through all the points common to two given conics, even when there are coincidences among these; and then speaking of a quadric as passing through all points common to two given quadrics when it meets every plane in a conic which passes through all the points common to the sections, by this plane, of the two given quadrics. It will be seen that this meets all cases.

Ex. 1. The necessary and sufficient condition that two quadrics $U = 0$, $V = 0$ should intersect in two conics, lying in planes $P = 0$, $Q = 0$, is the existence of an identity of the form $\lambda U + \mu V + \nu PQ = 0$, where λ, μ, ν are constants. And this remains true when the two conics coincide, the quadrics touching one another at all points of this.

Ex. 2. Three quadric surfaces, which have no common curve, meet in eight points. Taking for points of reference a self-polar tetrad for one of the quadrics, the common points may be found by substituting, in both the other two quadric equations, the coordinates $x = \theta + \phi$, $y = 1 - \theta\phi$, $z = 1 + \theta\phi$, $t = \theta - \phi$. The result is two equations, $(2, 2) = 0$ and $[2, 2] = 0$, which are both polynomials of the second order in each of θ and ϕ. The elimination of ϕ between these leads to an equation for θ by the solution of which the intersections can be found. Geometrically, regarding $(\theta, \phi, 1)$ as coordinates in a plane, the statement is that two quartic curves in a plane, which have common two double points, have eight other common points. The result may be regarded as always true, by proper interpretation of multiple intersections.

Ex. 3. Every quadric through seven points of space of which no four lie in a plane, and, therefore, no three lie in a line, has an equation of the form $\lambda U + \mu V + \nu W = 0$, wherein U, V, W can be supposed to be of the respective forms

$$yz + (a_1 x + b_1 y + c_1 z)\,t,\ zx + (a_2 x + b_2 y + c_2 z)\,t,\ xy + (a_3 x + b_3 y + c_3 z)\,t.$$

We can now prove that $U = 0$, $V = 0$, $W = 0$ have eight common points, so once more proving that all quadrics through seven given points have an eighth point common. In fact $V=0$, $W=0$ both contain the line $x=0$, $t=0$; thus they also intersect in a cubic curve, meeting this line in two points. This curve will meet $U=0$ in six points; these, with the two points in which $U=0$ is met by the line $x=0$, $t=0$, are the eight points in question, and, of these six, two are on the line $y=0$, $z=0$. If we put $x=\theta t$ in $V=0$, $W=0$, we find $a_2x+b_2y+(c_2+\theta)z=0$, $a_3x+(b_3+\theta)y+c_3z=0$, from which we have the expression of the cubic curve by the parameter θ. Substituting this expression in $U=0$, the determination of the four points of intersection of the three quadrics, beside the given points of reference (1, 0, 0, 0), (0, 1, 0, 0), etc., is reduced to the solution of a quartic equation.

Upon the curve of intersection of two general quadrics, there can be taken such seven independent points as we have supposed. Thus, the curve of intersection of two general quadrics is such that, of its intersections with another quadric, one is determined by the assignment of the other seven.

Ex. 4. Shew that if three quadric surfaces have a line in common, they have four common points not lying on this. And, if three quadric surfaces have a conic in common, they have two common points not lying thereon.

Ex. 5. If all possible quadric cones be constructed which contain six arbitrary fixed points of general position, prove that the vertices of these cones lie upon a surface meeting an arbitrary line in four points (Weddle's surface). Shew that this surface contains the fifteen lines joining the pairs of the six given points, and the ten lines of intersection of the pairs of planes each containing three of the points; and contains also the cubic curve passing through the six given points. Any line drawn through one of the six given points meets the surface in only two other points.

Ex. 6. The vertices of the quadric cones containing seven arbitrary fixed points of general position lie upon a curve which meets an arbitrary plane in six points. This curve has for chords the twenty-one joins of the seven points, and also the seven joins of these points to the further point common to all the quadrics containing these seven given points. (Hesse, *Crelle's J.*, XLIX, 1855, p. 279.)

Theorems for eight associated points. Some results may be given in regard to a set of eight points which are such that every quadric through seven of them also passes through the eighth.

Ex. 1. If A, B, C, M, D', L be six points of a conic it follows, from Pascal's theorem, that the point (CM, LA) is on the line joining the point (AB, MD') to the point $(BC, D'L)$.

Now let A, B, C, D be any points of a quadric surface, of which l and m are two generators of the same system. Let these generators meet the section of the quadric by the plane ABC respectively in

L and M, and let the generator of the quadric, of the opposite system, which passes through D, meet this section in D'. The point (CM, LA) is then on the line of intersection of the planes Cm, lA; the point (AB, MD') is the point where the line AB meets the plane mD, and may be described as the point (AB, mD); so the point $(BC, D'L)$ is the point (BC, Dl). Thus the line (Cm, lA) meets the join of the points (AB, mD), (BC, Dl).

Ex. 2. Let A, B, C, D and A', B', C', D' be any general set of associated points. There are then three linearly independent quadrics passing through these, and a quadric can be put through A, B, C, D to have $A'B'$ and $C'D'$ as generators of the same system. Denoting the points, respectively, by the numbers 3, 4, 5, 8 and 1, 2, 6, 7, and identifying the lines l, m of Ex. 1 respectively with the generators 12, 67, we thus reach the result that, if 1, 2, 3, 4, 5, 6, 7, 8 be a general set of associated points, the line (123, 567) intersects the join of the two points (34, 678), (45, 812).

Ex. 3. If we then remark that this join lies on the plane 345, we see that this join is also met by the line (345, 781). Again, the line (234, 678) contains the point (34, 678), and the line (456, 812) contains the point (45, 812). Thus the four lines

(123, 567), (234, 678), (345, 781), (456, 812)

are all met by the line joining the points, (34, 678), (45, 812). These four lines are, however, only interchanged among themselves by cyclical interchange of the numbers; it follows that they are met also by the join of the points (45, 781), (56, 123), and so on, in all by eight such joining lines. Thus the four lines are generators of the same system of a quadric.

Ex. 4. Let 2, 3, 4, 5, 7, 8 be any six points, and 1 a further point. From the last point let the transversal be drawn to the lines 23, 57, meeting these, respectively, in P and P'. Similarly, let Q, on 78, and Q', on 34, be on a line through 1; and R, on 45, with R', on 82, be also on a line through 1. The lines QR', RP', PQ' are then generators of a quadric, of which $Q'R$, $R'P$, $P'Q$ are also generators. This quadric meets the lines 23, 34, 45, 57, 78, 82, respectively, in other points P_1, Q_1', R_1, P_1', Q_1, R_1'. It can be shewn that the three lines P_1P_1', Q_1Q_1', R_1R_1' meet in a point. Denoting this by 6, the points 1, 2, ..., 8 are eight associated points. The proof follows easily from Ex. 2.

Ex. 5. For the determination of the eighth point of an associated set when seven are given, we have, by what precedes, two theorems: (1), the cubic curves determined by the two sets of six points, A, B, C, D, A', C'; A, B, C, D, B', C' have, respectively, chords meeting in the eighth point, D', required. The chord of the first curve is that drawn from B', that of the second curve is that drawn from A'. Or, as we have seen (Ex. 18, p. 139, above) that a single cubic curve exists passing through five given points, having a given line as chord, we can regard D' as the common point of the various cubic curves constructed each to pass through five of the seven given points and have the join of the other two as chord. (2), if A, B, C, D be a self-polar tetrad in regard to a quadric, and every two of A', B', C' be conjugate in regard to this quadric, the point D' is the pole of the plane $A'B'C'$ in regard to this. Hesse has given a linear construction for D' founded on this last fact (see *Ges. Werke*, pp. 46, 48, 680. See also Cayley, *Papers*, I, p. 425 (1849), for another view of the problem); it depends upon a theorem of plane geometry: In a plane let A, B, C be a self-polar triad in regard to a conic, and P, U; Q, V; R, W be three pairs of conjugate points in regard to this conic; then the polar lines, respectively l, m, n, of U, V, W, can be linearly constructed. Let BU, AV meet in O; let PO meet AC in Y, and QO meet BC in X; further, let XV meet AQ in H, and YU meet BP in K. Also let AC meet HK in L, and BC meet HK in M. Then PL is the polar of U, and QM the polar of V. Similarly, the line n can be constructed.

This being assumed, and the points A, B, C, D, A', B', C' given as above, let the six lines DA', $B'C'$, DB', $C'A'$, DC', $A'B'$ meet the plane ABC, respectively, in U, P, V, Q, W, R. The triad ABC will be self-polar in regard to the conic in which the quadric meets the plane ABC, and the polar line of DA' will be a line, l, lying in this plane and in a plane which contains $B'C'$. Thus the polar line of U in regard to this conic will be the line l, and this will contain P. Similarly V, Q will be conjugate points in regard to this conic, the polar line, m, of V containing Q; and W, R will be such that the polar line, n, of W, contains R.

The lines l, m, n being constructed as explained, the point D', required, is the intersection of the planes $(l, B'C')$, $(m, C'A')$, $(n, A'B')$.

Ex. 6. If the curve of intersection of two general quadric surfaces be projected on to a plane from any point of itself, it becomes a cubic curve. Conversely, it can easily be shewn that a curve in a plane, expressed by the vanishing of a homogeneous cubic polynomial in the three coordinates, may be regarded as arising by the projection of the curve of intersection of two quadric surfaces. Bearing in mind that the latter curve meets every generator, of a quadric surface on which it lies, in two points, the theorem that seven points in space determine an eighth is at once seen to be in correspondence with the theorem, for a plane, that all general cubic curves with eight common points pass through a ninth. We may construct this ninth point as follows: let A, B, C, M, N and P, Q, R be eight points of a plane, and D a point not lying in this plane; let Ω be a quadric containing A, B, C and the lines DM, DN (other than the plane pair ABC, DMN); let DP, DQ, DR meet this quadric (beside at D) in P_1, Q_1, R_1. Now let the cubic curve in space, which contains the five points D, A, P_1, Q_1, R_1 and has BC for a chord, meet Ω again in O_1. Then DO_1 meets the plane in the ninth point required. For this problem see also Cayley, *Papers*, IV, pp. 495–504 (1862). Also p. 178, below.

Ex. 7. A further way of regarding the theorem that quadrics through seven points have an eighth common point should be referred to briefly. Taking five arbitrary points, of which no four lie in a plane, there are five linearly independent quadrics passing through these, say $U_1=0$, $U_2=0$, ..., $U_5=0$. If four of the points be the points of reference and the fifth be (1, 1, 1, 1), the functions $U_1, \dots, U_5$ are in fact arbitrary linear functions of the six functions

$$\xi=y(t-z),\quad \eta=z(t-x),\quad \zeta=x(t-y),$$
$$\xi'=(t-y)z,\quad \eta'=(t-z)x,\quad \zeta'=(t-x)y,$$

which are equivalent to five in virtue of the identity

$$\xi+\eta+\zeta=\xi'+\eta'+\zeta',$$

and are further connected by the relation $\xi\eta\zeta=\xi'\eta'\zeta'$. We may regard the ratios of any five independent linear functions of ξ, η, ζ, ξ', η', ζ' as coordinates of a point of space of four dimensions. Thereby, to any point of our original three-fold space, other than certain exceptional points for which all of $\xi, \eta, \dots, \zeta'$ are zero, there corresponds a point of the three-fold locus, Σ, lying in the four-fold space, for which $\xi\eta\zeta=\xi'\eta'\zeta'$; conversely, save for exceptional points, to any point of this three-fold locus, Σ, there corresponds the point of the original space given by

$$x/t=(\eta'-\zeta)/(\xi-\xi'),\quad y/t=(\zeta'-\xi)/(\eta-\eta'),\quad z/t=(\xi'-\eta)/(\zeta-\zeta').$$

To the general quadric of the original space which passes through the given five points there corresponds, then, in the four-fold space, the intersection of the locus Σ with the locus expressed by an equation of the form $A\xi+B\eta+C\zeta=A'\xi'+B'\eta'+C'\zeta'$, wherein the relation $\xi+\eta+\zeta=\xi'+\eta'+\zeta'$ is understood. Let such a three-fold locus, or space, expressed by a linear equation, be, for distinctness, denoted by σ. To the quadrics of our original space passing through seven given points, consisting of the five chosen fundamental points and two other points, say A and P, will correspond the spaces σ, in the four-fold space, which pass through two given points, say $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ and $(\lambda, \mu, \nu, \lambda', \mu', \nu')$, of the locus Σ. These will contain all points of the line joining these two points of Σ; and, as we see by substituting expressions of the forms $\alpha+\phi\lambda, \dots, \gamma'+\phi\nu'$ for the coordinates in the equation of Σ, there is a third point of Σ lying on this line, through which, then, all these spaces σ pass. This is the point corresponding to the eighth intersection of the quadrics in the original space which pass through the seven given points.

The correspondence can be followed in detail. We limit ourselves to the result in the four-fold space which corresponds to the fact that the eighth point lies on the chord, drawn from P, to the cubic curve containing the five fundamental points and the point A. Let this point be (a, b, c, d); this cubic curve is then given by

$$x = a(\theta - a)^{-1}, \quad y = b(\theta - b)^{-1}, \quad z = c(\theta - c)^{-1}, \quad t = d(\theta - d)^{-1},$$

for varying values of θ, the points $(1, 1, 1, 1)$, (a, b, c, d) arising respectively for $\theta = 0$, $\theta = \infty$. It is easily seen that the locus in the four-fold space to which this corresponds is the line given by the three simultaneous equations $\xi/\alpha = \xi'/\alpha'$, $\eta/\beta = \eta'/\beta'$, $\zeta/\gamma = \zeta'/\gamma'$. This line passes through the point $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$, and lies entirely on the locus Σ. Through this line, and the point $(\lambda, \mu, \nu, \lambda', \mu', \nu')$, corresponding to P, there passes a plane; this plane meets the locus Σ, in fact, in a conic, as we see from the fact that any line meets the locus Σ in three points; the equations of the conic may easily be found. This conic is met by the line, given by the equations $\xi/\alpha = \xi'/\alpha'$, etc., in two points; it corresponds to the chord drawn from P in the original space, to meet the cubic curve; by its construction it contains the point $(\lambda, \mu, \nu, \lambda', \mu', \nu')$, and the line joining $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ to this point meets the conic again in the point of Σ which corresponds to the eighth intersection of the quadrics in the original space.

The locus Σ will be found to be of great interest to us. If, instead of $\xi, \eta, \zeta, \xi', \eta', \zeta'$, we use coordinates

$$x_1 = \tfrac{1}{2}(\eta + \zeta - \xi), \qquad x_2 = \tfrac{1}{2}(\zeta + \xi - \eta), \ \ldots,$$
$$x_4 = \tfrac{1}{2}(\xi' - \eta' - \zeta'), \ \ldots, x_6 = \tfrac{1}{2}(\zeta' - \xi' - \eta'),$$

its equations take the forms $\Sigma x_r^3 = 0$, $\Sigma x_r = 0$, $r = 1, \ldots, 6$.

The set of eight associated points was considered by von Staudt, *Beiträge zur Geometrie der Lage*, III, 1860, pp. 372 ff. The theorem of Ex. 2, above, was given by Hesse; see *Ges. Werke*, p. 680, and various papers in *Crelle's J.* The theorem of Ex. 3 was given by Zeuthen; see *Acta Math.* XII, 1889, p. 363. See also Dobriner, *ibid.* p. 339; and Schroeter, *Acta Math.* XIV, 1890. The locus Σ considered in Ex. 7 was studied by Segre; see *Atti Acc. sc. Torino*, XXII, 1886, 7; p. 547.

CHAPTER IV

THE GENERAL CUBIC SURFACE; INTRODUCTORY THEOREMS

In this book we regard the theory of a cubic surface as best treated in connexion with a certain figure in space of four dimensions. But the preliminary theorems for a general cubic surface are so intimately related with other results included in this volume that it seems desirable to give some account of them.

The theorem of a double-six of lines. It is convenient to give at once an independent proof of a theorem which arises implicitly below. Suppose that a, b, c, d, e are five lines, no two meeting one another, which have a common transversal, f'. There is then, beside f', a common transversal of every four of the five lines. The theorem referred to is that the five transversals so arising have themselves a common transversal. Denote by e' the transversal of a, b, c, d, beside f', by d' the transversal of a, b, c, e beside f', and so on. We are to prove that the five lines a', b', c', d', e' are all met by a line. It is clear that no two of these lines intersect; if, for instance, a' and b' were in one plane, every two of c, d, e, all of which meet a' and b', would intersect.

We prove that a', b', c', d', e' have a common transversal by shewing that the common transversal of a', b', c', d', other than e, coincides with the common transversal, other than d, of a', b', c', e'. This we do by shewing that the common transversal, other than e, of a', b', c', d', meets a', b', c', respectively, in three points which can be constructed from the lines a, b, c, a', b', c', f' only. In fact, denoting the nine intersections,

$$(b', c),\ (b, c'),\ (c', a),\ (c, a'),\ (a', b),\ (a, b'),\ (a, f'),\ (b, f'),\ (c, f'),$$

respectively by

$$A,\ A',\ B,\ B',\ C,\ C',\ U,\ V,\ W,$$

we shew that the common transversal of a', b', c', d', other than e, meets a' on the plane (A, A', U), meets b' on the plane (B, B', V), and meets c' on the plane (C, C', W).

Let f denote the common transversal, other than e, of a', b', c', d', and let this meet the lines a', b', c', respectively, in X, Y, Z. Consider the two quadric surfaces (b, c, e) and (b', c', d'), having, respectively, as generators of the same system, the two triplets of lines indicated. As the line e meets b', c', d', it is a generator of the second quadric as well as of the first; as d' meets b, c, e, it is a generator of the first quadric as well as of the second. As these two quadrics have thus two intersecting lines as common generators, their remaining common points lie in a plane. Of such points, not lying on e or d', are the points A, or (b', c), and A', or (b, c'); so also the point U, or (a, f'), is such a point, since the line a, meeting b', c', d', is a generator of the second quadric, and f', meeting b, c, e, is a generator of the first quadric. Lastly, the point X, or (a', f), is common to the two quadrics, since a' meets b, c, e, and f meets b', c', d'. Thus X, where f meets a', lies on the plane (A, A', U). By considering the two quadrics (c, a, e) and (c', a', d'), we similarly shew that Y lies on the plane (B, B', V); and, by considering the two quadrics (a, b, e) and (a', b', d'), that Z lies on (C, C', W).

For the common transversal, other than d, of a', b', c', e', we proceed in the same way, using e', d instead of d', e, respectively. The planes (A, A', U), (B, B', V), (C, C', W) arise once more. The theorem is thus proved.

Such an aggregate of twelve lines

$$\begin{matrix} a, & b, & c, & d, & e, & f, \\ a', & b', & c', & d', & e', & f', \end{matrix}$$

consisting of two sextuples, wherein each line meets five lines of the sextuple to which it does not belong, but is skew to the other lines of its own sextuple, and is also skew to a corresponding line of the other sextuple, is called a double-six. We shall commonly speak of the lines of the double-six which form a sextuple as a *row* of the double-six.

From the twelve lines of the double-six, fifteen other lines can be defined, as follows: associating the lines $a, b, \ldots, f, a', b', \ldots, f'$, respectively, with the numbers $1, 2, \ldots, 6, 1', 2', \ldots, 6'$, let the line of intersection of the plane of a, b', with the plane of a', b, say, of the plane $[1, 2']$ with the plane $[1', 2]$, be denoted by c_{12}; there will be fifteen such lines, obtained by taking every pair of the numbers $1, 2, \ldots, 6$. Fifteen other lines can also be obtained from the double-six: we may join the point of intersection, $(1, 2')$, of the lines a, b', to the point of intersection, $(1', 2)$, of the lines a', b; this construction can be made in fifteen ways. We may denote the joining lines by symbols c_{rs}'.

It has incidentally been shewn above that the four points $(2, 3')$, $(2', 3)$, $(1, 6')$, $(1', 6)$ lie in one plane, this being the fact that the points A, A', U, X are in one plane. Thus the line c_{23}' intersects the line c_{16}'. By a precisely similar argument it follows that, if no two of the numbers r, s, i, j are the same, the lines c_{rs}', c_{ij}' intersect one another. Thence, by the dual argument, the plane of two intersecting lines being considered instead of their point of intersection, it follows, if no two of the numbers r, s, i, j are the same, that any two lines c_{rs}, c_{ij} intersect one another.

We have now two sets, each of (12 + 15, or) twenty-seven lines, one set consisting of $a, b, \ldots, f, a', b', \ldots, f'$, together with the lines c_{rs}, the other set consisting of the same twelve lines together with the lines c_{rs}'. Let us consider the former set. It is easy to see that the lines of this set are symmetrical as regards their intersections with one another: each line of the twenty-seven intersects ten others of the set, and these consist of five pairs of mutually intersecting lines, so that in the plane of any two intersecting lines of the set is a third line of the set. For instance, the line 1 (or a) is met by the ten lines

$$2', c_{12};\ 3', c_{13};\ 4', c_{14};\ 5', c_{15};\ 6', c_{16},$$

which consist of five pairs of intersecting lines. Similarly, the line c_{12} is met by the lines of the five pairs, each of two intersecting lines,

$$1, 2';\ 1', 2;\ c_{34}, c_{56};\ c_{35}, c_{64};\ c_{36}, c_{45}.$$

Further, it is possible to arrange the twenty-seven lines in two sets of, respectively, twelve and fifteen lines, in other than the original way, so that the twelve lines form a double-six, and the fifteen are deducible from these by the same rule as were the lines c_{rs} from the original double-six; there are in fact thirty-six such arrangements. Writing, for greater clearness, 23 instead of c_{23}, and so on, the following twelve lines

$$\begin{matrix} 1, & 1', & 23, & 24, & 25, & 26 \\ 2, & 2', & 13, & 14, & 15, & 16 \end{matrix}$$

are at once seen to form a double-six; further, the lines formed from these, as were the lines c_{rs} from the original double-six, are easily seen to be the other fifteen lines of the original twenty-seven. Such a double-six as this can be constructed in fifteen ways. Another double-six, typical of twenty double-sixes, is given by

$$\begin{matrix} 1, & 2, & 3, & 56, & 64, & 45 \\ 23, & 31, & 12, & 4', & 5', & 6', \end{matrix}$$

and the remaining fifteen lines arise from this by the same rule as before. We see then how to obtain 1 + 15 + 20, or thirty-six, double-sixes.

Reciprocity of the figure. Considering the original double-six, it can be shewn that there exists a quadric surface in regard to which the six lines 1, 2, ..., 6, of one row, are, respectively, the polar lines of the corresponding lines, 1′, 2′, ..., 6′, of the other row. In regard to this quadric surface, then, the line c_{rs} is the polar of c_{rs}'. To prove this, consider the fifteen triplets of lines each consisting of two lines of the first row of the double-six taken with that one of the lines c_{rs} which intersects both; such a triplet is given, for instance, by the three lines 1, 2, c_{12}. With these fifteen triplets consider also the similar triplets each formed from two lines of the second row of the double-six taken with that one of the lines c_{rs} which intersects both, such, for instance, as $3'$, $4'$, c_{34}. Then it can be seen that a definite quadric exists, possibly degenerating into two planes, containing the combination formed by any triplet of the first set of fifteen triplets, taken with any triplet of the second set. For instance, the two triplets 1, 2, c_{12} and $1'$, $2'$, c_{12} both lie in the plane-pair $[1, 2']$, $[1', 2]$, whose line of intersection is the line c_{12}; again, the two triplets 1, 2, c_{12} and $2'$, $3'$, c_{23} lie on the plane-pair of which one plane contains the lines 1, $2'$, c_{12} and the other plane contains the lines 2, $3'$, c_{23}; while two triplets such as 1, 2, c_{12} and $3'$, $4'$, c_{34} consist of three lines 1, 2, c_{34}, of which no two intersect, together with three other such lines, $3'$, $4'$, c_{12}, of which, however, each intersects all the first three, so that the six lines all lie on a quadric surface. Assuming the statement verified, there will be, in all, 15 . 15 or 225 quadrics; it is then, further, the case that the equations of these quadrics, expressed in point coordinates, are linear functions of *nine* of them. For there are three linearly independent quadrics passing through a triplet consisting of two skew lines and a transversal of these: Let P, Q, R, S denote four triplets each containing two lines of the first row of the double-six, and P', Q', R', S' denote four triplets each containing two lines of the second row of the double-six, and let (S, S') denote the quadric containing the triplets S and S', and so in general; then, if the quadrics (S, P'), (S, Q'), (S, R') be linearly independent, the quadric (S, S') is a linear function of these; if the quadrics (P', P), (P', Q), (P', R) be linearly independent, the quadric (S, P') is a linear function of these; similarly, with a like hypothesis, the quadric (S, Q') is expressible by (Q', P), (Q', Q), (Q', R), and the quadric (S, R') by (R', P), (R', Q), (R', R); finally, every one of the quadrics is thus seen to be expressible by the nine obtained by taking P, Q, R with P', Q', R', provided that, in every case, the three quadrics (S, P'), (S, Q'), (S, R') are linearly independent. A detailed scrutiny shews that this is so in general.

There is, however, a single quadric envelope which is inpolar to nine linearly independent quadrics (above, p. 52). Thus there exists a quadric envelope, in general unique, which is inpolar to the 225 quadrics spoken of. This we may denote by Σ; it will sometimes be spoken of as the Schur quadric associated with the double-six.

Now take the three pairs of triplets

$$1, 2, c_{12} \text{ and } 1', 2', c_{12};\quad 1, 2, c_{12} \text{ and } 2', 3', c_{23};\quad 1, 3, c_{13} \text{ and } 1', 2', c_{12},$$

lying, respectively, in the three plane-pairs

$$[1, 2', c_{12}], [1', 2, c_{12}];\quad [1, 2', c_{12}], [2, 3', c_{23}];\quad [1, 2', c_{12}], [1', 3, c_{13}];$$

each of the pairs of planes being conjugate to one another in regard to the quadric Σ, the pole of the plane $[1, 2']$ is seen to be the common point of the planes $[1', 2]$, $[2, 3']$, $[1', 3]$, namely, the point $(1', 2)$. Similarly, the pole of the plane $[1, 3']$ is the point $(1', 3)$. The two planes both contain the line 1, the two points both lie on the line $1'$. Thus the lines $1, 1'$ are polars of one another in regard to the quadric Σ, as are, similarly, 2 and $2'$, 3 and $3'$, etc.

The quadric Σ is in fact determined by the six lines 1, 2, 3, $1'$, $2'$, $3'$, only. These six lines are the edges of a skew hexagon, A, B', C, A', B, C', wherein A, A' are, respectively, the points (b', c), (b, c'), or $(2', 3)$, $(2, 3')$, etc., and the quadric Σ is such that opposite edges of this hexagon are polar lines of one another. Thus the point A, the intersection of the edges AB' and AC', is the pole of the plane $A'BC$, etc. Let the three planes $A'BC$, $B'CA$, $C'AB$ meet in the point D, so that AD is the intersection of the planes $AB'C$, $AC'B$, etc. Then D is the pole of the plane ABC, and A, B, C, D are a self-polar tetrad of points in regard to Σ; while A', B', C' are three points of which every two are conjugate to one another in regard to Σ. These two conditions determine the quadric. It will, however, be seen to be of advantage to have considered the quadric from the point of view we have taken.

Another determination of the double-six. It can be shewn that, if we have six lines, a, b', c, a', b, c', forming a skew hexagon, as here, and another line, d, which meets one set of alternate edges, a', b', c', of the hexagon, then a double-six is determined, which will, therefore, be that here discussed with the same notation for the lines. The proof of this is instructive, and may be given. Consider the aggregate of lines which meet the three skew lines a', b', c', these being the generators, of the other system from these, of the quadric determined by a', b', c'. Let us regard two of these lines, say (x) and (y), as *corresponding*, when they are both met by some transversal of the three other alternate edges, a, b, c, of the hexagon. We can then see that to any (x) there correspond two lines (y), and, to any (y), two lines (x); for when a line (x), meeting a', b', c', is taken, there are two transversals of the four lines (x), a, b, c; if one of these, say (z), be taken, there is a definite transversal (y), beside (x), of the four lines (z), a', b', c'; similarly when (y) is taken first. A line (x), meeting a', b', c', may be defined by a parameter, x, as, for instance, by lying in a definite plane, of equation $U + xV = 0$, passing through a'; the correspondence between the lines (x) and (y) may then be represented by an equation connecting two parameters, x and y; this equation, by the geometrical definition, will be rational, of the second order in each of the parameters, and symmetrical in regard to these. If y_1 and y_2 be the two values of y given by this equation, corresponding to a value of x, it may, or may not, happen, that the two values corresponding to the value, y_1, of x, are x and y_2; this will happen, however, for all values of x if it happen for a particular value. In such case the three values x, y_1, y_2 will be one set of a general involution of triads of values (Vol. II, pp. 135 ff.). If then we can shew that there exists one set of three transversals of a', b', c' with the property that every two of them are met by a transversal of a, b, c, it will follow that, if we take the line d, arbitrarily, to meet a', b', c', and then the lines e, f, also to meet a', b', c', by the condition that e meets the transversal, f', of a, b, c, d, and f meets the other transversal, e', of a, b, c, d, then e and f will also meet a transversal, d', of a, b, c. We shall then have constructed a double-six, clearly unique, from the lines a, b, c, a', b', c', d. It is, however, easy to find one set of three transversals of a', b', c' with the property that every two of them are met by a transversal of a, b, c. For, let O and ϖ be, respectively, a point and a plane which are pole and polar of one another in regard to both the quadrics (a, b, c), (a', b', c'), defined, respectively, by the generators indicated; then, through the point where the line a' meets the plane ϖ, there passes a generator, say a_1', of the quadric (a', b', c'), which meets b' and c', and is the harmonic image of a' in regard to O and ϖ (see above, p. 80); similarly, through the point where the line a meets ϖ, there passes a generator, say a_1, of the quadric (a, b, c), which meets b and c as well as a. With an analogous notation in general, since b' and c' both meet a, it follows that b_1' and c_1', which both meet a', b', c', also both meet a_1. Three transversals of a', b', c' which are such that every two of them are met by a transversal of a, b, c, are, therefore, given by a_1', b_1', c_1'. Thus the proof of the existence of the double-six is completed.

The Steiner trihedral pairs. It has been shewn above that the quadrics containing the pairs of triplets of lines such as r, s, c_{rs} and $r', s', c_{r's'}$ are linearly dependent from nine such quadrics. These may in fact be taken to be those arising from the lines 1, 2, 3, 1′, 2′, 3′, c_{23}, c_{31}, c_{12}, for which the nine quadrics are all plane-pairs. Let the triplets in question, namely

$$2, 3, c_{23};\ 3, 1, c_{31};\ 1, 2, c_{12};\ 2', 3', c_{23};\ 3', 1', c_{31};\ 1', 2', c_{12},$$

be denoted, respectively, by p, q, r, p', q', r', and the planes [2, 3′], [2′, 3], [3, 1′], [3′, 1], [1, 2′], [1′, 2] be denoted, respectively, by α, α', β, β', γ, γ'; the quadrics containing the pairs of triplets

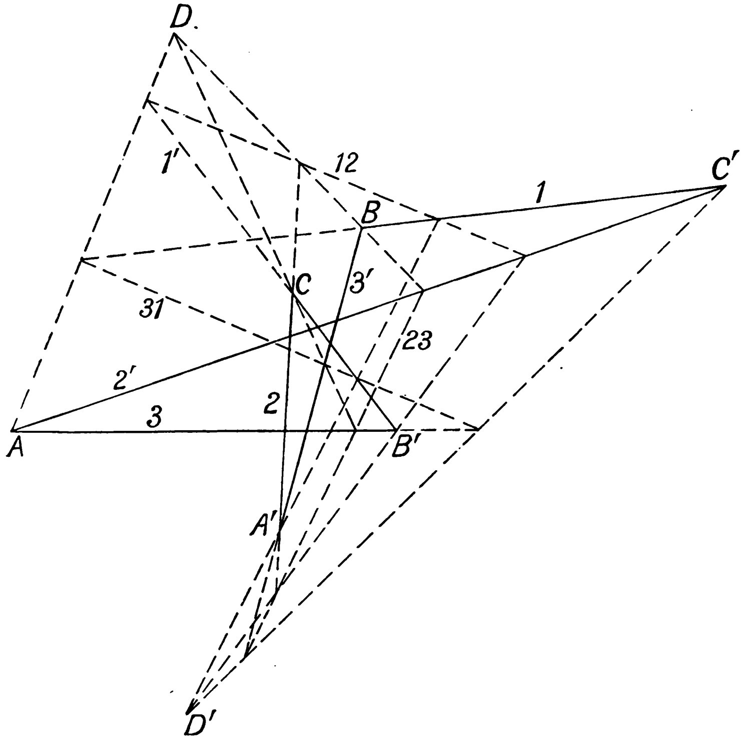
$$q, r';\ q', r;\ r, p';\ r', p;\ p, q';\ p', q;\ p, p';\ q, q';\ r, r'$$

are then, respectively, the plane-pairs

$$\beta, \gamma;\ \beta', \gamma';\ \gamma, \alpha;\ \gamma', \alpha';\ \alpha, \beta;\ \alpha', \beta';\ \alpha, \alpha';\ \beta, \beta';\ \gamma, \gamma'.$$

The nine lines are the intersections of the three planes α, β, γ

with the three planes α', β', γ'. If, as above, we introduce the point D, the intersection of the planes α, β, γ, and the lines DA, DB, DC, joining this to the points $(2', 3)$, $(3', 1)$, $(1', 2)$, respectively, and also introduce the point D', the intersection of the planes α', β', γ', and the lines $D'A'$, $D'B'$, $D'C'$, joining this, respectively, to the points $(2, 3')$, $(3, 1')$, $(1, 2')$, we shall in the aggregate obtain the figure consisting of all the intersections of the six planes α, β, γ, α', β', γ'. This contains twenty points, fifteen lines and six planes, of which each point lies in three of the lines and in three of the planes, each line contains four of the points and lies in two of the planes, while each plane contains ten of the points and five of the lines, so that the incidences can be denoted by the symbol



$$20\,(.,3,3)\,15\,(4,.,2)\,6\,(10,5,.).$$

The twenty points are the six points A, B', C, A', B, C', the points D, D', and the three sets of four points such as, for example, the intersections of c_{12} with 1, $2'$, $1'$, 2. The fifteen lines are the lines 2, $3'$, ..., c_{12}, originally considered, together with the six such as DA; each of these six lines contains three points of intersection of two of the nine original lines; for instance, DA contains the three points $(2', 3)$, $(1, c_{23})$, $(1', c_{12})$; and the nine lines consist of three coplanar triads all in perspective from D, and consist also of three coplanar triads in perspective from D'. The figure is a section, by a three-fold, of the figure in space of four dimensions considered on pp. 214, 215 of Vol. II.

Relatively to the nine original lines, the two triads of planes, α, β, γ, and α', β', γ', may be spoken of as a pair of *Steiner trihedrals.*

Examples of the configuration of twenty-seven lines. *Ex.* 1. It appears, from what has been said, that, from the twenty-seven lines, $a, b, \ldots, f, a', b', \ldots, f', c_{rs}$, a set of six lines can be chosen of which no two intersect, in seventy-two ways. Such a set of six lines being called a sextuple, it will be found that there are sixteen sextuples containing any one of the twenty-seven lines, but that no two sextuples have four lines in common. Conversely, any set of four lines, from the twenty-seven, of which no two intersect, lies in one of the sextuples.

Further, it will be found that two non-intersecting lines can be chosen from the twenty-seven in 216 ways, three lines of which no two intersect in 720 ways, four lines of which no two intersect in 1080 ways, and five lines of which no two intersect in 648 ways. Such a set of five skew lines has at least one common transversal, also one of the twenty-seven lines, and may have two such transversals. There are no sets of seven, or more, lines, among the twenty-seven, of which no two intersect.

Ex. 2. There are in all 135 points of intersection of two lines from the twenty-seven, and any two such intersecting lines are both met by a third line. Of such triads of three coplanar lines from the twenty-seven there are in all 45.

Ex. 3. The twenty-seven lines have been seen to arise from five non-intersecting lines with a common transversal. Now consider the surface which is represented by the vanishing of a general homogeneous polynomial of the third order in the space coordinates; this is called a cubic surface, being met by an arbitrary line in three points, the substitution of values of the forms $x_0 + \lambda x_1$, $y_0 + \lambda y_1$, etc., for the coordinates, leading to a cubic equation in λ. The conditions that a given line should lie entirely on the surface are four in number, all linear in the coefficients of the equation of the cubic surface; thus the conditions for the cubic surface, that it should contain one given line and five other skew lines which meet this are, in all, nineteen in number. The cubic surface, whose equation contains twenty terms, can thus be determined to contain these six lines. The cubic surface will then equally contain the other twenty-one lines constructed as above from these six, since each additional line is such as to meet four of those already taken.

Conversely, we may expect that any cubic surface, of general type, contains such a set of twenty-seven lines. For the figure of five non-intersecting lines with a common transversal is itself of amplitude nineteen, depends, that is, on nineteen constants, as does the equation of a general cubic surface. There are four constants to determine the transversal, five more to fix the points where the other lines meet this, and two more to determine each of these lines, say, by fixing the point where this meets an arbitrary plane. The coefficients in the equation of the cubic surface are then functions of the constants determining the figure; and we may expect that the latter can be chosen to correspond to arbitrary values of the former. But, for this to be so, the nineteen constants of the figure must enter as nineteen effectively independent constants in the expressions, in terms of them, of the coefficients in the equation of the cubic surface. For this it is sufficient that no change in the figure of the six lines, of arbitrarily limited magnitude, shall be possible without a change in the cubic surface.

We may approach the algebraical theorem thus suggested, that a cubic surface expressed by the vanishing of a general cubic polynomial contains twenty-seven lines, in a somewhat different way. There are four algebraic conditions to be satisfied for a line to lie entirely on the surface; and a line depends on four constants. Let us assume thus that the cubic surface contains at least one line; it is then easy to infer the existence of the other lines. Taking the one line to be given by $x=0$, $y=0$, a general plane through this, of equation $x+\lambda y=0$, will meet the cubic surface further in a conic; it is at once proved that the condition for this to break up into two lines is a quintic equation for λ. By the solution of this equation we then have ten further lines on the surface, and the proof of the remaining sixteen, and that there are no more, can be completed. Or, we may initiate the investigation by seeking a plane which meets the cubic surface in three lines; the elimination of one coordinate between the equation of the cubic surface and the equation of the plane leads to a cubic polynomial in the three remaining coordinates; the three coefficients in the equation of the plane are to be chosen so that this cubic polynomial breaks up into three linear factors. In such way it can be proved that for a cubic surface whose equation has real coefficients there are at least three lines whose equations have real coefficients.

It has been shewn that the twenty-seven lines can be constructed from the figure consisting of a skew hexagon together with a transversal of three alternate sides of this hexagon. This figure, we easily see, is also of amplitude nineteen, depending on nineteen constants.

Another figure depending on nineteen constants is that of four lines of which no two intersect together with a point of general position. For a cubic surface to contain this figure requires however only seventeen conditions for the coefficients in the equation of the surface.

Ex. 4. The proof of the reciprocity of the double-six can be obtained also as follows: Consider the figure consisting of three non-intersecting lines, say a, b, c, together with three other non-intersecting lines, say d', e', f', of which every one meets all of the first three. It can be proved that this figure is obtainable, by incidences only, from five suitably chosen points of the figure, or, also, from five suitably chosen planes of the figure. Hence it can be shewn that any two such figures can be regarded as related to one another (in the sense explained in Vol. I, p. 150) so that to every point of one corresponds a point of the other; or can be regarded also as related to one another in such a way that to every point of one figure corresponds a plane of the other figure (v. Staudt, *Beiträge z. Geom. d. Lage*, I, p. 6). When the two figures are formed by the lines (a, b, c, d', e', f') and (a', b', c', d, e, f), of a double-six, it can be shewn that they can be related in the latter of the two ways referred to, with the further limitation that the relation is involutory, a point of either figure being the pole in regard to a quadric surface of the corresponding plane of the other figure. (See E. Stenfors, *Die Schläflische Konfiguration von zwölf Geraden*, Helsingfors, 1921.)

Ex. 5. An interesting special double-six has been discussed by Dr W. Burnside (*Proc. Camb. Phil. Soc.*, XVI, 1911, p. 418). Let A, B, C, D, E be five general points, taken in a definite order, forming a skew pentagon in three dimensions. From each of these points let the transversal be drawn to meet the two lines constituted by, (a), the opposite edge of the pentagon, and, (b), the join of the two vertices of the pentagon which are contiguous to that vertex from which the transversal is drawn. It can be shewn that the five transversals so obtained are all met by two lines. It can further be shewn that if the five original points be taken in all possible orders, just six such pairs of lines, each meeting five transversals described as above, are obtained. And, then, that these six pairs of lines constitute a double-six. This double-six then has the property of being unaltered by a correspondence of the three-fold space to itself, of the kind described Vol. I, p. 150, determined by the fact that the five fundamental points correspond to these same points taken in a different order. It is essentially the only possible double-six unaltered by such a (1, 1) point correspondence, or *collineation*.

The fact that the five transversals spoken of, drawn one from each of the five given points, are all met by two lines, is deducible from the theorem (above, page 68, Ex. 26) that there exists a focal system in which each of the five given points is the pole of the plane containing this and the two contiguous vertices of the pentagon; for, by this theorem, there exists also another focal system in which each point is the pole of the plane joining this point to the opposite edge of the pentagon. Analytically, if the five points be represented by the symbols A, B, C, D, E, subject to

$$A+B+C+D+E=0,$$

and m, m' be the two real numbers for which $m+m'=1$ and $mm'=-1$, it will easily be found that the thirty points obtainable by all symbols of the form $mA+B+C$, which, in another order, are the same as those obtainable by the symbols $m'A+B+C$, lie in fives upon twelve lines forming the double-six in question.

If we also take the ten points whose symbols are the differences, such as $A-B$, of the original symbols, these can be regarded, in ten ways (Vol. II, p. 212), as consisting of two triads in perspective, together with their centre of perspective and three points lying on their axis of perspective. By joining each of these ten points, regarded as centre of perspective, to the three points lying on the corresponding axis of perspective, we obtain in all fifteen new lines. These are in fact the lines c_{rs}, of our preceding theory, to be associated with the double-six above obtained.

If we take four of these ten points, which do not lie in a plane, as fundamental points, X, Y, Z, T, defined by

$$5X=A-E,\quad 5Y=B-E,\quad 5Z=C-E,\quad 5T=D-E,$$

the original five points may be defined in terms of these, namely by

$$A=4X-Y-Z-T,\quad \ldots,\quad E+X+Y+Z+T=0.$$

With coordinates, x, y, z, t, referred to X, Y, Z, T, the twelve lines of the double-six may be shewn to be, respectively, those given by the symbols

$$\begin{gathered}
a\,(y, z, x;\ z, y, t),\quad b\,(z, x, y;\ x, z, t),\quad c\,(x, y, z;\ y, x, t),\\
d\,(t, x, z;\ x, t, y),\quad e\,(t, y, x;\ y, t, z),\quad f\,(t, z, y;\ z, t, x);\\
a'\,(t, x, y;\ x, t, z),\quad b'\,(t, y, z;\ y, t, x),\quad c'\,(t, z, x;\ z, t, y),\\
d'\,(y, z, t;\ z, y, x),\quad e'\,(z, x, t;\ x, z, y),\quad f'\,(x, y, t;\ y, x, z),
\end{gathered}$$

where, for example, the symbol $a\,(y, z, x;\ z, y, t)$ means the line given by the equations $my+z+x=0$, $mz+y+t=0$. If the first row of these be denoted by $1, 2, \ldots, 6$, and the second row by $1', 2', \ldots, 6'$, respectively, the other fifteen lines are then given, respectively, by

$$\begin{gathered}
c_{35}\,(x, y+z);\ c_{26}\,(t, y+z);\ c_{34}\,(t, z+x);\ c_{15}\,(t, x+y);\ c_{14}\,(y+z, x+t),\\
c_{16}\,(y, z+x);\ c_{45}\,(y, z+t);\ c_{56}\,(z, x+t);\ c_{46}\,(x, y+t);\ c_{25}\,(z+x, y+t),\\
c_{24}\,(z, x+y);\ c_{13}\,(z, y+t);\ c_{12}\,(x, z+t);\ c_{23}\,(y, x+t);\ c_{36}\,(x+y, z+t),
\end{gathered}$$

where, for example, the symbol $c_{35}(x, y+z)$ means the line given by

$$x=0,\quad y+z=0.$$

It is easy, now, to verify that these twenty-seven lines all lie on the locus given by the equation

$$x^3+y^3+z^3+t^3-(x+y+z+t)^3=0,$$

and that the lines, $1', \ldots, 6'$, of the second row of the double-six, are, respectively, the polar lines of the lines, $1, \ldots, 6$, of the first row of the double-six, in regard to the quadric whose equation is

$$x^2+y^2+z^2+t^2+(x+y+z+t)^2=0.$$

It may also be verified that one form of the equation of the cubic surface is

$$(my+z+t)(mz+x+t)(mx+y+t)$$
$$=(m'y+z+t)(m'z+x+t)(m'x+y+t).$$

The system of twenty-seven lines here considered is not the most general possible. The lines of the double-six meet in pairs in six points lying on a plane in ten different ways, these being the planes containing the sets of three of the five points A, B, C, D, E; the other fifteen lines meet in threes in ten points, as we have seen, the three concurrent lines lying in a plane. The cubic surface involved is that called by Clebsch the *diagonal* surface.

Geometrical definitions of a cubic surface. Suppose we are given any four lines, a, b, c, d, of which no two intersect, and a point, O, of general position. Let the lines be considered as two couples, a, b and c, d. If now a plane be drawn through O, and its intersections with the lines a, b be joined by a line, and its intersections with c, d be joined by another line, these two joining lines will intersect, in a point, Q, of the plane. We may consider the locus of this point Q for all possible planes drawn through the point O. The locus is in fact one of which there are three points upon an arbitrary line, l. For, consider the two quadrics (a, b, l) and (c, d, l); as they have a common generator, their common tangent planes, other than those containing the line l, form a cubic developable, of which, then, three planes pass through O. Let one of these three planes meet the line l in the point Q; we prove that this is a point of the locus spoken of. The plane in question, touching the quadric (a, b, l), contains a generator of this which meets the lines a, b, l, and, hence, passes through the point Q where the plane meets l. Similarly, a transversal of the lines c, d can be drawn, in this plane, passing through Q. Thus Q is a point of the locus. We thus obtain three points of the locus upon l, and, conversely, can shew that there are no others. When the line l, in place of being general, passes through the point O, two of the planes of the developable through O contain the line l; but there are still two points, Q, upon l, beside O, each an intersection of a line meeting a, b with a line meeting c, d. We may, thus, speak of the locus of Q as a cubic surface. Evidently this surface contains the four lines a, b, c, d; for, for instance, from any point of the line a, a transversal can be drawn to c and d, and the plane through O which contains this transversal will meet b in a point which may be joined to the point already taken on a. Also, the surface contains the transversal drawn from O to a and b; for, from any point of this, a line can be drawn to meet c and d, through which, and the former transversal, there passes a plane containing O. By a similar argument, the surface contains the transversal drawn from O to c and d.

Ex. The figure consisting of four non-intersecting lines and a point, depends on nineteen constants. If the four lines be divided into two pairs, and the two transversals be drawn from the point, one to each of these pairs, we have a new figure, depending on the same constants. And there are three such, corresponding to the three ways in which the four lines can be divided into two pairs.

The conditions that a homogeneous cubic polynomial in the four space coordinates should vanish for every point of each of the four given lines, and also for the given point, are only seventeen in number; but if the cubic polynomial is further to vanish at every point of either of the two transversals drawn from the given point to two pairs of the four given lines, two further conditions are necessary, and only one such cubic polynomial is possible. We expect then that upon the locus represented by the vanishing of such a cubic polynomial, the figure of four lines and two transversals of two pairs of these, will be obtained. We shall in fact identify this figure, below, as arising from the figure of twenty-seven lines previously discussed.

Next, suppose we are given five non-intersecting lines, a, b, c, d, e, which have a common transversal; this transversal may, for the present, be denoted by l; later it will be called f'. Then, the five lines being regarded as consisting of the two couples a, b and c, d, beside the line e, consider the two quadrics (a, b, e) and (c, d, e). As these have in common the two intersecting generators l and e, their common points, other than those on these generators, lie on a plane, and, therefore, on a conic; and their common tangent planes, other than those containing these generators, pass through a point, and, therefore, envelop a quadric cone. Let O be the vertex of this cone. Evidently the planes joining O to the six lines a, b, c, d, e, l are planes of this cone; further, the plane of the two transversals drawn from any point P, of the line e, one to the pair a, b, the other to the pair c, d, is a common tangent plane of the two quadrics, and passes through O; while, conversely, any tangent plane of the cone meets e in a definite point, P.

Now let any line, say m, be drawn through O. The two quadrics, (a, b, m) and (c, d, m), having common the generator m, meet also in a cubic curve, of which the lines a, b, c, d, as well as m, are chords. From any point, Q, of this curve, can be drawn a generator of the quadric (a, b, m), meeting the lines a, b, m, and also a generator of the quadric (c, d, m), meeting the lines c, d, m; thus Q can be constructed by drawing a plane through the line m, and, therein, joining the points where this plane meets a and b, and also the points where this plane meets c and d, and taking the intersection of these joins; the locus of such points, Q, for all such planes through m, is the cubic curve in question. As, however, the line m passes through the vertex, O, of the quadric cone above found, there are two planes through m which touch this cone; and, in each of these, by what has been said, the point Q, so obtained, is a point, P, of the line e. The cubic curve which arises from the line m has, therefore, beside a, b, c, d, m, also the line e as chord.

We thus obtain an infinite number of cubic curves having the given lines a, b, c, d, e as chords, each having as chord also a particular line, m, drawn through a point O, this point being constructed from a, b, c, d, e by a certain procedure. The aggregate of the ∞^2 points, Q, of all these cubic curves is evidently that found by drawing an arbitrary plane through O and, therein, finding the intersection of the line, joining the points where a, b meet the plane, and the line joining the points where c, d meet the plane. It is thus a cubic surface, Ω, containing a, b, c, d, the point O, and the transversals drawn from O, to meet a, b, and to meet c, d. The cubic curves obtained, having a, b, c, d and e as chords, all lie on this cubic surface, Ω; moreover, since, in any plane through O, an infinite number of lines, m, can be drawn through O, there will be an infinite number of these cubic curves passing through the point, Q, determined as above by this plane. Also, the surface Ω contains the line e, because, as we have shewn, through every point, P, of this line, there passes a plane, also containing O, for which P satisfies the condition by which points Q are determined. The surface also contains the line l, every point of which satisfies the condition of being the point Q for the plane joining O to this line.

Every four of the five lines a, b, c, d, e has, beside l, another transversal; we prove now that all these lie on the surface Ω. For the transversal, e', of a, b, c, d, this is clear, all points of this line satisfying the condition for the point Q of the plane (O, e'). Consider the transversal, d', of a, b, c, e: from the point, P, where this meets e, draw the transversal to c and d, meeting d in D; then, the join of any point, Q, of d', to D, meets c as well as d, and the point

Q lies on the line d' which meets a and b. Thus, this point Q satisfies the condition for the point Q of the plane (D, d'); and this plane passes through O because it contains the transversals from P to a, b and to c, d; by the construction made, this plane contains the line c. Denoting, similarly, by a', b', c' the transversals, beside l, respectively of (b, c, d, e), (c, a, d, e), (a, b, d, e), the point O is thus incidentally shewn to lie on the four planes $[a, b']$, $[a', b]$, $[c, d']$, $[c', d]$, which, therefore, meet in a point; these planes are in fact all tangent planes of both the quadrics (a, b, e) and (c, d, e). The line of intersection of the planes $[a, b']$, $[a', b]$, which we may denote by c_{12}, is also on the surface Ω, being the transversal drawn from O to a and b; so is the line of intersection, c_{34} say, of the planes $[c, d']$, $[c', d]$. Thus O may be defined, when c_{12}, c_{34} are given, as the intersection of these lines.

We have constructed the point O, and the cubic surface Ω, from the four lines a, b, c, d, a transversal, l, of these, and a further line, e, which meets l. Conversely, suppose a, b, c, d, and the point O, to be given, and l to be one of the transversals of a, b, c, d. Then the quadric cone, having O for vertex, which touches the five planes joining O to a, b, c, d, l, is definite; thence the quadric surface having this cone as enveloping cone and containing the two lines a, b as generators is also definite (just as, dually, the quadric containing a given conic and also two non-intersecting lines each meeting the conic once, is definite); so also, the quadric having the specified cone as enveloping cone and containing the two lines c, d as generators, is definite. Of both these quadrics the line l is a generator; these quadrics will then have common another generator, meeting l. This is the line e, which is thus constructed when a, b, c, d and O are given. However, the lines a, b, c, d have, beside l, another common transversal, already named, e'. With this, and a, b, c, d, O, we then find a further line, say f, which meets e'. As then we proceeded from a, b, c, d, e, with the common transversal l, to find O, and the cubic surface, so we can construct O, and the surface, from a, b, c, d, f, with the common transversal e'; it is easy to see that the same surface is found, and that the lines a', b', c', d', before defined, as meeting, respectively, (b, c, d, e), (c, a, d, e), (a, b, d, e), (a, b, c, e), all meet f. Thus a', b', c', d', e' have a common transversal, and, incidentally, the theorem of the double-six is obtained.

In the construction of O, and the cubic surface, from the lines a, b, c, d, e, and their common transversal, l, we have proceeded in an unsymmetrical manner in two respects, first in separating e from a, b, c, d, and then in taking the two pairs a, b and c, d from these. There are in all $5 . 3$, or fifteen, ways in which we may proceed; these lead to fifteen points O, but the cubic surface finally obtained is the same in all cases, there being only one such surface which contains the six lines a, b, c, d, e and their common transversal l, when these are general. For these involve the existence of fifteen other lines on the surface, which can be constructed from these lines alone, and the common points, of two cubic surfaces, lying in a general plane cannot be more than nine in number. If, now denoting the line l by f', we associate the lines $a, b, \ldots, e, f$; $a', b', \ldots, e', f'$, respectively, with the numbers $1, 2, \ldots, 5, 6$; $1', 2', \ldots, 5', 6'$, and denote the line of intersection of the planes $[1, 2']$, $[1', 2]$ by c_{12}, and so on, the fifteen points O will be the various points of intersection (c_{rs}, c_{ij}), where r, s, i, j are four different numbers from $1, 2, \ldots, 6$.

Ex. 1. Considering any plane through the line c_{12}, drawn to contain two further intersecting lines of the cubic surface, and any plane through the line c_{34}, drawn to contain two further intersecting lines of the cubic surface, it is the fact that either of the two former lines, meeting c_{12}, is met by one of the two latter lines, meeting c_{34}. For, otherwise, the line of intersection of these two planes, already meeting the cubic surface in the point (c_{12}, c_{34}), would meet the surface in more than three points. For instance, the plane drawn through c_{12} may contain the lines $1'$, 2, and the plane drawn through c_{34} may contain the lines 3, $4'$; then the lines $1'$, 3 meet, as also do the lines 2, $4'$. Any line of the surface meeting c_{12}, other than the lines c_{34}, c_{56}, say, for instance, the line 2, is thus met by four of the lines, other than c_{12} and c_{56}, which meet c_{34}; and the other line, $1'$, lying in the plane $[2, c_{12}]$, is met by the other four lines, beside c_{12} and c_{56}, which meet c_{34}; further, these two ranges of each four points, respectively on the lines 2 and $1'$, are in perspective from the point (c_{12}, c_{34}).

Prove that there are in all 1080 such pairs of ranges of four points of intersection, in perspective from a point of intersection of two lines of the cubic surface.

Ex. 2. Prove further that, upon the line containing the three points (c_{12}, c_{34}), $(1', 3)$, $(2, 4')$, there is a point at which intersect the planes containing, respectively, the three sets of lines $(2, 1', c_{12})$, $(4', 3, c_{34})$, (c_{24}, c_{13}, c_{56}). These nine lines, arranged in the three sets $(2, 4', c_{24})$, $(1', 3, c_{13})$, (c_{12}, c_{34}, c_{56}), lie in three other planes, making with the three former planes a pair of Steiner trihedrals. Shew also that of such lines as that containing the three intersections (c_{12}, c_{34}) $(1', 3)$, $(2, 4')$, each containing three intersections, there are, in all, 720.

Ex. 3. Prove that there exist 120 pairs of Steiner trihedrals.

The definition of a cubic surface in general. The preceding geometrical definition of a cubic surface, which has led us to the figure of twenty-seven lines previously discussed, forms a sufficient basis for the development of all the properties. Greater freedom of argument will however be available in some cases if it may be assumed that the surface so obtained has the generality of the locus expressed by the vanishing of a complete homogeneous cubic polynomial in the four coordinates of a point. We have given reasons for expecting this to be so; in what follows we shall assume that it is so. But it may be well to recognise explicitly that in so doing we omit an algebraical discussion which should logically be given.

The cubic curves of the surface which belong to a particular double-six. In the preceding discussion we have obtained a system of cubic curves lying on the cubic surface, each the intersection of two quadrics (a, b, m), (c, d, m), where m is an arbitrary line passing through the point (c_{12}, c_{34}), which was called O. Through an arbitrary point, Q, of the surface, there pass, we have seen, infinitely many such curves, these corresponding to lines, m, lying in the plane through O which contains the transversals from Q, respectively to a, b and to c, d. Through two points, Q and Q', of the surface, each of which determines such a plane, there passes one such cubic curve, corresponding to the line of intersection of these two planes. Thus two of these cubic curves intersect in one point only.

The line c_{12}, meeting a and b, and meeting the line m at O, is a generator of the quadric (a, b, m); thus, through the lines a, b, c_{12}, and any one of the cubic curves, there can be described a quadric surface. Conversely, any quadric containing a, b, c_{12} has, at O, a generator, m, of the same system as a, b, and this quadric therefore meets the cubic surface again in one of the cubic curves, which, again, lies on the quadric (c, d, m). It is clear, if we employ the algebraic symbols, that the points common to a quadric surface and a cubic surface lie on a sextic curve, that is, a curve meeting an arbitrary plane in six points; for the quadric meets the plane in a conic, of which the points are expressible by quadratic functions of a parameter, and the cubic surface meets the plane in a cubic curve, from the equation of which, by substitution of quadratic polynomials for the coordinates, a sextic equation results. Thus, when a quadric surface is drawn through the three lines a, b, c_{12}, the cubic curve in which this quadric further meets the cubic surface constitutes the whole of the residual intersection. By a similar argument, a quadric surface can be drawn through this cubic curve whose residual intersection with the cubic surface consists of the lines c, d, c_{34}. From this, by the symmetry between the lines of the cubic surface, we infer that, if r and s denote any particular two of the six lines a, b, ..., e, f, the cubic curves which we have obtained on the cubic surface are all obtainable as the residual intersections with the surface of the quadric surfaces, drawn through the three lines r, s, c_{rs}. Thence it appears, from what has been said (p. 162 above), that we are to regard the sets of three lines such as r', s', $c_{r's'}$, consisting of any two of the lines a', b', ..., e', f', and their associated line $c_{r's'}$, as particular forms of the cubic curves under discussion. But then, as the lines a', b', ..., f' are quite similar to the lines a, b, ..., f, it follows that there exists on the cubic surface another system of cubic curves, each the residual intersection with the surface of a quadric drawn, say, through the three lines $1'$, $2'$, c_{12}.

This new system of cubic curves is, however, obtainable from the former system; for it is the case that any cubic curve of the former system and any cubic curve of the latter system are, together, the complete intersection of the cubic surface with a quadric. To prove this, it is sufficient, after what has been said, to prove that, if we take the cubic curves γ and γ', of which γ is the residual intersection with the cubic surface of any quadric drawn through the lines $1'$, $2'$, c_{12}, and γ' is the residual intersection with the cubic surface of any quadric drawn through the lines 1, 2, c_{12}, then the curves γ, γ' lie on a quadric surface. This will follow if we shew that γ, γ' have five points in common, other than those on the specified lines; for, then, the quadric surface described to contain these five points, and also two other points of each of the curves, γ, γ', as then meeting each of the curves in seven points, will contain both these curves. Now, the common points of γ, γ' are the points of the cubic surface which lie on the common curve of the two general quadrics described as above, respectively through 1, 2, c_{12} and through $1'$, $2'$, c_{12}. These quadrics, having the line c_{12} in common, meet otherwise in a cubic curve; this curve passes through the two points $(1, 2')$, $(1', 2)$, which are on both quadrics, and also meets the common generator c_{12} in two points; beside the four points given by these, which lie on the cubic surface, the cubic curve will meet the cubic surface in five other points, not lying, when the quadrics drawn through the lines 1, 2, c_{12} and $1'$, $2'$, c_{12} are not special, on any of these lines. This proves the result enunciated.

From this we see that, given a single cubic curve, γ, of either system, the curves of the other system are obtainable by drawing the most general quadric surface through the curve γ. Such a quadric is identified by two points of the cubic surface not lying on γ, which also, we have seen, identify a cubic curve of the other system.

It follows now, as in a previous section (above, p. 162), that the quadric surfaces whose intersection with the cubic surface breaks up into cubic curves, one of each system, are, when expressed in point coordinates, linearly deducible from nine of them, which may be taken to be the same nine as those taken in the section referred to. These quadrics are then all outpolar to the same quadric, Σ, that, namely, in regard to which the six pairs of lines, (1, 1′), ..., (6, 6′), are polar lines of one another; and, by what we have seen, this is the same statement as that any quadric containing any cubic curve of either system is outpolar to Σ.

A particular consequence is that the five points above shewn to be common to two of the cubic curves, γ and γ', of different systems, are such that the plane which contains any three of them has, for its pole in regard to Σ, a point lying on the join of the other two points; or, in a phraseology previously employed (above, p. 37), that these five points are a self-conjugate pentad in regard to Σ. To prove this, consider the general quadric surface passing through these five points; it is a linear aggregate of five such quadrics, when expressed by its equation in point-coordinates; these may then be taken to consist of two quadrics through each of the two cubic curves which intersect in the five points, together with the quadric which contains both these curves, there being, as we know, three linearly independent quadrics through any cubic curve in space. We thus infer that every quadric through the five points is outpolar to the quadric Σ. This leads to the statement made, since a particular quadric through the five points is the plane-pair consisting of the plane through three of these points taken with any plane whatever through the other two.

Ex. 1. For the case of two degenerate cubic curves of the same system constituted, respectively, by the lines a, b, c_{12}, and by the lines c, d, c_{12}, we saw that, if a plane be drawn through the point O, or (c_{12}, c_{34}), which is common to these degenerate cubics, and the join of the two points where this plane meets the lines a, b be taken, as also the join of the two points where this plane meets the lines c, d, then these two joins meet in a point of the cubic surface, all points of the surface being obtainable by varying the plane drawn through O. We consider now a theorem which is a generalisation of this result: If any two cubic curves in space, of the cubic surface, belonging to the same system, be taken, intersecting in a point, Q, of the surface, and a variable plane be drawn through Q, meeting one of these curves again, say, in M and N, and the other in M' and N', then the lines MN, $M'N'$ meet in a point of the cubic surface, all points of which are thus obtainable by varying the plane. We base this theorem upon a property of the *plane* cubic curve in which the cubic surface meets an arbitrary plane, of which we give the proof immediately (Ex. 2); this property is that, if A, B, C, Q be any four fixed points of this plane cubic curve, and any conic be drawn through these points, meeting the plane cubic again in the two points M and N, then the third point in which the line MN meets the plane cubic again is the same whatever conic be taken. Assuming this result, let ω denote, in the theorem enunciated for the cubic curves in space on the cubic surface, the plane cubic curve in which the cubic surface is met by the plane which is drawn through Q; and let A, B, C be the points in which this plane curve ω is met by the lines a, b, c_{12}, of the cubic surface. The quadric surface containing these three lines, and also one of the two cubic curves considered, say γ, which pass through Q, will meet the plane in a conic containing the points Q, A, B, C, and also containing the two points, M and N, in which the plane meets the curve γ, beside Q. All these points are on the curve ω. Let K be the third intersection of the line MN with the curve ω. By the property of a plane cubic curve which we have assumed, this point K lies also on the line $M'N'$. Conversely, through any point, K, of the cubic surface, a single chord can be drawn of any assigned cubic curve in space, lying on the cubic surface, which passes through Q. This chord and the line KQ determine a plane. The theorem enunciated is therefore proved.

Ex. 2. Of the property, for a plane cubic curve, assumed in Ex. 1, which is a very particular case of a general theorem for plane curves, we may here give the following proof. The plane curve which is the section of the cubic surface by a plane, is met by an arbitrary line of that plane in three points; it can thus be represented by the vanishing of a homogeneous cubic polynomial in the three coordinates of a point on that plane. We have assumed that the cubic surface is that given by the vanishing of a general cubic polynomial in the four space coordinates; the plane curve will, therefore, if the plane be general, be expressed by the vanishing of a quite general cubic polynomial in the three coordinates, containing, homogeneously, ten independent coefficients. Let this equation be $f=0$, and let U, V be two other cubic polynomials in the three coordinates which both vanish at eight points of the curve $f=0$. The equation $U+\lambda V=0$ then represents a cubic curve, in the plane, passing through these eight points. We assume that the common points of the curves $U=0$, $V=0$ are nine in number, an assumption, moreover, easily verified in the application of the theorem made below. These curves will, therefore, intersect in a further point beside the given eight points, and the curve $U+\lambda V=0$ will pass through this for all values of λ. The value of λ is however particularised when a point, other than the specified ninth point, is assigned upon the curve $U+\lambda V=0$, and this curve is then uniquely determined; in particular, by taking this assigned point to lie on $f=0$, we see that the curve $f=0$ can be represented in the form $U+\lambda V=0$, if λ be properly chosen. (Cf. Ex. 6, p. 156.)

Now, suppose that through the four points A, B, C, Q, of the cubic curve $f=0$, two conics $S_1=0$, $S_2=0$ are drawn, meeting $f=0$ again, respectively, in the points M_1, N_1 and the points

M_2, N_2; let $P_1=0$, $P_2=0$ be the respective equations of the lines M_1N_1 and M_2N_2. Then the cubic polynomials S_1P_2 and S_2P_1 both vanish at the eight points A, B, C, Q, M_1, N_1, M_2, N_2 of the curve $f=0$. Thus, by what has been said, there is an identity of the form $f=\mu S_1P_2+\nu S_2P_1$, in which μ, ν are constants. This shews, however, that the lines $P_1=0$, $P_2=0$ meet on the curve $f=0$, which proves the property in question.

The general property, of which the above is a particular case, may be enunciated for any plane curve, $F=0$, which has no multiple points, as follows: let A, B, C, ..., T be any set of points on $F=0$; let $S_1=0$, $S_2=0$ be any two curves, in this plane, of the same order, both of which contain this set of points and meet the curve $F=0$, respectively, in the further points M_1, N_1, ..., U_1 and M_2, N_2, ..., U_2, the number of points in these two further sets being in fact the same. Now, let $P_1=0$ be any curve whatever drawn through the points M_1, N_1, ..., U_1, having with the curve $F=0$ the further intersections H, K, ... The property then is that there exists a curve, of the same order as $P_1=0$, whose complete intersection with $F=0$ consists of the two sets of points

$$M_2, N_2, \ldots, U_2, \ H, K, \ldots.$$

Each of the two sets $(M_1, N_1, \ldots, U_1)$ and $(M_2, N_2, \ldots, U_2)$ is said to be a *residual* of the first set, $(A, B, C, \ldots, T)$, and these two sets are therefore said to be *coresidual*. The set $(H, K, \ldots)$ is then a residual of the set $(M_1, N_1, \ldots, U_1)$. The property may then be stated by saying that any residual of one of two coresidual sets is equally a residual of the other.

Ex. 3. The property for plane curves just explained has an analogue for surfaces, in which a set of points of the plane curve is replaced by a curve on the surface, and residual sets of points by residual curves on the fundamental surface, that is, curves which, together, are the complete intersection of this surface with a further surface. We first illustrate this by considering a particular case for curves on the cubic surface. We denote the equation of the cubic surface by $F=0$. Through any line of this surface let two planes be drawn, $S_1=0$, $S_2=0$; the residual intersections, in general, conics, may be denoted by M_1, M_2, respectively. Let an arbitrary quadric, say $P_1=0$, be drawn through the conic M_1, and the residual intersection of this quadric with the cubic surface be denoted by H. Then it is the case that this curve H forms, with the conic M_2, the complete intersection of the cubic surface with another quadric, say $P_2=0$. In general terms, the curve H, which is a residual of one of the two coresidual conics M_1, M_2, is equally a residual of the other.

To prove this, remark, first, that the curve H is a quartic curve, meeting an arbitrary plane in four points; for the aggregate of this curve and the conic M_1, being the intersection of a quadric surface and a cubic surface, meets an arbitrary plane in six points. Next, this quartic curve, H, meets every generator of the quadric $P_1=0$ in two points; these, with the point where the generator meets the conic M_1, are the three intersections of the line with the cubic surface, when this does not contain the line entirely. A quartic curve, on a quadric, meeting every generator in two points, is however the complete intersection of this quadric with another quadric, and a quadric can be drawn through the curve and an arbitrary point. Assuming this fact, which is proved below, consider the four intersections of the quartic curve with the plane of the conic M_2; as the curve is on the cubic surface these intersections, if not all on the conic M_2, can only be on the line of intersection of the planes of the two conics M_1 and M_2, and, therefore, as the curve is on the quadric $P_1=0$, can only be where the conic M_1 meets this line, which requires a particular quadric P_1. Thus, in general, the curve will meet the conic M_2 in four points. Therefore a quadric can be drawn through the quartic curve to contain also the conic M_2; for, we have only to describe a quadric through the quartic curve which shall meet this conic in a fifth point other than those four in which the quartic curve already meets the conic.

To prove the fact stated, that if a quartic curve, H, lying on a quadric surface, P, meet every generator of the quadric in two points, then a quadric can be drawn through the curve and any other arbitrary point, we may argue as follows: Let the arbitrary point be O, and take eight points, $A_1, A_2, \ldots, A_8$, of general position, upon the quartic curve; describe the quadric surface, P', through O and these eight points; this will meet the quadric P in a quartic curve, H'. We prove that this coincides with the curve H. For, first, this curve H' meets every generator of the quadric P in two points, as we see by considering how many points of this curve, not on this generator, lie on an arbitrary plane drawn through this; this plane meets the quadric P, further, in a line, and meets the quadric P' in a conic. Wherefore, if we project the two curves H, H', on to an arbitrary plane, from one of the eight points, $A_1, \ldots, A_8$, which are common to both curves, say from A_1, we obtain two plane cubic curves having nine points in common, namely the projections of the seven points $A_2, \ldots, A_8$, and the points where the generators of the quadric P, at the point A_1, meet this plane. We have proved above (Ex. 2) that, of the nine intersections of a given plane cubic curve (here the projection of the curve H) with another such curve, only eight can be taken arbitrarily upon the first curve, the remaining one being then determinate. In our present case, by retaining the same positions for $A_1, A_2, \ldots, A_7$, and changing the position of A_8, we can, if the curve

H' is distinct from H, assume the ninth intersection of two such plane cubic curves quite arbitrarily upon one of them. We infer therefore that H' coincides with H. The fact stated is then proved.

Ex. 4. The general property referred to, at the beginning of Ex. 3, may be stated for a cubic surface (to which however it is not restricted) as follows: Suppose that, through any curve, l, on the cubic surface, there pass two surfaces of equal order, $S_1 = 0$, $S_2 = 0$, whose residual intersections with the cubic surface are, respectively, the curves M_1 and M_2; let any surface whatever, $P_1 = 0$, be drawn through the curve M_1, having, as its residual intersection with the cubic surface, a curve H; then a surface, of the same order, exists, whose complete intersection with the cubic surface consists of the curves H and M_2. It seems desirable to enunciate this property; but we give only indications of the proof. This depends on the fact that, if $F = 0$ be the equation of the cubic surface, and $S = 0$ the equation of another surface, then any surface containing the complete intersection of $F = 0$ and $S = 0$ has an equation of the form $\Theta F + PS = 0$; here Θ, P are homogeneous polynomials in the coordinates, of respective orders such as to make the left side of the equation of the order of the surface which is to be expressed. In our case take for S the surface $S_1 = 0$, whose intersection with the cubic surface consists of the curves l and M_1; consider the composite surface consisting of the surfaces $S_2 = 0$ and $P_1 = 0$, with equation $S_2 P_1 = 0$; as $S_2 = 0$ contains the curve l, as well as the curve M_2, and $P_1 = 0$ contains the curve M_1, as well as the curve H, this surface $S_2 P_1 = 0$ contains the intersection, (l, M_1), of the surfaces $F = 0$, $S_1 = 0$; there is thus, by what we have stated, an identity of the form $S_2 P_1 = \Theta F + P_2 S_1$, wherein Θ, P_2 are appropriate polynomials. The surface expressed by $P_2 = 0$ thus meets the cubic surface, $F = 0$, on the portion of the intersection of this with $S_2 P_1 = 0$ which does not lie on $S_1 = 0$. Wherefore, the curves M_2 and H lie on the surface $P_2 = 0$; it was the existence of such a surface which was to be proved.

The development of the properties of coresidual curves on a cubic surface is of great importance, but would carry us too far. The reader may consult a paper, *Proc. Lond. Math. Soc.* XI, 1912, p. 287.

Ex. 5. The quadrics which pass through the curve of intersection of two given quadrics meet an arbitrary line in pairs of points of an involution. From the result of Ex. 3, it follows that the conics in which the cubic surface is met by the planes drawn through a line of the surface, say l, meet this line in pairs of points of an involution. Of this result another proof arises below. The double points of the involution are two important points of the line; in fact it will be found below that the line touches at these two points the quartic surface known as the Hessian of the cubic surface.

Ex. 6. The theorem stated in Ex. 5 is easily obtained analytically; and another result which includes this as a particular case may be obtained at the same time. Let $F=0$ be the equation of the cubic surface, and x_1, y_1, z_1, t_1 be the coordinates of any particular point, P_1. The quadric surface whose equation is

$$[x_1\partial/\partial x + y_1\partial/\partial y + z_1\partial/\partial z + t_1\partial/\partial t]F = 0$$

is called the polar quadric of the point P_1 in regard to the cubic surface. It is such that the line joining P_1 to any point, P, of the curve of intersection of the cubic surface with this polar quadric of P_1, meets the cubic surface in two coincident points at P, as we may easily see by considering the equation in λ obtained by substituting in $F=0$, for the coordinates, the values

$$x+\lambda x_1,\ y+\lambda y_1, \text{ etc.}$$

The general result referred to is that, if we take any particular line lying on the cubic surface, the ∞^3 quadrics constituted by the polar quadrics of all points P_1 meet this line in pairs of points of an involution. That this involution includes the pairs of points of intersection, with the line, of conics lying on the cubic surface, in planes passing through the line, is clear; for a point, P, where such a conic meets the line, arises twice over in the intersections with the cubic surface of any line through P lying in the plane of the conic. To prove the result, let the coordinates be so taken that the line of the cubic surface which is considered is that given by $x=0$, $y=0$; the equation of the cubic surface is then of the form $xU+yV=0$, where $U=0$, $V=0$ represent quadric surfaces. The polar quadric, by the form of its equation, is unchanged by any particular choice of coordinates (cf. Vol. II, p. 104). In the present case, the equation of the polar quadric of P_1 is

$$x_1U+y_1V+xDU+yDV=0,$$

where D represents the operator by which the polar quadric, $DF=0$, is formed. The polar quadric thus meets the line $x=0$, $y=0$ in the pair of points given by $x_1(U)+y_1(V)=0$, where (U), (V), containing terms in z^2, zt, t^2, represent the result of putting $x=0$, $y=0$ in U and V. We know that the pairs of points given by this equation, when x_1, y_1 vary, belong to an involution (Vol. II, p. 111, Ex. 3). The general plane through the line $x=0$, $y=0$, say $y-\lambda x=0$, meets the cubic surface on the quadric cone expressed by $[U]+\lambda[V]=0$, where $[U]$, $[V]$ are obtained from U, V respectively, by substituting λx for y. This cone meets $x=0$, $y=0$ in a pair of points given by $(U)+\lambda(V)=0$, belonging to the same involution as before.

The double points of the involution, being harmonic in regard to the intersections of the line with a polar quadric, are conjugate to one another in regard to all polar quadrics. But there exist pairs of points with this property not lying on a line of the cubic surface, as we shall see (see below, under *Hessian surface*).

Ex. 7. A result given above suggests another way in which we may approach the theory of a cubic surface and the lines lying thereon. Suppose that we are given two cubic curves, γ and γ', in three dimensions, having a point in common, say O. An arbitrary plane, drawn through O, will meet these curves both in two further points, say, respectively, A, B, and A', B'. We may consider the locus of the point of intersection, P, of the two lines AB, $A'B'$, as the plane drawn through O varies. It follows from what has preceded that the locus is a cubic surface containing the curves γ, γ'. But this may be proved directly. For, take nine points, other than O, on each of these cubic curves. The nineteen points thus obtained suffice to determine a cubic surface, defined as the locus expressed by a cubic polynomial in the four space coordinates. This surface will contain both the cubic curves, as meeting each in ten points, nine being the number of intersections of a cubic surface with a cubic curve. There are, in general, ten common chords of two cubic curves in space, as has been seen (above, p. 141); the quadric cones projecting the cubic curves γ and γ' from O have four common generators, which are among these ten. There remain, then, six other common chords of these two cubic curves, say, a, b, c, d, e, f; and each of these chords, as meeting the cubic surface in four points, will lie entirely thereon. It can be shewn that, taking two of these common chords, say a and b, there is a single line meeting both, and also meeting each of the curves γ, γ', in such a way that the four points of meeting are distinct. This line will then also lie on the cubic surface. In general, four curves in space, of respective orders m, n, p, q, are met by $2mnpq$ lines (see below); this would give eighteen lines meeting both a and b and also meeting each of the curves γ, γ' once. We can, however, with more careful examination, see that seventeen of these lines are such as to meet the cubic surface in less than four points; it cannot, therefore, be inferred that these lie on the surface. For, let the line a meet the curves γ and γ', respectively, in the points A_1, A_2 and A_1', A_2', and the line b meet γ and γ' respectively, in B_1, B_2 and B_1', B_2'. Then, one of the seventeen lines is the transversal from O to a and b; two more are the joins of A_1 to B_1' and B_2', and there are eight such lines in all; if the quadric be described containing the curve γ' and its two chords, a, b, there will be a generator of this through A_1, of the opposite system from these, which will meet the curve γ', as well as a, b and the curve γ; this is another of the seventeen lines, and there are four such, respectively through A_1, A_2, B_1, B_2. There are four similar lines definable by the quadric containing the curve γ and its chords a, b. Seventeen lines are thus accounted for, each of which meets the cubic surface in three points only. There is, thus, as stated, one line meeting the curves γ, γ' and the lines a, b, in four distinct points, and, therefore, lying on the cubic surface. We may denote this by c_{12}. By taking every couple of the six lines $a, b, \ldots, e, f$, we obtain in all fifteen such lines of the cubic surface; the theory of the surface can then be developed from this point.

To prove that there are $2mnpq$ lines each meeting four given curves, of respective orders m, n, p, q, we may assume that a surface of order N is met by a curve in space, of order k, in Nk points. If the surface and curve are defined by algebraic equations this is an algebraic theorem, which needs proof. Assuming this, however, consider first the particular case of the result under discussion which asserts that there are $2m$ lines meeting one curve of order m and three lines. The transversals of these three lines generate a quadric surface, which, by the assumption made, meets the curve of order m in $2m$ points; from each of these there is one of the lines desired, meeting the curve of order m and the three given lines. Hence we have the result that the lines meeting a curve of order m, and also meeting two lines, generate a surface of order $2m$. By the assumption made, this surface will meet a curve of order n in $2mn$ points; this is, then, the number of lines meeting two given curves of orders m and n, and also meeting two given lines. Wherefore, the lines meeting two given curves of orders m and n, and also meeting a line, generate a surface of order $2mn$. By the assumption made, this surface will meet a curve of order p in $2mnp$ points; this is, then, the order of the surface formed by the lines meeting three given curves of respective orders m, n, p. Hence, this surface meeting a curve of order q in $2mnpq$ points, there are $2mnpq$ lines meeting four curves of respective orders m, n, p, q.

Definition of a cubic surface by three related star-figures. We have explained in Vol. I (p. 148) what is meant by two related plane systems, wherein, to any point of one plane there corresponds a point of the other, and to a range of points in one plane there corresponds a related range of points of the other. Suppose the points and lines of one of these planes are joined to a point, O, not lying in this plane, respectively by lines and planes; and, similarly, that the points and lines of the other plane are joined to a point, O', not lying in this plane, respectively by lines and planes. We then obtain two star-figures, or central-systems, with vertices, or centres, respectively at O and O', which are said to be related to one another, any line, or flat pencil of lines, passing through O, corresponding to a line, or related flat pencil of lines, passing through O', and conversely.

Such a relation between two star-figures can be set up by means of a cubic curve which contains the two vertices O and O'. It is only necessary to make correspond to any plane through O, which will meet the cubic curve in two further points, the plane joining O' to the chord of the cubic curve which contains these points; then, any line through O, which meets an infinite number of chords of the curve, whose ends are pairs of an involution thereon, will correspond to the line through O' which meets all these chords. It may be proved that any flat pencil of lines through O gives rise in this way to a related flat pencil of lines through O'; and, conversely, that any two related star-systems may be regarded as arising in this way from a cubic curve in space.

Now suppose we have three star-systems, of vertices O, O' and O'', of which any two are related to one another, so that, to any plane through O, there corresponds a definite plane through O' and a definite plane through O''. We may then consider the locus of the point of intersection of three such related planes. By what has been said, this comes to supposing that we have two cubic curves which meet in O, one of these containing O', the other containing O'', and that we draw through O an arbitrary plane, meeting the first cubic curve in the further points A', B' and the second curve in the further points A'', B'', and then consider the locus of the intersection of the lines $A'B'$ and $A''B''$. We are thus brought back to the definition of a cubic surface which has been considered (Ex. 7, p. 183); and it appears, since it has been shewn that, on a cubic surface, there can be drawn one cubic curve of a definite system through two arbitrary given points, that a cubic surface can be generated by the intersection of corresponding planes of three related star-systems in which the vertices, O, O', O'', are any three points of the surface.

By choosing a proper system of coordinates, related to a cubic curve containing the points O and O', the corresponding planes through these points, intersecting in a chord of this curve, can (above, p. 131) be supposed to have the equations

$$x-y(\theta+\phi+\omega)+z(\theta\omega+\phi\omega+\theta\phi)-t\theta\phi\omega=0,$$
$$x-y(\theta+\phi+\omega')+z(\theta\omega'+\phi\omega'+\theta\phi)-t\theta\phi\omega'=0,$$

or say $\quad \xi u+\eta v+\zeta w=0, \quad \xi u'+\eta v'+\zeta w'=0,$

wherein $\xi:\eta:\zeta, =1:\theta+\phi:\theta\phi$, have ratios which vary as these planes vary, but $u, =x-y\omega$; $v, =-y+z\omega$; $w, =z-t\omega$; $u', =x-y\omega'$, etc., are fixed. The planes $u=0$, $v=0$, $w=0$, of the star-figure of vertex O, correspond then, respectively, to the planes $u'=0$, $v'=0$, $w'=0$, of the star-figure of vertex O'. When we take any general system of coordinates the equations of corresponding planes of the star-figures (O), (O') will remain of the forms $\xi u+\eta v+\zeta w=0$,

$\xi u' + \eta v' + \zeta w' = 0$, and the plane of the star-figure (O'') which corresponds to these will be of the form $\xi u'' + \eta v'' + \zeta w'' = 0$, where u'', v'', w'' are fixed planes through O'', corresponding respectively to u, v, w. The equation of the cubic surface is thus

$$\begin{vmatrix} u, & v, & w \\ u', & v', & w' \\ u'', & v'', & w'' \end{vmatrix} = 0.$$

This form gives rise to several remarks. In the first place, considering the three quadrics which are represented by $vw' - v'w = 0$, $wu' - w'u = 0$, $uv' - u'v = 0$, the last two of these have in common the line given by $u = 0$, $u' = 0$, and, beside this, intersect in a cubic curve which lies on the first. Thus this cubic curve lies on the cubic surface. It is evidently the cubic curve joining the points O and O'. Similarly, the cubic curve of the surface joining the points O and O'' is that common to the three quadrics $vw'' - v''w = 0$, $wu'' - w''u = 0$, $uv'' - u''v = 0$. We may, however, without altering the equation of the cubic surface, replace u, v, w, respectively, by $au + bu' + cu''$, $av + bv' + cv''$, $aw + bw' + cw''$, where a, b, c are any constants; and at the same time make similar replacements for u', v', w', and for u'', v'', w''. Thereby we obtain an infinite number of cubic curves lying on the surface; it may be seen that these are such as those previously described as belonging to one system. Any one of these curves is obtained by taking two of the rows of the modified determinantal equation of the cubic surface; by dealing similarly with the *columns* of the determinant, another system of cubic curves of the surface is obtained. Moreover, as any minor of the determinant, of two rows and columns, belongs as well to two rows as to two columns, we see that any cubic curve of the first system and any cubic curve of the second system are together the complete intersection of the cubic surface with a quadric, as was previously proved. Further, the equations of all quadric surfaces so arising are linear functions of those nine of them which correspond to the nine minors of the determinant, of two rows and columns, as also has been shewn above.

Ex. 1. The six lines which form one row of the double-six associated with the present way of writing the equation of the cubic surface are the common chords of one system of cubic curves. To find them we are to express that the planes $\xi u + \eta v + \zeta w = 0$, $\xi u' + \eta v' + \zeta w' = 0$, $\xi u'' + \eta v'' + \zeta w'' = 0$, meet in a line; this requires an identity, in regard to the coordinates, of the form $\Sigma \xi (\lambda u + \lambda' u' + \lambda'' u'') = 0$, in which the three parameters λ, λ', λ'' are independent of the coordinates; this again, leads to four equations all linear both in ξ, η, ζ and in λ, λ', λ''. By elimination of λ, λ', λ'', from each set of three of these equations, we thence obtain cubic polynomials in ξ, η, ζ; the values of these for which these polynomials vanish solve the problem. The other row of the double-six is to be similarly found from the columns of the determinant.

Ex. 2. The nine independent quadrics, each containing a cubic curve, on the cubic surface, of each of the two systems, are outpolar to a single quadric, Σ; in regard to this quadric, as has been seen, the lines of one row of the double-six are polar lines of those of the other row, respectively. This quadric, Σ, is inpolar to any quadric which, expressed in point coordinates, is a linear function of the nine quadrics. Now, the cubic surface being $F=0$, let $DF=0$ be the polar quadric (above, p. 182) of any point (x_1, y_1, z_1, t_1), where the operator D is a sum of such terms as $x_1\partial/\partial x$. This is a linear function of the nine quadrics referred to; for, with the above determinant form for F,

$$DF=\begin{vmatrix} Du, & Dv, & Dw \\ u', & v', & w' \\ u'', & v'', & w'' \end{vmatrix} + \begin{vmatrix} u, & v, & w \\ Du', & Dv', & Dw' \\ u'', & v'', & w'' \end{vmatrix} + \begin{vmatrix} u, & v, & w \\ u', & v', & w' \\ Du'', & Dv'', & Dw'' \end{vmatrix},$$

and each of Du, Dv, etc., is a constant, being the values of u, v, etc. for the point (x_1, y_1, z_1, t_1); the minors, in the determinants, multiplying these constants, are the nine quadrics in question.

We infer, therefore, that the quadric Σ is inpolar to every polar quadric. The expression for F by which we have deduced this is connected with one of the double-sixes of the lines on the cubic surface, and the two systems of cubic curves associated therewith; the quadric Σ is also related to this double-six. There is such a quadric for every one of the double-sixes, and, therefore, there are in all thirty-six such Schur quadrics. All these will then be inpolar to the polar quadrics, that is, inpolar to the four quadrics

$$\partial F/\partial x=0, \ \ldots, \ \partial F/\partial t=0.$$

Hence, the *tangential* equations of the Schur quadrics are all linear functions of at most six of them.

We have already (above, p. 182) considered pairs of points which are conjugate to one another in regard to all the polar quadrics of the cubic surface. These form, in fact, pairs of so-called corresponding points of a certain quartic surface, the Hessian of the cubic surface. Each such pair may be regarded, tangentially, as a degenerate quadric which is inpolar to the polar quadrics; thus the tangential equations of all such point-pairs may be expressed linearly by those of six of them. The Schur quadrics, in tangential form, may then be expressed by such six point-pairs.

Ex. 3. For the cubic surface whose equation is expressed by

$$ax^3+by^3+cz^3+dt^3+eu^3=0,$$

where u is expressible in terms of x, y, z, t by means of the relation

$$ax + by + cz + dt + eu = 0,$$

the general polar quadric is at once seen to be expressible as a linear function of the four $x^2 - u^2 = 0$, $y^2 - u^2 = 0$, $z^2 - u^2 = 0$, $t^2 - u^2 = 0$, namely to be of the form

$$Ax^2 + By^2 + Cz^2 + Dt^2 + Eu^2 = 0,$$

with $A + B + C + D + E = 0$. Thus a pair of points, (x, y, z, t, u), $(x_1, y_1, z_1, t_1, u_1)$, which are conjugate to one another in regard to all these quadrics must be such that

$$xx_1 = yy_1 = zz_1 = tt_1 = uu_1;$$

they must, therefore, be such that both of them satisfy the equation

$$ax^{-1} + by^{-1} + cz^{-1} + dt^{-1} + eu^{-1} = 0,$$

which, conversely, is also a sufficient condition. It may easily be verified that the polar quadric of either of the two points, in regard to the cubic surface, is a quadric cone whose vertex is at the other point; this, as may easily be seen, is the same thing as that the two points lie on a surface whose equation can be expressed by the vanishing of a determinant of four rows and columns in which the elements of the first row are $\partial^2 F/\partial x^2$, $\partial^2 F/\partial x \partial y$, etc., those of the second row are $\partial^2 F/\partial y \partial x$, $\partial^2 F/\partial y^2$, etc., and so on. For the form of F under discussion, this equation reduces to that given, $ax^{-1} +$ etc. $= 0$. As has been said, the surface is called the Hessian. With this form for F, the equation of the Hessian can be seen to be capable, when u is expressed in terms of x, y, z, t, of the form

$$\lambda + bcX + adP + caY + bdQ + abZ + cdR = 0,$$

where

$X = yz^{-1} + y^{-1}z$, $P = xt^{-1} + x^{-1}t$, etc., and $\lambda = a^2 + b^2 + c^2 + d^2 - e^2$. This shews, however, that any seven points of the Hessian, of coordinates (x_r, y_r, z_r, t_r), for $r = 1, \ldots, 7$, are connected, when $A_1, \ldots, A_7$ are suitably chosen, by the equation

$$\sum_{r=1}^{7} A_r (ux_r + vy_r + wz_r + pt_r)(ux_r^{-1} + vy_r^{-1} + wz_r^{-1} + pt_r^{-1}) = 0,$$

holding for all values of u, v, w, p. This is the equation connecting the tangential equations of any seven pairs of corresponding points on the Hessian.

It may also be proved that the tangential equation of a general quadric which is inpolar to all the polar quadrics of the cubic surface is a linear function of the six quadrics such as

$$u^2 + v^2 + w^2 + p^2 - \lambda vw/bc = 0, \ldots, \; u^2 + v^2 + w^2 + p^2 - \lambda up/ad = 0,$$

where λ has the same value as above. Every Schur quadric is thus expressible by these six quadrics.

The representation of the cubic surface on a plane. In the case of curves we have distinguished between those, such as the conic and the cubic curve in space, of which the coordinates of a point are expressible by rational functions of a parameter, and those, much the larger class, for which this is not so. The examples of the latter class which we have met with most often are the cubic curve in a plane, and the quartic curve which is the intersection of two quadrics. In the case of surfaces a similar distinction is observable: there are surfaces of which the coordinates of a point are expressible as rational functions of two parameters, these parameters being themselves definable each as a rational function of the coordinates of a point of the surface; but for the majority of surfaces this is not possible; nor is it easy, as it is for curves, to give a general criterion sufficient to ensure its possibility. A quadric is a surface of which any point is given by the intersection of two generators, each of which is determined by a single parameter, and the expression in question is thence possible. We now proceed to remark on the fact that a cubic surface also has this character. In general, the two parameters, in terms of which the points of such a surface are expressible, may be interpreted as ratios of the coordinates of a point in a plane; but it is possible, in an infinite number of ways, to set up a correspondence, between any point of a plane and some other point and conversely, expressible by rational equations. Thus, when a surface is representable on a plane in one way, it is so representable in an infinite number of ways. In general, any such representation must be expected to have *singular* elements; a point of the surface may correspond, not to a point, but to a curve, of the plane; for instance, when a quadric is projected on to a plane from a point of the quadric, the point gives rise to a line of the plane; or, a curve of the surface may lead only to a point in the plane, as we shall see in the case of the cubic surface. The various representations possible for a given surface are to be distinguished by the character of these singular elements.

The generation of the cubic surface by the intersection of corresponding planes of three related star-systems leads at once to a representation of the surface upon a plane. For we may regard the planes of one, and therefore of all, of the star-systems, as being related to the points of a plane. With symbols, the cubic surface being obtained, as above, by the locus of the intersection of three planes $\xi u + \eta v + \zeta w = 0$, $\xi u' + \text{etc.} = 0$, $\xi u'' + \text{etc.} = 0$, we may represent this intersection by a point of coordinates ξ, η, ζ, in a plane. These are then proportional to quadric functions of the coordinates, x, y, z, t, of the point of the cubic surface; conversely, x, y, z, t, obtained from the three equations such as $\xi u + \text{etc.} = 0$, are proportional to cubic polynomials in ξ, η, ζ, say, respectively, $\vartheta, \phi, \psi, \chi$.

The correspondence between the planes $\xi u + \text{etc.} = 0$, $\xi u' + \text{etc.} = 0$, was obtained by the fact that these intersect in a chord of a certain cubic curve, say γ, each point of the curve being an intersection of corresponding rays of two star-systems; through each point of this curve there pass then an infinite number of pairs of corresponding planes of these two star-systems, and the cubic curve is the locus common to the three quadric surfaces $vw' - v'w = 0$, $wu' - w'u = 0$, $uv' - u'v = 0$. Another cubic curve, γ', common to $vw'' - v''w = 0$, $wu'' - w''u = 0$, $uv'' - u''v = 0$, determines the correspondence between the planes $\xi u + \text{etc.} = 0$, $\xi u'' + \text{etc.} = 0$. These curves meet in a point, O, where $u = 0$, $v = 0$, $w = 0$. From any general point, P, of the cubic surface, a chord can be drawn to each of the curves γ, γ'; the plane of these chords is the plane $\xi u + \text{etc.} = 0$, passing through the point O, and this determines the two corresponding planes of the other two systems, and the ratios $\xi : \eta : \zeta$. In particular, when P is on the curve γ, the chord of γ' drawn from P determines the plane $\xi u + \text{etc.} = 0$, and this determines the chord of the curve γ through which the plane $\xi u' + \text{etc.} = 0$ also passes. Thus, any point of the cubic surface determines a single determinate point, (ξ, η, ζ), of the plane. But, conversely, when P is on a line which is a common chord, not passing through O, of the curves γ, γ', the point (ξ, η, ζ) arises equally well from any point of this chord. We have already remarked that there are six such common chords (above, p. 183); thus there are six points of the plane (ξ, η, ζ) which correspond to lines, say a, b, c, d, e, f, of the cubic surface. This result is obtainable also by considering the sections of the cubic surface by planes which pass through a line, say l; if one such plane have the equation $Ax + By + Cz + Dt = 0$, its section with the cubic surface corresponds to the plane curve

$$A\vartheta + B\phi + C\psi + D\chi = 0,$$

which contains the three points of the plane corresponding to the intersections of the line l with the surface. A point of the plane (ξ, η, ζ), other than one of these three points, which is common to two such plane curves $A\vartheta + \text{etc.} = 0$, $A'\vartheta + \text{etc.} = 0$, corresponds, therefore, to *two* different points of the cubic surface, lying respectively on the planes $Ax + \text{etc.} = 0$, $A'x + \text{etc.} = 0$. Two plane cubic curves have, in fact, nine common points. Thus the four curves $\vartheta = 0$, $\phi = 0$, $\psi = 0$, $\chi = 0$ have six points in common; these are the six points referred to.

The loci in the plane (ξ, η, ζ) which correspond to the known loci of the cubic surface can now be enumerated. If, through the point O, there be drawn an arbitrary line, and $\xi_1 u + \eta_1 v + \zeta_1 w = 0$, $\xi_2 u + \eta_2 v + \zeta_2 w = 0$ be two planes which contain this, the corresponding lines of the other two star-systems are determined; to any plane through the first line correspond definite planes through the two other corresponding lines, and the locus of the intersection of these three planes, when the first plane varies, is a cubic curve. The points of this correspond to the points $(\xi_1 + \theta\xi_2, \eta_1 + \theta\eta_2, \zeta_1 + \theta\zeta_2)$, of the plane (ξ, η, ζ), obtained by varying θ. Such cubic curves of the cubic surface, of which, for instance, one particular case is the curve common to the three quadrics $uv' - u'v = 0$, $uv'' - u''v = 0$, $u'v'' - u''v' = 0$, thus correspond to the lines of the plane (ξ, η, ζ). The cubic curves of the cubic surface which are of the same family as the curves γ, γ', on which the vertices of the star-systems lie, meet the cubic curves just obtained, each in five points, and, moreover, meet each of the lines $a, b, \ldots, f$ in two points. The cubics of the cubic surface of the family of γ and γ' thus correspond to curves, in the plane (ξ, η, ζ), which are of the fifth order and have a double point at each of the six singular points of this plane. A particular case of a cubic of the same family as γ, γ' is the aggregate of the lines a', b', c_{12}; a line a' meets five of the lines a, b, c, d, e, f, and is a chord of cubics of the family complementary to γ, γ'. Thus the line a' corresponds to a curve, in the plane (ξ, η, ζ), which contains the five singular points corresponding to b, c, d, e, f, and meets a line of this plane in two points; namely, this curve is the conic containing the five singular points. The line c_{12}, which meets a, b and also a', b', corresponds to the join of the two singular points (a), (b); the degenerate quintic curve corresponding to (a', b', c_{12}) consists of two conics and this line. The fifteen lines c_{rs} thus correspond to the joins of the six singular points.

More generally, it appears that, if we use the symbol U to denote, either an arbitrary line in the plane (ξ, η, ζ), or the corresponding curve of the cubic surface, of the family not meeting any of the lines a, b, c, d, e, f, and use the symbols $A_1, A_2, \ldots, A_6$ to denote, either the six singular points of the plane, or the corresponding lines $a, b, \ldots, f$ of the cubic surface, then a curve of order λ, in the plane, which has a general multiple point of order λ_r at the point A_r $(r = 1, \ldots, 6)$, may be associated with the symbol

$$\lambda U - \lambda_1 A_1 - \ldots - \lambda_6 A_6;$$

and that this will denote then, equally, a curve of the cubic surface of order $3\lambda - \lambda_1 - \ldots - \lambda_6$. It is necessary to prove (see below, Ex. 5), but it is stated here for clearness, that this curve is obtainable by the following process: Draw, through a set of λ cubic curves of the cubic surface, chosen arbitrarily from among those of the family to which U belongs, any surface, Ω, of sufficiently high order to contain these, and consider the curve which is the residual intersection of this surface with the cubic surface; through this residual curve draw the general surface, Ω', of the same order as Ω, which contains the line a as a λ_1-fold line, the line b as a λ_2-fold line, and so on; the remaining intersection of this second surface, Ω', with the cubic surface, is a curve which is represented by the symbol $\lambda U - \lambda_1 A_1 - \ldots - \lambda_6 A_6$; and there is a family of such curves when the conditions for the surface Ω' do not determine it uniquely. The curves of the family U are the simplest example of the statement, these being determined, as we have seen, by quadric surfaces all passing through an arbitrary cubic curve of the complementary family. The lines of the cubic surface such as c_{12} form another example; on the plane (ξ, η, ζ) this line, passing through A_1 and A_2, is representable by $U - A_1 - A_2$; on the cubic surface, we have seen, the aggregate (c_{12}, a, b) form a particular curve of the family of curves U. The representation of a conic, on the plane, passing through $A_2, A_3, \ldots, A_6$, will be $2U - s + A_1$, where s denotes $A_1 + \ldots + A_6$; it has been seen that this conic represents the line a' of the cubic surface. That this follows from the rule stated above we may see by drawing quadrics through two of the curves U, and considering a quartic surface through the residual intersection of these quadrics with the cubic surface; this residual intersection consists of two cubic curves of the complementary system; the quartic surface through these two cubic curves, and through the lines b, c, d, e, f, will further meet the cubic surface in a locus of order $3 . 4 - 2 . 3 - 5$, that is, in a line. This is then the line a', which, meeting the quartic surface in the five points in which it meets the lines b, c, d, e, f, lies entirely thereon. The representation, upon the plane (ξ, η, ζ), of a plane section of the cubic surface, being by a cubic curve through the singular points, will be associated with the symbol $3U - s$. This is the same as

$$(U - A_1 - A_2) + (2U + A_1 - s) + A_2,$$

in accordance with the fact that, on the cubic surface, the lines c_{12}, a', b form a particular plane section. We may easily interpret the symbol $3U - s$ directly, on the cubic surface, in accordance with the rule given above; for instance by using a sextic surface passing through three cubic curves of the family complementary to U.

Ex. 1. With some loss of symmetry we may obtain the representation of a cubic surface upon a plane in what is perhaps a more natural way, as follows. Assume that the surface possesses two non-intersecting lines; then, from any point, P, of the surface, a definite transversal can be drawn to these two lines, and this will meet an assumed fixed plane in a definite point, Q; conversely, from Q we can in general pass back to P without ambiguity, also by drawing a transversal, and considering its third intersection with the cubic surface. In this representation, all the points of the cubic surface which lie on a line, lying on the cubic surface, which is a transversal of the two given lines, correspond to the same point, Q, of the plane. Through either of the two assumed lines there pass five planes whose further intersection with the cubic surface consists of two lines, as has been remarked; the other of the two assumed lines thus meets at least one of each of these five pairs of coplanar lines. That is, there are five lines on the cubic surface meeting both the two assumed lines. Denote the two assumed lines by a' and b', and one of the five lines meeting both these by c_{12}; take coordinates so that the line a' is $x=0$, $y=0$, the line b' is $z=0$, $t=0$, and the line c_{12} is $y=0$, $t=0$; these leave a certain arbitrariness in x and z. It can then be shewn that the equation of the cubic surface is of the form

$$x\{z(ht-h'y)+t(gt-a'x-g'y)\}$$
$$+y\{z(bz+ft-f'y)+t(ct-c'y)\}=0.$$

Now take a point (ξ, η, ζ), in a plane, such that

$$\xi:\eta:\zeta=xt:yz:yt;$$

this point then corresponds to a definite line meeting the lines a', b', and so to a definite point of the cubic surface; and points of a line, $l\xi+m\eta+n\zeta=0$, correspond to the points on a cubic curve of the cubic surface, lying on a quadric $lxt+myz+nyt=0$, which is the general quadric containing the lines a', b', c_{12}. Conversely, we find, for the point (x, y, z, t) of the cubic surface,

$$x:y:z:t=\xi\phi:\zeta\phi:\eta\psi:\zeta\psi,$$

where $\phi=0$, $\psi=0$ represent conics in the plane (ξ, η, ζ). Thus a plane section of the cubic surface corresponds to a cubic curve, in the plane (ξ, η, ζ), with the equation

$$(A\xi+B\zeta)\phi+(C\eta+D\zeta)\psi=0.$$

This, whatever A, B, C, D may be, has six definite points: the intersections of the conics $\phi=0$, $\psi=0$; and the two points, $\xi=0$, $\zeta=0$, on $\psi=0$, and $\eta=0$, $\zeta=0$, on $\phi=0$. The representation can now be compared in detail with that given above.

Ex. 2. The symbol, for a cubic curve of the cubic surface, of the family complementary to U, represented on the plane by a quintic curve having a double point at each of the singular points, is $5U-2s$. We have $5U-2s+U=2(3U-s)$, in accord with the fact that two complementary cubic curves are together the intersection of the cubic surface with a quadric.

Ex. 3. A conic on the cubic surface obtained by a plane drawn through the line a, is associated with the symbol $3U-s-A_1$, and leads, in the plane (ξ, η, ζ), to a cubic curve passing through all the singular points, but having a double point at A_1. A conic, on the cubic surface, in a plane drawn through the line a', is associated with the symbol $U - A_1$, and corresponds to a line, in the plane (ξ, η, ζ), passing through the point A_1.

Ex. 4. On the cubic surface there are seventy-two families of cubic curves, each having as chords the six lines of one row of one of the double-sixes. For the row consisting of the lines $4', 5', 6', c_{23}, c_{31}, c_{12}$ (see above, p. 161), the corresponding family of cubic curves is represented by $2U - A_4 - A_5 - A_6$, giving, in the plane (ξ, η, ζ), conics passing through the three singular points A_4, A_5, A_6. For the row consisting of $1, 2, 3, c_{56}, c_{64}, c_{45}$, the cubic curves are represented by $4U - s - A_1 - A_2 - A_3$. For the row consisting of $2, 2', c_{13}, c_{14}, c_{15}, c_{16}$, the cubic curves are represented by $3U - s + A_1 - A_2$. All other cases are easily deducible from these.

Ex. 5. Two curves of the cubic surface associated, respectively, with the symbols $\lambda U - \Sigma\lambda_r A_r$, $\mu U - \Sigma\mu_r A_r$, meet one another in $\lambda\mu - \Sigma\lambda_r\mu_r$ points. Further, a curve of the preceding symbol is one of a family of which each curve is identified by the assignment of $\frac{1}{2}(n + m)$ points of it, so that the family is said to be of this *freedom*. Also the curves of this family are of *genus* $1 + \frac{1}{2}(n - m)$. Here $n = \lambda^2 - \Sigma\lambda_r^2$, $m = 3\lambda - \Sigma\lambda_r$, and the genus, π, of a curve, without actual multiple points, may be defined by saying that, of the intersections of a surface therewith, if the surface be of sufficiently high order, all but π can be chosen at will. For these matters, and the establishment of the general ideas, reference may again be made to a paper, *Proc. Lond. Math. Soc.* XI, 1912, p. 287; but see, also, Clebsch, *Die Geometrie auf der Flächen dritter Ordnung*, *Crelle's J.*, LXV, 1866.

Ex. 6. In the preceding theory, the cubic curves in the plane (ξ, η, ζ) which represent the plane sections of the cubic surface have been supposed to arise from the cubic surface. We may, however, start with six quite arbitrary points of the plane, and, taking the general cubic curve passing through these, whose equation will be of the form $A\vartheta + B\phi + C\psi + D\chi = 0$, consider the locus of a point in three dimensions whose coordinates are given by

$$x : y : z : t = \vartheta : \phi : \psi : \chi.$$

This locus will be a cubic surface, as the point (ξ, η, ζ), by which $\vartheta, \ldots, \chi$ are expressed, describes the plane. For $\vartheta, \phi, \psi, \chi$ we may then take any four particular cubic polynomials vanishing in the six given points, which are linearly independent of one another, the consequent modification of x, y, z, t being only a linear transformation of these. A particular case arose in Ex. 1 above, where the cubic curves were formed with the help of conics. Another case arises by remarking that there are two cubic curves passing through five of the given points which both have a double point at the remaining point. If we take ξ, η so that these both vanish at this sixth point, but not at any of the other five, the equations of these two cubic curves may be supposed to be $f=0$, $f_1=0$, where $f=a\xi^2+2h\xi\eta+b\eta^2$, and $f_1=a_1\xi^2+2h_1\xi\eta+b_1\eta^2$, in which a, h, b, a_1, h_1, b_1 are all linear homogeneous polynomials in ξ, η, and the third coordinate, ζ. From these we deduce two other cubic polynomials, $(af_1-a_1f)/\eta$ and $(fb_1-f_1b)/\xi$, which, equated to zero, represent cubic curves passing simply through all the six points. The four cubics are linearly independent, since an identity

$$A(af_1-a_1f)/\eta+B(fb_1-f_1b)/\xi-Cf_1+Df=0,$$

in which A, B, C, D are constants, involving the identity

$$f/f_1=(Aa\xi-Bb\eta-C\xi\eta)/(Aa_1\xi-Bb_1\eta-D\xi\eta),$$

would require f, f_1 to have a common factor linear in ξ, η, ζ. We thus obtain the cubic surface expressed by the formulae

$$x=(af_1-a_1f)/\eta, \quad y=(fb_1-f_1b)/\xi, \quad z=-f_1, \quad t=f,$$

the point, (ξ, η, ζ), corresponding to any point, (x, y, z, t), of this, being obtainable by solving the equations

$$az+a_1t+\eta x=0, \quad bz+b_1t-\xi y=0,$$

which are linear in ξ, η, ζ.

Ex. 7. Given seven lines of which no two intersect, having a common transversal, there exists a single point such that the planes joining this to the eight lines touch a quadric cone (p. 150, above, Ex. 4). Given six skew lines with a common transversal, the locus of a point, such that the planes joining it to the seven lines touch a quadric cone, is a cubic curve, having the six lines as chords but not meeting the transversal (above, p. 136). Given five skew lines with a common transversal, the locus of a point, such that the planes joining it to the six lines touch a quadric cone, is a cubic surface on which the six lines lie. The analytical proof of this leads naturally to the formulae of Ex. 6.

Let the common transversal be taken for $z=0$, $t=0$, the planes zxy, txy being any planes drawn through this. A quadric cone which touches the line $z=0$, $t=0$ will meet the planes zxy, txy in two conics touching one another on this line, say at the point $(\eta, -\xi, 0, 0)$. The equation of a plane being $lx+my+nz+pt=0$, the tangential equations of these conics are of the respective forms

$$n(l\eta-m\xi)-(al^2+2hlm+bm^2)=0,$$
$$p(l\eta-m\xi)-(a_1l^2+2h_1lm+b_1m^2)=0;$$

thus, if we denote $al^2+2hlm+bm^2$, and $a_1l^2+2h_1lm+b_1m^2$, respectively, by $\phi(l, m)$ and $\phi_1(l, m)$, and denote $\phi(\xi, \eta)$ and $\phi_1(\xi, \eta)$, respectively, by f and f_1, the coordinates, l, m, n, p, of the general tangent plane of the quadric cone which touches the line $z=0$, $t=0$ at the point $(\eta, -\xi, 0, 0)$, satisfy the relation

$$(l\eta - m\xi)(nf_1 - pf) - [\phi(l, m)f_1 - \phi_1(l, m)f] = 0.$$

This, however, breaks up into the equation $l\eta - m\xi = 0$, and the equation

$$l\frac{af_1 - a_1f}{\eta} + m\frac{fb_1 - f_1b}{\xi} - nf_1 + pf = 0,$$

which is thus the equation of the vertex of the cone. Herein the quantities $\xi, \eta, a, h, b, a_1, h_1, b_1$ enter homogeneously to the third order in the aggregate, the expression $(af_1 - a_1f)/\eta$, for example, being of the first order in ξ and η, and of the second order in the other six parameters.

Now, any line meeting the transversal $z=0$, $t=0$ is given by two planes $px+qy+rz=0$ and $px+qy+st=0$, so that the coordinates of the general plane containing this line are of the forms $(\lambda+\mu)p$, $(\lambda+\mu)q$, λr, μs. For this to be a tangent plane of the quadric cone we require

$$\lambda r(p\eta - q\xi) - (\lambda+\mu)\phi(p, q) = 0, \quad \mu s(p\eta - q\xi) - (\lambda+\mu)\phi_1(p, q) = 0;$$

thus, the condition for the cone to touch the line is

$$-p\eta + q\xi + \frac{1}{r}\phi(p, q) + \frac{1}{s}\phi_1(p, q) = 0.$$

This is a linear condition for the parameters $\xi, \eta, a, h, b, a_1, h_1, b_1$. Thus, when the cone is to touch seven such lines, beside their transversal, it is determinate, in general. When the cone is to touch six such lines, and their transversal, the six parameters, $a, h, \ldots, b_1$ are, in general, definite linear homogeneous functions of ξ, η; and the vertex of the cone, whose coordinates are then homogeneous cubic polynomials in ξ, η, describes a cubic curve. It may be shewn that this line has the six skew lines as chords, but does not meet their transversal; it has already been remarked (Ex. 8, p. 135) that the chords of a cubic curve which meet a line are projected, from a point of the curve, by planes which touch a quadric cone touching the line. Lastly, when the cone is to touch five skew lines and their common transversal, the six parameters $a, h, \ldots, b_1$ are expressible linearly by ξ, η and a third parameter, say ζ; the vertex of the cone then describes a cubic surface, upon which the five lines and their transversal may be shewn to lie.

Derivation of a plane quartic curve from a cubic surface. The converse of the result of the preceding example (7) is that, with the notation we have employed, the six planes joining an arbitrary point of a cubic surface to the lines b, c, d, e, f, of one row of a double-six, and to the line, a', of the other row, which meets all these, touch a quadric cone. This result finds its proper place in the considerations to which we now pass.

First, it must be made clear what is meant by the *tangent plane* of the cubic surface at any point, and by the *inflexional lines* of the surface at this point. If the equation of the surface be $f=0$, the three points in which the line joining two points (x_0, y_0, z_0, t_0), (x, y, z, t), meets the surface, are obtainable by solving the cubic equation in λ which arises by substituting in the equation $f=0$, for the coordinates, respectively $x_0+\lambda x$, $y_0+\lambda y$, etc. If D denote the operator $x\partial/\partial x_0 + y\partial/\partial y_0 + z\partial/\partial z_0 + t\partial/\partial t_0$, and f_0 denote what f becomes when x_0, y_0, z_0, t_0 are put for the coordinates, this equation is $f_0 + \lambda D f_0 + \frac{1}{2}\lambda^2 D^2 f_0 + \frac{1}{6}\lambda^3 D^3 f_0 = 0$. If (x_0, y_0, z_0, t_0) be on the cubic surface, or $f_0=0$, one root of this cubic equation is zero, for all points (x, y, z, t); namely, every line through (x_0, y_0, z_0, t_0) meets the surface once at that point. If (x, y, z, t) lie on the plane whose equation is $Df_0=0$, every line drawn in this plane, through the point (x_0, y_0, z_0, t_0), meets the surface twice at this point. This is the plane which is called the tangent plane of the surface at this point; the curve in which it meets the cubic surface has a double point at this point. Now consider the quadric surface whose equation is $D^2 f_0 = 0$; it is easy to prove that it passes through the point (x_0, y_0, z_0, t_0), and has, for tangent plane at this point, the tangent plane of the cubic surface. This plane therefore contains two lines of this quadric. But when a point, (x, y, z, t), satisfies both $Df_0=0$ and $D^2 f_0=0$, the above cubic equation in λ has three zero roots. We see thence that the generators at (x_0, y_0, z_0, t_0), of the quadric $D^2 f_0 = 0$, meet the cubic surface in three coincident points at (x_0, y_0, z_0, t_0); these are then the tangent lines at this point of the two branches of the section of the cubic surface by its tangent plane. They are the lines called the inflexional lines of the surface at (x_0, y_0, z_0, t_0); they lie in the tangent plane at this point. More generally, when (x_0, y_0, z_0, t_0) is not necessarily a point of the cubic surface, the plane $Df_0=0$ is called the polar plane of (x_0, y_0, z_0, t_0) in regard to the cubic surface, and the quadric $D^2 f_0 = 0$ is called the polar quadric; it is easy to see that, if Δ denote what the operator D becomes when x, y, z, t are interchanged, respectively, with x_0, y_0, z_0, t_0, the polar plane can be given by $\Delta^2 f=0$, and the polar quadric by $\Delta f=0$; thus, if a point lie on the polar plane of another point, this lies on the polar quadric of the first. In particular, the polar quadric of a point (x_0, y_0, z_0, t_0) meets the cubic surface in points at which the tangent plane passes through (x_0, y_0, z_0, t_0). Further, it may be remarked that if two curves on a cubic surface intersect in a point O, the plane determined by the tangent lines of these curves at O is the tangent plane of the surface.

We take now a point, O, of the cubic surface, not lying on any of the lines of the surface. Then we consider the curve of the surface which is the locus of points, P, of the surface, whereat the tangent plane passes through O; as we have seen, this is the intersection of the cubic surface with the polar quadric of O; moreover, this polar quadric passes through O, has the same tangent plane as the cubic surface at O, and contains the two inflexional lines of the surface at O. The curve in question, being the intersection of the cubic surface with a quadric, is of the sixth order; by what is said, it passes through O, having there a double point, its two tangent lines at this point being the inflexional lines of the cubic surface. Thus an arbitrary plane through O meets the curve in four other points, and the curve is projected from O, on to an arbitrary plane, into a quartic curve. Conversely, it may be stated, any general quartic curve in a plane can be shewn to be capable of being thus derived. We shall denote the sextic curve on the cubic surface by ϑ.

If an arbitrary quadric cone be drawn with vertex O, four of its twelve points of intersection with ϑ will be at O, and there will be eight others; it may happen that these eight other points coincide in pairs, the quadric cone touching the curve ϑ in four points, not at O. When this is so, we may, for brevity, speak of the cone as a contact cone. In such case, if a further arbitrary quadric cone be drawn, with vertex at O, to pass through the four points of contact, with the curve ϑ, of the contact cone in question, the remaining four intersections, with ϑ, of this further cone, are also points of contact with ϑ, of a contact cone; this we speak of as being of the same *system* as the former. There are in all sixty-three different systems of contact cones, with the same vertex, O. In particular such a contact cone may consist of two planes; then either it consists of the tangent plane of the cubic surface at O, taken with the plane joining O to one of the twenty-seven lines of the surface; or it consists of the planes joining O to two non-intersecting lines of the surface. There are 216 pairs of non-intersecting lines of the surface, but it will be seen that these arrange themselves in sets of six pairs giving degenerate contact cones of the same system.

We approach the proof of these statements by considering, first, degenerate contact cones consisting of the tangent plane at O taken with the plane joining O to a line of the surface. When the curve ϑ is projected from O on to a plane, ϖ, the tangent line of the resulting quartic curve, at any point, is evidently the intersection with ϖ of the tangent plane of the cubic surface at the corresponding point, P, of the curve ϑ, this tangent plane, by hypothesis, passing through O. In particular, by taking P at O, the tangent plane of the cubic surface at O gives rise to a tangent line of the quartic curve; but there are two ways of approach of P to O, along the two branches at O of the curve ϑ; thus the tangent plane of the cubic surface at O touches the plane quartic curve in two points, lying, namely, on the inflexional lines of the cubic surface at O. The plane joining O to any one, say l, of the lines of the cubic surface, equally gives a double-tangent of the plane quartic curve; the points of contact are the projections of the two points of the line l at which the plane Ol is a tangent plane of the cubic surface, namely the points in which l is met by the conic in which the plane Ol cuts the surface. We thus see that the plane quartic curve has twenty-eight double-tangents. Now, let any plane be drawn through the line l, meeting the cubic surface also in a conic, say σ. The six intersections of this plane with the curve ϑ will contain the two points of this we have just found lying on the line l, which do not lie on σ, in general; and there will be four other points, which will lie on the conic σ. We have remarked that, when two curves of the cubic surface have distinct tangent lines at a point where the curves intersect, the plane of these lines is the tangent plane of the surface. Thus, the tangent plane of the cubic surface at any one of the four intersections of the curve ϑ and the conic σ, touches both these curves; while, by definition of ϑ, this plane passes through the point O. The quadric cone joining O to the conic σ is thus a contact cone, touching the curve ϑ in four points. This cone will meet the plane ϖ in a conic having four points of contact with the plane quartic curve, and, therefore, no further intersections with this. We have remarked that, in general, the conic σ does not contain the two points where the curve ϑ meets the line l; the curve ϑ, however, lies on a quadric, the polar quadric of O; the section of this quadric by the plane of σ is a conic; this conic contains the two points where ϑ meets l, and the four points where ϑ meets σ. This conic also meets the two generators at O of the polar quadric of O. Thus, on projection to the plane ϖ, the four points of contact of the conic arising by projection of σ, with the plane quartic curve, lie on a conic whose other four intersections with the quartic are the points of contact with this of two of its double-tangents. Of the contact cones, $O\sigma$, joining O to a conic σ, there is an infinite number, each defined by a plane through the line l; and, as an arbitrary line through O meets the cubic surface again in two points, two such cones pass through this line. In particular, there are five planes through the line l for which the conic σ breaks up into two lines, and the cone $O\sigma$ becomes two planes; we may also consider the plane-pair consisting of the plane Ol and the tangent plane at O as a sixth degenerate contact cone. Thus, in the plane ϖ, for the plane quartic curve, the result is that two of the contact conics, of the system derived from the line l, pass through an arbitrary point of the plane, and six of them break up into two lines, each a bitangent of the plane quartic curve. This system of contact conics may remind us of the conics in a plane which touch four given lines, which is indeed a particular case, arising when the quartic curve breaks up into four lines. In the particular case, all the conics are inpolar to any conic passing through three particular points, which form a triad self-polar in regard to all the conics touching the four lines. In the general case now before us, it is also true that the contact conics of the system considered are all inpolar to a conic, which is unique; in special this is true of the six contact conics which break up into two lines. Thus, as a pair of lines inpolar to a conic intersect thereon, we have the result that the six points of intersection of the pairs of bitangents which are particular contact conics of the system, lie upon a conic, to which all the proper contact conics of the system are inpolar. Such a set of six pairs of bitangents is called a Steiner system; there are evidently twenty-seven systems arising in this way. The proof of several of the statements here made has not been given; but the character of this proof will be clear after the consideration of the thirty-six cases of contact conics associated with pairs of non-intersecting lines of the cubic surface, to which we now pass.

Consider one of the thirty-six double-sixes of lines of the cubic surface; with this, as we have seen, are associated two families of cubic curves on the surface. Through the point O, and an arbitrary point, say H, of the curve ϑ, a definite curve of one of these families can be drawn; this curve is projected from O by a quadric cone, whose residual intersection with the cubic surface is, as we have seen, a cubic curve of the other family; and these two curves cross one another at five points, as was also proved. Of these five points, O is evidently one; at any other point of crossing of these curves, which lie both on the cone and on the cubic surface, the plane of the tangent lines of the two curves is as well a tangent plane of the cone, passing through O, and a tangent plane of the cubic surface; thus the point of crossing lies on the curve ϑ. Wherefore, the quadric cone touches the cubic surface, and the curve ϑ, at the assumed point H, and also at three other points, say P, Q and R, and is such a contact cone as was in view. It may be remarked, too, that, in virtue of what has previously been proved (above, p. 177), the polar line of OH, in regard to the Schur quadric belonging to the double-six under consideration, lies in the plane PQR, the five points O, H, P, Q, R forming a self-conjugate pentad in regard to this quadric, while the contact cone is outpolar to this quadric. The system of contact cones, of vertex O, of the curve ϑ, obtained in this way, consists of the cones projecting from O the cubic curves, through O, of either of the two families belonging to the double-six; and, since an arbitrary line through O meets the cubic surface in two further points, and a cubic curve of the family is determined by O and another point of the surface, we see that there are two such cones containing, as generator, an arbitrary line drawn through O. Now, suppose that a and a' are two lines of different rows of the double-six, not intersecting one another; then, one cubic curve, of those associated with the double-six, consists of the line a', together with the conic in which an arbitrary plane drawn through the other line, a, meets the cubic surface, other than a itself; a degenerate cubic of the complementary family is, similarly, constituted by the line a taken with a conic of the cubic surface lying in a plane through a'. Hence, a degenerate contact cone is constituted by the two planes Oa, Oa'. Also, if the conics in the planes Oa, Oa' meet a and a', respectively, in A, B and A', B', it is easily seen, in accordance with a remark made above, that A, B, A', B' form a self-polar tetrad in regard to the Schur quadric belonging to the double-six. On the other hand, the contact cones from O are all inpolar to a definite quadric cone with O as vertex. For, such a contact cone is determined by one of the cubic curves, associated with the double-six, which pass through O; and this curve is determined by its tangent line at O. The equation of the cone thus contains rationally a parameter, determining this tangent line in the tangent plane at O of the cubic surface. We have remarked, however, that there are two of these contact cones possible, containing an arbitrary line drawn through O. The parameter thus enters to the second order in the equation of the tangent cone; this equation is, therefore, of the form $\lambda^2 U + \lambda V + W = 0$, where λ is the parameter, and U, V, W are cones with vertex at O. By considering the tangential equation of this cone, we see that there is a single definite cone, with O as vertex, which is outpolar to all the contact cones. In particular, this single cone, say Ω, is outpolar to all the plane-pairs such as Oa, Oa'. Hence we have the result that if a, a'; b, b'; ..., f, f' be the pairs of lines in the six *columns* of a double-six, and O be an arbitrary point of the cubic surface, the six transversals from O, to a and a'; to b and b'; etc., lie upon a quadric cone, Ω.

Ex. 1. A similar argument is applicable to the case of the contact cones obtained by joining O to the conics of the cubic surface lying in planes through a line, l, of the surface. Hence we have the result that, if C_1, C_2, ..., C_5 be the points of intersection of the five pairs of lines of the cubic surface, of which a pair lie in a plane through the line l, then the quadric cone containing the five lines OC_r contains also the line in which the plane Ol meets the tangent plane at O. It will be recalled that it has been proved that the five planes joining O to one from each of the five pairs of intersecting lines meeting l, touch a quadric cone which touches the plane Ol (above, p. 195).

It is known, too, that the lines joining the pairs of the five points C_r, and the planes containing the threes of these, meet an arbitrary plane in ten points, and ten lines, of which each point is the pole of one of the lines in regard to a certain conic in that plane (Vol. II, p. 218). When the plane is any plane through the line l, the conic in question is the conic of the cubic surface which lies in this plane; and any cubic curve through the five points C_r meets this plane in points which form a self-conjugate triad in regard to this conic (A. C. Dixon, *Quart. J. of Math.*, XXIII, 1889, p. 343).

Ex. 2. We may deal with the theory of the contact cones of the curve ϑ with analysis. It will be sufficient here to consider the case of contact cones associated with a double-six. As before, let O be any point of the cubic surface, and a, a' be two non-intersecting lines of the surface. Let the conics of the surface which lie in the planes Oa, Oa' meet the lines a, a', respectively, in A, B and A', B'. Choose coordinates so that the planes Oa, Oa', OAA' are, respectively, $y=0$, $z=0$, $x=0$, the plane $t=0$ being any plane through the line AA'. The line a, in the plane $y=0$, will have an equation $x+\lambda t=0$; as it does not intersect the line a', it must intersect the conic of the cubic surface which lies in the plane Oa', or $z=0$, the point of intersection being on the line $y=0$, $z=0$; this conic may then be taken to have the equations

$$z=0, \quad x^2+\lambda xt+mxy+nyt=0.$$

The line a' being, similarly, $z=0$, $x+\mu t=0$, the conic in the plane Oa has equations of the forms $y=0$, $x^2+\mu xt+rxz+kzt=0$. Hence we find, for the equation of the plane OBB', $\xi=0$, where $\xi=(\lambda-\mu)x+y(n-m\mu)+z(r\lambda-k)$; and, for the equation of the tangent plane of the cubic surface at O, $P=0$, where

$$P=\lambda\mu x+n\mu y+k\lambda z.$$

Then the equation of the cubic surface is $C=0$, where

$$C=t^2P+tQ+R,$$

in which

$$Q=(\lambda+\mu)x^2+(n+m\mu)xy+(k+r\lambda)xz+2fyz,$$

$$R=x^2(x+my+rz)+yz(ux+vy+wz),$$

f, u, v, w being constants. Taking, then, $U=2tP+Q$, so that $U=0$ is the polar quadric of O, we can verify the identity

$$4PC=U^2-x^2\xi^2+4yz\psi,$$

where

$$\psi=x^2(n-m\lambda)(r\mu-k)+P(ux+vy+wz)-fQ+f^2yz.$$

This identity is the same as

$$4PC = U^2 - \Phi^2 + 4yz\Psi,$$

wherein $\Phi = x\xi + 2\lambda yz$, $\Psi = \psi + \lambda x\xi + \lambda^2 yz$, and λ is an arbitrary parameter. The curve ϑ, for which $C = 0$, $U = 0$, thus lies on the quartic cone, of vertex O, given by

$$\Phi^2 - 4yz\Psi = 0, \text{ or } x^2\xi^2 - 4yz\psi = 0,$$

and $\Phi = 0$ represents the general quadric cone, of vertex O, containing the points A, B, A', B'. A general contact cone is therefore given by $\Psi = 0$.

It is clear from the first form of the identity that the points of the curve given by the intersection of the quadrics

$$U + x\xi + 2\lambda yz = 0, \quad U - x\xi - 2\lambda^{-1}\psi = 0$$

lie either on the cubic surface, or on the tangent plane, $P = 0$. It can be seen that these quadrics have in common a line, through O, lying in the tangent plane, variable with λ. The common curve of these quadrics, however, lies on the cone $\Psi = 0$, which thus contains a cubic curve, of the cubic surface, passing through O. Another such cubic curve is obtained from the quadrics

$$U - x\xi - 2\lambda yz = 0, \quad U + x\xi + 2\lambda^{-1}\psi = 0.$$

Further elementary results will be found in papers, *Proc. Lond. Math. Soc.*, IX, 1910, p. 145, and p. 205, the latter of which gives valuable references to the literature. The theory of the contact cones was originally obtained with the help of multiple theta functions, and belongs to any curve whatever which is not rational or elliptic. See Clebsch, *Crelle's J.*, LXIII, 1864, p. 189; and, for a general introduction, the writer's *Abel's Theorem* (Cambridge, 1897), pp. 377—392. The result we have proved that, of the points of contact, H, P, Q, R, of a contact cone with the curve ϑ, the three latter are determined when the first is given arbitrarily, arises from the fact that the curve ϑ is of *genus* 3 (see above, p. 194). The number 4, of points of contact, is the value of $2p-2$ when $p=3$; the number 63, of systems of contact cones, is the value of $2^{2p}-1$ when $p=3$; the number 6, of degenerate contact cones of any system is the value of $2^{p-2}(2^{p-1}-1)$ when $p=3$. For a general sextic curve obtained by the intersection of any cubic surface with any quadric surface, there are, similarly, quadrics which touch the curve in six points, the value of the genus, p, in this case, being 4. An interesting geometrical introduction to this case is given by Milne, *Proc. Lond. Math. Soc.*, XXI, 1922, p. 373; see also P. Roth, *Monatsh. f. Math. u. Phys.*, XXII, 1911, pp. 64—88.

The equation of the cubic surface referred to two Steiner trihedrals. Consider a cubic surface defined by the vanishing of a general cubic polynomial in the coordinates x, y, z, t, and assume that this surface contains at least one line, and thence that there are five planes through this line whose further intersection with the surface consists of a pair of lines (as above, p. 167); let the coordinates be taken, with the point $x = 0$, $y = 0$, $z = 0$ on the surface, so that one line of the surface is $x = 0$, $t = 0$, so that another pair of lines of the surface which meet this are $t=0$, $y=0$ and $t=0$, $z=0$, and so that a further plane, containing a pair of lines beside $x=0$, $t=0$, is given by $t+x=0$, these lines being, respectively, $t+x=0$, $v=0$, and $t+x=0$, $w=0$, where v, w are homogeneous polynomials in x, y, z. Then the equation of the cubic surface is necessarily of the form $t^2P+t\phi+xyz=0$, where ϕ is a homogeneous quadratic polynomial in x, y, z, and P is linear in these; and, as $t+x=0$ intersects the surface in the lines for which $v=0$, $w=0$, we have $\phi=xP+yz-vw$. The equation is, therefore, $t^2P+t(xP+yz-vw)+xyz=0$. It is at once seen, however, that this is identically the same as

$$(t+x)(t+by)(t+cz)=t(t+x+qv)(t+x+rw),$$

in which b, c, q, r are constants, provided that these constants can be so chosen that $bc=qr$, and also so that there is the identity

$$x+qv+rw-by-cz+bcP=0\,;$$

if we assume general values for v, w, P, as linear functions of x, y, z, write $b=\lambda q$, $c=\lambda^{-1}r$, and equate coefficients of x, y, z in this identity, two of the resulting equations determine q and r uniquely in terms of λ, and the substitution of these in the third equation leads to an equation for λ of the fourth order. Thus the form in question is possible. In more general terms, the equation of the cubic surface is reducible to the form $UVW=U'V'W'$, in which $U=0, \dots$ are six planes. By considering the geometrical meaning of this (as will appear more particularly in a moment), we see that these are the planes of a Steiner trihedral pair, previously explained (above, pp. 164, 174); so that there are 120 ways in which the equation of the cubic surface can be reduced to this form.

Now assume this form; and let $U_0, V_0, \dots$ be the values of $U, V, \dots$, respectively, at an arbitrary general point, (x_0, y_0, z_0, t_0), of the surface, which does not lie on a line of the surface; put $u=U/U_0$, $v=V/V_0$, etc.; then, in virtue of $U_0V_0W_0=U_0'V_0'W_0'$, we can suppose the equation of the surface to be $uvw=u'v'w'$. The six planes $u=0$, $v=0$, ... must be expressible by four of them; thus we can suppose the existence of two identities of the forms

$$a_1u+b_1v+c_1w=a_1'u'+b_1'v'+c_1'w',$$
$$a_2u+b_2v+c_2w=a_2'u'+b_2'v'+c_2'w',$$

wherein, since $u, v, \dots$ reduce to unity at (x_0, y_0, z_0, t_0), the constants are such that

$$a_1+b_1+c_1=a_1'+b_1'+c_1',\quad a_2+b_2+c_2=a_2'+b_2'+c_2',$$

and, therefore, such that

$$(a_1+\sigma a_2)+(b_1+\sigma b_2)+(c_1+\sigma c_2)=(a_1'+\sigma a_2')+(b_1'+\sigma b_2')+(c_1'+\sigma c_2'),$$

whatever σ may be. We may, however, in three ways, choose σ so that

$$(a_1 + \sigma a_2)(b_1 + \sigma b_2)(c_1 + \sigma c_2) = (a_1' + \sigma a_2')(b_1' + \sigma b_2')(c_1' + \sigma c_2').$$

But, when we have six quantities $\lambda, \mu, \nu, \lambda', \mu', \nu'$ which satisfy the two relations $\lambda + \mu + \nu = \lambda' + \mu' + \nu'$, and $\lambda\mu\nu = \lambda'\mu'\nu'$, if we put

$$\alpha = (\mu' - \nu)/(\lambda - \lambda'), \quad \beta = (\nu' - \lambda)/(\mu - \mu'), \quad \gamma = (\lambda' - \mu)/(\nu - \nu'),$$

and, then, $\alpha' = 1 - \alpha$, $\beta' = 1 - \beta$, $\gamma' = 1 - \gamma$, it can at once be verified that $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are, respectively, proportional to $\beta\gamma', \gamma\alpha', \alpha\beta', \beta'\gamma, \gamma'\alpha, \alpha'\beta$. Hence, in our case, the identities connecting $u, v, \ldots$ are capable of the forms

$$\beta_r\gamma_r'u + \gamma_r\alpha_r'v + \alpha_r\beta_r'w = \beta_r'\gamma_r u' + \gamma_r'\alpha_r v' + \alpha_r'\beta_r w',$$

for $r = 1, 2, 3$, these being equivalent to two independent equations. If, then, we put

$$\xi_r = \beta_r\gamma_r'u, \quad \eta_r = \gamma_r\alpha_r'v, \quad \zeta_r = \alpha_r\beta_r'w,$$
$$\xi_r' = \beta_r'\gamma_r u', \quad \eta_r' = \gamma_r'\alpha_r v', \quad \zeta_r' = \alpha_r'\beta_r w',$$

the equation of the cubic surface is capable of the forms

$$\xi_r\eta_r\zeta_r = \xi_r'\eta_r'\zeta_r',$$

where $\xi_r, \eta_r, \ldots$ are subject to two linear identities, of which one is

$$\xi_r + \eta_r + \zeta_r = \xi_r' + \eta_r' + \zeta_r'.$$

With these equations we can at once specify the equations of the twenty-seven lines of the cubic surface. In the first place, the line of intersection of any one of the planes $\xi_r = 0$, $\eta_r = 0$, $\zeta_r = 0$ with any one of the planes $\xi_r' = 0$, $\eta_r' = 0$, $\zeta_r' = 0$ is evidently a line of the surface, the same line for the three values of the suffix r, since the plane $\xi_r = 0$ is the same as $u = 0$, etc. Denoting such a line by $\xi\xi'$, $\xi\eta'$, etc., we may represent these lines, respectively, with a notation previously employed, by the scheme

$$\begin{array}{lllccc} \xi\xi', & \xi\eta', & \xi\zeta' \equiv & c_{12}, & 1, & 2' \\ \eta\xi', & \eta\eta', & \eta\zeta' & 1', & c_{31}, & 3 \\ \zeta\xi', & \zeta\eta', & \zeta\zeta' & 2, & 3', & c_{23} \end{array}$$

and may thence recover the original definition of the trihedral pair. In the next place, if X_r', Y_r', Z_r', respectively, denote $\xi_r', \eta_r', \zeta_r'$, in some order, the line given by $\xi_r = X_r'$ and $\eta_r = Y_r'$ is, by the identity connecting the six planes, also on the plane $\zeta_r = Z_r'$; and thus lies on the surface. This line may be represented by $(X', Y', Z')_r$; by taking all three values of r, and all permutations for X_r', Y_r', Z_r', so that, for example, $(\xi', \zeta', \eta')_r$ stands for $\xi_r = \xi_r'$, $\eta_r = \zeta_r'$, $\zeta_r = \eta_r'$, we obtain eighteen lines. With the previous notation they may be identified as follows:

$(\eta', \xi', \zeta')_r \equiv c_{14}, c_{15}, c_{16}$; $(\zeta', \eta', \xi')_r \equiv c_{24}, c_{25}, c_{26}$; $(\xi', \zeta', \eta')_r \equiv c_{34}, c_{35}, c_{36}$
$(\xi', \eta', \zeta')_r \equiv c_{56}, c_{64}, c_{45}$; $(\zeta', \xi', \eta')_r \equiv 4, 5, 6$; $(\eta', \zeta', \xi')_r \equiv 4', 5', 6'$,
where the meaning is, that, in each case, the three lines on the right arise, respectively, for $r = 1, 2, 3$.

The Cremona form of the equation of the cubic surface. Remarking the identity

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3(y + z)(z + x)(x + y),$$

and putting

$$\xi = x_2 + x_3, \quad \eta = x_3 + x_1, \quad \zeta = x_1 + x_2,$$
$$\xi' = -(x_5 + x_6), \quad \eta' = -(x_6 + x_4), \quad \zeta' = -(x_4 + x_5),$$

or

$$2x_1 = \eta + \zeta - \xi, \; \ldots, \; 2x_4 = \xi' - \eta' - \zeta', \; \ldots,$$

the two equations $\xi\eta\zeta = \xi'\eta'\zeta'$, $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$, lead to the two

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0.$$

The cubic surface may then be supposed to be given by these two equations, taken with another necessary linear equation connecting $x_1, \ldots, x_6$. (Cremona, *Math. Annal.*, XIII, 1878.)

The planes $x_1 = 0, \ldots, x_6 = 0$, save for permutations, are determinate, in association with a particular double-six of lines of the cubic surface. For fifteen of the lines of the surface are given each by a set of three planes such as $x_1 + x_2 = 0$, $x_3 + x_4 = 0$, $x_5 + x_6 = 0$. The other twelve lines, which are $5, 5', c_{6m}$ and $6, 6', c_{5m}$ (for $m = 1, 2, 3, 4$), form a double-six, and can be found in terms of the coefficients in the remaining equation connecting $x_1, \ldots, x_6$, by the solution of a quadratic equation.

The Sylvester form of the equation of the cubic surface. The Cremona form may be regarded as a particular case of a form $\Sigma a_r x_r^3 = 0$, subject to the relation $\Sigma x_r = 0$, with another linear relation connecting $x_1, \ldots, x_6$; that namely in which the coefficients $a_1, \ldots, a_6$ are equal. As Reye has shewn (*Crelle's J.*, LXXVIII, 1874), there exists a set of six planes $x_1 = 0, \ldots$, corresponding to any arbitrary line, for which the more general form is possible. We assume this form, in order to reach a remarkable form, as a sum of *five* cubes, due to Sylvester (1851; *Math. Papers*, I, p. 195); the method we follow can in particular be carried out starting from the Cremona form. It will be seen that it is supposed that the cubic surface is quite general.

We assume, that is, the cubic surface given by the equations

$$\sum_{r=1}^{6} a_r x_r^3 = 0, \quad \sum_{r=1}^{6} x_r = 0, \quad \sum_{r=1}^{6} h_r x_r = 0,$$

of which the last is the second necessary condition connecting $x_1, x_2, \ldots$; in this we may, evidently, replace $h_1, h_2, \ldots$ by $h_1+\lambda$, $h_2+\lambda, \ldots$, where λ is arbitrary. We take the most general case possible, assuming that no two of $h_1, h_2, \ldots, h_6$ are equal; this is necessary for the method we employ. Then, $\phi(\sigma)$ denoting the product of the six factors $\sigma-h_r$, and $\phi'(\sigma)=d\phi(\sigma)/d\sigma$, none of the quantities $\phi'(h_r)$ is zero. We can also suppose that the sum, of six terms, $\Sigma a_r^{-1}\phi'(h_r)$, does not vanish. For, the quintic polynomial in λ expressed by $\Sigma a_r^{-1}\phi'(h_r+\lambda)$ vanishes only for five values of λ, unless it vanish identically—and that is not so; this we see by equating to zero the coefficients of the various powers of λ, and eliminating, from the six resulting equations, the six coefficients a_r^{-1}; the result is the product of the differences of $h_1, \ldots, h_6$. To ensure that $\Sigma a_r^{-1}\phi'(h_r)$ does not vanish we have, then, only to increase each of $h_1, h_2, \ldots$ by a suitable value of λ; this we suppose to be done. Now we consider the expression

$$\psi(\sigma), = -\phi(\sigma)\sum_{r=1}^{6}\frac{a_r^{-1}\phi'(h_r)}{\sigma-h_r};$$

by the condition just secured, this is a quintic polynomial in σ; let its roots be $k_1, \ldots, k_5$. It is necessary for the method we employ to assume that *no two of these roots are equal, nor any of them equal to any of* $h_1, \ldots, h_6$.

Next, let
$$F(\sigma)=\phi(\sigma)\sum_{r=1}^{6}\frac{x_r}{\sigma-h_r},$$

so that, by the conditions connecting $x_1, \ldots, x_6$, this is a cubic polynomial in σ. The equation $F(\sigma)=0$, when $x_1, \ldots, x_6$, as well as σ, are variable, is the equation of a plane of a cubic developable; as $F(h_r)=\phi'(h_r)x_r$, this is the cubic developable of which $x_1=0$, $x_2=0, \ldots, x_6=0$ are planes.

Now consider, as depending upon σ, the fraction

$$[F(\sigma)]^3/\phi(\sigma)\psi(\sigma),$$

of which the numerator is of the ninth order, and the denominator is of the eleventh order, and has its roots all different. Expressing this in partial fractions we can infer that

$$\sum_{r=1}^{6}\frac{F^3(h_r)}{\phi'(h_r)\psi(h_r)}+\sum_{s=1}^{5}\frac{F^3(k_s)}{\phi(k_s)\psi'(k_s)}=0;$$

as $F(h_r)=\phi'(h_r)x_r$ and $\psi(h_r)=-a_r^{-1}[\phi'(h_r)]^2$, the first sum is $-\Sigma a_r x_r^3$. If we put

$$\xi_s=[\phi(k_s)]^{\frac{1}{3}}\sum_{r=1}^{6}\frac{x_r}{k_s-h_r},$$

so that $\xi_s = 0$ represents one of five particular planes of the cubic developable containing the six planes $x_r = 0$, and also put

$$\mu_s = [\phi(k_s)]^{\frac{1}{2}}/\psi'(k_s),$$

we are finally able to infer that

$$\sum_{r=1}^{6} a_r x_r^3 = \sum_{s=1}^{5} \mu_s \xi_s^3,$$

where also $\sum_{s=1}^{5} \mu_s \xi_s = 0$; for

$$\sum_{s=1}^{5} \mu_s \xi_s = \sum_s \frac{\phi(k_s)}{\psi'(k_s)} \sum_r \frac{x_r}{k_s - h_r}, \; = \sum x_r \sum_s \frac{\phi(k_s)}{\psi'(k_s)(k_s - h_r)},$$

and we have only to consider the expression of $F(\sigma)/\psi(\sigma)$ in partial fractions, regarded as depending on σ. The relation $\Sigma \mu_s \xi_s = 0$ is the only necessary relation among the five planes $\xi_s = 0$.

This method of deducing Sylvester's form was given by Beltrami (*Rendic. Lomb.*, Milan, XII, 1879, p. 24). Sylvester's own enunciation was accompanied by indications of a proof, depending on the properties of the Hessian surface, which has been developed by Clebsch (*Crelle's J.*, LIX, 1861, p. 193); the nature of this proof will be understood from the properties of the Hessian developed below on the basis of Sylvester's form. The reader may also consult Gordan, *Math. Annal.*, V, 1872, p. 341. Also Segre and Sturm, *Archiv d. Math. u. Phys.*, X, 1906, pp. 209, 216. And, generally, Steiner, *Gesamm. Werke*, II, p. 657; Sturm, *Synthetische Untersuch. ü. Fl. dr. Ordn.*, Leipzig, 1867; Cremona, *Mémoire de géom. pure sur les surf. d. trois. ordre, Crelle's J.*, LXVIII, 1868, and Grassmann, *Crelle's J.*, XLIX, 1855, p. 37.

The polars in regard to a cubic surface. The Hessian surface. We have already explained (above, p. 182) what is meant by the polar quadric, and by the polar plane, of a point in regard to a cubic surface. In regard to the polar quadrics, it is convenient to notice here, (*a*), that the polar quadrics of all points of a line are a system of quadrics all passing through the curve of intersection of any two of them, as is clear because the polar quadric of a point is expressed by an equation which is linear in the coordinates of this point; and, (*b*), that the polar quadrics of all points of a plane have common the eight points of intersection of any three of these quadrics. When the polar quadric of one point passes through another point the polar plane of this other passes through the first point; there are thus eight points of which a given plane is the polar plane. In regard to polar planes, we consider similarly, (*a*), the envelope of the polar planes of the points of a given line. These planes are expressed by an equation of the form

$$[(\lambda x_1 + \mu x_2)\partial/\partial x + \ldots]^2 f = 0,$$

that is, of the form $\lambda^2 P + 2\lambda\mu Q + \mu^2 R = 0$, where P, Q, R are linear in the current coordinates x, y, z, t. These are the tangent planes of a quadric cone, whose equation is $RP = Q^2$. But it may happen, as will be seen, that the cone degenerates, all these polar planes having a line in common. Further, (b), we consider the envelope of the polar planes of the points of a given plane, say ϖ. It is clear that the *coordinates* of such a polar plane are homogeneous quadratic functions of *three* variable parameters. From this it follows (as will be seen below, p. 222) that these polar planes all touch a particular surface of the third order, which has four double points. The polar quadrics of the points of this cubic surface, taken in regard to the original cubic surface, can then be proved to touch the plane, ϖ, from which we began.

The polar quadric of a point, (x_0, y_0, z_0, t_0), may be a cone, of vertex (x_1, y_1, z_1, t_1). The condition for this is the coexistence of four equations, linear both in regard to (x_0, y_0, z_0, t_0), and in regard to (x_1, y_1, z_1, t_1), as we easily see; and these equations are symmetrical in regard to both these points. If we denote the values, at (x_1, y_1, z_1, t_1), of $\partial^2 f/\partial x^2, \partial^2 f/\partial x \partial y, \ldots, \partial^2 f/\partial t^2$, wherein $f=0$ is the equation of the cubic surface, respectively by $f_{11}, f_{12}, \ldots, f_{44}$, these four equations are $x_0 f_{r1} + y_0 f_{r2} + z_0 f_{r3} + t_0 f_{r4} = 0$, for $r = 1, \ldots, 4$. We infer, then, that the necessary and sufficient condition is that both (x_0, y_0, z_0, t) and (x_1, y_1, z_1, t_1) should be points of a quartic surface whose equation is expressible by the vanishing of the determinant, of four rows and columns, whose general element is f_{rs}. This is the Hessian surface of the cubic surface, of which (x_0, y_0, z_0, t_0) and (x_1, y_1, z_1, t_1) are then said to be *corresponding* points. (Cf. p. 188, above.) It is important to recognise that this surface is independent of the choice of coordinates by which it is expressed.

The Hessian may be formed by the rule for any cubic surface, however special. When the surface is sufficiently general to be expressible in Sylvester's form

$$ax^3 + by^3 + cz^3 + dt^3 + eu^3 = 0,$$

with the condition

$$ax + by + cz + dt + eu = 0,$$

the polar plane of a point (X, Y, Z, T, U), where also

$$aX + \ldots + eU = 0,$$

is easily found to be

$$aX^2x + bY^2y + cZ^2z + dT^2t + eU^2u = 0;$$

and this equation is also that of the polar quadric of (x, y, z, t, u) when $X, Y, \ldots$ are regarded as current coordinates. Hence the equation of the Hessian surface is

$$ax^{-1} + by^{-1} + cz^{-1} + dt^{-1} + eu^{-1} = 0,$$

the relation connecting corresponding points being

$$x_0x_1 = y_0y_1 = z_0z_1 = t_0t_1 = u_0u_1.$$

If $F = 0$ denote the Hessian, the tangent plane, at any point $(x_0, y_0, \ldots)$ of this Hessian, calculated by the formula

$$[x\partial/\partial x_0 + y\partial/\partial y_0 + z\partial/\partial z_0 + t\partial/\partial t_0]F = 0,$$

where, in F, the coordinate u_0 has been replaced by

$$-e^{-1}(ax_0 + by_0 + cz_0 + dt_0),$$

may easily be seen to be

$$axx_0^{-2} + \ldots + dtt_0^{-2} + euu_0^{-2} = 0.$$

If $(x_1, y_1, \ldots)$ be the point corresponding to $(x_0, y_0, \ldots)$, this equation is the same as $axx_1^2 + \ldots + dtt_1^2 + euu_1^2 = 0$. Thus the tangent plane of the Hessian surface at any point is the polar plane, in regard to the cubic surface, of the corresponding point of the Hessian. This result may be obtained without the special form.

Now consider the relations of the five Sylvester planes, $x = 0, \ldots, u = 0$, to the Hessian surface. If these are clear, since the Hessian may be formed from the original form of the cubic surface, the method of reduction of this to the Sylvester form may be inferred. The five planes, $x = 0, \ldots, u = 0$, meet in threes in ten points. It is easily seen that every line drawn through one of these meets the Hessian in two coincident points there. The Hessian has, therefore, these ten points as double points. The five planes also meet in pairs in ten lines, each of which, evidently, contains three of these double points; these ten lines lie wholly on the Hessian. In any one of the ten planes there are four of the lines, and six of the double points, lying at the intersections of the lines. The polar quadric, in regard to the cubic surface, of any one of the ten points, breaks up into two planes, containing the line of intersection of two of the five planes and harmonic therewith. For instance, the polar quadric of the point $x = 0$, $y = 0$, $z = 0$, is $dT^2t + eU^2u = 0$. By this condition the ten points are definable directly from the equation of the cubic surface.

Ex. 1. Prove that the vertices of a pair of Steiner trihedrals (above, p. 204) are corresponding points of the Hessian surface.

Ex. 2. It has been seen that there are eight points of which a given general plane is the polar plane, in regard to the cubic surface. Prove that, upon the line joining any two of these eight points, there is an infinite number of pairs of points, forming an involution, such that the polar planes, in regard to the cubic surface, of the two points of a pair, are the same. Also, that the polar planes of all points of such a joining line pass through another line; the axial pencil of these planes, taken twice over, is then a degenerate form of the quadric cone in general obtained by the envelope of the polar planes of the points of a line. Prove also that, if the polar planes of all points of a line, with respect to the cubic surface, pass through another line, then two of the four intersections of the first line with the Hessian surface are corresponding points of this surface.

Ex. 3. Let P, P' be any pair of corresponding points of the Hessian, and let the line joining these meet the Hessian again in the points M and N. Let M', N' be the points of the Hessian respectively corresponding to M and N. Prove, (1), that the line $M'N'$ is the intersection of the tangent planes of the Hessian at P and P'; (2), that the polar plane of every point of the line PP', with respect to the cubic surface, contains the line $M'N'$, and (3), that the line $M'N'$ touches the Hessian at M' and N'.

Ex. 4. The cubic surface which is the envelope of the polar planes, in regard to the cubic surface, of the points of a plane, touches the Hessian at the points of a sextic curve. This curve is the locus of the points corresponding to the points of the plane curve in which the given plane cuts the Hessian.

Ex. 5. Let an arbitrary line meet the Hessian in points P, Q, R, S; and let P', Q', R', S' be the points respectively corresponding to these. The polar quadrics of P', Q', R', S', with respect to the cubic surface, are cones, with vertices respectively at P, Q, R, S. Prove that there are two planes, through the line $PQRS$, which are conjugate to one another in regard to each of these four cones. Then, that these two planes, taken with the four planes each containing three of the points P', Q', R', S', are six planes, say $\chi_r = 0$, in terms of which the equation of the cubic surface is expressible in the form $\Sigma c_r \chi_r^3 = 0$, $(r = 1, 2, \ldots, 6)$. (Reye, *Crelle's J.*, LXXVIII, 1874.)

Ex. 6. Every line of the cubic surface touches the Hessian in two points; these are corresponding points; and the twenty-four points thus arising from the lines of a double-six are the intersections, of the Schur quadric associated with the double-six, with the curve, of order twelve, which is the intersection of the cubic surface and its Hessian. When the cubic surface is $\Sigma a x^3 = 0$, with $\Sigma a x = 0$, in Sylvester's form, the fifty-four points so arising satisfy the equation $\Sigma a x^{-3} = 0$.

Ex. 7. If $u = ax + by + cz + dt$, $v = lx + my + nz + pt$, the Hessian of the cubic surface whose equation is $x^3 + y^3 + z^3 + t^3 + u^3 + v^3 = 0$, is given by

$$\begin{aligned} xyzt + uxyzt\,\Sigma\, a^2 x^{-1} + vxyzt\,\Sigma\, l^2 x^{-1} & \\ + uv\,[\Sigma\,(dl - ap)^2 yz + \Sigma\,(bn - cm)^2 xt] &= 0. \end{aligned}$$

When $d = 0$, so that the four planes $x = 0$, $y = 0$, $z = 0$, $u = 0$ meet in a point, if we put X, Y, Z, U, V, respectively, for $p^2 x$, $p^2 y$, $p^2 z$, $p^2 u$, $p^2 v$, this reduces to the form

$$tV\,[\Sigma\, l^2 YZ + U\Sigma X\,(bn - cm)^2] + (t + V)\,[XYZ + U\Sigma a^2 YZ] = 0.$$

The six common generators of the two cones obtained by equating to zero the coefficients of tV and $t+V$, in this equation, which have a common vertex, are then six intersecting lines lying on the Hessian surface. We infer that the cubic surface in question is not general enough to be given by the Sylvester form, as a sum of five distinct cubes. The line $V=0$, $t=0$ also lies on this Hessian.

Reduced forms for twenty-three cases of cubic surfaces are given by Cayley, *Papers*, VI, p. 359. It may be a curious problem for the reader to enquire what is the fewest number of cubes by which these forms can severally be represented. That any cubic polynomial, in x, y, z, t, may be represented as a sum of cubes only, each of a linear function of x, y, z, t, with proper coefficients, is clear; as a very particular case it may be shewn there is always an unique representation by a selection from the following twenty cubes: four cubes such as x^3; four cubes such as $(y+z+t)^3$; six cubes such as $(y+z)^3$ and six cubes such as $(y-z)^3$.

The representation of a general cubic surface upon a quadric surface. As both a cubic surface and a quadric surface may be represented upon a plane, it follows that a cubic surface can be represented on a quadric surface. Let A, B, C, D, E be five points of a quadric; these lie in threes in ten planes, giving, in all, ten conics lying on the quadric; there are two generators of the quadric at each of the five points; there are also two cubic curves lying on the quadric, containing the five points (above, p. 129). The representation of the cubic surface upon the quadric can be made in such a way that the twenty-seven lines of the cubic surface correspond to the twenty-seven elements of the quadric which we have enumerated, namely 5 points, 10 conics, 10 generators and 2 cubic curves.

For let the cubic surface be given by $\xi\eta\zeta=\xi'\eta'\zeta'$, where ξ, η, ζ are such that $\xi+\eta+\zeta=\xi'+\eta'+\zeta'$ and $a\xi+b\eta+c\zeta=a'\xi'+b'\eta'+c'\zeta'$. Then, for any σ, we have the relation

$$(a+\sigma)\xi+(b+\sigma)\eta+(c+\sigma)\zeta=(a'+\sigma)\xi'+(b'+\sigma)\eta'+(c'+\sigma)\zeta'.$$

There are two values of σ for which

$$(a+\sigma)(b+\sigma)(c+\sigma)=(a'+\sigma)(b'+\sigma)(c'+\sigma);$$

let these be σ_1 and σ_2; put $\xi_1=(a+\sigma_1)\xi$, and $\xi_2=(a+\sigma_2)\xi$, etc. We then have $\xi_r\eta_r\zeta_r=\xi_r'\eta_r'\zeta_r'$, $\xi_r+\eta_r+\zeta_r=\xi_r'+\eta_r'+\zeta_r'$, for $r=1$ and $r=2$. Now take x, y, z, t by the definitions

$$x/t=(\eta'-\zeta)/(\xi-\xi'),\quad y/t=(\zeta'-\xi)/(\eta-\eta'),\quad z/t=(\xi'-\eta)/(\zeta-\zeta'),$$

and let $x'=t-x$, $y'=t-y$, $z'=t-z$. We easily verify that

$$\xi:\eta:\zeta:\xi':\eta':\zeta'=yz':zx':xy':y'z:z'x:x'y,$$

which involve the two relations $\xi\eta\zeta=\xi'\eta'\zeta'$, $\xi+\eta+\zeta=\xi'+\eta'+\zeta'$. The remaining relation which limits $\xi, \eta, \ldots$ requires, however, that

$$ayz'+bzx'+cxy'=a'y'z+b'z'x+c'x'y,$$

and expresses that the point (x, y, z, t) lies on a quadric surface containing the five points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$.

Denote these points, respectively, by A, B, C, D, E, the generators of one system of the quadric which contains these points, respectively, by a_1, b_1, c_1, d_1, e_1, and those of the other system, through these points, respectively, by a_2, b_2, c_2, d_2, e_2. Further, let the cubic curve lying on the quadric, passing through the five points, which meets the generators of the first system each in *one* point be denoted by γ_1, and the cubic curve on the surface, through the five points, which meets the generators of the second system each in one point, be denoted by γ_2. Then, denoting the line of the cubic surface for which $\xi = 0$, $\xi' = 0$ by (ξ, ξ'), etc., the line which lies on the planes $\xi = \xi'$, $\eta = \eta'$, $\zeta = \zeta'$ by (ξ', η', ζ'), that which lies on the planes $\xi_1 = \xi_1'$, $\eta_1 = \eta_1'$, $\zeta_1 = \zeta_1'$ by $(\xi', \eta', \zeta')_1$, and so on, it may be verified that the twenty-seven lines of the cubic surface correspond to the twenty-seven elements of the quadric surface which have been specified, in accordance with the following scheme:

(ξ, ξ'), the point A ; (ξ', η), the plane ABD ;
(ζ, ξ'), the plane CAE ; (ξ, η'), the plane ABE ;
(η, η'), the point B ; (η', ζ), the plane BCD ;
(ζ', ξ), the plane CAD; (η, ζ'), the plane BCE ;
(ζ, ζ'), the point C.

(ξ', η', ζ'), ABC ; (η', ζ', ξ'), E ; (ζ', ξ', η'), D; (ξ', ζ', η'), ADE;
(ζ', η', ξ'), BDE ; (η', ξ', ζ'), CDE ; $(\xi', \eta', \zeta')_1, \gamma_2$; $(\eta', \zeta', \xi')_1, d_2$;
$(\zeta', \xi', \eta')_1, e_2$; $(\xi', \zeta', \eta')_1, a_1$; $(\zeta', \eta', \xi')_1, b_1$; $(\eta', \xi', \zeta')_1, c_1$;
$(\xi', \eta', \zeta')_2, \gamma_1$; $(\eta', \zeta', \xi')_2, d_1$; $(\zeta', \xi', \eta')_2, e_1$; $(\xi', \zeta', \eta')_2, a_2$;
$(\zeta', \eta', \xi')_2, b_2$; $(\eta', \xi', \zeta')_2, c_2$.

A particular double-six of lines of the cubic surface corresponds then to the two rows of elements on the quadric surface given by

$$\begin{array}{cccccc} A, & B, & C, & D, & e_2, & e_1 \\ BCD, & CAD, & ABD, & ABC, & \gamma_2, & \gamma_1\,; \end{array}$$

and, if the two rows of lines of this double-six be denoted, respectively, by $l_1, \ldots, l_6$ and $m_1, \ldots, m_6$, and the line which meets the two l_r, m_s, and the two l_s, m_r, be denoted by n_{rs}, the respective elements $n_{1h}\,(h = 2 \ldots 6)$, $n_{2k}\,(k = 3 \ldots 6)$, $n_{3r}\,(r = 4 \ldots 6)$, $n_{4s}\,(s = 5, 6)$, n_{56}, are given by

ABE, CAE, ADE, a_1, a_2; BCE, BDE, b_1, b_2; CDE, c_1, c_2; d_1, d_2; E.

Further, the cubic curve of the cubic surface, obtained by the intersection of this with the general quadric containing the lines m_2, m_3, n_{23} (see above, p. 176), is found to correspond to the conic on the quadric surface given by $Px' + Qy' + Rz' = 0$, that is, to the general conic of the quadric surface passing through the point $(1, 1, 1, 1)$. While, the cubic curve of the cubic surface which is the intersection of this with the general quadric containing the lines l_2, l_3, n_{23} corresponds to the sextic curve on the quadric given by its intersection with the cubic surface $Lx'yz + My'zx + Nz'xy = 0$, or, what is the same thing, with the cubic surface given by $Lx^{-1} + My^{-1} + Nz^{-1} + Kt^{-1} = 0$, wherein $L + M + N + K = 0$. As is seen in the following note, this is the general cubic surface with four double points, at the points A, B, C, D, which contains the point $(1, 1, 1, 1)$.

Ex. Determine the representation on the quadric of the other cubic curves of the cubic surface.

Note on the cubic surface with four double points. The present chapter has been devoted to the fundamental properties of the general cubic surface. Many special cubic surfaces have interesting properties; but there seems special reason for devoting some remarks to that surface which has, in fact, the greatest number of possible double points—as it presents itself in various ways. The surface was first studied by Cayley, *Papers*, I, p. 183 (1844).

The equation $x^{-1} + y^{-1} + z^{-1} + t^{-1} = 0$ clearly represents a cubic surface having each of the four points such as $(1, 0, 0, 0)$ as a double point. The six lines joining the pairs of these points lie entirely on the surface; and there are three other lines, each meeting an opposite pair of these six lines, which also lie on the surface. For instance, one of these three lines is given by $y + z = 0$, $x + t = 0$; the plane $x + t = 0$ meets the surface in this line, and in the line $x = 0$, $t = 0$ counting doubly. The three new lines lie in the plane $x+y+z+t=0$; and are the joins of the three pairs of opposite intersections of the four lines in which this plane meets the four fundamental planes, $x = 0$, etc. The surface contains no other lines than the nine which have been mentioned; if each of the six lines joining two of the double points be counted four times, these nine lines count as twenty-seven.

If we take three quantities, ξ, η, ζ, which we may regard as coordinates in a plane, defined by

$$\xi : \eta : \zeta = x(y+z) : y(z+x) : z(x+y),$$

where x, y, z, t are the coordinates of a point of the cubic surface, so that, as we may easily verify

$$\xi^{-1} : \eta^{-1} : \zeta^{-1} = tx - yz : ty - zx : tz - xy,$$

then we deduce

$$x^{-1} : y^{-1} : z^{-1} : t^{-1} = \xi - \eta - \zeta : \eta - \zeta - \xi : \zeta - \xi - \eta : \xi + \eta + \zeta,$$

which evidently satisfy the equation of the cubic surface. By these formulae, there is established a birational correspondence between the points of the cubic surface and the points of the plane (ξ, η, ζ). In this plane there are four lines $\xi \pm \eta \pm \zeta = 0$, each of which corresponds to one of the double points of the cubic surface, while the lines $\xi = 0$, $\eta = 0$, $\zeta = 0$ may be regarded as corresponding to the three lines of the surface lying in the plane $x + y + z + t = 0$. In general, to the plane cubic curve in which the surface is met by the plane $Ax + By + Cz + Dt = 0$, there corresponds, in the plane, the cubic curve $AYZT + \ldots + DXYZ = 0$, where X, Y, Z, T are, for brevity, written for $\xi - \eta - \zeta, \ldots, \xi + \eta + \zeta$. This latter is the general cubic curve passing through the six intersections of the four lines $X = 0$, $Y = 0, \ldots$. Conversely, to a general line of the plane, $u\xi + v\eta + w\zeta = 0$, there corresponds, on the cubic surface, the curve in which this is met by the quadric cone

$$(v + w)yz + (w + u)zx + (u + v)xy = 0;$$

this curve lies on the cubic surface

$$ux^{-1} + vy^{-1} + wz^{-1} + (u + v + w)t^{-1} = 0,$$

which is another cubic surface having the same four double points as the original. The curve of intersection of the original cubic surface with that represented by $ax^{-1} + by^{-1} + cz^{-1} + dt^{-1} = 0$ corresponds to the line, in the plane, given by

$$(d + a - b - c)\xi + (d + b - c - a)\eta + (d + c - a - b)\zeta = 0,$$

that is to say, the portion of this curve of intersection other than the six joins of the four fundamental points, so corresponds. It may be convenient to describe as the fundamental cubic curves, of the original cubic surface, these cubic curves, which are thus given by its intersection with other cubic surfaces having the same double points. Two of these curves have one common point, other than the four double points, through which they all pass; and one of these curves can be drawn through two arbitrary points of the original surface.

There is, upon any one of these fundamental cubic curves, a particular point; in fact the tangent planes of the cubic surface at all points of this curve meet the curve again in the same point, which is the point in question. For the curve given by the intersection of the cubic surface with the surface

$$ux^{-1} + vy^{-1} + wz^{-1} + (u + v + w)t^{-1} = 0,$$

this particular point is of coordinates $(x_0, y_0, z_0, 1)$, where

$$x_0 = (w - u)(u - v)/(w + u)(u + v),$$

etc.; and, with this, the statement may be verified directly, the equation of the tangent plane of the surface at (X, Y, Z, T) being $xX^{-2} + \ldots + tT^{-2} = 0$. But the fact will appear from the argument below. Assuming the truth of this, it follows that the quadric cone of vertex $(x_0, y_0, z_0, 1)$, which projects the cubic curve from this point, touches the cubic surface at every point of this curve. Moreover, as x_0, y_0, z_0, are clearly unchanged by the substitution of u^{-1}, v^{-1}, w^{-1}, respectively, for u, v, w, there is another quadric cone of vertex $(x_0, y_0, z_0, 1)$ which touches the cubic surface along a cubic curve, this being the curve lying on the surface

$$u^{-1}x^{-1} + \ldots + (u^{-1} + v^{-1} + w^{-1})\, t^{-1} = 0.$$

These two cubic curves, together, constitute the complete locus of points of the cubic surface whereat the tangent planes pass through $(x_0, y_0, z_0, 1)$, and may be spoken of as *conjugate* fundamental curves. The point $(x_0, y_0, z_0, 1)$ may be taken to be any point of the cubic surface, the ratios of u, v, w being then determinable from the equations

$$u(1 + x_0^{-1}) + v(1 + y_0^{-1}) + w(1 + z_0^{-1}) = 0,$$
$$u^{-1}(1 + x_0^{-1}) + v^{-1}(1 + y_0^{-1}) + w^{-1}(1 + z_0^{-1}) = 0.$$

The cubic surface is, therefore, such that the tangent cone, of order four, drawn to it from any general point of itself, breaks up into two quadric cones, each meeting the surface in a cubic curve.

The cubic surface contains also another important system of curves, those, namely, which correspond to the conics, in the plane (ξ, η, ζ), which touch the four lines $\xi \pm \eta \pm \zeta = 0$. The equation of such a conic is $a^{-1}\xi^2 + b^{-1}\eta^2 + c^{-1}\zeta^2 = 0$, where a, b, c are any constants for which $a + b + c = 0$. The corresponding curve on the cubic surface satisfies the equation

$$a^{-1}x^2(y + z)^2 + b^{-1}y^2(z + x)^2 + c^{-1}z^2(x + y)^2 = 0\,;$$

or, as $tx(y + z) = -yz(x + t)$, the curve satisfies the equation

$$a^{-1}(y + z)(x + t) + b^{-1}(z + x)(y + t) + c^{-1}(x + y)(z + t) = 0,$$

equivalent with $a^2(yz + xt) + b^2(zx + yt) + c^2(xy + zt) = 0$. This equation represents a quadric surface, passing through the four double points of the cubic surface, and touching the four lines in which the plane $x + y + z + t = 0$ meets the planes $x = 0, \ldots, t = 0$. The curves in which these quadric surfaces meet the cubic surface are then sextic curves, with four double points (at the fundamental points). Two of these curves pass through an arbitrary point of the cubic surface. These curves have the property, as will appear below, or may be verified directly, that the tangent line of the curve, at any point of the curve, is one of the inflexional lines of the surface at this point (see above, p. 197); it meets the surface at this point in three coincident points, and does not meet the surface at any other point. They are, therefore, called inflexional curves; and there are no other inflexional curves on the surface than these.

Consider, now, the relation of three points of the cubic surface which lie on a line. The third of these is such that all plane sections of the surface which contain the first two equally contain the third. A plane section of the cubic surface, however, corresponds, in the plane (ξ, η, ζ), to a cubic curve through the six points of intersection of the lines $\xi \pm \eta \pm \zeta = 0$. Thus, three points of the cubic surface which are in line correspond to three points of the plane, say P, Q, R, which have the property that all cubic curves through the six fundamental points, and through P, Q, necessarily pass through R; the existence of such a property for plane cubic curves has already been remarked (pp. 156, 178). Of such cubic curves, however, a degenerate one consists of the line $\xi + \eta + \zeta = 0$ taken with the conic defined by P, Q and the three intersections of the other three fundamental lines; and there are three other such degenerate cubic curves. The point R is thus the point through which pass the four conics each defined as passing through the three intersections of three of the fundamental lines, and also through P and Q. With a phraseology previously employed, in which P and Q are taken as Absolute points, and the conics are described as circles, R is the focus of the parabola touching the four fundamental lines (Vol. II, p. 82). That is, if the unique conic, of the form $a^{-1}\xi^2 + b^{-1}\eta^2 + c^{-1}\zeta^2 = 0$, touching the four fundamental lines, which touches PQ, be taken, R is the intersection of the two other tangents drawn to this conic, one from P and one from Q. A particular case, which we employ here, is that if P and Q coincide on the line PQ, the cubic curves considered touching this line at the point of coincidence, then R is the point of contact, with the conic touched by the line PQ, of the other tangent drawn to this conic from the point of coincidence. For distinctness, let the conic, of the form $a^{-1}\xi^2 + b^{-1}\eta^2 + c^{-1}\zeta^2 = 0$, (with $a + b + c = 0$), which touches a given line PQ, be called the conic *corresponding* to this line. There is then, upon the line, a particular point, the point of contact of the line with its corresponding conic; and, when P and Q coincide at this point, the point R also coincides with these. Thus we see that, upon any one of the fundamental cubic curves of the cubic surface, there is a particular point, at which the tangent line of the curve meets the surface in three coincident points; this point lies on a particular one of those sextic curves of the surface, previously spoken of, which correspond to the conics $a^{-1}\xi^2 + \text{etc.} = 0$ $(a + b + c = 0)$, and touches this curve at this point. Moreover, through any point of such a sextic curve, there passes a fundamental cubic curve of the cubic surface, touching the sextic curve at this point. That these sextics are inflexional curves, in the sense explained, is clear. We see, also, that, if we consider the third point of intersection, with the cubic surface, of the tangent line of a fundamental cubic curve, at any point of this, the locus of this third point is the inflexional curve of the cubic surface which touches this cubic curve. Denoting the point of a cubic curve of the surface where it touches an inflexional curve as the inflexional point of the curve, and recalling that, through any point of the plane, two conics of the set

$$a^{-1}\xi^2 + \text{etc.} = 0 \quad (a+b+c=0)$$

can be drawn, we see that two of the fundamental cubic curves of the cubic surface can be drawn through any point of the surface to have this point as their inflexional point. In the plane, the two tangents of the conics in question which pass through any point, are conjugate lines in regard to all these conics; and, if their equations be $u\xi + v\eta + w\zeta = 0$, and $u_1\xi + v_1\eta + w_1\zeta = 0$, we have $uu_1 = vv_1 = ww_1$. This relation then holds for the cubic curves in question through any point of the surface; and these curves may be spoken of as conjugate.

To any point, P, of the plane (ξ, η, ζ), there corresponds a cubic curve, passing through the six intersections of the four fundamental lines, which has a double point at P. For, if we determine the locus of the points of contact of the tangents from P to all the conics $a^{-1}\xi^2 + \text{etc.} = 0$ $(a+b+c=0)$, we find, by elimination of u, v, w from the equations

$$u\xi + v\eta + w\zeta = 0, \quad u\xi_0 + v\eta_0 + w\zeta_0 = 0, \quad u^{-1}\xi + v^{-1}\eta + w^{-1}\zeta = 0,$$

wherein (ξ_0, η_0, ζ_0) are the coordinates of P, the equation

$$\xi(\eta\zeta_0 - \eta_0\zeta)^{-1} + \eta(\zeta\xi_0 - \zeta_0\xi)^{-1} + \zeta(\xi\eta_0 - \xi_0\eta)^{-1} = 0,$$

which represents the curve in question, as is easily verified. If P moves on a line, the various cubic curves so found all pass then through the point where the line touches the conic which corresponds to it. Now, such a cubic curve corresponds to a plane section of the cubic surface having a double point, that is, to the section of the cubic surface by a tangent plane. We can thus infer that the tangent planes of the cubic surface, at all the points of one of its fundamental cubic curves, pass through the inflexional point of the curve, as was stated above.

Ex. Another result may be referred to. To any point, (ξ_0, η_0, ζ_0), of the plane, there is associated a conic which is the envelope of the polar lines of the point in regard to all the conics

$$a^{-1}\xi^2 + \text{etc.} = 0 \quad (a+b+c=0);$$

in particular, this conic touches the two lines through (ξ_0, η_0, ζ_0) which are conjugate to one another in regard to all these conics, these being the tangents at (ξ_0, η_0, ζ_0) of the two of these conics which pass through this point. The equation of this conic is

$$(\xi_0\xi)^{\frac{1}{2}} + (\eta_0\eta)^{\frac{1}{2}} + (\zeta_0\zeta)^{\frac{1}{2}} = 0;$$

it may be referred to, here, as *the envelope conic* for this point. Correspondingly, on the cubic surface, there is associated to any point, (x_0, y_0, z_0, t_0), a locus given by the equation

$$[\xi_0 x(y+z)]^{\frac{1}{2}} + \text{etc.} = 0.$$

The rational form of this, if we use the relations

$$tx^2(y+z) = -xyz(x+t) \text{ and } tyz(x+y)(x+z) = xyz(tx-yz),$$

which follow from the equation of the cubic surface, is at once found to be

$$yz(\eta_0 - \zeta_0)^2 + xt(\eta_0 + \zeta_0)^2 + \text{etc.} = 0.$$

This is the same as $PT = U^2$, where

$$P = x(\eta_0 + \zeta_0)^2 + y(\zeta_0 + \xi_0)^2 + z(\xi_0 + \eta_0)^2, \quad T = x + y + z + t,$$
$$U = x(\eta_0 + \zeta_0) + y(\zeta_0 + \xi_0) + z(\xi_0 + \eta_0);$$

the equation, therefore, represents a quadric cone, passing through the double points of the cubic surface, whose vertex is on the plane $T = 0$, which, moreover, touches this plane. The vertex of the cone is given by

$$xt^{-1} = \frac{(\zeta_0+\xi_0)(\xi_0+\eta_0)}{(\zeta_0-\xi_0)(\xi_0-\eta_0)}, \; yt^{-1} = \frac{(\xi_0+\eta_0)(\eta_0+\zeta_0)}{(\xi_0-\eta_0)(\eta_0-\zeta_0)}, \; zt^{-1} = \frac{(\eta_0+\zeta_0)(\zeta_0+\xi_0)}{(\eta_0-\zeta_0)(\zeta_0-\xi_0)}.$$

Two cones can be drawn to touch the plane $T = 0$, passing through the four double points, with vertex at this point; the other is evidently obtainable by writing ξ_0^{-1}, η_0^{-1}, ζ_0^{-1} for ξ_0, η_0, ζ_0, respectively. The cone meets the cubic surface in a sextic curve with a double point at each of the four double points of the surface; indeed, such a sextic can similarly be shewn to arise from any conic of the plane. The particular sextic in question touches the three lines of the cubic surface which lie in the plane $T = 0$; and, also, touches the two fundamental cubic curves of the cubic surface lying on the enveloping quadric cones drawn to the surface from (x_0, y_0, z_0, t_0). Let this sextic be described, here, as the envelope sextic associated with this point, and these two cubic curves as the associated cubic curves.

From any point, (x_1, y_1, z_1, t_1), of the cubic surface, there can be drawn four lines, each to meet both these cubic curves, other than the four lines to the double points of the surface, and the one line to the point (x_0, y_0, z_0, t_0); for, these two cubic curves are projected from (x_1, y_1, z_1, t_1) by cubic cones, having, in all, nine common generators. It can be shewn, now, that, when (x_1, y_1, z_1, t_1) is on the sextic associated with (x_0, y_0, z_0, t_0), these four lines become two pairs of coincident lines, each of which then meets these cubic curves, respectively, in two points with the property that the tangent lines of these curves at these points lie in a plane. This plane is then a plane of the developable formed by the common tangent planes of the two associated cubic curves; it meets the cubic surface in a plane cubic curve which touches both these cubic curves; this plane cubic section also passes through (x_1, y_1, z_1, t_1), and touches the associated sextic at (x_1, y_1, z_1, t_1). If the tangent line be drawn at any point, *H*, of one of the two associated cubic curves, there are, we know, four tangent lines of the other cubic curve which meet the former tangent line; the plane determined by the former tangent line and one of these four other tangent lines of the associated cubic curve, meets the cubic surface in a plane cubic curve which, as the point *H* of the first cubic varies, envelops the associated sextic.

To prove this result, consider, in the plane (ξ, η, ζ), the points, say (ξ_0, η_0, ζ_0) and (ξ_1, η_1, ζ_1), corresponding, respectively, to the points (x_0, y_0, z_0, t_0), (x_1, y_1, z_1, t_1) of the cubic surface, of which the latter is, at first, taken quite arbitrarily. There are two conics of the system $a^{-1}\xi^2 + \text{etc.} = 0\ (a+b+c=0)$ in regard to both of which the points (ξ_0, η_0, ζ_0), (ξ_1, η_1, ζ_1) are conjugate to one another. For, if we consider the conic associated with the point (ξ_1, η_1, ζ_1), whose equation is $(\xi_1\xi)^{\frac{1}{2}} + (\eta_1\eta)^{\frac{1}{2}} + (\zeta_1\zeta)^{\frac{1}{2}} = 0$, there are two tangents of this conic passing through the point (ξ_0, η_0, ζ_0); and, $u\xi + v\eta + w\zeta = 0$ being one of these, there is a particular conic of the system $a^{-1}\xi^2 + \text{etc.} = 0\ (a+b+c=0)$ for which this line is the polar of (ξ_1, η_1, ζ_1), namely $\xi_1^{-1}u\xi^2 + \eta_1^{-1}v\eta^2 + \zeta_1^{-1}w\zeta^2 = 0$. Consider one of these two conics, of the system in question, in regard to which (ξ_0, η_0, ζ_0) and (ξ_1, η_1, ζ_1) are conjugate: denote this by Σ_1. The polar of (ξ_0, η_0, ζ_0) in regard to Σ_1, taken with the two lines through (ξ_0, η_0, ζ_0), say l and l', which are conjugate to one another in regard to all the conics $a^{-1}\xi^2 + \text{etc.} = 0\ (a+b+c=0)$, determine a self-polar triad in regard to Σ_1; if then the two tangents be drawn to Σ_1 from (ξ_1, η_1, ζ_1), meeting the line l, say, in *P* and *M*, and meeting the line l' in *Q* and *N*, it is easily seen that the two lines *PQ* and *MN* are tangents of Σ_1, which, by the construction, touches *PN* and *QM* (see Vol. II, Ex. 31, p. 58, and Ex. 5, p. 28). The three points, (ξ_1, η_1, ζ_1), *P*, *Q*, thus correspond, on the cubic surface, to a line drawn from (x_1, y_1, z_1, t_1) to meet the two conjugate cubic curves of the cubic surface associated with (x_0, y_0, z_0, t_0), as, likewise, do the three points, (ξ_1, η_1, ζ_1), *M*, *N*. The two other such transversals drawn from (x_1, y_1, z_1, t_1) similarly arise from the other conic, Σ_2, in regard to which (ξ_0, η_0, ζ_0) and (ξ_1, η_1, ζ_1) are conjugate. Now, it may happen that the conics Σ_1, Σ_2 coincide; namely, when the envelope conic associated with (ξ_1, η_1, ζ_1) contains (ξ_0, η_0, ζ_0), that is, when $(\xi_0\xi_1)^{\frac{1}{2}} + (\eta_0\eta_1)^{\frac{1}{2}} + (\zeta_0\zeta_1)^{\frac{1}{2}} = 0$. Then the conics Σ_1, Σ_2 become the same conic, $\alpha^{-1}\xi^2 + \beta^{-1}\eta^2 + \gamma^{-1}\zeta^2 = 0$, with $\alpha = (\xi_0\xi_1)^{\frac{1}{2}}$, etc., as is easily seen. When this is so, the point (ξ_1, η_1, ζ_1)

is on the envelope conic arising from (ξ_0, η_0, ζ_0), and the point (x_1, y_1, z_1, t_1) is on the sextic curve associated with (x_0, y_0, z_0, t_0). This establishes the result which has been stated.

The theorem that the tangent line of a cubic curve on the cubic surface meets the surface again on the inflexional curve touching the cubic curve, proved above, was given by Laguerre, *Nouv. Ann.*, XI, 1872, p. 342 (*Oeuvres*, II, pp. 275, 281). The representation of the inflexional curves by conics touching four lines was remarked by Lie (see *Geom. d. Berührungstransf.*, I, 1896, p. 342). The theorem given in the example above is derived from F. Morley (*An extension of Feuerbach's theorem, Nat. Ac. of Sc., U.S.A.*, 1916 (2), p. 171). For further references in regard to the cubic surface with four double points, the reader may consult the author's *Multiply-periodic functions* (Cambridge, 1907), pp. 139—151, where the surface arises as touching the Kummer surface at every point of its intersection with it; also Neuberg, *Arch. d. Math. u. Phys.*, XI, 1907, p. 228, and Servais, *Bull. Sc. Acad. r. de Belgique*, VIII, 1922, pp. 50—66 and 103—123.

The Steiner quartic surface. We may consider the surface which is dual to the cubic surface just discussed. As the equation of the tangent plane of the cubic surface is $x_0^{-2}x + y_0^{-2}y + \ldots = 0$, the tangential equation of this cubic surface is $l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} + p^{\frac{1}{2}} = 0$, and the equation of the quartic surface, dual to the cubic surface, is $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} + t^{\frac{1}{2}} = 0$. This meets each of the planes $x = 0$, $y = 0$, etc., in a conic, at every point of which it touches the plane, any two of these four conics having a point of contact, on the line of intersection of their planes. The surface has also three double lines, such as that given by $y - z = 0$, $t - x = 0$; these meet in the point $(1, 1, 1, 1)$, which is a triple point of the surface. Every tangent plane of the surface meets it in a curve breaking up into two conics; the four intersections of these consist of the point of contact of the plane, and the three points where the plane meets the double lines of the surface. The surface can be represented, on a plane (ξ, η, ζ), by means of the formulae

$$x = (\xi - \eta - \zeta)^2, \quad y = (\eta - \zeta - \xi)^2, \quad z = (\zeta - \xi - \eta)^2, \quad t = (\xi + \eta + \zeta)^2,$$

which are equivalent to

$$\xi\,(t + x - y - z) = \eta\,(t + y - z - x) = \zeta\,(t + z - x - y).$$

Conversely, let $U = 0$, $V = 0$, $W = 0$, $P = 0$, be any four general conics in a plane, expressed in any variables. Then the surface expressed by $X : Y : Z : T = U : V : W : P$ is essentially the same as the above; namely, we can find four linearly independent linear functions of U, V, W, P which, in terms of suitable coordinates in the plane, have the forms $(\xi - \eta - \zeta)^2, \ldots, (\xi + \eta + \zeta)^2$. For, there are two tangential conics which are inpolar to all of $U = 0, \ldots, P = 0$; let coordinates be taken so that the four common tangents of these two tangential conics are $\xi - \eta - \zeta = 0, \ldots, \xi + \eta + \zeta = 0$; the four original conics are then outpolar to the three point-pairs expressed tangentially by $v^2 - w^2 = 0$, $w^2 - u^2 = 0$, $u^2 - v^2 = 0$, these being the opposite pairs of intersections of the four lines. The general conic, $a\xi^2 + 2f\eta\zeta + \ldots = 0$, outpolar to these point-pairs, is $\xi^2 + \eta^2 + \zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0$; as each of $\xi^2 + \eta^2 + \zeta^2$, $\eta\zeta$, $\zeta\xi$, $\xi\eta$ is expressible as a linear function of the four squares in question, the result stated is proved. Incidentally, we remark, that, taking X, Y, Z, T proportional to $\eta\zeta$, $\zeta\xi$, $\xi\eta$, $\xi^2 + \eta^2 + \zeta^2$, the equation of the surface is capable of the form $Y^2Z^2 + Z^2X^2 + X^2Y^2 - XYZT = 0$, which puts in evidence the double lines of the surface.

Thus, if U, V, W, P be any four linearly independent homogeneous quadratic functions of three variables, ξ, η, ζ, the envelope of the plane $Ux + Vy + Wz + Pt = 0$ is a cubic surface with four double points. If U_1, U_2, U_3, respectively, denote $\partial U/\partial\xi$, $\partial U/\partial\eta$, $\partial U/\partial\zeta$, and U_{11}, U_{12}, ... denote $\partial^2 U/\partial\xi^2$, $\partial^2 U/\partial\xi\partial\eta$, ..., and so on, and u_{rs} denote $U_{rs}x + V_{rs}y + W_{rs}z + P_{rs}t$, the point of contact of the plane corresponding to a particular set of values of ξ, η, ζ is given by the three equations $U_r x + V_r y + W_r z + P_r t = 0$, and the locus of this point of contact, that is, the surface enveloped by the plane, is given by the equation, obtained by eliminating ξ, η, ζ from these three equations,

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix} = 0,$$

where $u_{rs} = u_{sr}$. The quadrics obtained by equating to zero the first minors of this determinant have four points in common; these are the double points of the cubic surface.

Ex. 1. For the general cubic surface, expressed, in Sylvester's way, by $\Sigma ax^3 = 0$, with $\Sigma ax = 0$, the envelope of the polar planes of the points of the plane which passes through the three points $(x_1, y_1, \ldots)$, $(x_2, y_2, \ldots)$, $(x_3, y_3, \ldots)$, (cf. p. 209), is expressible in the form above, with

$$u_{11} = axx_1^2 + \ldots + euu_1^2, \text{ and } u_{12} = axx_1x_2 + \ldots + euu_1u_2, \text{ etc.}$$

Ex. 2. It follows from the dual relation of the cubic surface with four double points, and the Steiner quartic surface, or may be proved directly, that, by the representation upon the plane, the inflexional curves of the Steiner surface are made to correspond to the conics, expressed by $a^{-1}\xi^2 + \text{etc.} = 0$ $(a + b + c = 0)$, which are inpolar to the conics to which the coordinates on the surface are proportional.

Ex. 3. The inflexional curves of the Steiner surface are rational quartic curves, having the three double lines of the surface as chords. Of the four points in which any such curve is met by a tangent plane of the surface, regarded as represented by the values of the parameter by which any point of the curve may be rationally expressed, two are harmonic conjugates in regard to the other two.

Ex. 4. The inflexional curves of the surface expressed by the equation $x^m + y^m + z^m + t^m = 0$ satisfy the equation

$$(ax^m)^{\frac{1}{2}} + (by^m)^{\frac{1}{2}} + (cz^m)^{\frac{1}{2}} = 0,$$

where a, b, c are any constants whose sum is zero. Verify this in the case of the cubic surface with four nodes and in the case of the Steiner surface.

Ex. 5. If $\phi_1 = 0, \ldots, \phi_6 = 0$ be any six conics in a plane, the equations $x_1/\phi_1 = x_2/\phi_2 = \ldots = x_6/\phi_6$ represent, in space of five dimensions, a surface (containing ∞^2 points), known as Veronese's surface. Evidently the Steiner surface may be regarded as arising by projection of this surface, *from a line.*

Ex. 6. The lines joining a point, O, of coordinates x, y, z, t, to the points $(a, 1, 1, 1)$, $(1, b, 1, 1)$, etc., meet the planes $x=0$, $y=0$, etc., respectively, in coplanar points. Prove that the locus of O is the Hessian of the cubic surface

$$\Sigma \frac{x^3}{a(a-1)} + \frac{u^3}{(e-1)(3-2e)} = 0,$$

where e and u are defined by

$$1 + \Sigma (a-1)^{-1} + (e-1)^{-1} = 0, \quad \Sigma x (a-1)^{-1} + u (e-1)^{-1} = 0.$$

When $a=b=c=d=-3$, taking for Absolute Conic the intersection of the plane $x+y+z+t=0$ with the quadric $x^2+y^2+z^2+t^2=0$, shew (after Steiner, 1845) that the cubic surface with four double points is the locus of a point such that the feet of the four perpendiculars drawn from it to the four planes $x=0$, $y=0$, etc. are coplanar. See Jessop, *Quartic Surfaces*, p. 190; also p. 36 above. And *Archiv d. Math. u. Phys.* IV, 1903, p. 275 (a reference for which the writer is indebted to Mr J. P. Gabbatt).

For the Steiner surface, the reader may consult Jessop, *Quartic Surfaces* (Cambridge, 1916), Chapter VII. For Veronese's surface, see Cayley, *Papers*, VI, p. 198 (1868), and Bertini, *Geometria projettiva d. iperspazi*, 1907, Chapters XIV, XV. See also Castelnuovo, *Atti d. r. Acc. d. Lincei*, III, 1894, p. 22, who proves, after Kronecker, that an irreducible algebraic surface which is met by ∞^2 planes in reducible curves, if not a ruled surface, is a Steiner quartic surface.

CORRECTIONS FOR VOLUMES I AND II

VOLUME I.

Page 28, line 7. *For* FCF′ *read* FRF′.

p. 39, l. 22. *For* S_{2n-1} *read* S_{2n-2}.

p. 47, l. 12 f. b. *For* insection *read* intersection.

p. 48, l. 8 f. b. *For* (2) *read* (3).

p. 65, l. 13. *For* $b+a$ *read* $0+a$.

p. 68, ll. 2, 3. Both diagonal elements should be $\alpha\beta-\alpha'\beta'$.

p. 74, l. 18. *For* meet *read* meets.

p. 75, l. 20. *For* $P=0$ *read* $P=O$.

p. 89, l. 16. *For* $C-mC'$ *read* $C-C'$.

p. 92, l. 10 f. b. *For* $\sigma\rho\sigma^{-1}B'$ *read* $\sigma\rho^{-1}\sigma^{-1}B'$.

p. 99, l. 11. *For* A *read* A′.

p. 102, l. 11 f. b. *For* any point *read* any point except B or C.

p. 108, l. 2 f. b. *For* B″ *read* B′.

p. 127, l. 14. *For* P and V *read* P_0 and V.

p. 155, l. 5 f. b. After *a*, *b*, *c*, *add* and A, B, C are not in line,

p. 156, l. 3 f. b. When *a*, *b*, *c* meet in a point, and A, B, C are in line, there *may* be an infinite number of solutions.

p. 174, ll. 16, 17 f. b. Interchange O′, U′, W′ and O_1', U_1', W_1'.

p. 174, last line. *For* $m_1'+n_2'-n_2'$ *read* $m_1'+m_2'-n_2'$.

VOLUME II.

p. 13, l. 8. *For* MacClaurin *read* Maclaurin.

p. 28, l. 16. *For* PH, PK *read* PQ, PQ′.

p. 28, l. 18. *For* pole *read* polar.

p. 49, l. 11 f. b. *For* C *read* O.

p. 58, l. 1. *For* intersecting *read* interesting.

p. 65, l. 13 f. b. *For* pole *read* polar.

p. 83, last l. *For* hyperbolas *read* hyperbolas.

p. 86, l. 2 f. b. *For* from (S or *read* from S (or.

p. 92, l. 17 f. b. *For* Σ, in A, B, C, D′ *read* Σ, and meets Σ in A, B, C, D′.

p. 96, l. 10 f. b. *For* any algebraic symbol *read* any algebraic symbol other than zero.

p. 102, l. 19. *For* we can *read* we can if L contain x.

p. 114, l. 18. See Morley, *Bull. Amer. Math. Soc.* I, 1895, p. 116, and *Proc. Camb. Phil. Soc.* XX, 1920, p. 119.

p. 119, last l. *Read*
$a^2l^2+b^2m^2-n^2+\lambda(l^2+m^2)$.

p. 120, l. 1. *Read* a^2l^2 *for* a^2b^2.

p. 125, last l. *Read* $bc'+b'c-2ff'$.

p. 130, l. 15. *For* S *read* S_1.

p. 140, l. 7. *For* conic *read* conics.

p. 167, l. 3 f. b. *For* his *read* this.

p. 169, l. 24. *For* $B^2+C^2-A^2-D^2$ *read* $A^2+D^2-B^2-C^2$, and so in ll. 25, 26. In the first terms of y_1 and z_1 *read* x *for* y and z.

p. 197, l. 23. *For* lines *read* segments.

p. 203, l. 13 f. b. *For* $-(A+\phi)$ *read* $-(A+\psi)$.

p. 205, l. 7 f. b. *Read* $\left(1-\frac{G^2}{CA}\right)^{\frac{1}{2}}$.

p. 212. In the left-hand diagram *join* the points A, E.

p. 231, l. 16. *For* three *read* the.

p. 234, l. 4. *For* (QR, P′Q) and (RP, Q′R) *read* (QR, P′Q′) and (RP, Q′R′), respectively. And so in ll. 5, 6.

p. 234, l. 5 f. b. *For* X′U *read* X′U′.

p. 237. Add reference to Cayley, ‘Sixth Memoir on Quantics,’ *Papers*, II, p. 583. Also to Beltrami, ‘Saggio di interpret. d. Geom. non-eucl.,’ *Giorn. d. Mat.* VI, p. 284 (*Opere*, I, p. 374), 1868; and ‘Theoria fond. d. spazii di curv. cost.,’ *Ann. d. Mat.* II, p. 232 (*Opere*, I, p. 406).

p. 238, l. 14 of ref. to p. 41, in Y (−D+D′+A′, B′, C′, D′, 0), *for* second D′ *read* D.

The author desires to acknowledge the kindness of Miss H. P. Hudson, London, of Professor Carslaw, Sydney, and especially of Professor Wilton, Adelaide, for the majority of these corrections.

INDEX

Absolute conic, and plane, 70
Apolar form, apolar to a system of binary forms of the same order, 136
Apolar linear complexes, 66
Associated points, eight, 148, 154, 157
Axes of confocal system of quadrics, 98–100
Axis of a plane in regard to a system of confocals, 93, 98; of a developable, 133

Beltrami, 208, 224
Bertini, 223
Bitangents of a plane quartic curve, 199
Burnside, 168

Carslaw, 224
Castelnuovo, 223
Cayley, 212, 214, 223, 224
Central systems, 11, 184
Centre of a quadric, 79
Centres of similitude of two spheres, 78
Chasles, 54, 67, 69, 111
Chords, ten, common to two cubic curves in space, 141
Circles, 72
Circular sections of a quadric, 79
Clebsch, 170, 194, 203, 208
Coaxial system of spheres, 74
Complex, linear, of lines, 56, 61; quadratic complex, tetrahedral complex, 99
Cone, quadric, defined, 4, 17; cone of revolution, 81
Confocal paraboloids, 95
Confocal quadric surfaces, 91 ff.; the three through a point not always distinct, 92; touching a line, 109; as dual of quadrics with a common curve, 119; leading to Poncelet's porism, 116
Congruence, linear, of lines, 65
Conics, meeting eight lines, 150
Conjugate lines, in regard to a quadric, 34, 59
Conjugate points and planes, in regard to a quadric, 32
Contact cones, for a cubic surface, leading to contact conics for a plane quartic curve, 198
Coordinates of a line, 56
Coresidual sets of points on a plane curve, 179
Coresiduation, theorem of, for plane curves, 179; for surfaces, 181
Correspondence between generators of a quadric determining cubic curve, 135
Corresponding points of the Hessian surface, 209
Cremona, 206, 208
Cubic curve in a plane, determination of the ninth intersection of two when eight are given, 156, 178, 217
Cubic curve, tangents belong to a linear complex, 68; generated by points of contact of tangent planes of confocals through a line, 107. General consideration of cubic curve in space, 125 ff.; drawn through five points of a quadric, 129; four tangents meet a line, 132; outpolar to a quadric, 146; constructed to satisfy six conditions, 129, 139–142; constructed to meet each of four lines in a pair of an involution given thereon, 145
Cubic developable, dual to a cubic curve, 130, 135
Cubic surface, general account, 159 ff.; theorem of coresiduation for curves upon, 181; curves upon, 191; equations of, 203–208; determined by six arbitrary points in a plane, 194; leading to a plane quartic curve, 196; Hessian of, 208; cubic surface with four double points, 214
Cubo-cubic transformation in space, 143
Curves of intersection of a quadric with confocals projected into confocal conics, 113
Cuspidal edge of a developable, 131
Cylinder, 81

Dandelin, 44
Deformation of quadric with rigid generators, 114

Developable, 103; of common tangent planes of confocals, 118; cubic, 130, 133, 135
Diagonal cubic surface, 170
Director sphere of quadric, 86
Dixon, A. C., 202; A. L., 118
Dobriner, 158
Double-six of lines, 159 ff.; determined by five arbitrary points, 168
Double tangents of plane quartic curve, 199
Dual character of a quadric, 4, 5, 30, 32

Elliptic quartic curve, eight tangents meet any line, 69, 151
Enveloping cones drawn from a point to confocals, 110
Equation of a quadric, 7

Feuerbach, 221
Focal conics, 92, 95
Focal distance property of conics, generalised, 96
Focal lines of contact cones drawn to confocal quadrics, 109
Focal system, 61 ff., 135
Four curves in space, number of lines meeting, 184
Four lines, condition they are tangent lines of a cubic curve, 145
Freedom of a system of curves on a surface, 194

Gabbatt, 223
Gaskin's theorem in three dimensions, 89
Generator of a developable, 131
Generators of a quadric, 5, 7; through the umbilici, 80; at right angles, 90
Genus of a curve in space, 194
Geodesics, 112
Gordan, 208
Grassmann, 208

Hesse, 44, 154, 158
Hessian pair of three points of a cubic curve in space, 137; Hessian of a cubic surface, 181, 187, 188, 208
Hexad, self-conjugate, 48, 50, 146
Hierholzer, 151
Hudson, 224

Inflexional curves of the cubic surface with four double points, 216; of Steiner's quartic surface, 222
Inflexional lines at any point of a cubic surface, 197
Inpolar, or outpolar, quadric, 52, 88
Intersection of two quadrics, 123
Invariant of two lines, 58; of two linear complexes, 66
Involution, general, of sets of four points of a cubic curve in space, 139
Ivory's theorem, 118

Jessop, 223

Kronecker, 223
Kummer's surface, 221

Laguerre, 221
Lie-Scheffers, 149; Lie, 221
Limiting points of two spheres, 74
Linear complex, 56, 61, 62, 66; linear congruence, 65
Lines of a quadric, 1; coordinates of a line, 56; invariant of two, 58; lines and planes at right angles, 70
Lüroth, 151

Middle point, 70
Milne, 203
Moebius and focal systems, 69
Moebius tetrads, 67, 137, 138
Morley, 221, 224

Neuberg, 221
Normals of a quadric, 86; of confocals, through a point, 104; at right angles to a given line, locus of, 106; of three confocals through a point are focal lines of enveloping cone, 109
Null system, 69

Osculating plane of cubic curve, 127
Outpolar, cubic curve, to a quadric, 146
Outpolar, or inpolar, quadric, 52, 88

Parabola in a plane touched by axes of confocals, 98
Paraboloid, 83
Parallel lines and planes, 70
Parallel tangent planes of confocals have their points of contact on a conic, 108

Parametric expression of a quadric, 7
Pascal, 44
Pentad, self-conjugate, 37, 50, 147, 177
Pentagon, skew, determines a focal system, 68
Perpendicular lines and planes, 70
Plane, equation of, 1
Plane sections of a sphere, 73
Plücker, 102
Point-pair of Hessian surface expressed by six point-pairs, 188
Polar line of a line in regard to a quadric, 28, 58, 69
Polar lines of a line in regard to confocals, locus of, 106
Polar plane of a quadric, from the equation, 22, 23; defined descriptively, 27
Polar point of a plane in regard to a quadric, 29, 30, 31
Polar quadric, in regard to a cubic surface, 182
Polar reciprocal in regard to a quadric, 93
Pole and polar, in a focal system, 61, 64
Poncelet, 10; porism for conics deduced from theory of confocal surfaces, 116
Principal planes of a quadric, 79; of enveloping cones drawn to confocals, 109

Quadric polynomial reduced to sum of squares, 15; representing a cone, 18; representing two planes, 19; representing a repeated plane, 20
Quadric surface defined by its lines, 2; by two conics and a line, 7; by two related central systems, 11; as representing a plane with two fundamental points, 13. Possible intersection of two quadric surfaces, 123; self-polar tetrad of two, 152; three intersect in eight points, 153, 154
Quartic curve, elliptic, intersection of two quadrics, 69, 151; rational, 85, 150, 222
Quartic curve, plane, derived from cubic surface, 196
Quartic surface determined by normals of quadric at points of plane section, 105; determined by tangents of a cubic curve, 132; containing umbilicar generators of a quadric, 93; Steiner's, 221

Radical plane of two spheres, 74
Rational curve in general, 149
Rational quartic curve, 85, 150, 222
Residual set of points on a plane curve, 179
Revolution, cone of, 81, 82
Reye, 49, 206, 211
Right angles, lines and planes at, 70
Right circular cone, 81, 82; circumscribing a quadric, locus of vertex, 95
Roth, 203

Schroeter, 158
Schur quadrics, 162, 168, 177, 187, 200, 211
Screw, 69
Section of a quadric by plane at right angles to a focal conic, 95
Segre, 158, 208
Self-conjugate pentad, 37, 50, 147, 177; hexad, 48, 50, 146
Self-polar figure, a general, 40
Self-polar tetrad in regard to a quadric, 33, 50, 52
Serret, P., 49
Servais, 221
Sextic curve in space, 154, 198
Similitude, centres of, for two spheres, 78
Spheres, 72; cutting at right angles, 76
Squares, quadric polynomial reduced to, 16; expression of a quadric by five, 38, 39; by six, 49
Star-systems, related, 11, 184
Staudt, von, 158, 168
Steiner points, for Pascal figure, 44
Steiner, trihedral pairs, 164, 174, 203; also 208; Steiner system of bitangents of a plane quartic curve, 200; Steiner's quartic surface, 221
Stenfors, 168
Striction, line of, on a quadric, 84
Sturm, 208
Sylvester, 64, 66, 206, 208, 210

Tangent cone, or enveloping cone, of a quadric, 26
Tangent of a cubic curve, 127, 132; tangent plane of a cubic surface, 197
Tangent plane of a quadric, 23, 27
Tangential equation of a quadric, 25
Telling, 148

Tetrad, self-polar, 33, 50, 52, 145; of points whose opposite joins are conjugate points in regard to a quadric, 35; of which opposite joins are at right angles, 78; two tetrads in perspective, 37; two related by reciprocation, 41, 53
Tetrahedral complex, 99, 134
Thomson and Tait, 132
Transformation, cubo-cubic, in space, 141
Trihedral, 164, 174, 203

Umbilicus of a quadric, 80; of a paraboloid, 83; umbilicar generators of a quadric lie on a quartic surface, 93

Veronese, 223

Wakeford, 148
Weddle's surface, 154
Wilton, 224

Zeuthen, 158

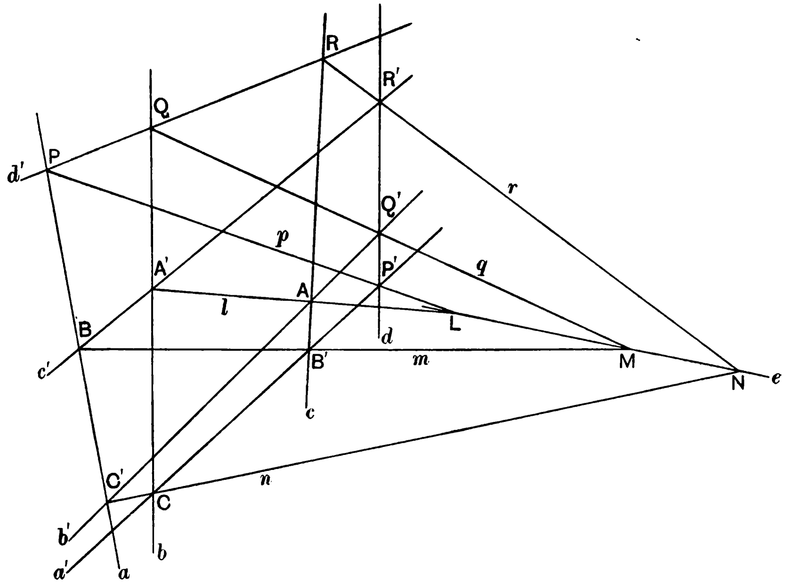
CAMBRIDGE : PRINTED BY W. LEWIS AT THE UNIVERSITY PRESS

PRINCIPLES OF GEOMETRY

CAMBRIDGE UNIVERSITY PRESS
LONDON : FETTER LANE, E.C. 4

LONDON : H. K. LEWIS AND CO., LTD.,
136, Gower Street, W.C. 1
NEW YORK : THE MACMILLAN CO.
BOMBAY }
CALCUTTA } MACMILLAN AND CO., LTD.
MADRAS }
TORONTO : THE MACMILLAN CO. OF CANADA, LTD.
TOKYO : MARUZEN-KABUSHIKI-KAISHA

ALL RIGHTS RESERVED


R
R′
Q
P
d′
r
Q′
p
q
A′
P′
A
l
L
B
d
c′
B′
m
M
N
e
c
C′
C
n
b′
b
a′
a

The Figure of fifteen lines and fifteen points, in space of four dimensions.
(See Ch. V.)

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., LL.D., F.R.S.,
LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY, AND FELLOW OF
ST JOHN'S COLLEGE, IN THE UNIVERSITY OF CAMBRIDGE

VOLUME IV

HIGHER GEOMETRY

BEING ILLUSTRATIONS OF THE UTILITY OF THE
CONSIDERATION OF HIGHER SPACE, ESPECIALLY
OF FOUR AND FIVE DIMENSIONS

CAMBRIDGE
AT THE UNIVERSITY PRESS
1925

PRINTED IN GREAT BRITAIN

PREFACE

THE present volume, the first written and the most revised, of the book, for which indeed, mostly, the earlier volumes were undertaken, still bears many marks of the difficulty of compressing the matter into brief compass. But the writer hopes that it may seem to the reader as remarkable as it does to him, that it should be possible to comprehend under one point of view, and that so simple, the introduction to nearly all the surfaces ordinarily studied in the geometry of three dimensions, as well as the usual line geometry. Chapters v, vi, vii seek to make clear that this is so. To these the earlier chapters are auxiliary. But Chapters ii and iv have been introduced as much for their own interest as for their illustrative value; the results obtained in these two chapters are not required in the subsequent pages. It is hoped that the Table of Contents, and the Index, may make it easy to use the volume. It will of course be understood that the volume is throughout intended to be introductory and illustrative; hardly anywhere is it complete.

To the Staff of the University Press grateful acknowledgments are due for their continued courtesy and care.

H. F. B.

1 *June* 1925

TABLE OF CONTENTS

CHAPTER I. INTRODUCTORY. RELATIONS OF THE GEOMETRY OF TWO, THREE, FOUR AND FIVE DIMENSIONS

PAGES

SECTION I. THEOREMS OF TWO AND THREE DIMENSIONS . . . 1—32

The conics touching the fives from six arbitrary lines of a plane . 1—4

Exx. 1—4. *Particular algebraic results; equation of triangularly circumscribed conic.* . . . . 4—6

Representation of a plane upon a quadric . . . . . 6—8

The generalised Miquel theorem . . . . 8—10

Ex. 1. *The conic of an arbitrary plane determined by joins of five points* 10, 11

Ex. 2. *Cubic curve through the five points gives a self-polar triad* . 11

Ex. 3. *Dual of theorem of Ex.* 1; *consequence* 11

Ex. 4. *Aggregate of conics thus arising for sets of five points of a cubic curve* 11, 12

Ex. 5. *Alternative proof of generalised Miquel theorem* . . 12

Ex. 6. *A plane theorem arising from Ex.* 5 12

The theory of inversion in a plane . . . . 12—14

Ex. 1. *Equations for inversion in a plane* 14

Ex. 2. *Relations between centres of inverse circles* . . . 14, 15

Ex. 3. *Inversion of one given circle into another* 15

Ex. 4. *Centres of similitude and circle of similitude of two circles* . 15

Ex. 5. *Circles through threes of four points; property of circles of similitude* 15, 16

Ex. 6. *Quadric touching four planes having double contact with another* 16

Ex. 7. *Relation of two points of contact in Ex.* 6 . . . 16

Ex. 8. *Relation expressed by means of a cubic curve* . . . 16, 17

Ex. 9. *Employment of cross-ratios* 17

Ex. 10. *A line, and its polar line, intersecting a quadric* . . 17

Ex. 11. *Five circles constructed from five given circles* . . 17, 18

Wallace's theorem for four circles and Moebius's figure of two tetrads . . . . 18—20

Ex. 1. *Dual of Moebius's figure* 20

Ex. 2. *A quartic surface considered by Bauer* 20, 21

Ex. 3. *A theorem of Steiner's for two triads of points* . . . 21

Additions to Wallace's theorem in a plane 21, 22

Ex. 1. *Pedal circles orthogonal to one circle* . . . . 22—24

Ex. 2. *Alternative to use of associated points* 24

Ex. 3. *Pedal properties in threefold space* 24, 25

Ex. 4. *Theorem of symmetry of five points, for pedal property* . 25

PAGES

Ex. 5. *Pedal property of circle generalised to rational curve of higher space* 25, 26
Ex. 6. *Generalised form of pedal property* 26
Ex. 7. *Algebraic proof of preceding results* 26, 27
Ex. 8. *A generalised orthocentre of four arbitrary points* . . 28
Ex. 9. *Theorem for two systems of conics* 28
Ex. 10. *A property of the cubic surface with four double points* . 28
Ex. 11. *Cubic curve in threefold space, three tetrads upon* . . 28, 29
Ex. 12. *Algebraic form of result of Ex.* 11 29
Theorems for circumscribed circles for any number of lines in a plane . . . . 29—31
Ex. 1. *Representation of points of quadric by transversals of two lines* 31, 32
Ex. 2. *Six conics triangularly inscribed to another* . . . 32
SECTION II. THEOREMS OF THREE AND FOUR DIMENSIONS . . . 32—40
Tetrahedral complex, determined by planes meeting three lines in fourfold space . . . . 32—34
Ex. *Algebraic formulation* 34
Analogous generation of a linear complex 34, 35
Ex. *Algebraic formulation*. 35, 36
Spheres as determined from sections of a quadric in fourfold space 36—38
Exx. 1, 2. *Two conjugate points of quadric in fourfold space* . 38
Ex. 3. *Representation of a circle by a line in fourfold space* . 38, 39
Ex. 4. *Two spheres cut everywhere at the same angle* . . . 39, 40
SECTION III. USE OF SPACE OF FIVE DIMENSIONS . . . . 40—64
The representation of lines of threefold space by points of fivefold space . . . . 40—42
Ex. 1. *Representation of two conjugate or apolar linear complexes* 42
Ex. 2. *Six linear complexes mutually conjugate* 42
Ex. 3. *General form of theorem of Ex.* 2 . . . . . 42, 43
The lines common to three, and to four, linear complexes . . 43, 44
Planes lying entirely on the quadric fourfold in the space of five dimensions 44, 45
The planes of the quadric fourfold treated with the symbols . . 45—47
The aggregate of the planes lying on a quadric in fivefold space . 47, 48
Ex. 1. *Representation of point and polar plane in regard to a focal system* 48
Ex. 2. *Representation of a pair of lines, polars in regard to a quadric* 48, 49
The representation of congruences of lines of the original space . 49, 50
Number of lines common to two congruences in threefold space . 50, 51
Ex. *Number of chords common to two curves* 51
The chords of a cubic curve in the threefold space. Veronese's surface . . . . 52—54

PAGES
Ex. 1. *The section of Veronese's surface by a fourfold space.* . 54
Ex. 2. *The representation in fivefold space of the conics of a plane* 54
Ex. 3. *Statement of some properties of Veronese's surface* . . 54, 55
Lie's correspondence between lines and spheres in space of three dimensions 55, 56
Ex. 1. *Algebraic formulation of the correspondence* . . . 56, 57
Ex. 2. *Comparison with deduction of sphere from fourfold space* . 57
Ex. 3. *The eight pairs of spheres which touch four spheres* . . 57, 58
Ex. 4. *The eight spheres touching four planes* 58
Ex. 5. *The double-six theorem expressed in terms of spheres* . 58
Generalisation of Wallace's theorem. The theorem of the double-six. A theorem for six lines with a common transversal . 58—60
Exx. 1—5. *Proofs of the theorems* . . . . 60—64
Note. Another proof of Wallace's theorem, and extensions . 64

CHAPTER II. HART'S THEOREM, FOR CIRCLES IN A PLANE, OR FOR SECTIONS OF A QUADRIC

The sections of a quadric which touch three given sections . . 65, 66
Ex. 1. *A pair of variable sections of a quadric touching one another and two fixed sections* 66
Ex. 2. *Construction of the circles touching three given circles* . 66
Ex. 3. *A theorem for circles through threes of four special points*. 66, 67
Ex. 4. *Circles cutting three given circles at equal angles* . . 67
Ex. 5. *Circles cutting four given circles at equal angles* . . 67
Ex. 6. *Circles cutting three given circles at given angles* . . 67
Ex. 7. *Common tangents, of three circles, which meet in a point* . 67, 68
Ex. 8. *Solution of generalised Malfatti's problem*. . . . 68, 69
Ex. 9. *Equations of sections touching three sections of a quadric* . 69, 70
Exx. 10—15. *Various algebraic results* 70, 71
The Hart circles of three circles when the four have a common orthogonal circle 71, 72
The Hart circles of three circles in general . . . . . 72—75
Ex. 1. *The various cases* 75
Ex. 2. *The point of contact of the Hart section* 75, 76
Ex. 3. *The bipolar character of the figure obtained* . . . 76, 77
Ex. 4. *Eight sections divided into four pairs* 77
Ex. 5. *Relations with metrical geometry* 78
Ex. 6. *The particular case of the Feuerbach circle* . . . 78—80
Ex. 7. *The same Hart section arises in different ways* . . . 80
Ex. 8. *The radius of the Hart section*. 80, 81
Ex. 9. *Four circles arising from the intersections of three circles* . 82
The bifocal property is necessary as well as sufficient . . . 82—85
Ex. 1. *A particular form of the equation of the quadric* . . 85, 86

PAGES
Exx. 2, 3. *Two algebraic results* 86
Ex. 4. *Relations between four sections and four tangent sections* . 86—88
Ex. 5. *The harmonically conjugate quadric in regard to tangent sections* 88
Ex. 6. *The Hart circle defined by angles of intersection* . . 88

CHAPTER III. THE PLANE QUARTIC CURVE WITH TWO DOUBLE POINTS

The generation of the curve. The four principal circles . . 89, 90
The curve as an envelope of circles. The director conic . . 90
Ex. 1. *Algebraic formulation* 90, 91
Ex. 2. *Two tangent circles have their points of contact on a circle*. 91
The four director conics are confocal 91, 92
The sixteen foci of the quartic curve. Are intersections of a principal circle and a director conic . . . . . . 92—94
Relation of distances of point of curve from three foci . . . 94
Construction of the plane quartic curve given four foci lying on a circle . . . . 94—96
Inversion of the curve into itself 96, 97
Ex. 1. *Cubic curves with two fixed points, arising by inversion* . 97
Ex. 2. *Equations for the case of the cubic curve* 97, 98
Ex. 3. *Case of quartic with four foci on a line* 98
Ex. 4. *The Cartesian, Limaçon and Cardioid* 98, 99
Angles of intersection of enveloping circles of the plane quartic which belong to different families . . . . 99—101
Exx. 1, 2. *Theorem for eight intersections of four circles taken in order* 101
Ex. 3. *Condition by angles of intersection for three circles to meet in a point* 101
Exx. 4, 5. *Angles between six planes, in regard to a quadric* . 102, 103
Ex. 6. *Theorem of intervals, for quadrics with ring contact* . . 103
Ex. 7. *Descriptive form of distance theorem for focal conics*. . 103
Ex. 8. *A theorem for the foci of an anallagmatic curve* . . 103

CHAPTER IV. A PARTICULAR FIGURE IN SPACE OF FOUR DIMENSIONS

The structure of the figure. Identity of numbers with those for lines of cubic surface 104, 105
Geometrical bearings of the figure. The group 105
The establishment of the figure . . . . 105—111
Exx. 1—4. *Some particular results. Connexion with theory of spheres* 111, 112

CHAPTER V. A FIGURE OF FIFTEEN LINES AND POINTS, IN SPACE OF FOUR DIMENSIONS; AND ASSOCIATED LOCI

PAGES

The figure of fifteen lines and points . . . . 113—115

Proof of the incidences with the help of the symbols . . . 115, 116

Ex. 1. *A property of five associated lines* 117

Ex. 2. *A generation of five associated lines* 117

Ex. 3. *A quadric surface obtained from six points in fourfold space* 117

Ex. 4. *Harmonic inversion of figure, with point and solid fundamental* 117

Ex. 5. *Dual of the figure. Harmonic inversion with two fundamental planes* . . . . 117—119

Ex. 6. *Harmonic inversion with line and plane fundamental* . 119

Ex. 7. *Separation of the figure into two parts, in ten ways* . . 119, 120

Ex. 8. *Application of inversion of Ex. 6 to quartic curve in fourfold space* 120

The associated line of a set of five is met by all planes meeting the first four lines . . . . 120—123

Interpolation in regard to quadric point-cone and line-cone . . 120—122

The six systems of planes 123, 124

The planes of the figure deduced with the help of the symbols . 124, 125

The equation of an associated quartic locus 125, 126

Ex. *The six sets of associated lines give related ranges on certain conics* 127, 128

The tangent solid of the quartic locus Σ 128, 129

The equations of the singular solids 129, 130

The intersections of a tangent solid of Σ with the fifteen fundamental lines . . . . 130—133

The intersections of a tangent solid of Σ with the planes of the six systems . . . . 134—136

Ex. 1. *Planes of a system meeting a line. Algebraic expression* . 136, 137

Ex. 2. *Equations for congruence of lines on tangent solid, by planes of a system* 137, 138

Ex. 3. *Particular algebraic results* 138

Ex. 4. *Intersection of locus Σ with tangent solid is a Kummer surface* 138, 139

Ex. 5. *Six conjugate linear complexes determined by six points* . 139, 140

Ex. 6. *Algebraic formulation of theorem of Ex. 5* 140

Ex. 7. *Number of non-intersecting pairs and threes of lines in the figure* 140

Ex. 8. *Other forms of congruence of lines in tangent solid, by planes of a system* 140, 141

Ex. 9. *Theorem for relation of a line to four mutually inscribed tetrads* 141, 142

PAGES

Ex. 10. *The locus Σ giving rise to Kummer surface, or to Plücker's Meridian Surface* 142, 143
Ex. 11. *Six arbitrary lines in fourfold space are met by five associated planes* 143
Ex. 12. *Solid through associated plane meets four planes in lines of a quadric* 143, 144
Ex. 13. *Joins of two triads met by solid in six planes touching a quadric cone* 144, 145
Ex. 14. *A plane in a singular solid determines a quadric cone* . 145
Ex. 15. *Aggregate of planes each joining corresponding points in three related planes* 145, 146
Exx. 16, 17. *Various algebraic results* 146, 147
Ex. 18. *Enveloping cone of Segre's locus from point of a particular line* 147
Ex. 19. *Trisecant planes of a quartic curve in fourfold space* . 147
Ex. 20. *Symmetrical tangential equation of Segre's quartic locus* . 147, 148
Ex. 21. *Quadric in regard to which Segre's figure is self-polar* . 148
Ex. 22. *Derivation of Segre's figure from fivefold space* . . 149
Ex. 23. *The projection of Segre's figure on to a threefold space* . 149—151

The dual of the figure which has been considered. The cubic locus *S* 151—158
Ex. 1. *Equations for the six lines of the locus S through an arbitrary point* 158
Ex. 2. *The equation of the Weddle surface*. 158, 159
Ex. 3. *Quadratic congruence obtained by projection of lines of S* . 159
Ex. 4. *Dual derivation of Weddle's surface* 159
Ex. 5. *The locus Σ, and its tangent solid, in terms of parameters* . 159, 160
Ex. 6. *Quartic surface with fifteen double points, changed to surface with five double points* 160
Ex. 7. *The locus S by projection from fivefold space* . . . 160
Ex. 8. *Theory of system of any four quadrics in threefold space* . 160

CHAPTER VI. A QUARTIC SURFACE IN SPACE OF FOUR DIMENSIONS. THE CYCLIDE

A quartic surface in space of four dimensions 161, 162
Preliminary algebraic consideration of the surface . . . 162—165
Ex. 1. *Surface projected on to plane by planes through a line of the surface* 165
Ex. 2. *Reduction of equations of two quadric point-cones* . . 165
Ex. 3. *Combination of harmonic inversions from points of a simplex* 165
Ex. 4. *Algebraic proof surface contains only sixteen lines* . . 166
Ex. 5. *Algebraic arrangement of lines* 166

Resumption of the descriptive theory of the quartic surface. The sixteen lines . . . . 166—168

PAGES

The self-polar pentad for the quartic surface . . . . . 168—170

Ex. 1. *Harmonic inversion of the lines of the figure* . . . 170

Ex. 2. *The lines meet in pairs in eight points of a face of the pentad* 170

Ex. 3. *A notation for the sixteen lines. Double-fours of lines* . 170, 171

The conics lying on the quartic surface 171, 172

The cubic surface in space of three dimensions; and the theory of inversion 172, 173

Ex. 1. *Equation for a cubic surface derived from fourfold space* . 173

Ex. 2. *The twenty-seven lines of the cubic surface obtained by projection* 174

Ex. 3. *A line meeting the Absolute conic inverts into an intersecting line meeting this conic* 174

The Cyclide, or quartic surface with a double conic, in threefold space 174, 175

Ex. 1. *Equation of Cyclide, and of enveloping spheres* . . . 175, 176

Ex. 2. *Cyclide as locus of circle, intersection of corresponding spheres of two systems* 176

Ex. 3. *Cyclide as locus of limiting points of fixed sphere and varying tangent plane of quadric* 176, 177

Ex. 4. *Enveloping spheres meet Cyclide on tangent planes of a quadric cone* 177

The five generations of the Cyclide, with confocal quadric surfaces 177, 178

Ex. 1. *The focal conics of the confocal principal quadrics* . . 178, 179

Ex. 2. *The angle between two enveloping spheres of different systems* 179

Ex. 3. *The five inversions of a Cyclide into itself.* . . . 179

Ex. 4. *Four points of double conic of Cyclide at which tangent planes coincide* 179, 180

Confocal Cyclides. Three through arbitrary point with given focal curve, cutting orthogonally . . . . 180—183

Ex. 1. *Algebraic formulation* 183

Ex. 2. *Lines in fourfold space which become normals of Cyclide* . 183—186

Ex. 3. *The lines of curvature of a Cyclide* . . . . . 186—188

Ex. 4. *The normals of Cyclide along line of curvature generate a developable surface* 188, 189

Ex. 5. *Sixteen lines of Cyclide are complete intersection with a quartic surface* 189

Ex. 6. *A line of curvature is inverted into a line of curvature* . 189

Ex. 7. *A criterion for a line of curvature* 189

Ex. 8. *Two orthogonal surfaces intersecting in a line of curvature on one* 189

Ex. 9. *Plane lines of curvature* 189, 190

Ex. 10. *Differential equation for inflexional curve, and for line of curvature* 190

In regard to some particular cases. The Dupin Cyclide . . 190—194

PAGES
Ex. 1. *An irrational form of the equation of a general Cyclide* . 194, 195
Ex. 2. *Equations for a Cyclide with two double points* . . . 195, 196
Ex. 3. *Equations for a Cyclide with four double points* . . 196—198
Ex. 4. *Maxwell's generation of Dupin's Cyclide generalised* . . 198
Ex. 5. *Inversion of Dupin's Cyclide into an Anchor Ring* . . 198—200
Ex. 6. *Two theorems for inversion of spheres* 200
Ex. 7. *The envelope of a sphere touching three spheres* . . . 200, 201
Ex. 8. *The inversion of Dupin's Cyclide into a Quadric cone of revolution* 201
Ex. 9. *Dupin's Cyclide given by its tangent planes* . . . 201, 202
Ex. 10. *Further equations for Dupin's Cyclide* 202

CHAPTER VII. RELATIONS IN SPACE OF FIVE DIMENSIONS. KUMMER'S SURFACE

Klein's figure in three dimensions, related to a figure in five dimensions . . . . 203—205
Exx. 1—4. *Various particular formulae* 205, 206
Ex. 5. *Origin and scheme for ten important quadric surfaces* . 206—208
A transformation of Segre's figure, in space of four dimensions, to a figure in five dimensions . . . . 208—211
Relation of the transformation to the theory of Kummer's surface 211, 212
Kummer's quartic surface as the locus of singular points of a quadratic complex of lines . . . . 212—217
Ex. 1. *The question of the covariance of the transformation* . 217
Ex. 2. *The tetrahedroid, and Wave Surface, of Fresnel* . . 217, 218
Kummer's surface in the geometry of space of five dimensions . 218—221
The tangent lines of a Kummer surface at a point . . . 221, 222
The inflexional directions at a point of the Kummer surface. . 222, 223
The associated quadratic congruences . . . . . . 223—225
The common singular points and planes of the six congruences . 225—227
The confocal quadrics . . . . 227—229
Singular planes through a more particular point 229, 230
The Inflexional (or Asymptotic) curves of the Kummer Surface . 230—234
The rationality of the Quadratic Complex, and of the Quadratic Congruence . . . . 234—237
The generation of a quadric in fivefold space. The tetrahedral complex in threefold space . . . . 237—240
The singular points and planes of a Quadratic Congruence. Caporali's theorem . . . . 240—243
Exx. 1—3. *Various algebraic results* 243, 244

CORRECTIONS TO VOLUME III 244

INDEX . . . . 245—250

CHAPTER I

INTRODUCTORY. RELATIONS OF THE GEOMETRY OF TWO, THREE, FOUR AND FIVE DIMENSIONS

THE present chapter consists of various examples of the interest and importance of the comparison of the geometry of spaces of different dimensions. The first section (pp. 1—32) is concerned with relations between theorems in two and in three dimensions. The second section (pp. 32—40) deals with the representation in four dimensions of some results belonging to ordinary space of three dimensions. The last section (pp. 40—64) deals with the employment of space of five dimensions for the consideration of properties arising both in three and in two dimensions. Some few references occur to space of any number of dimensions.

SECTION I. THEOREMS OF TWO AND THREE DIMENSIONS

The conics touching the fives from six arbitrary lines of a plane. Let three lines, p, q, r, be given in a plane, as well as a fourth line containing two points, I, J; let any conic be drawn touching the four lines; let σ be the conic, through the points I, J, which contains the three intersections of the lines p, q, r; then this conic σ passes through the point, S, in which intersect the tangents from I, J to the former conic. Or, in other words (Vol. II, p. 81), the circle through the intersections of three tangents of a parabola contains the focus of this parabola. Thus if four lines be given, beside the line which contains the points I, J, the conic touching the five lines being then definite, the four conics, all through I, J, each containing the intersections of three of the four given lines, meet in a point, namely, the point, S, in which the tangents from I, J to the former conic intersect (Vol. II, p. 82); namely, these are four circles meeting in the focus of the parabola. If now, finally, five lines be given, beside the line containing the points I, J, there will be five parabolas, each a conic touching the last line and four of the others, and five foci, S_1, S_2, ..., S_5. It is the case that the circle containing any three of these foci passes through the other two, that is, that the seven points S_1, ..., S_5, I, J lie on a conic. Of this theorem a proof was given by Clifford ("A synthetic proof of Miquel's theorem," *Math. Papers*, 1882, p. 38), with the help of certain particular cubic curves.

A circle through the focus, S, of a parabola is a conic containing the intersections of three tangents of the parabola, namely IJ, IS, JS, and may be said to be triangularly circumscribed to the parabola. Conversely, any conic triangularly circumscribed to a conic and meeting a tangent of this in points I, J, contains the intersection, S, of the tangents to this from I and J; it may, therefore, be regarded as a circle through the focus of a parabola, when I and J are taken as the Absolute points. Thus the theorem above referred to may be stated by saying that, if a, b, c, d, l' and l be six arbitrary lines given in a plane, and I, J be two arbitrary given points of the line l, and five conics be defined each as touching l and four of the lines a, b, c, d, l', then there exists a conic passing through I and J which is triangularly circumscribed to these five conics. The symmetry may suggest that this latter is also triangularly circumscribed to the conic which is defined by touching the five lines a, b, c, d, l'; this is in fact the case. We thus have six conics, each touching five of the six given lines; and each of the six lines touches five of the conics. There cannot be two conics through the points, I, J, of l, both triangularly circumscribed to the six conics; such a conic, if existent, is defined by the points I, J and three of the conics touching the line IJ, namely as containing the intersections, S_1, S_2, S_3, of the tangents from I, J to these three conics, respectively.

To prove this symmetrical result we may proceed as follows: Denote the plane of the six given lines, a, b, c, d, l, l' by ϖ. Draw through each of the lines a, b, c, d an arbitrary plane, denoting the intersections of these in threes by A, B, C, D, of which D is the intersection of the planes through a, b, c, and so on. It is assumed that A, B, C, D are not in a plane. Through the six points consisting of A, B, C, D and the two arbitrary points I, J, of the line l, there can be put a definite cubic curve, which we denote by γ. This will meet the plane ϖ in another point beside I and J; say, in E. Then, through the five points A, B, C, D, E there can be drawn another cubic curve, γ', to have the line l' for chord, meeting this, suppose, in the points I', J' (Vol. III, p. 139). We prove that the conic, ω, containing the five points I, J, E, I', J', is triangularly circumscribed to the six conics touching the fives of the six given lines a, b, c, d, l, l'. It is thus independent of the planes drawn through a, b, c, d.

The conic ω is the intersection with the plane ϖ of the quadric, Ω, defined by the nine points A, B, C, D, E, I, J, I', J'; the cubic curves, γ, γ', each meeting Ω in seven points, lie on this quadric. Cubic curves lying on a quadric are of two families, since such a curve meets all generators of the surface, of one system of generators, in one point, and all generators of the other system in two points;

two cubics of the same family have four common points, but two cubics of different families have five common points (Vol. III, p. 139). Thus the curves γ, γ' are of different families on Ω.

We prove now, first, that the conic ω is triangularly circumscribed to the conic touching the lines a, b, c, d, l; namely by obtaining triads of points of ω whose joins touch this conic (a, b, c, d, l). Let P be any point of ω; through this, and the other four points, A, B, C, D, of Ω, there can be drawn, lying on Ω, a cubic curve of the same family as γ (Vol. III, p. 129). This curve will meet ω in two further points, say Q and R. Either of these, with A, B, C, D, determines the cubic curve, and so determines the other two of the three points, P, Q, R, of ω. Thus, as P varies on ω, the triads P, Q, R form an involution of sets of three points thereon, and the lines QR, RP, PQ all touch a conic (Vol. II, p. 135); this conic, which we may denote by λ, is evidently triangularly inscribed in ω. We prove that λ is the conic touching a, b, c, d, l, by considering different positions of P. When P is at E, the line QR is the line l; thus λ touches l. In general the cubic curve through A, B, C, D, P, Q, R is projected from P by a quadric cone, which may be defined as that containing PA, PB, PC, PD and a particular one of the two generators of Ω at that point P. But when P is at one of the two intersections of the line d with ω, this cone degenerates, becoming the aggregate of the plane ABC, which contains d, and the plane joining PD to the particular generator at P spoken of; one of the two lines PQ, PR, in which the cone meets the plane ϖ, is thus the line d. Therefore the conic λ touches d. By a similar argument it touches a, b, c.

To prove that the conic ω is triangularly circumscribed to the conic touching a, b, c, d and l', we describe a cubic curve through a varying point, P, of ω and through A, B, C, D, lying on Ω, but of the same family as the curve γ'.

That the conic touching a, b, c, and both of l and l', is triangularly inscribed to ω, we likewise prove by obtaining an involution of sets of three points lying on ω. For this, let the points in which the lines DA, DB, DC meet the plane ϖ be denoted, respectively, by A_1, B_1 and C_1, and consider the conics drawn through the four points A_1, B_1, C_1 and E. The sets of three points other than E in which these conics meet ω are then sets of such an involution (Vol. II, p. 138), and the three joins of the points of such a set are tangents of a conic, which we denote by δ. One conic through A_1, B_1, C_1 and E consists of the two lines B_1C_1 and A_1E; thus δ touches B_1C_1, which is the line a. Similarly δ touches b and c. Again, the conics through A_1, B_1, C_1 and E may be defined by quadric cones, of vertex D, containing DA, DB, DC and DE; one such cone, however, is the cone which contains the cubic curve γ, and this meets the conic ω in the points I, J. Thus the conic δ touches the line l. Another such cone is that containing the curve γ', which meets ω in I' and J'. Thus δ also touches l'. It is thus shewn that the conic touching a, b, c and l, l' is triangularly inscribed in ω. An analogous argument proves ω to be triangularly circumscribed to the *three* conics touching l, l' and, respectively, b, c, d; c, a, d; a, b, d.

The theorem stated is thus completely established. It has been remarked that the conic obtained, triangularly circumscribed to the six primary conics, is uniquely determined by its intersections, I, J, with one of the six given lines, l. It is clear, however, from the reasoning given that the conic is also determinate when E is given; its intersections with l being on the tangents from E to the conic touching a, b, c, d, l, and its intersections with l' being on the tangents from E to the conic touching a, b, c, d, l', while the conic passes through E. Thus, whatever be the planes, drawn through a, b, c, d, by which A, B, C, D are determined, the cubic curve drawn through E, A, B, C, D, to have l for chord, meets l in the same two points, I, J; and similarly for l'.

Ex. 1. If E, I, J be three points of a cubic curve in space, of which A, B, C, D are four other points, the two axial pencils of planes joining EI and IJ to A, B, C, D are related to one another; and the former is related to the range on EI determined by the four planes containing A, B, C, D, the latter being similarly related to the range which these four planes determine on IJ. Wherefore, there is a conic, in the plane EIJ, touching the four lines in which this plane is met by the four planes containing A, B, C, D, and also touching EI, IJ. By parity of reasoning this same conic touches also the line EJ.

Ex. 2. If, in the construction above, the lines a, b, c, d, l, lying in the plane $t=0$, be taken to be

$$x=0,\ y=0,\ z=0,\ ax+by+cz=0,\ lx+my+nz=0,$$

the planes DBC, DCA, DAB, ABC being

$$x=0,\ y=0,\ z=0,\ ax+by+cz-t=0,$$

and the point E be $(\xi, \eta, \zeta, 0)$, prove that the points I, J lie on the conic in the plane $t=0$ given by

$$x^{-1}\xi(mb^{-1}-nc^{-1})+y^{-1}\eta(nc^{-1}-la^{-1})+z^{-1}\zeta(la^{-1}-mb^{-1})=0;$$

this lies on the cone, of vertex D, containing the cubic curve through A, B, C, D, E which has the line l for chord.

Ex. 3. Hence infer that if the six given lines of the original plane be of equations

$$x=0,\ y=0,\ z=0,\ a_r x+b_r y+c_r z=0,\ (r=1, 2, 3),$$

then the general conic which is triangularly circumscribed to the six conics touching the fives of these lines has for equation

$$\begin{aligned}&\Delta(a_1x+b_1y+c_1z)(a_2a_3xYZ+b_2b_3yZX+c_2c_3zXY)\\&\quad+(A_1X+B_1Y+C_1Z)(A_2A_3Xyz+B_2B_3Yzx+C_2C_3Zxy)=0,\end{aligned}$$

where Δ is the determinant (a_1, b_2, c_3), and $A_1, B_2, \ldots$ are the minors of $a_1, b_2, \ldots$ therein, while (X, Y, Z) is an arbitrary point, such that $(a_1^{-1}A_1X, b_1^{-1}B_1Y, c_1^{-1}C_1Z)$ is the point E, or (ξ, η, ζ), of the foregoing theory. This equation, it may easily be seen, is unaltered by cyclical change of the suffixes, X, Y, Z being unaltered; it is derived from a form obtained algebraically by Mr F. P. White (*Camb. Phil. Proc.* XXII, 1924, p. 11). If we take

$$X'=A_1X+B_1Y+C_1Z,\quad Y'=A_2X+B_2Y+C_2Z,\quad Z'=A_3X+\ldots,$$

and also

$$x'=a_1x+b_1y+c_1z,\quad y'=a_2x+\ldots,\quad z'=a_3x+\ldots,$$

the equation is also capable of the form

$$\begin{aligned}&x'(B_1C_1x'Y'Z'+B_2C_2y'Z'X'+B_3C_3z'X'Y')\\&\qquad+\Delta X(b_1c_1X'y'z'+b_2c_2Y'z'x'+b_3c_3Z'x'y')=0,\end{aligned}$$

and of two other forms, obtained from this by cyclical interchange of a, b, c, without change of X', Y', Z'.

Further, if we take the three points P_1, P_2, P_3, where P_r is of coordinates $(a_r^{-1}A_rX, b_r^{-1}B_rY, c_r^{-1}C_rZ)$, and denote the line $a_rx+b_ry+c_rz=0$ by l_r, and the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ by A_1, B_1, C_1, and then define a conic, S_1, as that through $(A_1, B_1, C_1, P_2, P_3)$, a conic, S_2, as that through $(A_1, B_1, C_1, P_3, P_1)$ and a conic, S_3, similarly, it will be found that the three pairs of intersections (S_1, l_1), (S_2, l_2), (S_3, l_3) lie on a conic through P_1, P_2, P_3, this conic, through these nine points, being the conic triangularly circumscribed to the six primary conics.

The reader may also be reminded of Taylor's theorem (Vol. II, p. 61), that the poles of an arbitrary line in regard to the six primary conics lie on another conic.

Ex. 4. Any six lines of a plane can in fact be regarded as the projections of six chords of a cubic curve, from a point of this curve, these chords having the property that every five of them have a common transversal. For, taking the six conics touching the fives of the given lines, let the conic which is triangularly circumscribed to these have its equation put in the form $xz-y^2=0$, as is possible in an infinite number of ways. Then with an arbitrary point for the intersection of planes $x=0$, $y=0$, $z=0$, this conic is on the cone projecting from this point the cubic curve whose points are given by $(\theta^3, \theta^2, \theta, 1)$. The tangent line of any conic given in the original plane, $t = 0$, may then be defined by a plane,

$$ux + vy + wz = 0,$$

through the point $(0, 0, 0, 1)$, and the tangential equation of the conic may be taken in the form

$$c'u^2 + a'v^2 - cw^2 - bvw - (a + a')\,wu - b'uv = 0,$$

in which u, v, w are tangential coordinates. The chord of the cubic curve joining the points, other than $(0, 0, 0, 1)$, in which the plane meets the curve, has for coordinates the six

$$wu - v^2,\ uv,\ -u^2,\ wu,\ vw,\ w^2;$$

thus the equation of the conic expresses that the tangent line of this is the projection of a chord of the cubic curve which belongs to the linear complex (a, b, c, a', b', c'). In particular, if

$$aa' + bb' + cc' = 0,$$

the conic is triangularly inscribed to the conic $xz - y^2 = 0$, as we easily verify (Cremona, *Giorn. d. Mat.*, x, 1872, p. 47). Whence, six conics triangularly inscribed to $xz - y^2 = 0$ can be regarded as having the equations $(r = 1, \ldots, 6)$

$$n_r'u^2 + l_r'v^2 - n_r w^2 - m_r vw - (l_r + l_r')\,wu - m_r'uv = 0,$$

each being obtained by projecting the chords of the cubic curve which meet a line of coordinates $(l_r, m_r, n_r, l_r', m_r', n_r')$. When the six conics touch the fives of six lines, so that every five of them have a common tangent, these lines are obtained by projection from six chords of the cubic, forming with the lines $(l_r, m_r, \ldots)$ a double six of lines. (Cf. Wakeford, *Proc. Lond. Math. Soc.*, xv, 1916, p. 340.)

It may be remarked, too, that the conic, $(l_r, m_r, \ldots)$, touching the projections of the chords of the cubic which meet the line $(l_r, m_r, \ldots)$, has, beside five of the originally given six lines, as a sixth tangent, the line $l_r'x + m_r'y + n_r'z = 0$. If we take the intersection of this with the remaining one of the six original lines, we obtain one of six points which lie on a conic. (Cf. Vol. III, p. 136, Ex. 8, and p. 201.)

Representation of a plane upon a quadric. Consider a quadric surface, Ω; and, upon this, a point, U, which is to be taken as centre of projection. Consider also a plane, ϖ. Let the generators at U, of the quadric Ω, meet the plane ϖ in the points I and J. Any plane meets the generators; and, thus, the conic, in which Ω is met by any plane which does not pass through U, is projected from U into a conic of the plane ϖ which passes through I and J. Regarding I and J as Absolute points, we may then speak of this conic as a circle. Conversely, any conic, in the plane ϖ, which passes through I and J, lies on a quadric cone of vertex U; this cone contains the generators UI, UJ; its remaining intersection with Ω is thus a curve of the second order, namely a plane section. Thus plane sections of the quadric Ω, not containing U, and circles of the plane ϖ, are transformable into one another by projection from U. Sections of Ω by planes through U project into lines of ϖ, and conversely.

Let α be any conic section of Ω not passing through U; and let A be the pole of the plane of α, in regard to Ω. Let α' be the circle of the plane ϖ which arises by projection of α from U. We prove that the line UA meets the plane ϖ in the pole, in regard to α', of the line IJ; that is, that the projection of A is the *centre* of the circle α'. Let the line in which the tangent plane at U, of the quadric Ω, is met by the plane of α, be denoted by u; and the point in which the line UA meets the plane of α be denoted by A_0. The line UA is the polar line of u, in regard to Ω; thus the polar plane of A_0 contains u, and A_0 is the pole of u in regard to the conic α; so that any line in the plane of α, through A_0, meets u in the harmonic conjugate of A_0, in regard to the two points in which this line meets the conic α. This relation persists after projection from U; and this proves the statement made. Conversely, however, in the correspondence between the points of the plane ϖ, and the points of the quadric Ω, the centre of the circle corresponds to the point, other than U, in which the line UA meets Ω.

We have explained (Vol. II, p. 166) how to measure the angle between two lines of a plane, say of equations $P+\lambda Q=0, P+\mu Q=0$, with respect to two other lines $P=0$, $Q=0$, by means of the ratio λ/μ. In particular, the angle between two lines lying in the tangent plane of the quadric Ω, at a point O, and meeting at this point, may be measured with respect to the generators of the quadric at this point. As each of these generators meets one of the generators at U, if O project from U into O' on the plane ϖ, the generators at O project into the lines $O'I$, $O'J$. Thus the angle in question is that between the lines in the plane ϖ, into which the two original tangents of Ω at O project, when measured with respect to the two Absolute points I, J of the plane ϖ. If O be one of the intersections of two plane sections of the quadric Ω, the other being Q, and the lines taken at O be the two tangents of these sections at O, the flat pencil in the tangent plane at O, consisting of these lines and the generators at O, is the section by the tangent plane at O of the axial pencil consisting of the two planes, which meet in OQ, and the two tangent planes of Ω drawn from the line OQ; for each of these tangent planes contains a generator at O and a generator at Q. Thus the angle in question is that of the planes of the two sections measured in regard to the two tangent planes of Ω drawn from OQ. If, in particular, the planes of the two sections are conjugate to one another, each containing the pole of the other, then they are harmonic in regard to the two tangent planes of Ω drawn through their line of intersection. Thus we reach the results that two circles in the plane ϖ intersect at the same angle at both their common points, this being the angle between the corresponding planes of the quadric Ω when suitably measured, and that the circles cut at right angles when these planes are conjugate to one another (cf. Vol. II, p. 193). By considering sections of Ω through U, all conjugate to a chosen plane section, we obtain the result that all lines through the centre of a circle cut this at right angles; by considering plane sections all passing through a point not on Ω, this being the pole of a certain section, we obtain the aggregate of all circles in the plane ϖ which cut a certain circle at right angles; by considering sections of Ω by planes through a line, we obtain a system of coaxial circles, whose limiting points are the projections of the two points in which Ω is met by the polar line of the given line, while sections by planes through this polar line give circles cutting at right angles those of the original system; thence, two points of Ω, lying on a line through the pole of a chosen section of Ω, project into two points which are conjugate in regard to the circle into which the chosen section projects; and so on, all the familiar properties of circles being easily interpretable. It may be remarked that the angle between two plane sections of Ω, which we have identified with the angle between two circles, is also the same as the interval between the points which are the poles of these plane sections, measured with respect to the intersections with Ω of the join of these poles.

The generalised Miquel theorem. We have given (Vol. II, p. 70) the theorem that if D, E, F be any points respectively on the joins, BC, CA, AB, of three points of a plane, then the three circles AEF, BFD, CDE meet in a point. We consider this result in particular from our present point of view. There is a corresponding theorem for any number of dimensions, capable of similar proof.

(Roberts, *Proc. Lond. Math. Soc.*, XII (1881), p. 117; *ibid.*, XXV (1893), p. 306; Grace, *Trans. Camb. Phil. Soc.*, XVI (1897), p. 168; Haskell, *Arch. d. Math. u. Phys.*, V (1903), p. 278.)

Suppose we have, in space of three dimensions, any three planes passing through a point, O, say $x = 0$, $y = 0$, $z = 0$, and take three arbitrary points: A on the line of intersection of $y = 0, z = 0$; B on the line of intersection of $z=0, x=0$, and C on the line $x=0, y=0$; and then take, on the plane OBC, the point D; on the plane OCA, the point E, and on the plane OAB, the point F; the planes AEF, BFD, CDE will meet in a point, say Q. If O is taken on an arbitrary given quadric, and A, B, C are the intersections with this quadric of the lines $y = 0$, $z = 0$, etc., while D, E, F are on the sections of this quadric by the planes $x = 0$, $y = 0$, $z = 0$, respectively, then the point Q also lies on this quadric. For, all quadrics through the seven points O, A, B, C, D, E, F have in common a further point; of such quadrics a degenerate one consists of the two planes $OBCD$, AEF; another of the planes $OCAE$, BFD, and a third of the planes $OABF$, CDE. As these degenerate quadrics all contain the point Q, so does the original. When, now, we project the points of the given quadric from O, on to an arbitrary plane, the plane $x = 0$ gives a line containing the projections of the points B, C, D, and so on, and the section of the quadric by the plane AEF gives a circle; the three such circles meet in the point which is the projection of Q. This proves the theorem of Miquel.

It is equally the case that, in space of three dimensions, if A, B, C, D be any four points, and points, P, Q, R, P', Q', R' be taken arbitrarily, respectively on the lines DA, DB, DC, BC, CA, AB, then the four spheres, each containing one of the four points A, B, C, D and also the three points on the joins of this to the other points, meet in a point. These are the spheres $APQ'R'$, $BQR'P'$, $CRP'Q'$, $DPQR$. A proof is as follows. In space of four dimensions, consider four threefolds, $x=0$, $y=0$, $z=0$, $t=0$, meeting in a point O; any three of these threefolds will meet in a line through O, and upon each of the four lines so obtained a point may be taken; let the point on the line $y = 0$, $z = 0$, $t = 0$ be A, the others being B, C, D, of which, for example, D is on $x = 0$, $y = 0$, $z = 0$. Upon the plane $y = 0$, $z = 0$, which contains the lines OA, OD, let the point P be taken, arbitrarily; and, similarly, the points Q and R be taken on the planes OBD, OCD, respectively, as also P', Q', R' on the respective planes OBC, OCA, OAB. Then consider the threefolds $APQ'R'$, $BQR'P'$, $CRP'Q'$, $DPQR$, which we denote, respectively, by $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\tau = 0$; and let T be their point of intersection. It can then be shewn that T lies on any quadric threefold constructed to contain the points $O, A, B, C, D, P, Q, R, P', Q', R'$. In fact the equation of a quadric threefold, in fourfold space, contains fifteen terms; thus a quadric through the eleven specified points will be of the form $\lambda_1 U_1 + \ldots + \lambda_4 U_4 = 0$, where $\lambda_1, \ldots, \lambda_4$ are arbitrary, and $U_1 = 0, \ldots, U_4 = 0$ are four such quadrics, which we suppose to be linearly independent. It follows that any quadric through the eleven specified points will also pass through the remaining $2^4 - 11$, or five, common points of these four quadrics. This assumes that these four quadrics have no common curve, or surface, and intersect in sixteen points. We can, however, at once specify four degenerate quadrics through the eleven points, namely $x\xi = 0$, $y\eta = 0$, $z\zeta = 0$, $t\tau = 0$; for instance $x = 0$ contains O, B, C, D, P', Q, R, and $\xi = 0$ contains A, P, Q', R'. Thus, every quadric through the eleven points contains the point, T, common to $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\tau = 0$. This is the result desired. The four other common points of these four quadrics, which equally lie on any quadric through the eleven points, are the points such as $\eta = 0$, $\zeta = 0$, $\tau = 0$, $x = 0$. It can be shewn now that, by projection from O upon any threefold space, the theorem enunciated in regard to the four spheres is obtained. But, for convenience, the discussion of the derivation of a sphere by projection from space of four dimensions is deferred to a later section of this chapter (p. 36). A similar proof holds in higher space; this depends on a theorem that, in space of n dimensions, all quadric $(n-1)$-folds through $\frac{1}{2}n(n+1)+1$ general points are expressible by n such quadrics, and pass through $2^n - 1 - \frac{1}{2}n(n+1)$ other points.

From the Miquel theorem in a plane we were able to infer (Vol. II, p. 71) that, if four arbitrary lines be given, the circles each containing the intersections of three of these lines, four in all, would meet in a point; namely, by supposing the points D, E, F, in the enunciation given above, to be in line. This theorem, ascribed to Wallace (Scoticus, *Leybourn's Math. Repos.*, I, 1806, p. 170), will be considered below (p. 18). If, in the corresponding theorem above considered, for four spheres in space of three dimensions, we suppose the six points P, Q, R, P', Q', R' to be in a plane, the four spheres there taken will still meet in a point, but this point will be on the plane containing the six points. This is obvious by considering the Wallace theorem for the four lines in this plane which are obtained by the intersections of this plane with the four planes BCD, CAD, ABD, ABC; the four spheres of the theorem meet this plane in circles. We cannot, therefore, go on to infer that the sphere through A, B, C, D passes through the point of intersection of the first four spheres, as in the plane case; the five spheres meet in fours in points lying one on each of the five planes involved. It will be seen below (p. 59) that the generalisation of the Wallace theorem, which thus does not hold in space of three dimensions, holds nevertheless in space of four dimensions, and in space of any even number of dimensions (see Grace, *as above*, p. 163).

Ex. 1. Through five points, A, B, C, D, E, in three dimensions, can be drawn five linearly independent quadrics. The conics in which these meet an arbitrary plane, ϖ, are also linearly independent, and determine a definite conic, σ, inpolar to all of these. Thus every quadric through A, B, C, D, E meets the plane ϖ in a conic which is outpolar to σ. Such a quadric is formed by the plane ABC taken with any plane through the line DE; hence the point in which ϖ is met by DE is the pole, in regard to σ, of the line in which ϖ is met by the plane ABC. Thus the system of ten points and lines, in which the joining lines and planes of A, B, C,

D, E meet the plane ϖ, is self-polar in regard to the conic σ A similar proof holds for the theorem obtained similarly by taking $(n+2)$ points in space of n dimensions (cf. Vol. II, p. 218), the number of linearly independent quadric $(n-1)$-folds which pass through $(n+2)$ general points, namely $\frac{1}{2}n(n+1)-1$, being one less than the number of terms in the equation of a quadric $(n-2)$-fold in a space of $(n-1)$ dimensions.

Ex. 2. A cubic curve through five points, A, B, C, D, E, meets an arbitrary plane, ϖ, in a triad which is self-polar in regard to the conic, σ, determined in ϖ by these five points, as in Ex. 1. For the three linearly independent quadrics, through the curve, meet ϖ in conics, outpolar to σ, all passing through the points of this triad. The corresponding theorem for a rational curve of order n, through $(n+2)$ points of space of n dimensions, can be similarly proved.

Ex. 3. The dual of the theorem referred to in Ex. 1 is that, if we have five planes, α, β, γ, δ, ϵ, and consider the ∞^4 quadrics touching these planes, then the enveloping cones to these quadrics, drawn from any point, E, are all inpolar to a particular quadric cone, say Σ, of vertex E, determined by the five given planes and the point E. Let the intersections of the four planes α, β, γ, δ be denoted by A, B, C, D, the point D being the intersection (α, β, γ), and so on. Then the line ED is the polar line of the plane joining E to the line (δ, ϵ), in regard to the cone Σ; and the line joining E to the point $(\beta, \gamma, \epsilon)$ is the polar line of the plane joining E to the line (α, δ); that is, the line joining E to the point where AD meets ϵ is the polar line of the plane EBC, in regard to Σ. Thus, considering the conic where the plane ϵ is met by the cone Σ, this conic, σ, is that before described, in regard to which the joining lines and planes of the five points A, B, C, D, E determine a polar system in the plane ϵ in which σ lies.

Regarding σ as an Absolute conic, the point E is the intersection of perpendiculars drawn from the points A, B, C, D, respectively to the planes α, β, γ, δ; and each pair of opposite joins of A, B, C, D, such as BC and AD, are at right angles to one another (cf. Vol. III, p. 78, Ex. 7). In regard to σ, the relation of the five points A, B, C, D, E is symmetrical.

Ex. 4. Five points of a given cubic curve, A, B, C, D, E, determine as above a conic, σ, in a given arbitrary plane. Consider what is the aggregate of all such conics when the five points of the cubic curve have all possible positions thereon. If the curve meet the plane in the points P, Q, R, any such conic, σ, must be such that P, Q, R are a self-polar triad in regard to σ, as we have seen. The five quadrics through A, B, C, D, E may be regarded as consisting of three quadrics containing the curve, together with two degenerate quadrics each consisting of the plane ABC taken with a plane through the line DE. Thus the point F, in which the line DE meets the plane PQR, must be the pole, in regard to σ, of the line in which the plane ABC meets the plane PQR. Conversely, then, taking any conic, σ, in regard to which P, Q, R are a self-polar triad, we can find five points A, B, C, D, E of the cubic curve, for which this is the appropriate conic. Namely, by taking A, B, C arbitrarily upon the curve, then finding the pole, F, of the plane ABC in regard to σ, and then drawing the chord, DE, of the curve which passes through F. The aggregate of the conics, σ, for all possible sets of five points of the curve, is thus the same as the aggregate of the conics of the plane PQR in regard to which P, Q, R are a self-polar triad.

Ex. 5. The proof of Miquel's theorem in a plane which was given in Vol. II, p. 70, was in effect by the polar system determined on the Absolute line by the conics through four points of the plane. This proof can be used also for the Miquel theorem for the four spheres through $APQ'R'$, $BQR'P'$, $CRP'Q'$, $DPQR$, in the notation used above. For it can be shewn that, if O be any further independent point, in the space of three dimensions, and we consider the four conics, in any plane ϖ, such as the conic σ in Ex. 1 above, arising, respectively, from the four sets of five points $OAPQ'R'$, $OBQR'P'$, $OCRP'Q'$, $ODPQR$, then these conics are linearly connected. Denote these conics by σ_1, σ_2, σ_3, σ_4. Thus, if three quadrics drawn, respectively, through $APQ'R'$, $BQR'P'$, $CRP'Q'$ have in common a conic, S, on the plane ϖ, then S, being outpolar to each of σ_1, σ_2, σ_3, is outpolar to σ_4; from this the proof can be completed.

Ex. 6. The theorem referred to in Ex. 5 is equivalent to the following: Let x, y, z, t be four lines in a plane, and A, B, C, D be four points of the plane, such that the line AD contains the point (y, z), the line BC contains the point (x, t), and so on, the six joins of the four points containing the six intersections, properly chosen, of the four lines. Further, let P, Q, R, P', Q', R' be arbitrary points respectively on the lines AD, BD, CD, BC, CA, AB. Then the triads P, Q, R and x, y, z, being in perspective from D, are polars of one another in regard to a conic, say σ_4; similarly, the triads P, Q', R' and y, z, t are polars of one another in regard to a conic, σ_1; and there are two other conics, σ_2, σ_3, determined similarly. These four conics are linearly connected.

The theory of inversion in a plane. Consider a particular correspondence of two points of space in three dimensions, which is of importance. Let L be a fixed point, and λ a fixed plane. To any point, P, we can then make correspond the point, P', of the line LP, which is harmonically conjugate to P, with respect to L and the point of LP which lies on λ; then, conversely, the point corresponding to P' is the point P itself. By this correspondence, to the points of a line, PQ, correspond the points of another line, $P'Q'$, and these two lines meet on the plane λ; also, to the points of a plane, PQR, correspond the points of another plane, $P'Q'R'$, these two planes meeting in a line of the plane λ. Hence, also, as a particular result, to the points of a quadric cone, of vertex O, correspond the points of another quadric cone, of vertex, say, O', these two cones having in common the points of a conic in the plane λ. The two cones have then in common, also, the points of another conic; the points of this second conic must then correspond to one another in pairs, since the cones have no points in common other than those of these two conics, and the points of the conic in the plane λ correspond each to itself. The plane of the second conic common to the two cones must, therefore, pass through L; and, hence, the tangent planes of the two cones at the points where the second conic meets the plane λ, which are points of contact of the cones, must pass through L, as well as through the vertices of both cones.

Now suppose that the plane λ is the polar plane of L in regard to a certain quadric, Ω. Then the point, P', which corresponds to a point, P, lying on the quadric, is the second intersection of LP with Ω; and, to a plane section of Ω, corresponds another plane section, lying on the quadric cone joining L to the first section, the two sections meeting in two points of the polar section, λ. Unless, indeed, the first section is in a plane passing through L, in which case the second section coincides with it. Let, now, U be a definite point of the quadric Ω, and ϖ a definite plane, and let us project the points of Ω from U upon ϖ, as before. Then the polar section, λ, of L, projects into a definite circle of ϖ, say $[\lambda]$, with centre at the projection of L, as we have seen; two points, P, P', of Ω, on a line through L, project into two points of ϖ, say $[P]$ and $[P']$; as every plane through the line LPP' gives a section of Ω which is conjugate to the polar section λ, the points $[P]$, $[P']$ lie on an infinite number of circles cutting the circle $[\lambda]$ at right angles; thus the points $[P]$, $[P']$ lie on a line through the centre of $[\lambda]$, and are inverse points in regard to this circle, being harmonic conjugates in regard to the points where the line $[PP']$ meets $[\lambda]$. This *inversion* is the same as that defined in Vol. II, p. 67. A line, or a circle, of the plane ϖ, is clearly changed by this process into a circle, or, in particular, into a line, the circles inverse to one another in regard to $[\lambda]$ being those corresponding to two plane sections of Ω which lie on a quadric cone of vertex L. If, on Ω, P and P' be two points of a line through L, either generator at P meets one of the generators at P', the point of meeting being the point of contact, of a tangent plane of Ω drawn through the line PP', and, therefore, on the polar section

λ; thus these generators correspond to one another in regard to L; from this it is easily seen that two circles in the plane ϖ intersect at the same angle as their inverse circles in regard to $[\lambda]$, the angle being measured as explained above. Again, any plane section, μ, of Ω, its pole M, and the enveloping cone of Ω along μ, whose vertex is M, are changed by the correspondence in regard to L, respectively, into a section μ', its pole M', and the enveloping cone of Ω along μ'; thus, in the plane ϖ, any circle and two points which are inverses to one another in regard to this circle, are changed, by inversion in regard to $[\lambda]$, into a circle and two points inverse to one another in regard thereto. It may also be remarked that if, instead of L, we take another point, L_1, lying on the line UL, and its corresponding polar section, λ_1, as base for the correspondence, the difference in the plane ϖ is that inversion takes place in regard to another circle, $[\lambda_1]$, having the same centre as $[\lambda]$.

Ex. 1. Taking coordinates in which the plane ϖ is $t=0$, the point of projection, U, is $x=0$, $y=0$, $z=0$, with $z=0$ for the tangent plane of Ω at this point, the generators being $x \pm iy = 0$, $z=0$, the equation of Ω is of the form $F=0$, where

$$F = x^2 + y^2 + 2gzx + 2fyz + cz^2 - 2zt.$$

If, then, L be (x_0, y_0, z_0, t_0), the point, P, of Ω, be (ξ, η, ζ, τ), and P' be $(\xi', \eta', \zeta', \tau')$, such that $\xi' = \xi + \lambda x_0, \ldots, \tau' = \tau + \lambda t_0$, these satisfying the equation of Ω, we find

$$\lambda = -2\left[\xi x_0 + \eta y_0 + g(\zeta x_0 + \xi z_0) + f(\eta z_0 + \zeta y_0) + c\zeta z_0 - \zeta t_0 - \tau z_0\right] F_0^{-1},$$

where F_0 is the value of F for (x_0, y_0, z_0, t_0). Thus $\xi', \eta', \zeta', \tau'$ are expressible as homogeneous linear functions of ξ, η, ζ, τ. Taking

$$R^2 = \left[(\xi z_0 - \zeta x_0)^2 + (\eta z_0 - \zeta y_0)^2\right] z_0^{-2} \zeta^{-2}, \quad k^2 = F_0 z_0^{-2}$$

we easily compute $\zeta'/\zeta = R^2 k^{-2}$ and

$$\frac{\xi' z_0 - \zeta' x_0}{\zeta'} = \frac{k^2}{R^2} \frac{\xi z_0 - \zeta x_0}{\zeta},$$

$$\frac{\eta' z_0 - \zeta' y_0}{\zeta'} = \frac{k^2}{R^2} \frac{\eta z_0 - \zeta y_0}{\zeta};$$

these are the formulae of inversion in the plane ϖ, in which (ξ, η, ζ) are the coordinates of a point, and (ξ', η', ζ') those of the inverse point, the circle, $[\lambda]$, of inversion having the equation

$$(xz_0 - x_0 z)^2 + (yz_0 - y_0 z)^2 - k^2 z^2 z_0^2 = 0.$$

Ex. 2. We can (Vol. II, p. 69, Ex. 4) state a relation between the centres of two circles of the plane ϖ which are inverses of one another in regard to the circle $[\lambda]$. But the relation in the space of three dimensions is simpler, these centres being the projections, from U, of the poles of two plane sections of Ω which lie on a quadric cone of vertex L.

Ex. 3. Two plane sections of a quadric, having two points in common, lie on two quadric cones (Vol. III, p. 10). Projecting these sections upon a plane from any point of the quadric, and at the same time the section by the polar plane of the vertex of one of the containing cones, we see that, if two circles be given in a plane, there are two circles of inversion by means of which one of the given circles is changed into the other.

Ex. 4. The *circle of similitude* of two given circles $[\alpha]$, $[\beta]$, of the plane ϖ (Vol. II, p. 132), may be considered from the present point of view. With a pair of plane sections, α, β, of Ω, intersecting in a line l, and the pair of tangent planes of Ω through the line l, an axial pencil of pairs of planes in involution, can be defined. Thereby, to the plane lU, joining l to the point U of Ω, there is defined a section, γ, passing through l, the two planes, lU, γ, being a pair of the involution. Prove that, if the quadric be projected from U, on to a plane ϖ, whereby the sections α, β become the circles $[\alpha]$, $[\beta]$, then the section γ becomes the circle, coaxial with $[\alpha]$ and $[\beta]$, passing through the centres of similitude of these, that is, the circle of similitude. The centres of similitude are the two intersections of common tangents of $[\alpha]$, $[\beta]$ which lie on the line joining the centres of these circles. The circles are supposed unequal.

Ex. 5. If we have four circles in a plane meeting, in threes, in four points, there are six circles of similitude, one for each pair. It can be shewn that these meet in a point. This follows easily if we assume (see below, Ex. 6) that, if A, B, C, D, U be points of a quadric, Ω, then the quadric, Σ, described to touch the planes BCD, CAD, ABD, ABC and to contain the generators of Ω at U, contains two other generators of Ω, say those at U'. For then, regarding Ω and Σ as defined by their tangent planes, and considering the degenerate tangential quadric which consists of all the planes through the points U and U', we have three quadrics such that all the tangent planes common to any two of them are equally tangent planes of the third, these consisting of all the planes through the generators of Ω (and Σ) at U and U'. The three pairs of tangent planes of these three quadrics which pass through any line are, therefore, three pairs of an axial involution. In particular, taking AB for the line, the pair of planes ABU, ABU' belong to the involution determined by two pairs, namely the pair ABC, ABD, and the pair which consists of the tangent planes to Ω drawn from AB. From this it follows that the sections of Ω by the planes ABC, ABD project from U, on to any plane, into two circles whose circle of similitude passes through the projection of U'. Thus, by a similar argument, the sections of Ω by the four planes BCD, ..., ABC

become, projected from U, four circles whose six circles of similitude all pass through the projection of U'.

Ex. 6. To prove that the quadric, Σ, containing the generators of a quadric Ω at a point U of this, and also touching four planes, α, β, γ, δ, whose four points of intersection lie on Ω, also contains two other generators of Ω, say at U', we use the fact that a quadric which touches the tangent planes of Ω drawn from one point, P, also touches the tangent planes of Ω drawn from another point, P'. If we denote the tangential equation of Ω, referred to the planes α, β, γ, δ, by $\Omega = 0$, and (ξ, η, ζ, τ), $(\xi', \eta', \zeta', \tau')$ be the coordinates of P, P', the equation of the quadric in question is then $\Omega - kPP' = 0$, where $P = l\xi + m\eta + n\zeta + p\tau$, $P' = l\xi' + \ldots + p\tau'$. If this quadric touch the planes α, β, γ, δ, its equation must contain no terms in l^2, m^2, n^2, p^2. Thus, if the coefficients of these terms in Ω be A, B, C, D, the quadric touching α, β, γ, δ and also the tangent cone of Ω drawn from the point (ξ, η, ζ, τ), touches also the tangent cone of Ω drawn from $(A\xi^{-1}, B\eta^{-1}, C\zeta^{-1}, D\tau^{-1})$. Now, in particular, let (ξ, η, ζ, τ) be on Ω; then we know, or it is easily verified, that, as Ω contains the four intersections of the planes α, β, γ, δ, the point $(A\xi^{-1}, \ldots, D\tau^{-1})$ is likewise on Ω (Vol. III, p. 55). Thence, the quadric in question, containing the tangent planes of Ω from a point of itself, contains the generators of Ω at this point; and it is proved that it then contains the generators at another point.

Ex. 7. At the point (ξ, η, ζ, τ) of the quadric

$$fyz + \ldots + uxt + \ldots = 0,$$

let the equation of the tangent plane be written

$$\lambda x + \mu y + \nu z + \varpi t = 0.$$

Using A, B, C, D, respectively, for fvw, gwu, huv, fgh, prove that the polar reciprocal of the quadric, in regard to the quadric

$$A^{-1}\xi\lambda x^2 + B^{-1}\eta\mu y^2 + C^{-1}\zeta\nu z^2 + D^{-1}\tau\varpi t^2 = 0,$$

is a quadric containing the generating lines of the original both at (ξ, η, ζ, τ) and at $(A\xi^{-1}, B\eta^{-1}, C\zeta^{-1}, D\tau^{-1})$.

Ex. 8. Prove that the quadrics $xy + hzt = 0$, $xy + kzt = 0$ are polar reciprocals of one another in regard to any quadric whose equation is of the form $ax^2 + by^2 + hkz^2 + abt^2 = 0$, and that tetrads of points of the first quadric exist which are self-polar in regard to the last, two points of the tetrad, which are conjugate in regard to the last quadric, being arbitrary. Prove, also, that the four points of such a tetrad, taken with the points to which the coordinates are referred, are eight associated points; and that the cubic curve containing the points of the tetrad and two of the reference points whose join is a generator, lies on the first quadric. Hence, also, if AC, AD be two intersecting generators of a quadric, and P, Q, R be any three points of the quadric, the cubic curves on the quadric defined, respectively, by P, Q, R, A, C and by P, Q, R, A, D, have, as their fifth intersection, a point S, such that another quadric can be constructed, touching the four planes QRS, ..., PQR, to contain the generators of the original quadric at C and at D (cf. Vol. III, p. 55).

Ex. 9. If we employ the *cross-ratio* of four lines (cf. Vol. II, p. 166), it is a fundamental property that, if A, B, C, D be four fixed points of a conic, the lines joining these to any point of the conic have an invariable cross-ratio, say $(A, B; C, D)$. It can be shewn that if two circles meet in points A, B, beside the Absolute points I, J, the circle of similitude of these circles has, for its value of $(A, B; I, J)$, the product of the values belonging to this on the two circles. From this it can be shewn at once that, if the circles defined by the triads of coplanar points (B, C, D), (C, A, D), (A, B, D), (A, B, C) be denoted, respectively, by S_1, S_2, S_3, S, then the circles of similitude of the pairs (S, S_1), (S, S_2), (S, S_3), supposed undegenerate, meet in a point. And the theorem of Ex. 5 is deducible.

It can be shewn that the two tangents drawn to one circle of a pair from any point of their circle of similitude are at the same angle as the tangents from this point to the other circle. Hence, from what is proved above, there is a point at which the four circles, drawn through the triads of four points of a plane, subtend equal angles. This result was given by Dr G. T. Bennett.

If V be the vertex of one of the two quadric cones which contain the sections of a quadric by two planes α, β, whose line of intersection is l, and the line joining V to a point, U, of the quadric, meet the quadric again in U_1, the plane lU_1 meets the quadric in a section projecting into the circle of similitude of the two circles into which the sections α, β project, the centre of projection being U.

If W be the vertex of the other quadric cone containing the sections α, β, the planes lU, lU_1 are harmonic conjugates in regard to the planes lV, lW.

If a conic meet the joins BC, CA, AB, of three points A, B, C, respectively in P, P'; Q, Q'; R, R': if QR, $Q'R'$ meet in X; RP, $R'P'$ meet in Y, and PQ, $P'Q'$ meet in Z: if H be any point of the conic, and the lines HX, HY, HZ, meet the conic again in X_1, Y_1, Z_1, respectively: then the lines AX_1, BY_1, CZ_1 meet in a point. We can deduce from this that the three circles of similitude of the three pairs of three arbitrary circles, in a plane, are coaxial.

Ex. 10. For any quadric, given in point coordinates $ax^2+\ldots=0$, let (ξ, η, ζ, τ), $(\xi', \eta', \zeta', \tau')$ be two points of the quadric, let $\phi=a\xi\xi'+\ldots$ be the polar form, linear and symmetrical in (ξ, η, ζ, τ) and $(\xi', \eta', \zeta', \tau')$, let Δ be the discriminantal determinant, and $\sigma=\phi/2\Delta$. If the tangent planes at (ξ, η, ζ, τ), $(\xi', \eta', \zeta', \tau')$ meet in a line which cuts the quadric in (x, y, z, t) and (x', y', z', t'), prove that $(xx'/A)^{-1}(\sigma-\xi\xi'/A)=(yy'/B)^{-1}(\sigma-\eta\eta'/B)=$ etc., where A, B, ... are the minors of the diagonal elements of Δ.

Ex. 11. Given a quadric, Ω, and five planes, α, β, γ, δ, ϵ, let a quadric be drawn through the four intersections of α, β, γ, δ to contain the section $\Omega\epsilon$; this quadric will have with Ω another plane section, say ϵ_1. Similarly a quadric can be drawn through the four intersections of α, β, γ, ϵ to contain the section $\Omega\delta$; this quadric will have with Ω another plane section, say δ_1. Shew that the sections δ_1, ϵ_1 lie on a quadric cone whose vertex is the intersection of the planes α, β, γ.

Projecting on to a plane, we have five given circles, (α), (β), (γ), (δ), (ϵ), and five other circles, (α_1), ..., (ϵ_1), such that (δ_1) and (ϵ_1) are inverses of one another in regard to the circle which cuts all of (α), (β), (γ) at right angles; and so on. There are ten pairs such as (δ_1), (ϵ_1), and each circle such as (ϵ_1) is the inverse of all the other four with appropriate circles of inversion.

Relation between Wallace's theorem for four circles, and Moebius's figure of two inscribed tetrads. We have given (Vol. II, p. 71) the theorem that, if four arbitrary lines be given in a plane, the four circles, each defined by the intersections of three of the lines, meet in a point. This result is ascribed to Wallace (1806; see *Camb. Phil. Proc.*, XXI, 1923, p. 348). We have also (Vol. I, p. 61) given the theorem, due to Moebius (1828; see *Ges. Werke*, I, 1885, p. 443), that it is possible for two tetrads of points in three dimensions to be so situated that every point of either tetrad lies on the plane determined by three points of the other tetrad. We give here some interesting results of which one incidental consequence is the essential identity of the figures arising in these two theorems. The order in which these results are given is chosen, however, with a further aim.

Let P_1, P_2, P_3, P_4 be any four points in a plane; through each of the six joins of these four points let an arbitrary plane be drawn, the plane through the join of P_i and P_j being denoted by α_{ij}. Every three of these planes, by their intersection, determine a point. Let the point of intersection of the planes, α_{23}, α_{31}, α_{12}, drawn through the joins of the three points P_1, P_2, P_3, be denoted by P_4'. There arise four such points, P_1', P_2', P_3', P_4'. We prove that these lie in a plane. This is, in fact, only a restatement of Moebius's theorem, as may be seen by comparison with p. 61 of Vol. I, where a diagram is given: denote the points P_1, P_2, P_3, P_4, respectively, by A, B, C and D'; let the point of intersection of the arbitrary planes drawn through BC, CA, AB be denoted by D, the point of intersection of the planes drawn through BC, BD', CD' be denoted by A', and similarly for B' and C'; consider then the line of intersection of the planes ABC, $A'B'C'$, denoting the points where this line is met, respectively, by BC, AD', CA, BD', AB, CD', by P, P', Q, Q', R, R'; then, the points B', C' are both on the plane which was drawn through AD', so that $B'C'$ meets AD', and, hence, $B'C'$ passes through P'; in the same way, $C'A'$ passes through Q', and $A'B'$ passes through R'; as P, P'; Q, Q'; R, R' are the points of intersection of a line with the pairs of opposite joins of the four points A, B, C, D', these three pairs of points are in involution; whence it follows (Pappus' theorem being assumed) that the three lines PA', QB', RC' meet in a point, say D_1; the point D, of the theorem, lies on the plane BCA', for both A' and D are on the same arbitrary plane drawn through the line BC; thus we infer that D_1 is the same as D; this, however, shews that the four points A', B', C', D are in one plane; this is what was to be proved.

The dual of the preceding theorem is as follows: From the point O, let there be drawn, in space of three dimensions, four arbitrary planes, α_1, α_2, α_3, α_4; upon the line of intersection of any two of these planes, say of α_1 and α_2, let there be taken an arbitrary point, which, for the case of these two planes, may be denoted by P_{12}. The three points so taken upon the lines of intersection of any three of the original planes determine a new plane; for example, the points P_{23}, P_{31}, P_{12} determine a plane which we may denote by α_{123}. There are four such planes; the theorem is that these meet in a point, say O'. The figure contains eight points, $O, P_{23}, \ldots, P_{34}, O'$, and eight planes, $\alpha_1, \ldots, \alpha_4, \alpha_{234}, \ldots, \alpha_{123}$; of these, four planes pass through every point, and four points lie in every plane. Moreover, the points are a system of *associated* points (Vol. III, p. 154), being the intersections of three linearly independent quadrics; for instance, the plane α_1 contains O, P_{12}, P_{13}, P_{14}, and the plane α_{234} contains P_{23}, P_{34}, P_{42}, O'; thus the plane-pair (α_1, α_{234}) is such a quadric, and two others, independent of this, are formed by the plane-pairs (α_2, α_{314}), (α_3, α_{124}). The dual theorem, that the eight planes of two Moebius tetrads touch three independent quadric envelopes, is also easily verified. (Cf. Vol. III, p. 138, Ex. 14.)

Now suppose that we have four lines in a plane, and also two points, I, J, in this plane. We may consider the four conics, all passing through I and J, each determined as also passing through the three intersections of three of the lines. The Wallace theorem is that these four conics have another common point. Take an arbitrary point, O, outside the plane, and describe a non-degenerate quadric containing O and the two lines OI, OJ. Consider the four planes joining O to the four lines of the plane; say, these are α_1, α_2, α_3, α_4; let the line of intersection of α_1 and α_2 meet the quadric, beside O, in the point P_{12}; and so on; the plane, α_{123}, containing the points P_{23}, P_{31}, P_{12}, will meet the quadric in a conic which projects from O, on to the plane, into one of the four conics of the Wallace theorem. The point, O', in which the four planes such as α_{123} intersect, will lie on this quadric, by the property of eight associated points. The Wallace theorem thus follows.

The theorem that the four planes $\alpha_{234}, \ldots, \alpha_{123}$ meet in a point, O', is capable of a suggestive analytical proof, as follows: Denote the equations of the four planes $\alpha_1, \ldots, \alpha_4$, drawn through O, respectively by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$; these will be connected by a linear identity, say, $ax_1 + bx_2 + cx_3 + dx_4 = 0$. Let $t = 0$ be the equation of any chosen plane not passing through O. The equation of any one of the four planes $\alpha_{234}, \ldots, \alpha_{123}$ may be expressed in terms of t and any three of x_1, x_2, x_3, x_4. We may then suppose the equations of these four planes to be, respectively,

$$k_{12}x_2 + k_{13}x_3 + k_{14}x_4 - at = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$k_{41}x_1 + k_{42}x_2 + k_{43}x_3 - dt = 0,$$

where x_1 does not appear in the first equation, ..., and x_4 does not appear in the last. Then the point P_{23}, for example, lying on $x_2 = 0$, $x_3 = 0$, lies on α_{234} and on α_{123}; thus it satisfies both the equations $k_{14}x_4 - at = 0$, $k_{41}x_1 - dt = 0$, as well as the identity $ax_1 + dx_4 = 0$. Thence we infer that $k_{14} + k_{41} = 0$. In general, then, in the same way, we have $k_{rs} + k_{sr} = 0$, and, with these identities, the four planes are adequately given by the equations. The condition that these four planes meet in a point is then given, if we add to the four equations the identity $ax_1 + \ldots + dx_4 = 0$, by the vanishing of a determinant of five rows and columns, in which the elements of the first row are $0, k_{12}, k_{13}, k_{14}, -a$, and so on, and the elements of the fifth row are $a, b, c, d, 0$. As this determinant is skew symmetrical, and of odd order, it vanishes identically. See, below, p. 61; and, for another proof of Wallace's theorem, see p. 64.

Ex. 1. Moebius's theorem, above identified with a figure from which Wallace's theorem follows, can be stated thus (cf. Vol. I, p. 61): If a line meet the joins of three points, A', B', C', respectively, in P', Q', R', and the joins of these to three other points, A, B, C, namely the lines $P'A$, $Q'B$, $R'C$, meet in a point, D', then the intersections of the line with the joins of A, B, C, say the points P, Q, R, when joined to A', B', C', will give three lines PA', QB', RC', which also meet in a point, D.

The dual result is that if the joins of a point, O, to three points A', B', C', meet the joins BC, CA, AB, of three points A, B, C, in three points which are in line, then the joins of O to A, B, C meet $B'C'$, $C'A'$, $A'B'$, respectively, in three points also in line. It may easily be found that O lies on a certain cubic curve, whose equation thus appears as capable of two forms.

Ex. 2. We may similarly, in three dimensions, consider the locus of a point whose joins to four given points meet four given planes in points lying on a plane. This locus is a quartic surface, considered by Bauer (*Sitzber. bayer. Ak.*, München, XVIII, 1888, p. 337). When the four given planes determine a tetrad of points in perspective with the four given points, the locus is the Hessian of a cubic surface. (Cf. Vol. III, p. 223.) In general, the reciprocity which holds for the case of Ex. 1 does not hold in three dimensions. If the four given points are in one plane, and are the poles of the four given planes in regard to a conic in this plane, then, with this conic as Absolute conic, the locus is that of a point such that the perpendiculars from it to the given planes have their feet on a plane; this locus is a cubic surface with four double points. (see below, p. 25).

Ex. 3. The reader may compare the result of Ex. 1 with the theorem given by Steiner (*Ges. Werke*, I, p. 157): If the perpendiculars drawn from three points, A', B', C', to the joins, BC, CA, AB, of three other points, meet in a point, so do the perpendiculars drawn from A, B, C to the joins $B'C'$, $C'A'$, $A'B'$, of the first three points. This will be found to be true in the more general form: Given two triads of points, A', B', C', and A, B, C, and also an arbitrary conic; if A', B', C' be in perspective with the triad formed by the poles of BC, CA, AB, taken in regard to this conic, then A, B, C are in perspective with the poles of $B'C'$, $C'A'$, $A'B'$. And generalisation is possible to two tetrads of points in three dimensions.

Additions to Wallace's theorem in a plane. Wallace's theorem has additions (Vol. II, p. 72), ascribed to Steiner (1828, cf. *Ges. Werke*, I, p. 197), to the effect that the feet of the perpendiculars, drawn from the Wallace point to the four lines, lie in line, and that the centres of the four circles which meet in the Wallace point, lie on a circle containing the Wallace point. We proceed now to obtain these results by projection from the figure in three dimensions. Let O, A, B, C be four points of a quadric; let P be any point of the section of the quadric by the plane ABC; and let I, J be the points in which the generators of the quadric at O meet the generators at P. Thus the planes OPI, OPJ are the tangent planes of the quadric drawn from the line OP, and OP is the polar line of IJ. Next, let U, V, W be the poles, lying in the tangent plane at O, respectively of the planes OBC, OCA, OAB. We can prove that the six points O, I, J, U, V, W are on a conic. For, if OI, OJ meet the section of the quadric by the plane ABC in I', J', the fact that this section contains the six points P, I', J', A, B, C involves (Vol. II, p. 29) that the planes joining O to the six lines $I'J'$, PI', PJ', BC, CA, AB touch a quadric cone; and the poles of these planes, respectively, are the points O, I, J, U, V, W. Conversely, if conics be taken, in the tangent plane at O, through the four points O, U, V, W, and one of these meet the generators at O in the points I, J, the second tangent plane of the quadric, drawn from the line IJ, touches the quadric in a point, P, of the section by the plane ABC. When we project the figure on to a plane, from the point P, the three sections OBC, OCA, OAB become circles, the projections of I, J being the Absolute points, their centres being the projections of U, V, W; and we have the result that, if three circles in a plane have a common point, their three other intersections being in line, then their centres determine a circle passing through their common point. When we project the figure from the point O, the section of the quadric by the plane OUP becomes a line, through the projection of P, which is at right angles to the line into which the conjugate section OBC projects, the Absolute points being again the projections of I and J. Let H be the point in which the sections OBC, OUP meet, other than O; also let O' be the point which, on the conic $OIJUVW$, is the harmonic of O in regard to I and J; it can be shewn that the line OH lies on the polar plane of O'. To prove this it is sufficient to shew that the polar line of OH, which lies in the tangent plane at O, and contains the pole, U, of the plane OBC, passes through O'. For this, consider the pole of the plane OUP. This pole is the intersection of the polar planes of O, U, P; that is, it lies in the tangent plane at O, in the plane OBC, and in the tangent plane at P; it is thus the intersection of the tangent line, at O, of the section OBC, with the line IJ; the tangent line, at O, of the section OBC, is however harmonic to OU in regard to OI, OJ. Thus the pole of the plane OUP is the point, of IJ, where this is met by the line, from U, which is harmonic to UO in regard to UI and UJ. This line contains O'; as it contains U, it is the polar line of OH. Now the point O' is determined by O, I, J and the conic $OIJUVW$ alone. Thus, as we have proved that the polar plane of O' contains the line (OBC, OUP), it follows that this plane also contains the lines (OCA, OVP) and (OAB, OWP); in other words the points, other than O, in which the sections OBC, OCA, OAB are respectively met by the planes OPU, OPV, OPW, lie in a plane through O. Wherefore, projecting from O, on to a plane, we have the result that if the perpendiculars be drawn to the joins of three points (the projections of A, B, C), from any point (the projection of P) on the circle containing these points, then the feet of these perpendiculars are on a line (Vol. II, p. 71). From this result, and the result obtained in regard to the centres of three circles which meet in a point, the additions to Wallace's theorem are obtained at once.

Ex. 1. If, with any two points, I, J, as Absolute points, the perpendiculars be drawn to the joins of three points, A, B, C, of a plane, from any point, P, the feet of these determine in general a circle, the pedal circle of P (Vol. II, p. 87). We have proved (Vol. II, p. 88) that, if P move on a line passing through the centre of the circle which contains A, B, C, the pedal circles have a common point (Bobillier, *Gergonne's Ann.*, XIX, 1828, p. 356; Fontené, *Nouv. Ann.*

de Math., v, 1905, p. 504; vi, 1906, p. 55). This point will then be the intersection of the pedal lines of the two points of intersection of the circle A, B, C with the line on which P moves. It can be shewn that these pedal lines are at right angles, and that their point of intersection is on the nine points circle of A, B, C. More generally, if P move on an arbitrary fixed line, there is a circle to which all the pedal circles are at right angles; this remark the writer owes to Mr J. H. Grace. It will now be shewn further that this remains true when P moves on a certain cubic curve, which in particular may degenerate into the aggregate of a line and a conic.

Let O, A, B, C be fixed points of a quadric, the poles of the planes OBC, OCA, OAB, lying in the tangent plane at O, being denoted by U, V, W. Let P be any other point of the quadric. Let the plane OPU meet the section OBC in H, beside O; similarly let OK, OL, where K, L are on the quadric, be the lines (OPV, OCA) and (OPW, OAB). When we project from O, on to a plane, and take for Absolute points, I, J, the points of the plane lying on the generators of the quadric at O, the pedal circle, in regard to the projections of A, B, C, of the point which is the projection of P, is the circle obtained as the projection of the section HKL of the quadric. To prove that the pedal circles, for certain positions of P, cut a fixed circle at right angles, we prove that the corresponding planes HKL pass through a fixed point of space; when this point is on the quadric the pedal circles meet in a point.

We can suppose the equation of the quadric, referred to O,A,B,C, to be

$$fyz + gzx + hxy + t(x + y + z) = 0,$$

where $t = 0$ is the plane ABC, and $x = 0$ the plane OBC, etc. The point U then has coordinates $(-f, w, v, fu)$, where $2u = g + h - f$, $2v = h + f - g$, $2w = f + g - h$. If the point P, on the quadric, be (ξ, η, ζ, τ), the point H, where the line (OBC, OUP) meets the quadric, beside O, is then found to be

$$\{0,\ w\xi + f\eta,\ v\xi + f\zeta,\ -(w\xi + f\eta)(v\xi + f\zeta)(\xi + \eta + \zeta)^{-1}\},$$

with similar coordinates for K and L; from these the plane HKL is found, expressed without τ, to be

$$x\xi X + y\eta Y + z\zeta Z + t(\xi + \eta + \zeta)(f\eta\zeta + g\zeta\xi + h\xi\eta) = 0,$$

where

$$X = (u\eta + g\zeta)(u\zeta + h\eta),\quad Y = (v\zeta + h\xi)(v\xi + f\zeta),$$
$$Z = (w\xi + f\eta)(w\eta + g\xi).$$

It will be shewn that when P moves on a section of the quadric by a fixed plane containing O and the pole of the plane ABC, the plane HKL passes through a definite point of the quadric; this leads to the theorem above referred to, given in Vol. ii, p. 88. More generally, regarding x, y, z, t, in the above equation of the plane HKL, as fixed, and regarding (ξ, η, ζ) as variable, these being taken as proportional to the coordinates of the projection of P, the equation will represent a locus for P such that the corresponding pedal circles cut a fixed circle at right angles; the Absolute points are given by

$$f\eta\zeta + g\zeta\xi + h\xi\eta = 0, \quad \xi + \eta + \zeta = 0.$$

It can be shewn, in particular, that if l, m, n be arbitrary, and x, y, z, t be given, with $q = vw + wu + uv$, by $ux = mn - t$, $vy = nl - t$, $wz = lm - t$ and

$$u^{-1}v^{-1}w^{-1}q^2t = q\,(mnu^{-1} + nlv^{-1} + lmw^{-1}) \\ - u\,(m-n)^2 - v\,(n-l)^2 - w\,(l-m)^2,$$

then the cubic relation connecting ξ, η, ζ is satisfied either by $l\xi + m\eta + n\zeta = 0$ or by $lf\xi^{-1} + mg\eta^{-1} + nh\zeta^{-1} = 0$. More particularly, if l, m, n be such that $lfu + mgv + nhw = 0$, the values for x, y, z, t, namely, $u^{-1}l\,(m+n)$, $v^{-1}m\,(n+l)$, $w^{-1}n\,(l+m)$, and $-(mn+nl+lm)$, correspond to a point lying on the quadric, and the equation $l\xi + m\eta + n\zeta = 0$ corresponds to a line through the centre of the circle ABC; we then have the result of Vol. II, p. 88.

Ex. 2. If four planes, $\alpha, \beta, \gamma, \delta$, be drawn through a point, O, of a quadric, the lines of intersection (β, γ), (γ, α), (α, β), (α, δ), (β, δ), (γ, δ) meeting the quadric again, respectively, in A, B, C, A', B', C', we have proved that the planes ABC, $AB'C'$, $BC'A'$, $CA'B'$ meet in a point, O', and inferred that O' is on the quadric by remarking that O, A, B, C, A', B', C', O' are a set of associated points. This follows from what is said above, p. 21; for let the poles of $\alpha, \beta, \gamma, \delta$ be, respectively, U, V, W, T; let the conic through O, U, V, W, T meet the generators at O again in I and J. Then the second tangent plane to the quadric from IJ has its point of contact on every one of the planes ABC, $AB'C'$, $BC'A'$, $CA'B'$.

Ex. 3. We may consider the generalisation to three dimensions of the pedal properties above given for a plane. A particular result, leading to a cubic surface with four double points, has been mentioned in Vol. III, p. 223. Defining the perpendicular from a point to a plane by means of an Absolute conic, ω, namely as the line, from this point, to the pole of the plane in regard to this conic, ω, the condition that the feet of the perpendiculars, drawn from a point E, to the planes containing four points A, B, C, D, should lie in a plane, is the same as that the quadric which touches these planes, and has the cone $E\omega$ as enveloping cone, should touch the plane of ω. To prove this, consider first, instead of the conic ω, a quadric, Ω, the perpendicular from E to a plane being the join of E to the pole of this plane in regard to Ω. Let the poles of the planes BCD, ..., ABC be denoted, respectively, by

A', ..., D', and let the lines EA', ..., ED' meet these planes, respectively, on a plane ϖ, whose pole is P. Let the polar plane of E be ϵ, and denote the planes BCD, ..., ABC by α, ..., δ, respectively. The line EA' then intersects the line (ϖ, α), which is the polar line, in regard to Ω, of the line PA'; in other words, the lines EA', PA' are conjugate. So, likewise, are the pairs (EB', PB'), (EC', PC'), (ED', PD'). The locus of a point whose joins to E and P are conjugate lines in regard to the quadric Ω is, however, that quadric, through the plane sections of Ω by the polar planes ϵ, ϖ, which contains both E and P, as is easy to see. This description of the quadric locus is, however, redundant, since any quadric, through the section of Ω by the polar plane ϵ, which contains also the pole E, will meet Ω in another conic and contain the pole of the plane of this. Thus the condition for E, in relation to A, B, C, D, is that the quadric, which contains A', B', C', D', and also the conic in which Ω is met by the polar plane, ϵ, of E, should also contain E. Or, the condition may be stated in dual form, recalling that the planes ϵ, ϖ meet each of α, β, γ, δ in a pair of lines which are conjugate in regard to Ω, and reciprocating in regard to Ω, by saying that the quadric which touches α, β, γ, δ, and has, for enveloping cone, that drawn from E to Ω, should touch the polar plane, ϵ, of E. When Ω consists of the planes containing the tangent lines of a conic, ω, the polar plane of E becomes the plane of ω, and the condition is as stated above; namely, the quadric touching α, β, γ, δ and the tangent planes of the cone $E\omega$, touches the plane of ω. This quadric has then, also, as generating lines, the two tangent lines of ω which pass through the point of contact of the quadric with this plane.

Ex. 4. If the perpendiculars, in regard to an Absolute conic ω, drawn from a point E to the four planes containing points A, B, C, D, meet these planes in points which lie on a plane, then each of the five points A, B, C, D, E is in the same relation to the other four. For let σ be the conic, in the plane of ω, in regard to which the joining lines and planes of the points A, B, C, D, E determine a polar system. Then, as we have seen (above, p. 11), the enveloping cone from E to any quadric touching the four planes BCD, ..., ABC, and touching also the plane of ω, meets this plane in a cone which is inpolar to σ. Thus, from Ex. 3 preceding, when the perpendiculars, in regard to ω, drawn from E to the four planes BCD, ..., ABC, meet these in points of a plane, the conic ω is inpolar to σ, and conversely. As ω is given, and σ is symmetrical in regard to A, B, C, D, E, the property in question is likewise symmetrical.

Ex. 5. The theorem of the symmetry of the points A, B, C, D, E in Ex. 4 is due to W. Mantel, *Wiskundige Apgaven*, 1899–1902, p. 396, No. 199. See H. W. Richmond, *Camb. Phil. Proc.*, Vol. XXII, Part I, 1924. In this paper Mr Richmond proves, for the rational curve of order n in space of n dimensions, in extension of the Wallace theorem for a circle in a plane, a result which, for the cubic curve in three dimensions, may be stated: The necessary and sufficient condition that a cubic curve should be such that, in regard to an Absolute conic ω, the perpendiculars from any point of the curve to the four planes containing any other four points of the curve, should have their feet on a plane, is that the conic ω should touch the three chords of the curve which lie in its plane. Let the three points of the curve which lie in this plane be P, Q, R; let σ be the conic, of the plane PQR, in regard to which the joining points and planes of five points, A, B, C, D, E, of the curve, form a self-polar system. We have shewn above (p. 12) that the only condition for σ is that P, Q, R should form a self-polar triad in regard thereto; also (Ex. 4, preceding) that the condition for A, B, C, D, E is that ω should be inpolar to σ. A conic, ω, inpolar to all conics, σ, for which P, Q, R form a self-polar triad, is given by any conic which touches PQ, QR, RP; and, conversely, only by such a conic. This proves the result.

Ex. 6. Prove that, if Ω be any quadric, the necessary and sufficient condition that the lines joining a point E to the poles, A', B', C', D', respectively, of the planes BCD, CAD, ABD, ABC, in regard to Ω, should meet these planes, respectively, in points lying on a plane, is that the lines $D'E$, $D'E_1$ should be conjugate in regard to Ω, where E_1 is the definite point such that the enveloping cones from E and E_1 to Ω are also enveloping cones of a quadric touching the four planes BCD, ..., ABC. It can be shewn (cf. Ex. 11 above) that, if D_1 be defined from D, and the four planes containing A, B, C, E, just as E_1 is here defined, then the line D_1E_1 contains D'. Hence the condition for E in regard to A, B, C, D, is that the line $D_1D'E_1$ should intersect the polar line of $D'E$. The condition that the lines joining D to the poles of the planes BCE, CAE, ABE, ABC, in regard to Ω, should meet these planes in points of a plane, is that the line $D_1D'E_1$ should intersect the polar line of $D'D$. One condition does not involve the other, in general; this will be so, however, when the points D', D, E are in line. It will also be so when Ω reduces to a conic.

Ex. 7. The results may be obtained analytically. As before, let A, B, C, D, E be any five points, and E_1 be determined from E by the condition that the quadric which touches the planes BCD, ..., ABC, and the tangent cone from E to the given quadric Ω, has E_1 as the vertex of the further common enveloping cone of the two quadrics, while D_1 is determined from D and Ω similarly, by a quadric touching the four planes joining A, B, C, E and the enveloping cone to Ω drawn from D; let coordinates be taken with respect to A, B, C and a further arbitrary point; and the tangential equation of Ω be $(A,B,C,D,F,G,H,U,V,W \between l,m,n,p)^2=0$. Then prove that the line D_1E_1 contains the pole, D', of the plane ABC, and meets the plane ABC in the point whose coordinates are

$$\frac{A\tau\tau' - U(\xi\tau' + \xi'\tau) + D\xi\xi'}{\xi\tau' - \xi'\tau}, \quad \frac{B\tau\tau' - V(\eta\tau' + \eta'\tau) + D\eta\eta'}{\eta\tau' - \eta'\tau},$$
$$\frac{C\tau\tau' - W(\zeta\tau' + \zeta'\tau) + D\zeta\zeta'}{\zeta\tau' - \zeta'\tau}, \quad 0,$$

where (ξ, η, ζ, τ), $(\xi', \eta', \zeta', \tau')$ are the coordinates of D and E, respectively. In order that the perpendiculars from E, in regard to Ω, drawn to the planes BCD, ..., ABC, should meet these in the points of a plane, we have to express that the lines $D'E$, $D'E_1$ should be conjugate in regard to Ω; that is that the polar plane of E should contain the point in which $D'E_1$ meets the plane ABC. If the point equation of Ω be $(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2=0$, this condition is that E should lie on the quartic surface whose equation, with x, y, z, t as current coordinates, is

$$\frac{A\tau t - U(\xi t + \tau x) + D\xi x}{\xi t - \tau x}(ax + hy + gz + ut) + \ldots$$
$$+ \frac{C\tau t - W(\zeta t + \tau z) + D\zeta z}{\zeta t - \tau z}(gx + \ldots + wt) = 0.$$

When coordinates are referred to A, B, C, D, so that $\xi = 0$, $\eta = 0$, $\zeta = 0$, this equation is

$$\frac{At - Ux}{x}(ax + hy + gz + ut) + \ldots + \frac{Ct - Wz}{z}(gx + \ldots + wt) = 0,$$

or

$$\frac{A(ax+hy+gz+ut)}{x} + \ldots + \frac{C(gx+\ldots+wt)}{z} + \frac{D(ux+\ldots+dt)}{t} - \Delta = 0,$$

where Δ is the discriminantal determinant of the point equation of Ω.

If Ω be a conic, lying in the plane $\lambda x + \mu y + \nu z + \varpi t = 0$, so that there are four equations such as $A\lambda + H\mu + G\nu + U\varpi = 0$, the point equation of Ω is $(\lambda x + \ldots + \varpi t)^2 = 0$. The equation then divides by $\lambda x + \ldots + \varpi t = 0$. The remaining cubic surface is given by

$$\lambda(\xi t - \tau x)^{-1}[A\tau t - U(\xi t + \tau x) + D\xi x] + \ldots + t = 0$$

or, referred to A, B, C, D, by

$$A\lambda x^{-1} + B\mu y^{-1} + C\nu z^{-1} + D\varpi t^{-1} = 0,$$

and is a cubic surface with four double points. The equation expresses that the point E_1, derived from E as in the preceding Example, is on the plane of the Absolute conic.

Ex. 8. In regard to the Absolute conic given by

$$\lambda x + \mu y + \nu z + \varpi t = 0, \quad Ax^2 + By^2 + Cz^2 + Dt^2 = 0,$$

the four points of reference for the coordinates are such that their opposite joins are at right angles; and they have an orthocentre, the intersection of the four perpendiculars from the points to the opposite planes, which is $(\lambda A^{-1}, \mu B^{-1}, \nu C^{-1}, \varpi D^{-1})$.

Ex. 9. The quadrics touching $x=0$, $y=0$, $z=0$, $t=0$ in regard to which the point (ξ, η, ζ, τ) is the pole of the plane $\lambda x + \mu y + \nu z + \varpi t = 0$, give, in this plane, ∞^2 conics. The quadrics of the form $Ax^2 + By^2 + Cz^2 + Dt^2 = 0$, which pass through (ξ, η, ζ, τ), give, in this plane, other ∞^2 conics. Prove that every conic of the latter system is outpolar to every conic of the former system.

Ex. 10. The surface $x^{-1} + y^{-1} + z^{-1} + t^{-1} = 0$ is the locus of a point from which the perpendiculars to $x=0$, $y=0$, $z=0$, $t=0$ have their feet on a plane, when a proper conic is taken as Absolute conic. Prove that, of such conics, there are ∞^2 in an arbitrary plane, $\lambda x + \mu y + \nu z + \varpi t = 0$, namely the intersections of this plane with quadrics touching $x=0$, $y=0$, $z=0$, $t=0$ which are such that the pole of this plane, in regard to the quadric, is

$$(\lambda^{-\frac{1}{2}}, \mu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \varpi^{-\frac{1}{2}}).$$

Ex. 11. It should be remarked in connexion with preceding theorems for a cubic curve, that, if three tetrads of points of the cubic be taken, say, A, B, C, D; A', B', C', D'; A'', B'', C'', D'', then the twelve planes, of which each contains three points of one tetrad, are all touched by a quadric. Further, taking any two other arbitrary points of the curve, the two tangent planes of the quadric, through the chord joining these two points, determine two further points of the curve, and hence a tetrad, of which the two other connecting planes also touch the quadric.

Let the planes $BCD, \ldots, ABC$ be denoted, respectively, by $\alpha, \beta, \gamma, \delta$, with a similar notation for the planes of the two other given tetrads, namely $\alpha', \beta', \gamma', \delta'$, and $\alpha'', \beta'', \gamma'', \delta''$. Since the quadrics touching the planes $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma'$ likewise touch δ', these quadrics, expressed tangentially, are ∞^2, and such a quadric exists touching also the planes joining D to the chords $A''B''$, $A''C''$. The enveloping cone from D to this quadric then touches the five planes $\alpha, \beta, \gamma, DA''B'', DA''C''$. But the lines $DA, DB, DC, DA'', DB'', DC''$ lie on a quadric cone (of vertex D, containing the curve). Hence the quadric cone touching the five planes $\alpha, \beta, \gamma, DA''B'', DA''C''$ also touches the plane $DB''C''$. Wherefore, the quadric constructed, touching the planes $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ and $DA''B''$, $DA''C''$, equally touches the plane $DB''C''$. As touching $\alpha', \beta', \gamma', \delta'$ and $DB''C''$, $DC''A''$, $DA''B''$, this quadric also touches the eighth plane, $A''B''C''$, or δ'', these eight planes being associated, Vol. III, p. 139. Similarly it touches the planes α'', β'', γ''.

Now, let A_3 be any further point of the curve, and let the tangent planes of the quadric from DA_3 meet the curve again in B_3 and C_3. Then, as above, the quadric also touches the plane DB_3C_3, and thence, also, the plane $A_3B_3C_3$. If, then, D_3 be taken arbitrarily, on the curve, and the tangent planes be drawn from A_3D_3 to this quadric, determining, by their further intersections, the points B_3', C_3' of the curve, it follows, as before, that this quadric also touches $A_3B_3'C_3'$, and, thence, also the plane $D_3B_3'C_3'$. This establishes the statement made, the two arbitrary points taken being A_3 and D_3.

Ex. 12. If the coordinates of a point of the cubic curve be written $(\theta^3, \theta^2, \theta, 1)$, and the points A, B, C, D be given by $a_0\theta^4 + a_1\theta^3 + \ldots + a_4 = 0$, or say $f = 0$, the points A', B', C', D' being given by $\phi = 0$, where $\phi = b_0\theta^4 + b_1\theta^3 + \text{etc.}$, and the points A'', B'', C'', D'' by $\psi = 0$, where $\psi = c_0\theta^4 + c_1\theta^3 + \text{etc.} = 0$, shew that the planes of every tetrad of points of the cubic, given by an equation of the form $\xi f + \eta\phi + \zeta\psi = 0$, touch the quadric whose tangential equation is

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4 \\ b_0, & b_1, & b_2, & b_3, & b_4 \\ c_0, & c_1, & c_2, & c_3, & c_4 \\ l, & m, & n, & p, & 0 \\ 0, & l, & m, & n, & p \end{vmatrix} = 0.$$

Theorems for the circumscribed circles when an indefinite number of lines is given in a plane. Recurring now to the theorem in which four arbitrary points were taken in a plane, and arbitrary planes were drawn through the joins of each two, we were able, with the use of Pappus' theorem, to shew that the four points, obtained as intersections of the planes through the joins of each triad of points, lie in a plane (above, p. 18). If we start with five arbitrary points, it can be shewn that the five planes, each obtained as above from four of these, all meet in a point; and for the proof of this the Propositions of Incidence only are sufficient. If we begin with six points, the six points so obtainable, one from each five of these, lie in a plane. And so on, indefinitely.

In a plane, α, let five points P_1, P_2, ..., P_5 be arbitrarily taken; and through the join of every two let an arbitrary plane be drawn, for instance the plane α_{12} through the join P_1P_2. The planes $\alpha_{23}, \alpha_{31}, \alpha_{12}$ determine a point, P_{123}, and so on. The four points P_{123}, P_{234}, P_{314}, P_{124} lie in a plane; this is what was proved above (p. 18). When we start with five points there arise five such planes. Denoting these by α_{1234}, ..., α_{2345}, we now prove that these five planes meet in a point.

In the plane α_{12} are five points, namely P_1, P_2, P_{123}, P_{124}, P_{125};

and the process followed gives another plane than α_{12} passing through the line joining any two of these points. Of these five points consider, at first, only the four P_1, P_{123}, P_{124}, P_{125}, and apply to these the process which was followed when we started with four points, P_1, P_2, P_3, P_4, lying in a plane α. Through the lines joining the respective six pairs, of the four points,

$$(P_{124}, P_{125}),\ (P_{125}, P_{123}),\ (P_{123}, P_{124}),\ (P_1, P_{123}),\ (P_1, P_{124}),\ (P_1, P_{125})$$

there pass, beside the plane α_{12}, the planes

$$\alpha_{1245},\quad \alpha_{1235},\quad \alpha_{1234},\quad \alpha_{13},\quad \alpha_{14},\quad \alpha_{15};$$

of these, however, the planes α_{1245}, α_{14}, α_{15} meet in P_{145}; the planes α_{1235}, α_{13}, α_{15} meet in P_{135}; and α_{1234}, α_{13}, α_{14} meet in P_{134}. Thus, the plane α_{1345}, which is the plane of the points P_{145}, P_{135}, P_{134}, contains the point of intersection of the planes α_{1245}, α_{1235}, α_{1234}. If we next consider the four points P_2, P_{123}, P_{124}, P_{125}, we similarly prove that the same point of intersection lies on the plane α_{2345}. That is, the five planes, $\alpha_{p,q,r,s}$, where p, q, r, s are every four of the numbers 1, 2, 3, 4, 5, meet in a point. This is what we desired to prove. The final point may be denoted by P_{12345}.

If we now begin with six points, $P_1, \ldots, P_6$, of the original plane, every five of these, by the same process, will give rise to a point. It can be shewn that the six points so obtained lie in a plane. The process gives rise, with others, to the four points P_{123}, P_{124}, P_{125}, P_{126}, all lying in the plane α_{12}. Beside this plane there pass, through the six joins of pairs of these points, namely

$$(P_{124}, P_{125}),\ (P_{125}, P_{123}),\ (P_{123}, P_{124}),\ (P_{123}, P_{126}),\ (P_{124}, P_{126}),\ (P_{125}, P_{126}),$$

respectively the planes

$$\alpha_{1245},\quad \alpha_{1235},\quad \alpha_{1234},\quad \alpha_{1236},\quad \alpha_{1246},\quad \alpha_{1256},$$

which, for brevity, we may denote, momentarily, by (45), (53), (34), (36), (46), (56), respectively. By what we have shewn, the points of intersection of threes of these planes, which are

$$(45, 53, 34),\quad (45, 46, 56),\quad (53, 56, 36),\quad (34, 36, 46),$$

lie in one plane. These points are, however, respectively, P_{12345}, P_{12456}, P_{12356}, P_{12346}. By a similar argument it follows that every four of the six points in question, whose symbols have two digits in common, are in one plane. From this, the fact that all six points are in one plane is clear.

If we start with seven points, $P_1, \ldots, P_7$, in the original plane, there will be seven planes, of which one arises, as in the case considered, from every six of these points. It can be shewn that these seven planes meet in a point. The proof may begin by considering the five coplanar points P_{12r}, where $r = 3, 4, \ldots, 7$. And so on.

We may consider the dual of these theorems, obtained by taking planes all passing through a point, and an arbitrary point on the line of intersection of any two of these planes. The case when four planes are drawn through the point was considered above. In particular, the points which are taken on the lines of intersection of the pairs of planes, may be the further intersections of these lines with a quadric drawn through the original point. When the number of planes originally taken is even, the theorem leads finally to a point; it can be shewn, as in the case of four planes, that this point is also on the quadric. When we begin by considering five planes through the original point, we obtain, by projection on to a plane, the part of the theorem proved above (p. 1) which is expressed by saying that the foci of the five parabolas, which touch five lines in fours, lie on a circle. When we begin by considering six planes through the original point, the result is that the six circles, obtained from every five parabolas, have a point in common. And so on. These theorems were given by Clifford (1870), *Math. Papers* (1882), p. 38, whose method of proof (in dual statement) is by a curve of order n with a fixed $(n-1)$-fold point.

Ex. 1. Another representation of the preceding results may be referred to. On a quadric surface, any generator of one system may be associated with a value of a parameter, say θ, and any point of the surface with the values of a pair of parameters, say θ and ϕ. Now take two fixed lines of space, say x and y, both meeting a line h; and associate the points of the line x with the values of θ, the point (h, x) corresponding to $\theta=0$, say, and two other assigned points of x corresponding to assigned values of θ; similarly, associate the points of the line y with the values of ϕ. Next, take a further line p, meeting h but not x nor y. Then any transversal of the three lines x, y, p, as meeting x and y in points, say θ and ϕ, corresponds to a definite point (θ, ϕ) of the original quadric; and the aggregate of all such transversals, which meet x and y in related ranges, corresponds to the points of a plane section of the original quadric; upon this plane section there is a point, say H, independent of the line p, which corresponds to the transversal h. Conversely, any plane section of the quadric passing through the point H, corresponds to such an aggregate of transversals of the fixed lines x, y and another line p, meeting h but not x nor y, wherein however the line p is not unique. Two such plane sections, α_1 and α_2, have, beside H, another common point, say P_{12}; this corresponds to the fact that two such reguli of lines meeting x and y, both containing h, have in common another line. If three such sections of the quadric be taken through H, and the three further intersections of pairs of these be P_{23}, P_{31}, P_{12}, the points of the plane section of the quadric, which contains these points P_{23}, P_{31}, P_{12}, have parameters

θ, ϕ connected by an equation, linear in both these; the points of this plane section thus correspond to a regulus of lines meeting x and y, in points forming two related ranges; this regulus does not contain h, but contains three lines corresponding, respectively, to P_{23}, P_{31}, P_{12}. The correspondence may be pursued when more than three sections of the quadric are taken through H. In particular we have the following result: Let h be a given transversal of two given non-intersecting lines, x and y. Consider five reguli of lines, each consisting of lines meeting x and y in related ranges, each regulus having h as one line. Two of these reguli have common a line beside h, say q_{ij}; the lines q_{12}, q_{23}, q_{31}, common to the pairs of three of the reguli, determine a further regulus of lines meeting x and y, say q_{123}, not containing h; the four reguli, q_{234}, q_{314}, q_{124}, q_{123}, so arising from four of the original reguli, have then a line in common, say q_{1234}; and the five lines, $q_{1234}, \ldots, q_{2345}$, all belong to a regulus of lines meeting x and y.

From this result, taking a section by an arbitrary plane, we obtain the figure considered above (p. 31), in which we have considered the circles through the intersections of the triads selected from five arbitrary lines of a plane; there is the slight generalisation, however, unless the plane of section be taken to contain the line h, that here, instead of five lines in a plane, we obtain five circles with a common point.

Ex. 2. Let S be the conic drawn to touch five given lines of a plane; let $S_1, \ldots, S_5$ be conics each touching four of these lines; suppose the six conics are all triangularly inscribed to another conic. Prove that $S_1, \ldots, S_5$ have a common tangent line (Wakeford, *Proc. Lond. Math. Soc.*, xv, 1916, p. 340, who uses the result to establish the theorem of a double six of lines).

SECTION II. THEOREMS OF THREE AND FOUR DIMENSIONS

Tetrahedral complex as determined by planes meeting three lines in four dimensions. We have considered in Vol. III (p. 99) the system of lines which are normals of the quadrics of a confocal system. These constitute a tetrahedral complex; this is an aggregate (∞^3) of lines, in space of three dimensions, determined by the single condition that the range of points, on every one of the lines of the complex, in which the line meets four given fixed planes, is related to a given range of four points. Or, what is the same thing (Vol. I, p. 30), by the condition that the planes, joining every one of the lines of the complex to the four points of intersection of the four given planes, form an axial pencil of four planes which is related to the same given range of four points.

We assume the elementary properties of space of four dimensions (cf. Vol. I, pp. 33, 37, etc.); that a general line meets a threefold space in a point; that a general plane meets a threefold space in a line; that two general planes have a point in common; and two general threefold spaces have a plane in common; that two non-intersecting lines determine a threefold space; that three non-intersecting lines have a common transversal line, which meets each of the three lines in the point in which this meets the threefold space determined by the other two. We also assume that through a plane, ϖ, there pass ∞^1 threefold spaces which meet any two lines in related ranges of points. When the two lines lie in a plane, say α, if P, P' be the points in which these lines meet one of the threefolds through ϖ, the line PP' meets ϖ, and lies in α, and so passes through the point common to ϖ and α; the ranges determined on the two lines by the threefolds are thus in perspective. From this it is easy to deduce the general statement.

Now suppose that, in the fourfold space, we have three non-intersecting lines, and also a (flat) threefold space, Ω, not containing any of the lines, nor their transversal. As the join of three arbitrary points, one on each of the lines, there is determined a plane, and there are ∞^3 such planes. We prove that the lines, in which the threefold Ω is met by these planes, constitute a tetrahedral complex in Ω. And, conversely, we shew that any tetrahedral complex in Ω can be determined in such a way.

Let the lines be a, b, c; let the points in which they meet Ω be A, B, C, these being supposed not to lie in line. Then the common transversal of a, b, c does not lie in Ω; let this transversal meet Ω in D. Also let this transversal meet a, b, c in A', B', C'. Consider a plane, ϖ, not lying in Ω, meeting a, b, c, in P, Q, R, respectively. The threefold space, ϖA, containing ϖ and A, as containing P, contains the line a, and hence also the point A'. Thus the four threefolds ϖA, ϖB, ϖC, ϖD meet the transversal line of a, b, c, respectively, in the points A', B', C', D; these points are independent of the plane ϖ. Next, let l be the line in which the plane ϖ meets Ω; the threefold space ϖA then meets Ω in the plane lA; and any line of Ω meets the plane lA in the point in which this line meets the threefold ϖA. Thus the planes lA, lB, lC, lD are met by any line of Ω in the range in which this line meets the threefold spaces ϖA, ϖB, ϖC, ϖD; and this range is therefore related to the fixed range A', B', C', D, by what we have seen. Wherefore, the lines l, in which Ω is met by the planes ϖ, constitute a tetrahedral complex, relative to A, B, C, D.

Conversely, given in Ω any tetrahedral complex relative to four points, A, B, C, D, of Ω, we can draw, in a fourfold space containing Ω, an arbitrary line through D, not lying in Ω; we can then take on this line three points, A', B', C', in such a way that the range A', B', C', D is related to the range determined, on any line of Ω, by the planes joining A, B, C, D to a line of the complex. Then the tetrahedral complex is that constituted by the lines of Ω which lie in planes meeting the three lines AA', BB', CC', of the fourfold space.

Ex. Let the fourfold space be referred to the five points A, B, C, D and C', and, in terms of the symbols of D and C', let the symbols of A' and B' be, respectively, given by $A'=D+C'$, $B'=D+\lambda C'$. A plane, ϖ, meeting the lines AA', BB', CC', contains three points with symbols, respectively,

$$A+p(D+C'),\quad B+q(D+\lambda C'),\quad C+rC',$$

so that a general point of this plane has a symbol

$$\xi[A+p(D+C')]+\eta[B+q(D+\lambda C')]+\zeta(C+rC');$$

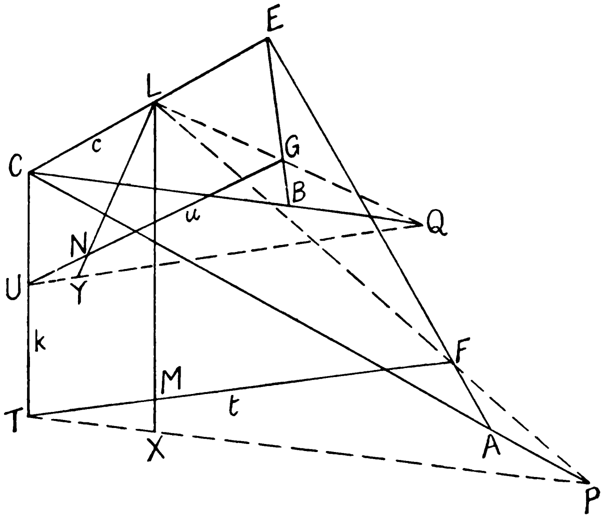
in particular, the points of this plane which lie in the space (A, B, C, D) are such that $\xi p+\eta q\lambda+\zeta r=0$; such points then have symbols $r\xi[A+pD]+r\eta[B+qD]-(\xi p+\eta q\lambda)C$, and are those of the line joining the two points

$$r(A+pD)-pC,\quad r(B+qD)-q\lambda C.$$

The coordinates of these points, relative to A, B, C, D, are $(r, 0, -p, rp)$ and $(0, r, -q\lambda, rq)$; and the line joining them has the coordinates $(-rq, rp, qp-qp\lambda, p, q\lambda, r)$; denoting these by $(l, \ldots, n')$, they satisfy $\lambda ll'+mm'=0$. This is the equation for the tetrahedral complex. It may be readily verified that the planes joining any line of it to A, B, C, D meet an arbitrary line in a range related to A', B', C', D.

Analogous generation of a linear complex. In the preceding section it was assumed that the common transversal of the three lines a, b, c was not in the threefold space Ω. We now shew that, if this be so, the planes meeting a, b, c meet Ω in the lines of a linear complex. The definition of a linear complex has been given in Vol. III, pp. 56 ff., and it has been shewn that this can be stated without the use of the algebraic symbols (Vol. III, p. 64): suppose we have, in the space of three dimensions, two planes, intersecting in a line TU; and, in one of the planes, a flat pencil of lines of vertex T, and in the other plane a flat pencil of lines of vertex U; these pencils being related to one another, and the ray TU, of the one, corresponding to the ray UT of the other. Then the aggregate of the lines joining a point, of one ray of one pencil, to any point of the corresponding ray of the other pencil, is a linear complex; and, conversely, any linear complex can be so determined.

Let c, t, u be three non-intersecting lines of a fourfold space; let k be the common transversal line of these, meeting them, respectively, in C, T, U. Let Ω be a threefold space containing the line k, but not the lines c, t, u. Take fixed points, E, F, G, arbitrarily upon c, t, u, and let EF, which does not lie in Ω, because c and t do not, meet Ω in A; similarly let EG meet Ω in B. Thus AB is a particular fixed line of Ω lying in a plane which meets the lines c, t, u, and we may regard Ω as determined by the line k and the points A, B. If now L be any point of the line c, the line LF, lying in the plane $CEFA$, will meet the fixed line CA, say in P; and the line LG, lying in the plane $CEGB$, will meet the fixed line CB, say in Q; and as L varies on the line c, the ranges (P), (Q), on CA and CB, will be related. In particular, when L is at C, both P and Q are also at C. Thus, the pencil $T(P)$, in the plane kA, is



related to the pencil $U(Q)$, in the plane kB; and these pencils have the ray TU in common. If now M be any point of the line t, the line LM, which lies in the plane TLF, meets the line TP, say in X; and as T, C, A, P are in Ω, so is X; again if N be any point of the line u, the line LN, which lies in the plane ULG, meets the line UQ, say in Y; and as U, C, B, Q are in Ω, so is Y. Thus XY is a line of Ω, in which this is met by a plane LMN which meets the given lines c, t, u; and X, Y are, respectively, upon the corresponding rays TP, UQ, of the two related pencils spoken of. Thus, as L, M, N vary, the line XY describes a linear complex in Ω, of which AB and PQ are particular lines.

Ex. Let the fourfold space be referred to the five points A, B, T, U, E, the space Ω being referred to the first four of these. We may then, without loss of generality, suppose the symbols of C, F, G, in terms of those of the fundamental points, to be $C = T + U$, $F = A + E$, $G = B + E$. If then the symbol of L be written $C + xE$, or $T+U+xE$, the point P, lying on LF and CA, has the symbol $P=T+U-xA$, and the point Q, lying on LG and CB, has the symbol $Q=T+U-xB$. Whence, if the symbols of M and N be written, respectively, $M=A+E+yT$, $N=B+E+zU$, the point X, lying on LM and TP, has the symbol $X=T(1-xy)+U-xA$; and the point Y, lying on LN and UQ, has the symbol

$$Y=U(1-xz)+T-xB.$$

That is, the coordinates of X and Y relatively to A, B, T, U are, respectively, $(-x, 0, 1-xy, 1)$ and $(0, -x, 1, 1-xz)$. From these we at once find that the line XY belongs to the linear complex expressed by $m+m'=0$.

The determination of the tetrahedral complex, and of the linear complex, by planes meeting three lines in space of four dimensions, are given by Segre, "Alcune considerazioni elementari," etc.,' *Rend. Circ. Mat. Palermo,* II, 1888, p. 45. Another way of regarding these complexes arises below, in Section III of this Chapter (p. 40).

Spheres as determined from sections of a quadric in fourfold space. We have seen that circles in a plane may be regarded as projections of plane sections of a quadric in threefold space, the centre of projection being on the quadric. The Absolute points of the plane, through which the circles pass, are on the generators of the quadric at the centre of projection.

Spheres in a threefold space, Π, may similarly be obtained by projection of sections of a quadric in fourfold space. Such a quadric, Ω, may be defined here as the locus of points (∞^3 in multiplicity) whose coordinates, in the fourfold space, satisfy a general homogeneous quadratic equation. Let $x=0$, $y=0$, $z=0$, $t=0$ represent four threefolds which intersect in a point, O, of the quadric Ω, and $v=0$ be another threefold. The equation of Ω will then have a form

$$ax^2+by^2+cz^2+dt^2+2fyz+2gzx+2hxy+2pxt+2qyt$$
$$+2rzt+2(Ax+By+Cz+Dt)v=0.$$

Any line drawn through O will meet Ω in one point at O. Lines can be drawn through O which meet the quadric in two coincident points at O; these, it is easily proved, lie in and generate a threefold space; for the given equation of Ω this is given by

$$Ax+By+Cz+Dt=0;$$

it is called the *tangent threefold* of Ω at O. If P be any point, other than O common to Ω and its tangent threefold at O, the line OP must lie entirely on Ω, because it meets Ω in two coincident points at O and in a further point P, while a line not lying on Ω meets this only in two points. Thus the common points of Ω and its tangent threefold at O are the points of an aggregate of lines through O. Now, an arbitrary threefold meets Ω in a quadric surface, lying in this threefold; an arbitrary plane, which may be regarded as the intersection of two threefolds, meets Ω in a conic; and, if any line, in the tangent threefold at O, but not passing through O, meet the quadric Ω in P and Q, the conic, in which the plane OPQ meets Ω, consists of the two lines OP, OQ. Thus, the intersection of Ω with the tangent threefold at O consists of a quadric cone, of vertex O, lying in this threefold, the ∞^1 generators of this cone being the intersections of Ω with the tangent threefold, constituting all the lines of Ω which pass through O. With the equation taken for Ω, this cone is the intersection of the threefold given by $Ax + By + Cz + Dt = 0$, with the locus represented by

$$(a, b, c, d, f, g, h, p, q, r \between x, y, z, t)^2 = 0.$$

This last consists of an ∞^2 of lines, joining O to the points of a quadric surface lying in the threefold $v = 0$.

If we project the points of Ω from O, on to an arbitrary threefold, Π, the quadric cone of ∞^1 lines of Ω, passing through O, will evidently give rise to a conic, σ, in Π, lying in the plane in which Π is met by the tangent threefold of Ω at O. Now consider an arbitrary threefold, Σ, not passing through O; this meets Ω in a quadric surface; and it is intersected by every line of Ω which passes through O. The section of Ω by Σ will thus project, from O, into a quadric surface in the threefold Π, with the property that it passes through the conic σ. Or, if we regard σ as the Absolute conic of the threefold space Π, the sections of Ω, by all threefold spaces not passing through O, project into spheres in Π.

The tangent plane of such a sphere, at any point, is the projection of the tangent plane of the quadric section of Ω at the corresponding point, say P, of Ω; this tangent plane at P is the intersection of the tangent threefold of Ω, at P, with the threefold, Σ, whose section of Ω is under consideration. But, in particular, if we consider the tangent plane of the sphere, in the space Π, at a point of the conic σ, the corresponding point, P, of Ω, lies on the tangent threefold of Ω at O, and the tangent threefold of Ω at P then passes through O. This tangent threefold of Ω at P contains also the pole point, say S, of the threefold Σ, in regard to Ω, and thus contains the line OS. The point in which OS meets the threefold Σ thus lies on the tangent plane at P of the quadric section (Ω, Σ). Wherefore, the tangent planes of the sphere, in the space Π, at all points of the conic σ, pass through the point in which the line OS, from O to the pole of the cutting threefold Σ, meets the space Π. In other words, the centre of the sphere is the projection of the pole of the cutting threefold Σ, in regard to Ω. In this description, we have assumed that the notion of a polar point, and polar threefold, in regard to the quadric Ω, is clear, from the analogy of the cases in two and three dimensions, without detailed explanation; the equation of the polar threefold of a point (x', y', z', t', v') is formed from the equation of Ω by the operator $x'\partial/\partial x + \ldots + v'\partial/\partial v$; and if the polar threefold of a point P contain a point Q, the polar threefold of Q contains P.

Consider two points of Ω, say O and Q; the tangent threefolds of Ω at these points will meet in a plane, and this plane will meet Ω in a conic. Every line, lying on Ω, which passes through O, will therefore meet this conic, and will intersect a particular line, lying on Ω, of those passing through Q, at a point of this conic. Through this point, which is on Ω, there pass ∞^1 lines, lying on Ω, to points such as Q. Thus every line of Ω meets ∞^2 others, and there are ∞^3 lines lying upon Ω. When the points O and Q are such that each lies on the tangent threefold of Ω at the other, the plane common to these two tangent threefolds passes through both O and Q, and contains the line OQ. This line, meeting Ω in two coincident points both at O and Q, lies entirely on Ω. It can be shewn that, in this case, the plane common to the two tangent threefolds meets Ω only in this line, OQ, taken twice over, and touches the cones, of lines of Ω, drawn from O and Q, along this line. Further, a tangent plane of the cone of lines of Ω, through O, is characterised by the fact that every threefold which contains this plane is a tangent threefold of Ω.

Ex. 1. Given the six equations

$$x^2+y^2+z^2+t^2+v^2=0, \qquad x_0^2+\ldots+v_0^2=0, \qquad x_1^2+\ldots+v_1^2=0,$$
$$x_0x_1+y_0y_1+\ldots+v_0v_1=0, \qquad xx_0+\ldots+vv_0=0, \qquad xx_1+\ldots+vv_1=0,$$

deduce that

$$[x(y_1z_0-y_0z_1)+y(z_1x_0-z_0x_1)+z(x_1y_0-x_0y_1)]^2=0.$$

Ex. 2. If the equation of Ω be written, as above,

$$\phi(x, y, z, t)+2v(Ax+By+Cz+Dt)=0,$$

the cone of lines of Ω from $(0, 0, 0, 0, 1)$ is given by $\phi(x, y, z, t)=0$, $Ax+\ldots+Dt=0$. If (x', y', z', t') be such that $Ax'+\ldots+Dt'=0$, and ϕ' denote $\phi(x', y', z', t')$, this cone is touched by the plane given by $Ax+\ldots+Dt=0$ and $x\partial\phi'/\partial x'+\ldots+t\partial\phi'/\partial t'=0$.

Ex. 3. It follows from what is said that a sphere may be represented by the threefold whose section with Ω projects into the sphere. The sphere may, therefore, equally be represented by the point which is the pole of this threefold in regard to Ω. A circle in the space Π, through which an infinite aggregate of spheres can be drawn, may thus be represented, in the fourfold space, by a plane, through which an infinite aggregate of threefolds can be drawn; or the circle can equally be represented by the line of the fourfold space which is the pole, in regard to Ω, of the plane; this being the locus of the poles, in regard to Ω, of the threefolds drawn through the plane, as we easily verify. A consequence of this representation will be that two lines of the fourfold space which have a point in common, being the polars in regard to Ω of two planes which lie in the same threefold space, will represent two circles of the threefold space which lie on the same sphere, that is, two circles which have two points in common. In particular, from the theorem that three lines in fourfold space have a common transversal, if no two of the lines meet, we infer that three circles in threefold space, of which no two lie on a sphere, are all met in two points by a properly chosen circle. (Darboux, *Compt. Rend.*, XCII, 1881, p. 447.)

Ex. 4. As an exercise in regard to the derivation of a sphere from a threefold section of a quadric in four dimensions, we may prove that two spheres cut everywhere at the same angle. By this we understand that the four planes, passing through the tangent line of the circle of intersection of the two spheres, at any point, P, of this circle, which consist of the two tangent planes of the spheres at P, together with the two tangent planes of the Absolute conic drawn from this tangent line, form an axial pencil which is related to that similarly arising for any other point of this circle of intersection. We shall assume that if four threefolds, in space of four dimensions, have a common plane, and an arbitrary threefold be taken, to meet this plane in a line, and each of the four threefolds in a plane, then the axial pencil of four planes so obtained is related to that obtained similarly by another arbitrary threefold; in fact, a line can be drawn (in many ways) meeting the four threefolds in points lying on the four planes of the axial pencil.

Consider two quadric sections of Ω, by two threefolds, Σ and Σ'; let ϖ be the plane common to these threefolds, and P a point of the conic in which this plane meets Ω; let T be the tangent threefold of Ω at P, and l the line in which T meets the plane ϖ. The range of threefolds which can be drawn through ϖ meets T in a range, or axial pencil, of planes, all passing through l; of these threefolds Σ and Σ' are two, and these meet T in the tangent planes at P of the quadric sections of Ω by Σ and Σ'. There are also, however, through ϖ, as through any other plane, two threefolds which touch Ω. By what is said above, these meet the tangent threefold T in planes which touch the cone of lines, from P, lying on Ω. Thus these planes touch the conic in which the cone of lines of Ω from P is met by the cone of lines of Ω drawn from any other point, O, of Ω. These planes thus project from O, on to any threefold Π, into planes touching the Absolute conic of the space Π. These planes are, however, by definition, the intersections of the tangent threefold T, at P, with the two tangent threefolds drawn to Ω from the plane ϖ, and these two threefolds are independent of the position of P upon the conic in the plane ϖ.

Thus it appears that the two spheres obtained by projection from O of the two sections $\Omega\Sigma$, $\Omega\Sigma'$, have the same angle of intersection at all their common points. This angle is a right angle when Σ, Σ' are conjugate in regard to Ω.

As many detailed applications of the geometry of space of four dimensions occur in subsequent pages of this Volume, we content ourselves here with these indications.

SECTION III. USE OF SPACE OF FIVE DIMENSIONS

The representation of the lines of space of three dimensions by points in space of five dimensions. A line, in space of three dimensions, depends on *four* parameters; it may be given, for example, by the two points in which it meets two specified planes, though this would fail if the line were in one of the two planes. It appears that the only representation which does not fail for any line is that given by Cayley* (see above, Vol. III, p. 56), wherein a line is represented by the ratios of six coordinates, l, m, n, l', m', n', connected by an equation $ll'+mm'+nn'=0$. If, then, these six coordinates are taken to be the coordinates of a point in space of five dimensions, the lines of the original threefold space are represented by the points of a quadric fourfold lying in the fivefold space. In the representation, considered above, in Section I, of the points of a plane by the points of a quadric surface in three dimensions, there are two points of the plane to which there correspond all the points of two lines of the quadric surface; in the representation (Vol. III, p. 189) of the points of a plane upon a cubic surface, there are, similarly, six points of the plane each corresponding to a line on the cubic surface. It is remarkable that no such singular elements occur in the correspondence between the lines of space of three dimensions, and the points of the fourfold quadric in space of five dimensions; to every element in either figure corresponds a definite element in the other. Moreover, the quadric fourfold is quite general; for, as in preceding cases (Vol. III, p. 15), any general quadratic relation in six homogeneously entering variables is reducible to the form $u^2+v^2+w^2-x^2-y^2-z^2=0$, and, by putting $l=u+x$, $l'=u-x$, $m=v+y$, $m'=v-y$, $n=w+z$, $n'=w-z$, this is of the form $ll'+mm'+nn'=0$.

Through any point of such a quadric fourfold, Ω, there pass ∞^3 lines which meet Ω in two points coinciding at this point; these lines, therefore, have no other point of intersection with Ω, unless they lie entirely thereon. These lines generate a tangent fourfold of Ω at this point. If $\lambda, \ldots, \lambda', \ldots$ be current coordinates, the equation of this tangent fourfold is found by acting upon the equation of Ω with an operator $\lambda\partial/\partial l + \ldots + \lambda'\partial/\partial l' + \ldots$. There are, however, ∞^2 lines, through any point of Ω, which lie entirely on Ω. The points of these lines constitute the aggregate of the points common to Ω and the tangent fourfold at the point considered. These lines meet any (flat) fourfold space in the points of a quadric surface, which lies in the threefold space in which this fourfold is met by the tangent fourfold at the point considered. If one point of Ω lie in the tangent fourfold at another point of Ω, this latter lies in the tangent fourfold at the former; two such points of Ω may be spoken of as *conjugate* to one another; they are such that the line joining them lies entirely on Ω. More generally, to any point of the fivefold space there corresponds a fourfold, given by operating on the equation of Ω by $\lambda\partial/\partial l + \ldots + \lambda'\partial/\partial l' + \ldots$, where $(l, \ldots, l', \ldots)$ are the coordinates of the point, and $(\lambda, \ldots, \lambda', \ldots)$ are current coordinates. This is called the *polar* fourfold of the point. Thus to any line of the fivefold space there corresponds a *polar threefold*, which is common to the polar fourfolds of all points of the line.

---

* Cf., also, Klein, *Ges. Math. Abh.*, I, pp. 107 ff.

When the equation of Ω is $ll' + mm' + nn' = 0$, the condition that two points of Ω, $(l, m, \ldots)$, $(\lambda, \mu, \ldots)$, should be conjugate is $\lambda l' + \lambda' l + \ldots = 0$. This is the condition that the two corresponding lines of the original threefold space should intersect one another. The coordinates of any point of the line joining the two conjugate points are of the forms $(\sigma l + \lambda, \sigma m + \mu, \ldots)$, for a proper value of σ; such points evidently represent the lines of the original threefold space which lie in the plane of the two intersecting lines, and pass through their point of intersection; they describe a line lying on Ω. The aggregate of all the lines of the threefold space, which meet a given line $(l, m, \ldots)$, is represented by the points of Ω lying on the tangent fourfold at the point $(l, m, \ldots)$.

Consider the lines of a linear complex in the original space, satisfying an equation $al' + \ldots + a'l + \ldots = 0$ (cf. Vol. III, p. 61). These are represented by the points of Ω lying on the fourfold represented by this linear equation; these points constitute a threefold quadric in this fourfold space. The lines, of a linear congruence, common to two linear complexes, are then represented by the points of Ω lying on a threefold given by two such equations, say $al' + \ldots + a'l + \ldots = 0$ and $Al' + \ldots + A'l + \ldots = 0$; these points constitute a quadric surface lying in this threefold space. This space is the intersection of the polar fourfolds of two points $(a, b, \ldots)$, $(A, B, \ldots)$; it is equally given by the intersection of the tangent fourfolds of Ω at the two points where Ω is met by the line joining these two points. Thus, the lines of a linear congruence, in the original space, consist of the lines which meet two properly chosen lines of this space. Incidentally we thus again reach the correspondence, above referred to (p. 31), between the points of a quadric surface, and the lines meeting two arbitrary skew lines. In particular, the lines of a linear complex, $(a, b, \ldots)$, of the original space, which meet an arbitrary line, $(l, m, \ldots)$, where $ll' + mm' + nn' = 0$, are represented by the points of Ω lying on the threefold intersection of the tangent fourfold of Ω, at the point $(l, m, \ldots)$, with the polar fourfold of the point $(a, b, \ldots)$; this threefold lies equally on the tangent fourfold of Ω at the point, other than $(l, m, \ldots)$, in which Ω is met by the join of the two points $(l, m, \ldots)$, $(a, b, \ldots)$. Thus, all the lines of a linear complex, $(a, b, \ldots)$, which meet an arbitrary line $(l, m, \ldots)$, equally meet another line, say $(\lambda, \mu, \ldots)$. This other line is that known as the *polar line* of the first in regard to the focal system associated with the linear complex (Vol. III, pp. 61, 64); its coordinates are of the forms $l - \rho a$, $m - \rho b, \ldots$, where ρ is such that these satisfy the equation of Ω, namely such that $(l - \rho a)(l' - \rho a') + \ldots = 0$, so that $\rho = (al' + a'l + \ldots)/(aa' + bb' + cc')$. In the space of five dimensions, the condition for the two points of Ω, that correspond to a pair of polar lines in regard to a linear complex of the original space, is that the line joining these points should pass through the pole, in regard to Ω, of the fourfold which defines the linear complex; or, that the tangent fourfolds of Ω at these two points should meet on this fourfold.

Ex. 1. Consider two linear complexes $(a, b, \ldots)$, $(A, B, \ldots)$, for which the linear invariant $aA' + a'A + \ldots$ is zero. In the space of five dimensions, they correspond to two fourfolds of which each contains the pole of the other, in regard to Ω; so that, in place of being spoken of as *apolar*, they may be spoken of as *conjugate*. A line from the pole of one of these fourfolds, to a point of Ω which lies on the other fourfold, evidently meets Ω again in a point also on this other fourfold. Thus, in the original space, the polar line, in regard to one of the complexes, of a line belonging to the other, is equally a line of the other (cf. Vol. III, p. 65).

Ex. 2. Prove that every two of the six linear complexes expressed by $l + l' = 0$, $l - l' = 0$, $m + m' = 0$, $m - m' = 0$, $n + n' = 0$, $n - n' = 0$ are conjugate. Also, that the poles of an arbitrary plane, in the original threefold space, in regard to the six focal systems determined by these, lie on a conic; and, that the six polar planes of an arbitrary point, in these focal systems, touch a quadric cone.

Ex. 3. If, instead of the coordinates $l, m, \ldots$, we use coordinates, x, y, z, u, v, w, such that $l = x + iu$, $l' = x - iu$, $m = y + iv$, $m' = y - iv$, $n = z + iw$, $n' = z - iw$, the equation of Ω takes the form

$$x^2 + y^2 + z^2 + u^2 + v^2 + w^2 = 0.$$

Two linear complexes may then be represented by

$$ax + by + cz + fu + gv + hw = 0 \text{ and } a'x + \ldots + h'w = 0.$$

With these forms the condition that these complexes should be conjugate is easily found to be $aa' + bb' + \ldots + hh' = 0$. Suppose now we have six linear complexes, $X_1 = 0, \ldots, X_6 = 0$, where $X_r = a_r x + b_r y + \ldots + h_r w$, of which every two are conjugate, so that $a_r a_s + \ldots + h_r h_s = 0$; without loss of generality we can suppose that also $a_r^2 + b_r^2 + \ldots + h_r^2 = 1$, $(r = 1, \ldots, 6)$. From these twenty-one conditions there can be inferred (see below) the twenty-one conditions $a_1 b_1 + a_2 b_2 + \ldots + a_6 b_6 = 0, \ldots, g_1 h_1 + g_2 h_2 + \ldots + g_6 h_6 = 0$, $a_1^2 + a_2^2 + \ldots + a_6^2 = 1, \ldots, h_1^2 + h_2^2 + \ldots + h_6^2 = 1$. Hence the equation of Ω can equally be written $X_1^2 + X_2^2 + \ldots + X_6^2 = 0$. Thus the property enunciated in Ex. 2 holds for any six linear complexes of which every two are conjugate.

To shew that the second form of the conditions follows from the first, consider the matrix, M, of six rows and columns, in which the r-th row consists of the elements $a_r, b_r, \ldots, h_r$. Using $\bar{M}$ for the transposed matrix, obtained from M by interchanging rows and columns, the first form of the conditions is expressed by $M\bar{M} = 1$ (cf. Vol. I, p. 67, and Vol. III, p. 71). This, however, gives $\bar{M} = M^{-1}$, and hence $\bar{M}M = 1$. This expresses the second form of the conditions.

Consider, now, *the lines common to three linear complexes* of the original threefold space. These are represented, in the fivefold space, by the points of Ω lying on the plane which is common to the three fourfolds which represent these linear complexes in the fivefold space. This plane meets Ω in the points of a conic. If $(l_1, m_1, \ldots)$, $(l_2, m_2, \ldots)$, $(l_3, m_3, \ldots)$, be three points of this conic, any other point of the conic has coordinates $(\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 l_3, \sigma_1 m_1 + \sigma_2 m_2 + \sigma_3 m_3, \ldots)$, provided $\sigma_1, \sigma_2, \sigma_3$ are subject to the quadratic condition which expresses that this point lies on Ω. In the original space the corresponding lines are, in general, those of one system of generators of a quadric surface (cf. Vol. III, pp. 58, 60; Exx. 4, 9). To any plane of the fivefold space there corresponds another plane; for the polar fourfolds, in regard to Ω, of any three points of the plane intersect in another plane, through which there passes the polar fourfold of every point of the original plane; the polar fourfold of any point of the second plane equally contains the first plane; the planes may be spoken of as polars of one another. Thus, to a conic determined on Ω by its intersection with a plane, there corresponds another conic also on Ω, of which every point is conjugate to every point of the former conic; so that the line joining any point of one conic to any point of the other conic lies entirely on Ω. The points of these two conics correspond, in the original threefold space, to the two systems of generators of a quadric surface. The two lines, of either of these systems of generators, which meet an arbitrary line of the threefold space, correspond to the two points of Ω, on one of these conics, which lie on the tangent fourfold of Ω at the point corresponding to the arbitrary line.

If we take four linear complexes, they will have two lines in common, in general; these correspond, in the fivefold space, to the two points in which Ω is met by the line common to the fourfolds representing the complexes. In particular, the two transversals of four skew lines of the threefold space require the consideration of the line common to four tangent fourfolds of Ω.

If we take three lines in the original space of which every two intersect, say $(l_1, m_1, \ldots)$, $(l_2, m_2, \ldots)$, $(l_3, m_3, \ldots)$, the coordinates $(\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 l_3, \sigma_1 m_1 + \sigma_2 m_2 + \sigma_3 m_3, \ldots)$ are those of a line whatever $\sigma_1, \sigma_2, \sigma_3$ may be. Thus, in the fivefold space, if three points of Ω be taken of which every two are conjugate, *the plane containing these points lies entirely upon* Ω. Two lines $(l_1, m_1, \ldots)$, $(l_2, m_2, \ldots)$, of the threefold space, which intersect, give rise to ∞^1 other lines, with coordinates of the forms $(l_1 + \sigma l_2, m_1 + \sigma m_2, \ldots)$, which pass through the point of intersection and lie in the common plane of the lines; these correspond to the points of Ω lying on the line joining the points $(l_1, m_1, \ldots)$, $(l_2, m_2, \ldots)$. There are, now, in the threefold space, two systems of lines, each ∞^2 in aggregate, all of which intersect both the given lines; namely, first, the lines through the point of intersection of the two given lines, and, second, the lines in the common plane of the two given lines. If $(l_3, m_3, \ldots)$ be a line of the first system, not lying in the plane of the two given lines, all lines of the first system have coordinates of the forms $(\rho_1 l_1 + \rho_2 l_2 + \rho_3 l_3, \ldots)$; and, if $(\lambda_3, \mu_3, \ldots)$ be a line of the second system, that is, a line in the plane of the two given lines, not passing through their point of intersection, all lines of the second system have coordinates of the forms $(\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 \lambda_3, \ldots)$. To these two systems of lines there evidently correspond, in the space of five dimensions, two planes lying entirely on Ω, both passing through the line of Ω which joins the two given conjugate points, $(l_1, m_1, \ldots)$, and $(l_2, m_2, \ldots)$. Considering the matter more generally, in the fivefold space, let x, y, z, u, t, v denote coordinates therein, of which $t = 0$ represents the tangent fourfold of Ω at the point $(0,0,0,0,0,1)$, lying on Ω, and $v = 0$ represents the tangent fourfold of Ω at the point $(0,0,0,0,1,0)$, also lying on Ω, the joining line of these two points being, therefore, $x = 0$, $y = 0$, $z = 0$, $u = 0$. These points are supposed not to be conjugate. The equation of Ω will then be of the form $tv - \phi = 0$, where ϕ is a homogeneous quadratic polynomial in x, y, z, u. As $\phi = 0$ is satisfied by an ∞^2 of values of the form $x/x' = y/y' = z/z' = t/t'$, this equation, $\phi = 0$, represents an aggregate of ∞^2 planes, all passing through the line $x=0$, $y=0$, $z=0$, $u=0$. The intersection of Ω with the tangent fourfold $t=0$ consists of the ∞^2 lines in which this fourfold meets these planes. The points common to Ω and the tangent fourfolds $t=0$, $v=0$, consist of the ∞^2 points of a quadric surface, lying in the threefold given by $t=0$, $v=0$; this surface is the intersection of Ω with the polar threefold of the line $x=0$, $y=0$, $z=0$, $u=0$. Now suppose we take coordinates so that the points $(0,0,0,0,0,1)$ and $(0,0,0,0,1,0)$, lying on Ω, are conjugate to one another. The equation of Ω then takes a form $tV+vT-\psi=0$, where ψ is a quadratic polynomial in x, y, z, u, but T, V are linear forms in x, y, z, u. The tangent fourfold at $(0,0,0,0,0,1)$ is now $T=0$, and that at $(0,0,0,0,1,0)$ is $V=0$. The points of Ω which are conjugate to both these points, which are given by $T=0$, $V=0$, $\psi=0$, then lie on two planes; the equations of these planes are, respectively, of the forms $x/x_1=y/y_1=z/z_1=t/t_1$, and $x/x_2=y/y_2=z/z_2=t/t_2$; they are the intersection of Ω with the threefold space common to the tangent fourfolds at the two given conjugate points of Ω. This threefold space is the polar of the line joining these points, and the planes in which it intersects Ω contain this line. In the original threefold space there are ∞^3 points, through each of which pass ∞^2 lines; and there are ∞^3 planes, in each of which there lie ∞^2 lines. There are thus, on Ω, in the fivefold space, two different systems of planes, each ∞^3. Through two points of the original threefold space there passes one line, and two planes meet in one line; but there is not, generally, a line through a given point which lies in a given plane. Thus, in the fivefold space, two planes of Ω, of the same system, have a point in common, but two planes of different systems do not, in general, intersect. In the original space, if a point lie in a plane, there is a range (pencil) of lines through this point which lie in this plane. Thus, in the fivefold space, if two planes on Ω, of different systems, have a point in common, they intersect in a line, lying on Ω, and every two points of the line are conjugate to one another.

The planes of the quadric fourfold treated with the symbols. The existence of the planes of the quadric fourfold Ω is immediately clear from the equations. The equation of Ω can be supposed to be $\lambda'^2+\mu'^2+\nu'^2=\lambda^2+\mu^2+\nu^2$, where, in terms of the original line coordinates, $\lambda=l-l'$, $\lambda'=l+l'$, etc. If $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ be nine quantities subject to the six equations $a_r^2+b_r^2+c_r^2=1$, $a_ra_s+b_rb_s+c_rc_s=0$ $(r, s=1, 2, 3)$, so that, as will appear in a moment, they are expressible by three parameters, the equation of Ω is identically satisfied by supposing

$$\lambda'=a_1\lambda+a_2\mu+a_3\nu, \quad \mu'=b_1\lambda+b_2\mu+b_3\nu, \quad \nu'=c_1\lambda+c_2\mu+c_3\nu;$$

these three equations connecting the six coordinates represent a plane. The coefficients, a_r, b_s, c_t, in these equations, are such that the square of the determinant (a_1, b_2, c_3) has the value unity, so that there are two cases, according as this determinant is $+1$ or -1. We can easily see that the three equations of the plane express, either that a line, in the original threefold space, passes through a definite point, or lies in a definite plane; this point, or plane, is determined by the coefficients a_r, b_s, c_t, that is, by the three parameters upon which these depend. For, the conditions that a line, $(l, m, \ldots)$, of the original threefold space, contains the point (ξ, η, ζ, τ), may be taken to be the three, $l'\tau + m\zeta - n\eta = 0$, $m'\tau + n\xi - l\zeta = 0$, $n'\tau + l\eta - m\xi = 0$, which, together, unless $\tau = 0$, involve

$$ll' + mm' + nn' = 0\,;$$

replacing $l, l', \ldots$ by $\frac{1}{2}(\lambda' + \lambda)$, $\frac{1}{2}(\lambda' - \lambda), \ldots$, we obtain three linear equations by which λ', μ', ν' may be expressed in terms of λ, μ, ν; thereby, explicit expressions, in terms of ξ, η, ζ, τ, are obtained, in place of the coefficients a_r, b_s, c_t, in the three equations above. The conditions that a line, $(l, m, \ldots)$, in the original threefold space, lies in a plane, of coordinates (u, v, w, p), are three similar equations, with ξ, η, ζ, τ replaced by u, v, w, p, respectively, but l, l', ... respectively replaced by l', l, ... (Vol. III, p. 57); thus, if we solve the three latter equations for λ', μ', ν', we obtain the same linear functions of λ, μ, ν as before, with two differences: the unimportant difference that ξ, η, ζ, τ are replaced by u, v, w, p, and the important difference, arising from the interchange of l, m, n with l', m', n', that the signs of λ', μ', ν' are all changed. Explicitly, with the notation of matrices (Vol. I, p. 67), the formulae in the former case are found to be

$$(\lambda', \mu', \nu') = (\tau - \omega)^{-1}(\tau + \omega)(\lambda, \mu, \nu), = (\tau + \omega)(\tau - \omega)^{-1}(\lambda, \mu, \nu),$$

where
$$\omega = \begin{pmatrix} 0, & -\zeta, & \eta \\ \zeta, & 0, & -\xi \\ -\eta, & \xi, & 0 \end{pmatrix},$$

while, in the latter case,

$$\begin{aligned}(\lambda', \mu', \nu') &= -(p - \psi)^{-1}(p + \psi)(\lambda, \mu, \nu),\\ &= -(p + \psi)(p - \psi)^{-1}(\lambda, \mu, \nu),\end{aligned}$$

where ψ is obtained from ω by replacing ξ, η, ζ, respectively, by u, v, w. It follows that the aggregate of the lines through a point, in the original space of three dimensions, corresponds to the aggregate of the points on a plane on Ω, given by such a set of equations as the former, in which the determinant, (a_1, b_2, c_3), of the coefficients, has the value $+1$; but the lines in a plane of the original space correspond to the points of a plane of Ω for which the corresponding determinant has the value -1. Conversely, when the nine coefficients

a_r, b_s, c_t, in the equations of a plane lying on Ω, in the fivefold space, are given, there are unique values for ξ/τ, η/τ, ζ/τ, or u/p, v/p, w/p, necessary to put the equations in the forms above (p. 205, below).

The properties of the planes of Ω, in regard to intersection, which we have deduced from consideration of the original threefold space, can be deduced directly from the above equations. If $(\lambda', \mu', \nu') = D(\lambda, \mu, \nu)$ denote a plane on Ω of the first system, depending on (ξ, η, ζ, τ), and the matrix replacing D for another plane of this system, depending on $(\xi_1, \eta_1, \zeta_1, \tau_1)$, be D_1, it is easy to verify that the matrix $D_1^{-1}D$ is also of the same form, say D_2, in terms of suitable parameters, $(\xi_2, \eta_2, \zeta_2, \tau_2)$. Now, the condition that the planes (D), (D_1) should intersect is, that we should be able to solve, for λ, μ, ν, the three equations $(D - D_1)(\lambda, \mu, \nu) = 0$, or $(D_1^{-1} D - 1)(\lambda, \mu, \nu) = 0$, or $(D_2 - 1)(\lambda, \mu, \nu) = 0$; namely is, that the determinantal equation for ρ, $|D_2 - \rho| = 0$, should have the root $\rho = 1$. In fact, it is easy to prove that the determinant $|D_2 - \rho|$ has the form $(1 - \rho)(\rho - e^{i\theta})(\rho - e^{-i\theta})$, with a proper value for θ. Thus two planes of the first system, or, similarly, two planes of the second system, lying on Ω, in the fivefold space, have a point in common. For an intersection of a plane, (D), of the first system, with a plane, $(-D_1)$, of the second system, we should, similarly, require the determinantal equation $|D_2 + 1| = 0$. With the notation above, this can only be satisfied if $\theta = \pi$; when this is so, however, all second minors of the determinant $|D_2 + 1|$ vanish, as is easily seen; the three linear equations for λ, μ, ν are then satisfied by an infinite aggregate (∞^1) of values. Thus, if a plane of the first system, on Ω, in the fivefold space, have a point common with a plane of the second system, it has a line of common points. In general, the value of θ is $2\tan^{-1}[(\xi_2^2 + \eta_2^2 + \zeta_2^2)^{\frac{1}{2}} \tau_2^{-1}]$; this is π when $\tau_2 = 0$.

The aggregate of the planes lying on the quadric Ω in fivefold space. We have introduced a plane on Ω as that containing three points of Ω of which every two are conjugate. Such a plane is, therefore, its own polar plane in regard to Ω. And the converse is true. In general, the plane given by the three equations

$$\lambda' = a_1\lambda + a_2\mu + a_3\nu, \quad \mu' = b_1\lambda + b_2\mu + b_3\nu, \quad \nu' = c_1\lambda + c_2\mu + c_3\nu$$

contains the three points $(1, 0, 0, a_1, b_1, c_1)$, $(0, 1, 0, a_2, b_2, c_2)$, $(0, 0, 1, a_3, b_3, c_3)$; the polar fourfolds of these points, in regard to Ω, whose equation is $\lambda'^2 + \mu'^2 + \nu'^2 = \lambda^2 + \mu^2 + \nu^2$, are, respectively, given by

$$\lambda = a_1\lambda' + b_1\mu' + c_1\nu', \quad \mu = a_2\lambda' + b_2\mu' + c_2\nu', \quad \nu = a_3\lambda' + b_3\mu' + c_3\nu';$$

the plane given by the aggregate of the three latter equations agrees with the former if, and only if, the coefficients a_r, b_s, c_t satisfy the six equations which characterise a plane that lies on Ω.

Through any line lying in a plane on Ω, there passes another plane on Ω, of the other system; the two planes are, together, the complete intersection of Ω with the polar threefold, in regard to Ω, of the line. Conversely, if any threefold be drawn through a plane on Ω, this threefold has for its polar line, in regard to Ω, a line lying on the plane; and the threefold meets Ω in a further plane. Thus, any plane on Ω meets, each in a line, ∞^2 planes of the other system. The ∞^2 lines on Ω, that can be drawn through any point, O, of Ω, arrange themselves in ∞^1 planes, on Ω, passing through this point, there being one plane of each system through every such line. These planes meet any arbitrarily taken fourfold, Π, each in a line; these lines lie in the threefold in which Π is met by the tangent fourfold of Ω at O. Two of the planes of the same system, through O, have no other common point, and give lines in Π which do not intersect; but two planes of different systems, through O, have a line in common, and meet Π in intersecting lines. Thus, in the threefold intersection of Π with the tangent fourfold of Ω at O, the lines obtained by the planes on Ω, of the two systems, which pass through O, are the generators of a quadric surface. If ϖ_1, ϖ_2 be two planes of the first system, from these, whose only common point is O, the plane, ϖ', of the second system, which passes through any line, l, lying in ϖ_2 but not passing through O, will not meet ϖ_1; for the line of meeting would intersect l in a point lying both on ϖ_1 and ϖ_2. Conversely, if ϖ' be any plane of the second system which does not meet ϖ_1, the plane, ϖ_2, of the first system, which contains a line, l, lying on ϖ', will meet ϖ_1 in a point, say O; and, then, in an infinite number of ways, a plane of the second system can be drawn through O to meet ϖ_1 and ϖ_2 in a line. We thus see how to pass from any one plane on Ω to every other plane on Ω; namely, by suitably repeating the process of drawing, through a plane of either system, a threefold which determines a plane of the other system as its residual intersection with Ω.

Ex. 1. Let Λ, Λ' be two conjugate fourfold sections of Ω. Consider the line in which Λ meets a plane, ϖ_1, of the first system on Ω; through this line there passes a plane, ϖ', of the second system; consider the line in which ϖ' meets Λ'; through this line there passes a plane, ϖ_2, of the first system. Prove that ϖ_2 is also obtained from ϖ_1 if the same construction is made with the interchange of Λ and Λ'. Two planes on Ω, of different systems, which meet in a line lying on Λ, correspond, in fact, to a point and its polar plane, in the original threefold space, taken in regard to the focal system given by Λ (cf. Vol. III, p. 66, Ex. 10).

Ex. 2. The representation, in the space of five dimensions, of the polar line of a line, l, of the original threefold space, in regard to a given quadric surface of this threefold space, is also interesting.

Let one point where l meets the quadric be the intersection of the generators p_1, q_1 of the quadric, and the other point, where l meets the quadric, the intersection of the generators p_2, q_2. The generators p_1, q_2, of opposite systems, meet in a point, and the generators p_2, q_1 meet in a point, the join of these two points being the polar line, l', of l, in regard to the quadric. Correspondingly, in the fivefold space, let L be any point of Ω; let the tangent fourfold of Ω at L meet the conic, on Ω, which represents one system of generators of the original quadric, in P_1 and P_2; to this conic there is a conjugate conic on Ω, as we have seen above; let this be met by the tangent fourfold of Ω at L in Q_1 and Q_2. Thus the lines P_1Q_1, P_2Q_2 lie on Ω, but do not intersect; and the planes LP_1Q_1, LP_2Q_2 lie on Ω, intersecting in L. These are of the same system. Through each of P_1Q_1, P_2Q_2 there pass also planes, on Ω, of the other system; their point of intersection, L', represents the polar line, l'.

We may remark, further, that the lines P_1Q_2, P_2Q_1 lie on Ω, and there are planes, on Ω, of the same system, through these, which intersect in L', and planes of the other system, on Ω, through these, which intersect in L. In other words, the construction, as stated, is valid without distinction of P_1 and P_2, or of Q_1 and Q_2. There are in the figure, six points on Ω, $(P_1, P_2, Q_1, Q_2, L, L')$; twelve lines on Ω, $(P_1Q_1, P_2Q_2, P_1Q_2, P_2Q_1, LP_1, LP_2, LQ_1, LQ_2, L'P_1, L'P_2, L'Q_1, L'Q_2)$; four planes on Ω of one system, $(P_1Q_1L, P_2Q_2L, P_1Q_2L', P_2Q_1L')$, and four others of the other system, $(P_1Q_1L', P_2Q_2L', P_1Q_2L', P_2Q_1L')$.

The representation of congruences of lines of the original space. The lines of the original threefold space, whose coordinates are subject to two (rational algebraic) conditions, are ∞^2 in aggregate; they are said to form a *congruence* of lines. We have, in particular, spoken of the linear congruence, of lines common to two linear complexes, which are the transversals of two fixed lines; they are such that one of the lines passes through any arbitrarily taken point, of general position, and one lies in any arbitrarily taken plane, of general position. The lines of a linear congruence are represented, in the fivefold space, by the points of a quadric surface lying on Ω, having, clearly, one point of meeting with every plane on Ω. A more general congruence of lines, in the original threefold space, is that formed by the chords of a (not plane) curve of this space; this, also, consisting of ∞^2 lines, is represented, in the fivefold space, by the points of a surface lying on Ω. We proceed, in illustration of general ideas, to find the order of this surface. In general, we mean by the *order of a locus*, of k dimensions, lying in space of n dimensions, the number of points of the locus which lie on a planar manifold of dimension $n - k$, when this number is the same for every such general manifold. Thus, the order of a surface, in space of five dimensions, will be the number of its points of intersection with an arbitrary planar threefold. We have seen that there are threefolds meeting Ω in two planes, of different systems; to determine the order of a surface lying entirely on Ω it is convenient to take such a threefold. This will meet the surface only on the two planes. Thus the order of a surface lying on Ω is the sum of the numbers of points in which this surface meets two planes of different systems on Ω, say h and h', respectively; that these numbers are both unaltered whatever two planes of the two systems be taken, appears from the possibility we have remarked, of passing from any plane on Ω to any other by means of intersecting threefolds. In the case of a surface, on Ω, which represents the chords of a curve in the original threefold space, the number, h, of points common to the surface and a plane of the first system on Ω, is the number of chords of the curve, in the original space, which pass through an arbitrary point of general position in this space; the number, h', of points of the surface on a plane of the second system, is the number of chords of the curve which lie in an arbitrary plane of general position. In the fivefold space there are two tangent fourfolds of Ω passing through a given threefold; their points of contact with Ω are the intersections of this with the polar line of the threefold in regard to Ω. Thus the order of a surface, lying on Ω, is also the number of points of the surface which are conjugate to two arbitrarily taken points of Ω; in the original threefold space, this number is that of the lines of the congruence, which corresponds to the surface on Ω, which belong to an arbitrary linear congruence. For the congruence consisting of the chords of a curve of order m, in the threefold space, the number, by what we have seen, is $h + \frac{1}{2}m(m-1)$, where h is the number of chords of the curve passing through an arbitrary point. If $p = \frac{1}{2}(m-1)(m-2)-h$, this number is $(m-1)^2 - p$.

Number of lines common to two congruences in threefold space. It can be shewn that two surfaces on Ω, of which the first meets general planes on Ω, of the first and second systems, respectively, in h and h' points, and the second meets general planes on Ω, of the first and second systems, respectively, in k and k' points, have in common $hk + h'k'$ points. In the original threefold space this is the statement that two congruences of lines, of which the first has h lines through an arbitrary point and h' lines in an arbitrary plane, and the second has k lines through an arbitrary point and k' lines in an arbitrary plane, have in common a number of lines given by $hk + h'k'$. There is a theorem, for algebraic curves lying on a quadric surface in space of three dimensions, that such a curve meets all the generators of the quadric, of the same system, in the same number of points; that, if these numbers, for the two systems of generators, be r and r', respectively, the order of the curve is $r+r'$; and that such a curve, say (r, r'), meets another curve, (s, s'), of the quadric, in $rs'+r's$ points; the theorem may be proved by projecting the curves into plane curves, from a point of the quadric, so obtaining a curve with two multiple points of orders r, r', and a curve with two multiple points of orders s, s'. We shall assume this result, in order to obtain the theorem above enunciated for the number of intersections of two surfaces lying on Ω, in the space of five dimensions. We may denote these surfaces, respectively, by (h, h') and (k, k'). Let O be a point of Ω, not lying on either surface, and Π be an arbitrary fourfold space. When we project the points of Ω, from O, upon Π, the planes on Ω which pass through O determine, in Π, the generators of a quadric surface, ω, as we have seen; this quadric lies in the threefold space, (Π, T), in which Π is met by the tangent fourfold, T, of Ω at O. The surface (h, h'), on Ω, being met by an arbitrary threefold in $h+h'$ points, will project into a surface in the fourfold space Π; and this, being met by an arbitrary plane in Π in $h+h'$ points, will be of order $h+h'$. The surface (h, h') will be met by the tangent fourfold T in a curve; this projects into a curve lying on the quadric ω, having the property of meeting the generators of the two systems of ω, respectively, each in h and h' points. We assume that two surfaces in fourfold space, of respective orders M and N, which have not an infinite number of common points (as those of a line, etc.), have MN common points. Thus the two surfaces, in Π, obtained by projection from O of two surfaces, (h, h') and (k, k'), upon Ω, have $(h+h')(k+k')$ common points. These, however, arise in part from only apparent intersections of the two surfaces (h, h'), (k, k') upon Ω, namely by lines through O meeting both these surfaces, but not in the same point. Every such line, meeting Ω in three distinct points, will lie entirely on Ω, and be, therefore, in the tangent fourfold T; such a line will give an intersection of the curves lying on the quadric ω, which are the intersections of ω with the surfaces, in the fourfold space Π, arising from projection of the surfaces (h, h'), (k, k'), on Ω. The number of such intersections, we have remarked, is $hk'+h'k$. The number of intersections on Ω of the surfaces (h, h'), (k, k') is thus $(h+h')(k+k')-hk'-h'k$, or $hk+h'k'$; as we desired to prove.

Ex. If h, k be the respective numbers of chords that can be drawn from an arbitrary point, to two given curves of orders m, n, in threefold space, the number of common chords of these curves is $hk+\frac{1}{4}(m-1)(n-1)mn$.

It may be remarked that the surface on Ω, corresponding to the chords of a curve, in space of three dimensions, meets a plane on Ω which corresponds to a point of the curve, in an *infinite* aggregate of points; these points form a curve of order $m-1$, if the original curve be of order m.

The chords of a cubic curve in the original threefold space. Veronese's surface. An important particular case of the preceding is that of the chords of a cubic curve in the original threefold space. Of these chords, one passes through an arbitrary point, and three lie in an arbitrary plane. The corresponding surface on Ω thus meets the planes on Ω, of the first and second systems, respectively, in one and three points, and is of order four, having four points of intersection with an arbitrary threefold. The axes of a cubic developable, in the original space, similarly give a surface on Ω for which the corresponding numbers are three and one. We may denote the former surface by V, and the latter by V'. That the surface V is of order four is equivalent, we have seen, to the fact that there are four chords, of a cubic curve in threefold space, which meet two arbitrary lines of this space. This may be verified directly, for example, from the fact that the chords of the curve which meet one line are projected from a point of the curve by the tangent planes of a quadric cone (Vol. III, p. 135, Ex. 8). The surface V meets ∞^1 planes of the first system on Ω, those corresponding to the points of the cubic curve in the original space, each in a conic; and two such conics intersect in the common point of their planes. But there are, in fact, ∞^2 conics lying on the surface V, whose planes do not lie on Ω. For, if an arbitrary line be drawn through a point of the cubic curve in the original space, there is an infinite aggregate of chords of the cubic curve meeting this line; these chords form one system of generators of a quadric surface, and meet the curve in the pairs of points of an involution on the curve. (Cf. Vol. III, p. 128.) We have seen that the lines of one system of generators, of a quadric surface, in the original space, correspond to the points of Ω lying on a conic; further, an involution on the cubic curve is determined by two pairs of points, that is by two chords of the curve; and two such involutions have a common pair of points. Hence there are, as stated, ∞^2 conics of Ω which lie on the surface V; one of these conics passes through any two chosen points of V, and any two of these conics have a common point. The conics on V, previously remarked, whose planes lie on Ω, arise among these.

In more detail, every quadric surface containing the original cubic curve, is determined by the chords of this curve which meet a line drawn through an arbitrary fixed point, K, of the curve. Thus, if we take any fixed plane, κ, lying on Ω, from among the ∞^1 which meet V in a conic, and take, in this plane, a point, P, not generally lying on V, then, the tangent fourfold of Ω, at this point, not only contains the plane κ, but also meets V in a conic; and, as

P varies in the plane κ, all the ∞^2 conics of V are obtained. When P is taken on V, that is, on the conic in which the plane κ meets V, the tangent fourfold of Ω, at P, contains, beside the plane κ, a conic whose plane lies on Ω; and all the ∞^1 such conics are obtainable in this way. Conversely, the plane of any one of the conics of V meets the plane κ in a point, and lies in the tangent fourfold of Ω at this point, which also contains the plane κ. If a line be taken in the plane κ, the tangent fourfolds of Ω, at the points of this line, will meet V in conics having a common point; and the tangent fourfold of Ω at this point will meet the plane κ in this line. This follows from what has been said, but is obvious by recurring to the original threefold space, wherein, corresponding to the line of κ, there is a flat pencil of lines passing through the point K of the cubic curve. Thus the points of V are in unique correspondence with the lines of the plane κ. Further, it can be shewn that the points of V which lie on an arbitrary (planar) fourfold give rise, in this correspondence, to lines of the plane κ which touch a conic; for this fourfold will meet the conic, on V, which corresponds to any point, P, of the plane κ, in two points; the tangent fourfolds of Ω at these two points of V will meet the plane κ in two lines passing through P. Thus, the coordinates of any point of V are proportional to quadratic polynomials in three parameters, the coordinates of any line of the plane κ. (Cf. Vol. III, p. 223, Ex. 5.) We may obtain the explicit expressions by reference to the original threefold space. The cubic curve being given by points $(\theta^3, \theta^2, \theta, 1)$, the point K being for $\theta = k$, a plane, through K and the chord (θ, ϕ) of the curve, has the equation, in terms of coordinates x, y, z, t in the threefold space, $u(x - yk) + v(y - zk) + w(z - tk) = 0$, where $\theta + \phi = -v/u$, $\theta\phi = w/u$. Thus the coordinates of the chord (θ, ϕ) are $(wu - v^2, uv, -u^2, wu, wv, w^2)$; these are the coordinates of a point of the surface V in terms of the parameters u, v, w. The tangent fourfold of Ω at this point of V, with $l, m, \ldots$ as current coordinates, has the equation

$$lwu + l'(wu - v^2) + mwv + m'vu + nw^2 - n'u^2 = 0.$$

The line which is the intersection of this with the plane κ may be found by substituting herein for l', m', n' from the equations of the plane κ; these are $l'\tau + m\zeta - n\eta = 0$, $m'\tau + n\xi - l\zeta = 0$, $n'\tau + l\eta - m\xi = 0$, where $\xi = k^3$, $\eta = k^2$, $\zeta = k$, $\tau = 1$. The result of the substitution is found to be

$$(k^2u + kv + w)\left[lu + m(v - ku) + n(w - kv)\right] = 0;$$

the first factor of this vanishes only when the point (u, v, w), of the surface V, is on the plane κ; the vanishing of the second factor determines uniquely the point (u, v, w), of V, when the line in the plane κ is given. The lines of the plane κ which correspond to the points of the surface V which lie on the polar fourfold, in regard to Ω, of the point (a, b, c, a', b', c'), touch the conic given tangentially by $awu + a'(wu - v^2) + bwv + b'vu + cw^2 - c'u^2 = 0$. As has already been remarked (Ex. 4, p. 6, above), when the point $(a, b, \ldots)$ is on Ω, this conic is triangularly inscribed to $xz - y^2 = 0$.

Ex. 1. The section of the surface V by any (planar) fourfold is a rational quartic curve; the conic just obtained arises from the tangent fourfolds of Ω at all the points of this curve. The values of u, v, w corresponding to a point of this curve can be expressed, in terms of a parameter θ, in the forms

$$u = a'\theta^2 - b\theta - c, \quad v = a'\eta\theta^2 + (a + a')\theta - c\xi,$$
$$w = -a'(\xi - \eta)\theta + b\xi + a + a',$$

where, for convenience, in place of b' and c', we have ξ, η, such that $b' = a'(\xi + \eta)$, $c' = a'\xi\eta$. From these values of u, v, w the coordinates of a point of the quartic curve can be expressed as quartic polynomials in θ.

Ex. 2. The conics of a plane may evidently be represented by the points of a fivefold space, the coefficients of the equation of the conic being taken as coordinates. More generally, if any conic in a plane be expressed as a linear function of six given conics in that plane (themselves not linearly connected), the coefficients in the expression may be interpreted as coordinates of a point in space of five dimensions. The condition that such a conic, supposed given in point coordinates, should be a pair of lines, is, then, a single equation of the third order in the coordinates; this represents a fourfold of the third order in the fivefold space, say M_4^3. The conditions that these two lines coincide reduce the number of independent coordinates further to two, the surface in the fivefold space representing this being essentially the surface V, of the fourth order, just discussed. This lies upon the M_4^3, and is a double surface thereon. See Segre, *Atti d. r. Acc. d. Sc. d. Torino*, XX (1884–85), pp. 367–384.

Ex. 3. The surface V was briefly considered by Cayley, *Papers*, VI, p. 198 (1868); at length by Veronese, *Mem. d. Acc. d. Lincei*, XIX (1883–84), and by Segre (see the reference in Ex. 2). Bertini, *Geom. d. iperspazi* (Pisa, 1907), gives, with others, the following results in regard thereto: A surface, in space of any number of dimensions, which contains ∞^2 conics, is the surface of Veronese, or one of its projections. (Cf. Darboux, *Bull. d. sc. math.*, IV, 1880: A surface, in threefold space, containing ∞^2 conics, is a quadric, a ruled cubic surface, or the quartic Steiner surface.) A surface, in space of any number, r, of dimensions, for which $r > 4$, with the property that every two tangent planes have a common point, is a cone, or is the surface of Veronese (Del Pezzo, *Rend. d. Cir. Mat. d. Palermo*, I, 1887). A surface, in space of any number of dimensions, which is not a cone, and is such that no line can be drawn through an arbitrary point of the space to meet the surface in two points, is the surface of Veronese. (Cf. Severi, *Rend. d. Palermo*, xv, 1901; and, *Mem. d. Acc. d. Torino*, LII, 1902.) A surface of order $r-1$, in space of r dimensions, is a rational ruled surface, or, for $r=5$, is the surface of Veronese (Del Pezzo, *Rend. d. Acc. d. Napoli*, 1885 and 1886). A surface, in space of r dimensions, which meets any (planar) $(r-1)$-fold in a rational curve, unless it is a rational ruled surface, is the Veronese surface, or one of its projections. (Cf. Picard, *Théorie des fonctions algébriques*, II, 1900, p. 59: A surface, in space of three dimensions, whose plane sections are rational curves, is a rational ruled surface, or the quartic Steiner surface.) Compare the theorem of Kronecker and Castelnuovo quoted in Vol. III, p. 223.

Lie's correspondence between lines and spheres in space of three dimensions. Sophus Lie has enunciated, with the help of the equations, a relation whereby, to a line of threefold space, there corresponds a sphere (or plane); and, conversely, to a sphere, there corresponds a pair of lines; with the property that, to two lines which intersect, there correspond two spheres which touch (1869. See Lie u. Scheffers, *Berührgstransftn.*, 1896, pp. 453 ff.; Klein, *Ges. Math. Abh.*, I, p. 96). The correspondence will be obtained here in a geometrical manner. We have seen above (Section II, p. 36) that, in a fourfold space, a section of a threefold quadric, by a planar threefold, projects, from a point of this quadric, on to a threefold, into a sphere. We may regard such a fourfold space as lying in a fivefold space, wherein the lines of a threefold space are represented by the points of a quadric fourfold.

Let Ω be a fourfold quadric, of fivefold space, whose points represent the lines of the original threefold space; let Π be an arbitrary (planar) fourfold space; this meets Ω in a threefold quadric, U. The tangent fourfold of Ω, at a point P, meets Π in a threefold, ϖ; the section of U by this threefold, ϖ, is a surface, which, projected from any point, O, of U, upon any threefold lying in Π, gives a sphere therein. It is sufficient, then, to regard the sphere as determined by the section of U by ϖ. This sphere is determined by the point P, that is, by the line of the original threefold space which is represented by P. From the threefold, ϖ, there can be drawn another tangent fourfold of Ω, touching this, say, in P'; thus, to the sphere arising from P, there corresponds also P'. The lines, of the original threefold space, which correspond to P and P', are, as we have seen above (p. 42), polars of one another in regard to the focal system, or linear complex, which is represented by the section of Ω by the fourfold Π. We have shewn that, to a line, of the threefold space, there can be made to correspond a sphere; a sphere, conversely, corresponding to a pair of lines, polars of one another in a certain focal system.

Now consider two points, P, Q, of Ω, which are conjugate to one another, and so represent intersecting lines of the original threefold space. We desire to see that the spheres, which correspond to these lines, touch one another. The line PQ lies wholly on Ω; it meets Π in a point, say H, also lying on Ω; this point is, therefore, on the quadric, U, given by Ω and Π. Let the tangent fourfolds of Ω at the points P, Q, H be, respectively, denoted by t_P, t_Q, t_H; the tangent planes, at H, of the sections (U, t_P) and (U, t_Q), are, respectively, the planes (Π, t_P, t_H) and (Π, t_Q, t_H). Now, it is easily seen that the tangent fourfolds of Ω, at the points of a line which lies thereon, form a range (pencil) of fourfolds all passing through a threefold; this threefold is then determined as the intersection of any two of the fourfolds. The threefolds (t_P, t_H) and (t_Q, t_H) are thus identical. Wherefore, the planes (Π, t_P, t_H) and (Π, t_Q, t_H) are identical. This shews that the spheres obtained by projection of the sections (U, t_P) and (U, t_Q), from any point, O, of U, touch one another at the point which is the projection of H. As a particular case, the point O may lie on t_Q; the projection of the section (U, t_Q) will then be, not a sphere but, a plane. This plane will touch the sphere obtained by projection of (U, t_P).

Ex. 1. We may take coordinates, x, y, z, t, u, v, in the fivefold space, such that $u = 0$ is the fourfold, Π, of the above statement; while the point, O, from which the projection is to be made, is given by $x = y = z = t = u = 0$; the tangent fourfold of Ω, at O, may be supposed to be $t = 0$; and the tangent fourfold of Ω at the point, other than O, in which the line $x = y = z = u = 0$ meets Ω, to be $v = 0$. Then, the equation of Ω will contain no term in v^2, and the coefficient of v in this equation will be a constant multiple of t; similarly, there will be no term in t^2, and the terms involving t will reduce to tv. Thus the equation of Ω will be of the form $-2vt + u^2 + uL + Q = 0$, where L and Q are respectively linear and quadratic polynomials in x, y, z only; by proper choice of x, y, z we may then suppose the equation to be

$$-2vt + u^2 + 2auz + x^2 + y^2 + z^2 = 0,$$

in which a is a constant. The surface obtained by the intersection of Ω with the fourfold Π, or $u = 0$, and the tangent fourfold of Ω at a point (x', y', z', t', u', v'), is then to be found from

$$u = 0, \quad -2vt + x^2 + y^2 + z^2 = 0, \quad -vt' - v't + au'z + xx' + yy' + zz' = 0.$$

The projection of this surface from O, by elimination of v from the two latter equations, satisfies the equation

$$-2t\,(xx' + yy' + zz' + au'z - v't) + t'\,(x^2 + y^2 + z^2) = 0;$$

this is of the form

$$x^2 + y^2 + z^2 - 2(fx + gy + hz)t + ct^2 = 0,$$

appropriate to a sphere, with

$$ft' = x', \quad gt' = y', \quad ht' = z' + au', \quad ct' = 2v'.$$

If f, g, h and c be given, and we seek (x', y', z', t', u', v'), expressing that this point is on Ω we first obtain

$$t'^2(f^2 + g^2 + h^2 - c) = u'^2(a^2 - 1),$$

and then, to each of the two solutions of this, a unique set of ratios $x' : y' : z' : t' : u' : v'$. Thus the two points of Ω, or the two lines of the original threefold space which correspond to a given sphere, are obtainable.

The condition that two spheres (f_1, g_1, h_1, c_1) and (f_2, g_2, h_2, c_2) should touch one another, which is (Vol. III, pp. 77, 78)

$$f_1f_2 + g_1g_2 + h_1h_2 \pm (f_1^2 + g_1^2 + h_1^2 - c_1)^{\frac{1}{2}}(f_2^2 + g_2^2 + h_2^2 - c_2)^{\frac{1}{2}} = \tfrac{1}{2}(c_1 + c_2),$$

leads, for the corresponding points $(x_1, y_1, \ldots)$ and $(x_2, y_2, \ldots)$ of Ω, to the condition

$$x_1x_2 + y_1y_2 + z_1z_2 + (z_1 + au_1)(z_2 + au_2) \pm u_1u_2(a^2 - 1) = v_1t_2 + v_2t_1;$$

if the lower of the ambiguous sign be taken this is the same as

$$x_1x_2 + y_1y_2 + z_1z_2 + a(u_1z_2 + u_2z_1) + u_1u_2 = v_1t_2 + v_2t_1;$$

this is the condition that the two points of Ω should be conjugate, or that the corresponding lines of the original threefold space should intersect. In words, if two spheres touch, then either of the lines corresponding to one of the spheres intersects one of the lines corresponding to the other.

Ex. 2. In a preceding section (II, p. 36), we have regarded a sphere in threefold space as arising by projection of a section, by a threefold, of a quadric threefold, U, in fourfold space. Two spheres will then touch if the corresponding quadric sections touch, that is, if the poles of the threefolds which determine these sections, taken in regard to U, lie on a line which touches U. With the figure now under consideration, let M be the pole, in regard to Ω, of the fourfold Π; then prove that, if P, Q be two points of Ω which are conjugate to one another, the plane MPQ meets Π in a line touching U, or (Ω, Π), at the point H, where the line PQ meets Π.

Ex. 3. Given four spheres, we can prove, either from the point of view of the preceding section (II, p. 36), or from the point of view of the present section, that there are eight pairs of spheres touching all of them. From the former point of view, we consider four threefold sections of a quadric U in fourfold space, and the enveloping cones of U at the points of each of these sections. The polar threefold of a point which is common to these four cones, in regard to U, will give a section touching the four given sections (cf. below, Chap. II). There are however sixteen common points of these four cones. From the present point of view, we assume that four lines in a threefold space, of general positions, have two common transversals; this is only equivalent to the statement that the tangent fourfolds at four points of the fourfold quadric Ω, in the space of five dimensions, meet in a line, which then intersects Ω in two points. To the four given spheres there correspond, in the manner explained, four pairs of lines. Every four lines, chosen from the available eight, which are so taken that no two belong to the same pair of lines, have two transversals; these give two spheres touching the four original spheres.

Ex. 4. The special case of the preceding example, that there are eight spheres touching four given planes, of a threefold space, may likewise be examined from both points of view. It will be sufficient to take the point of view of the present section. To a point of Ω lying in the tangent fourfold of Ω at O, there corresponds, not a sphere, but a plane. To four points of Ω lying in this tangent fourfold at O will then correspond four planes. These four points determine a threefold, from which only one tangent fourfold can be drawn to touch Ω, beside the tangent fourfold at O. This gives rise to one sphere touching the four planes, in the original threefold space. Conversely, a given plane of this space leads to two points of Ω, lying in the tangent fourfold at O, whose join passes through the pole of Π in regard to Ω. From this the eight spheres touching four given planes can be inferred.

Ex. 5. The figure of a double six of lines in threefold space is obtained by starting from five lines having a common transversal (Vol. III, p. 159). Thus, if five planes be taken, and a properly chosen one of the spheres touching every four of these planes, the five spheres so obtained touch another sphere (J. H. Grace, *Camb. Phil. Transactions*, XVI, 1898, p. 167).

Generalisation of Wallace's theorem. The theorem of the double six. A theorem for six lines with a common transversal. We consider now a set of six connected theorems. We state them, in the first instance, for space of five dimensions, remarking, later, on their meaning in ordinary threefold space.

1. In fivefold space, let six arbitrary fourfolds be given passing through a point which lies on a quadric, Ω. Every four of these meet in a line, having a further intersection with Ω. Thus, from five of these fourfolds we obtain five further points of Ω; and these five points determine a fourfold, passing through them. In all, then,

we have six such fourfolds. The theorem is that these six fourfolds meet in a point, and that this point lies in Ω.

2. Dually, let six arbitrary points be taken on the tangent fourfold at any point of Ω. Four of these points determine a threefold, from which can be drawn, to Ω, a further tangent fourfold, beside the original. Thus, from five of the points we obtain five further tangent fourfolds of Ω; and these meet in a point, not generally on Ω. In all, then, we have six such points. These six points lie on a fourfold; and this touches Ω.

3. Through a point of Ω let five fourfolds be taken, each of which touches Ω, at a point other than that through which they all pass. The line of intersection of every four of these meets Ω in a further point. In all there are five such points, lying on Ω. The theorem is that the fourfold determined by these five points touches Ω. Thus, also, the tangent fourfolds of Ω at these five points have for their intersection a point which lies on Ω.

4. Dually, let five points be taken, on a tangent fourfold of Ω, which are also on Ω itself; any four of these determine a threefold, from which another tangent fourfold can be drawn to Ω. In all there are five such new tangent fourfolds. The point of intersection of these is on Ω. Thus, also, the points of contact of the five fourfolds are in another tangent fourfold of Ω.

5. Through a point of Ω let six fourfolds be taken each of which touches Ω, at a point other than that through which they all pass. By theorem (3) above, every five of these fourfolds determine a further tangent fourfold of Ω. In all there will be six such further tangent fourfolds. The theorem is that these meet in a point, and that this point is on Ω. Thus, their six points of contact are in another tangent fourfold of Ω.

6. Dually, let six points be taken on a tangent fourfold of Ω which are also on Ω itself. By theorem (4) above, every five of these determine a new point of Ω. The six new points of Ω so found, lie on a fourfold; and this touches Ω.

In regard to theorem (1), we recall the proof above given (p. 19) of the theorem of Wallace, that the four circles containing the triads of intersections, of the sets of three out of four arbitrary lines, in a plane, meet in a point. A corresponding theorem, for the loci which correspond to circles, holds in space of any *even* number of dimensions (J. H. Grace, *Camb. Phil. Trans.*, XVI, July 1897, p. 163; Kühne, *Crelle*, CXIX, 1898, p. 186; cf. *Camb. Phil. Proc.*, XXII, Part I, 1924). That this is so will be clear from the proof of theorem (1). In regard to theorem (4), we notice that, when points of a quadric Ω, in space of five dimensions, represent lines of a threefold space, this is equivalent to the theorem of a double six of lines; the theorem (4) is equivalent to (3). The theorem

(6), in virtue of (4), is a particular case of (2); and the equivalent theorem (5), in virtue of (3), is a particular case of (1). This theorem (6), however, is equivalent to the following, for lines in space of three dimensions: Consider six lines, in space of three dimensions, which have a common transversal, no two of the lines intersecting; from every five of these, by the theorem of the double six, let there be found another line. Then the six new lines so found have a common transversal. Let the six lines be denoted here by a, b, c, d, e, f, their common transversal being t. It can be shewn that the locus of a point, such that the planes joining it to the seven lines all touch a quadric cone, is a cubic curve, say ϑ, having the six lines a, b, c, d, e, f as chords; dually, the planes meeting the seven given lines in the points of a conic constitute a cubic developable, say Θ, of which the six lines a, b, c, d, e, f are axes (Vol. III, p. 195, Ex. 7). Further, if, with the fixed lines a, b, c, d, e, and their common transversal, t, we define a double six of lines, and denote the completing line of this by f_1 (Vol. III, p. 159), then f_1 is both a chord of the curve ϑ and an axis of the developable Θ. In fact, the points where ϑ meets f_1 are on the planes of Θ which meet in f, and the points where ϑ meets f are on the planes of Θ which meet in f_1. Thus we have a tetrad of points of ϑ determined by a tetrad of planes of Θ. We may, however, select any five of the lines a, b, c, d, e, f, and with these, and t, form a double six. Thereby we shall obtain six such tetrads of points and planes, associated with ϑ and Θ. But any two tetrads of points of a cubic curve are both self-polar in regard to a proper quadric (Vol. III, p. 148). We can thus infer that there exists a quadric in regard to which Θ is the polar reciprocal of ϑ, and the six pairs of lines such as f and f_1 are mutually polar lines. Then, from the fact that the lines a, b, c, d, e, f have a common transversal, it follows that the six lines $a_1, \dots, f_1$ have also a common transversal. The theorem was given by Mr J. H. Grace, and this proof by Mr E. K. Wakeford (cf. *Proc. Lond. Math. Soc.*, XXI, 1922, p. 127).

In order to prove the six theorems of fivefold space enunciated above, it is clearly sufficient, after what has been said, to prove (1) and (3). Proofs are given in the examples following.

Ex. 1. Let $x_1 = 0, \dots, x_6 = 0$ be any six fourfolds, of fivefold space, passing through the same point, so that there exists an identity of the form $a_1x_1 + \dots + a_6x_6 = 0$. On the line of intersection of any four of these, say on $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, let another arbitrary point be taken, say, for this case, P_{1234}. The five points so arising, by taking the fours from $x_1, \dots, x_5$ only, determine a fourfold, say Π_6. We first prove, as in a preceding case, that the six fourfolds thus obtained, $\Pi_1, \Pi_2, \dots, \Pi_6$, meet in a point.

Let $v = 0$ be any fourfold not passing through the point common to the original six fourfolds. We can suppose the equation of Π_6 to be expressed in terms of $x_1, \ldots, x_5$ only, and v, say $\xi_6 = 0$, where

$$\xi_6 = a_{61}x_1 + \ldots + a_{65}x_5 - a_6v.$$

The point P_{1234}, for which $x_1 = x_2 = x_3 = x_4 = 0$, will then be such that $a_{65}x_5 - a_6v = 0$, as well as $a_5x_5 + a_6x_6 = 0$. If we similarly express the equation of Π_5, without use of x_5, in the form $\xi_5 = 0$, where

$$\xi_5 = a_{51}x_1 + \ldots + a_{54}x_4 + a_{56}x_6 - a_5v,$$

the point P_{1234} will also be given by $a_{56}x_6 - a_5v = 0$. By comparison of these two forms for P_{1234} we infer that $a_{56} = -a_{65}$. With similar forms for the equations of $\Pi_1, \ldots, \Pi_4$ we prove, in the same way, that in general $a_{rs} = -a_{sr}$ $(r, s = 1, \ldots, 6)$. If then we add to the six equations $\xi_1 = 0, \ldots, \xi_6 = 0$ the identical equation

$$a_1x_1 + \ldots + a_6x_6 = 0,$$

we have seven equations in $x_1, \ldots, x_6, v$ whose coefficients form a skew-symmetrical system; the determinant of these, being of odd order, vanishes identically. This proves that $\Pi_1, \ldots, \Pi_6$ meet in a point; and there is, evidently, a corresponding theorem in space of any *odd* number of dimensions.

Ex. 2. In space of n dimensions, all $(n-1)$-dimensional quadrics which pass through $\frac{1}{2}n(n+1)+1$ general points are expressible linearly, when their equations are in point-coordinates, by n linearly independent quadrics passing through these points; all such quadrics, therefore, pass through $2^n - 1 - \frac{1}{2}n(n+1)$ other points. In particular, in space of five dimensions, all quadrics (of dimension four) through sixteen given general points are expressible linearly by five such quadrics. In the preceding example, the original point, through which the fourfolds $x_1 = 0, \ldots, x_6 = 0$ pass, and the fifteen points such as P_{1234}, lie on the degenerate quadric expressed by $x_6\xi_6 = 0$. For $x_6 = 0$ contains, beside the original point, the ten points P_{rst6}, in which r, s, t are any triad from 1, 2, 3, 4, 5; and $\xi_6 = 0$ contains the five points P_{rstk}, in which r, s, t, k are any set of four from 1, 2, 3, 4, 5. Similarly, each of the quadrics $x_r\xi_r = 0$ contains the same sixteen points. These six quadrics are equivalent to five in virtue of the identity $\Sigma x_r\xi_r = 0$, but, in general, are otherwise linearly independent. Hence, any quadric through the original point and the fifteen points such as P_{1234} is capable of being written in the form $\lambda_1x_1\xi_1 + \ldots + \lambda_6x_6\xi_6 = 0$, wherein, in virtue of $\Sigma x_r\xi_r = 0$, only the differences $\lambda_r - \lambda_s$ are definite. Every quadric through the sixteen points named thus passes through the point common to $\xi_1 = 0, \ldots, \xi_6 = 0$. Thus, conversely, if we suppose the original point, $x_1 = 0, \ldots, x_6 = 0$, to be upon an arbitrary given quadric, Ω, and determine the fifteen points such as P_{1234} as the points of intersection, other than the original point, of the lines such as

$x_1 = x_2 = x_3 = x_4 = 0$ with Ω, then the final point $\xi_1 = \xi_2 = \ldots = \xi_6 = 0$ will also lie on Ω. The $2^5 - 16 - 1$, or fifteen, further intersections of the quadrics $x_r\xi_r = 0$ are the intersections of the fifteen lines, such as $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$, each with a fourfold, such as $x_5 = 0$ or $x_6 = 0$, these giving the same point. There is in fact exact reciprocity, which we can put in evidence by the equations: Let the minors of the elements of the last row of the vanishing skew-symmetrical determinant, of seven rows and columns, which we have considered above, be denoted, respectively, by $A_1, \ldots, A_6, \Delta$; thus Δ is the skew-symmetrical determinant of six rows and columns whose general element is a_{rs}. And let A_{rs} denote the minor of a_{rs} in Δ itself. Then it is easy to see that $A_{rs} = -A_{sr}$, and

$$A_s = a_1A_{1s} + a_2A_{2s} + \ldots + a_6A_{6s},\ (r,\ s = 1,\ 2,\ \ldots,\ 6).$$

Hence we find $\Delta x_s = A_{1s}\xi_1 + \ldots + A_{6s}\xi_6 + A_sv$, and $A_1\xi_1 + \ldots + A_6\xi_6 = 0$. Further, the final point, $\xi_1 = \ldots = \xi_6 = 0$, is given by

$$x_r/v = A_r/\Delta\ (r = 1,\ \ldots,\ 6).$$

Ex. 3. If desired, the proof of the form $\lambda_1x_1\xi_1 + \ldots + \lambda_6x_6\xi_6 = 0$, for the quadric Ω, may be obtained directly. The quadric, passing through $x_1 = x_2 = \ldots = x_6 = 0$, is necessarily capable of an equation of the form

$$\Sigma\,(g_rx_r^2 + 2g_{rs}x_rx_s) + 2v\,(\lambda_1a_1x_1 + \ldots + \lambda_6a_6x_6) = 0,\ (r, s = 1, 2, \ldots, 6);$$

here, for symmetry, we allow terms in a redundant coordinate, say x_6, and $a_1, \ldots, a_6$ are as taken above, in the equations of $\xi_1 = 0, \ldots, \xi_6 = 0$. The condition that this quadric contain the point P_{1234}, for which $x_1 = x_2 = x_3 = x_4 = 0$, $-x_5/a_6 = x_6/a_5 = v/a_{56}$, is one of fifteen equations, which are all of the form

$$2g_{rs} = g_ra_sa_r^{-1} + g_sa_ra_s^{-1} - 2a_{rs}\,(\lambda_r - \lambda_s);$$

if we regard these equations as determining the coefficients g_{rs}, and notice the identities

$$\sum_{r,s}^{1\ldots6}\,[g_rx_r^2 + (g_ra_sa_r^{-1} + g_sa_ra_s^{-1})\,x_rx_s], = (\Sigma g_ra_r^{-1}x_r)\,(\Sigma a_sx_s), = 0,$$

$$\sum_{r,s}\,a_{rs}\,(\lambda_r - \lambda_s)\,x_rx_s + v\,\Sigma\lambda_ra_rx_r = -\sum_r\lambda_rx_r\sum_s\,(a_{rs}x_s - a_rv),$$

the equation of the quadric is obtained in the form specified.

Ex. 4. With a loss of symmetry the algebra may be simplified. We may suppose the fourfold $v = 0$ to be taken so as to be the same as $\xi_6 = 0$, so that $a_{16}, a_{26}, \ldots, a_{56}$ are zero; and may take $\lambda_6 = 0$. That is, we may begin by supposing Ω to have an equation of the form

$$\sum_{r,s}^{1\ldots5}\,c_{rs}x_rx_s - (x_1 + x_2 + \ldots + x_5)\,v = 0,$$

the sixth plane drawn through the original point of Ω being of equation such as $c_1x_1 + \ldots + c_5x_5 = 0$. It is then easy to compute the forms of $\xi_1, \ldots, \xi_5$; and to obtain the result of Ex. 2 as the result of the vanishing of a skew-symmetric determinant of five rows and columns.

Ex. 5. The theorems (3), (4) above, equivalent to the double six theorem in threefold space, may be proved independently of this, as follows: Let X, Y, Z, U, T be points of the quadric Ω, lying in the tangent fourfold of Ω at a point V'. Let T' be the other point of Ω, beside V', which is conjugate to the four points X, Y, Z, U. Take, for the fundamental points of the fivefold space, the six points X, Y, Z, U, V', T', and for coordinates x, y, z, u, v, t; so that $x = 0$ contains all these points except X, and so for $y = 0$, $z = 0$, $u = 0$, the last containing all these points except U; while $v = 0$ contains all these points except V', and is the tangent fourfold of Ω at T'; and $t = 0$ contains all these points except T', and is the tangent fourfold of Ω at V'. The equation of Ω will then be of the form $\phi - \Delta tv = 0$, where ϕ is of the form

$$2fyz + 2gzx + 2hxy + 2pxu + 2qyu + 2rzu,$$

and Δ is the discriminantal determinant of ϕ. The equation $\phi = 0$ represents the quadric, lying on Ω, in the threefold $t = 0$, $v = 0$; this quadric we denote by H. Let the minors in Δ be A, B, C, D, F, G, H, P, Q, R.

The coordinates of the point T, lying in the tangent fourfold, $t = 0$, of Ω, at V', we denote by $(\xi, \eta, \zeta, \omega, 2, 0)$. The quadric H contains then the point $(\xi, \eta, \zeta, \omega, 0, 0)$; thus this quadric also contains (Vol. III, p. 55) the point $(\xi_1, \eta_1, \zeta_1, \omega_1, 0, 0)$, where

$$\xi_1 = A\xi^{-1}, \quad \eta_1 = B\eta^{-1}, \quad \zeta_1 = C\zeta^{-1}, \quad \omega_1 = D\omega^{-1}.$$

Therefore Ω contains the point $(\xi_1, \eta_1, \zeta_1, \omega_1, 0, 2)$. We prove that this is, in fact, the final point of the theorem under consideration.

Let X' be the point of Ω, beside V', which is conjugate to Y, Z, U, T; so Y' the point of Ω, beside V', conjugate to Z, X, U, T; and Z' the point of Ω, beside V', conjugate to X, Y, U, T; and U', the point of Ω, beside V', conjugate to X, Y, Z, T. We can shew that the coordinates of these points are, respectively,

$$X'(A, H, G, P, \xi_1, \xi);\quad Y'(H, B, F, Q, \eta_1, \eta);$$
$$Z'(G, F, C, R, \zeta_1, \zeta);\quad U'(P, Q, R, D, \omega_1, \omega).$$

We have defined T' as the point of Ω, beside V', conjugate to X, Y, Z, U. We then shew that the point $(\xi_1, \eta_1, \zeta_1, \omega_1, 0, 2)$ is conjugate to X', Y', Z', U', T'.

The point of coordinates (A, H, G, P, ξ_1, ξ) is at once verified to lie on Ω. The tangent fourfold at this point is found to be

$2x - t\xi_1 - v\xi = 0$. This contains the points Y, Z, U, all lying on $x = v = t = 0$; and it contains T, or $(\xi, \eta, \zeta, \omega, 2, 0)$. A precisely similar verification is possible for Y', Z', U'. Again, the tangent fourfold at $(\xi_1, \eta_1, \zeta_1, \omega_1, 0, 2)$ is expressed by

$$\xi_1 (hy + gz + pu) + \eta_1 (hx + fz + qu) \\ + \zeta_1 (gx + fy + ru) + \omega_1 (px + qy + rz) = \Delta v;$$

it is at once seen that this contains the points X', Y', Z', U', T'. This proves what has been stated.

Note to p. 20. Wallace's theorem, that the four conics, through two points, I, J, of a plane, each containing the intersections of three out of four lines of the plane, all have another common point, may also be proved by remarking that, if four arbitrary planes be drawn, one through each line, meeting in threes in X, Y, Z, T, then the cubic curve through I, J, X, Y, Z, T meets the plane in another point. For the four conics are the projections of this cubic, from X, Y, Z, T, respectively.

More generally if, in n-fold space, there be given two points I, J, and also $(n+2)$ *primes* (spaces of $(n-1)$-dimensions), and we consider the $(n+2)$ rational curves of order n, each passing through I, J and the $(n+1)$ intersections of $(n+1)$ of these primes, then these $(n+2)$ curves have $(n-1)$ further common points. For these curves are all projections of a curve of order $(n+1)$, in space of $(n+1)$ dimensions, defined as passing through I, J and through the intersections of $(n+2)$ spaces of dimension n, arbitrarily drawn through the given $(n+2)$ primes.

For generalisation of this to the proof of Clifford's chain of theorems (above, p. 31), see F. P. White, *Camb. Phil. Proc.*, XXII, 1925. For instance, that the foci of the five parabolas which touch the fours of five given lines, lie on a circle, arises from the fact that the chords of a rational quartic curve, in fourfold space, which meet a plane meeting the curve in two points, meet this plane in the points of a conic passing through these two points.

Wallace's theorem is also a particular case of the theorem that plane cubic curves through eight points have another common point (Vol. III, p. 217).

CHAPTER II

HART'S THEOREM, FOR CIRCLES IN A PLANE, OR FOR SECTIONS OF A QUADRIC

GIVEN three lines in a plane, there are four circles touching them; these circles, we know, are all touched by another circle, the nine-points circle (Feuerbach's theorem; see Vol. II). In other words, given three lines, we can add to them a circle such that the four, these lines and the circle, are all touched by four other circles.

In the present chapter we shew how, given any three circles in a plane, we can add to them another circle, which we call the Hart circle, such that the four circles are all touched by four other circles (Hart, *Quart. J. of Maths.*, IV (1861), p. 260).

The three original circles are in fact touched by eight other circles, as we shall prove. There are fourteen ways of choosing, from these eight, four circles which all touch another circle. In six of these ways, the four circles chosen have a common orthogonal circle; and the four circles consisting of the original circles, and their Hart circle, have also a common orthogonal circle.

We have shewn that circles in a plane may be regarded as projections of plane sections of a quadric. We prove the results enunciated as theorems for such plane sections. This appears greatly to increase the interest and clearness of the matter.

The sections of a quadric which touch three given sections. Let the lines of intersection of the pairs of the planes of three sections of a quadric be OX, OY, OZ. Let the sections in the planes YOZ, ZOX, XOY be named, respectively, α, β, γ; let the common points of β, γ, on the line OX, be A and A'; the common points of γ, α, on OY, be B and B'; and the common points of α, β, on OZ, be C and C'. Let the poles of the planes YOZ, ZOX, XOY, in regard to the quadric, be, respectively, P, Q, R, so that the plane PQR is the polar plane of O. The conics β, γ, having two common points, lie on two quadric cones; the vertices of these cones lie on the line QR, and are harmonic conjugates in regard to Q and R; the polar plane of either vertex, in regard to the quadric, contains the line of intersection of the planes of the sections, and also contains the other vertex. Any plane section of the quadric, which touches both the sections β, γ, is a tangent plane of one of the two cones. When the sections are projected on to any plane, from any point of the quadric, they become two curves which we have agreed to call circles, the projections of Q and R being the centres of the circles; the projections of the vertices of the two cones which contain β and γ are the centres of similitude of these circles. Each centre of similitude is the centre of a circle, coaxial with the two circles, in regard to which these two circles are inverses of one another.

Consider, then, the six quadric cones, two containing every pair of the three sections α, β, γ; by Pascal's theorem, for the section of the quadric by the plane PQR, we see that these vertices lie, in threes, on four lines in this plane. Let l be one of these lines, containing the vertices of three of these cones, each cone containing two of the sections α, β, γ. Consider two of these three cones; as they have one common plane section (one of α, β, γ), they will have another common plane section; they will therefore have two common tangent planes, which must intersect in the line l containing the vertices of the two cones. This shews that a tangent plane drawn from l to one of the three cones whose vertices are on l, is equally a tangent plane of the other two cones. This plane will then meet the quadric in a section touching the three conics α, β, γ. As there are four lines l, we thus construct eight sections of the quadric touching the three sections α, β, γ.

Ex. 1. If two fixed sections α, β, of a quadric, which meet in A, B, be both touched by two other sections ξ, η, which also touch one another, and the planes of ξ, η meet the line AB in P and Q, the range A, B, P, Q is related to the range A, B, P', Q', similarly obtained from two other such sections, ξ', η'. It is understood that the sections ξ, η touch the same one of the cones which contain α and β; if V be the vertex of this cone, the theorem is obtained by considering, on the plane of the polar of V, the section both by the quadric and by the cone. We then have two conics which touch at two points, and a pair of tangent lines, to one of these, drawn from a variable point of the other.

Ex. 2. The problem of drawing a circle to touch three given circles in a plane is sometimes called Apollonius's problem. Deduce from what is said above the following solution of this, given by Gergonne (*Ann. d. Math.*, VII, 1817, p. 289): Find O, the centre of the common orthogonal circle of the three given circles; take one of the four lines which contain three of the centres of similitude of pairs of the circles, and let U, V, W be the respective poles of this line in regard to the three given circles. Then the joining lines OU, OV, OW contain the points of contact with the given circles of two of the circles which touch them all. (Cf. Poncelet, *Prop. Proj.*, I, 1865, p. 138.)

Ex. 3. Prove that there are four circles touching three lines in a plane. Also, that the circle through the centres of three of these circles, and the circle containing the three intersections of the lines, are two circles which have two common tangents, meeting in the centre of the fourth of the original circles.

Ex. 4. Regarding two circles in a plane as projections of plane sections, α, β, of a quadric, prove that a circle cutting these circles at equal angles is the projection of a section of the quadric by a plane passing through the vertex of one of the two quadric cones containing the sections α, β. Hence, given three circles in a plane, prove that a circle cutting these three circles all at equal angles belongs to one of four systems of coaxial circles, of which the radical axis contains three of the centres of similitude of pairs of the three given circles.

Ex. 5. Consider four plane sections of a quadric, and the vertices of cones containing pairs of these sections, twelve points in all. The plane containing the poles of three of the four given planes contains, we have seen, six of these vertices. Omitting the four such planes, prove that the twelve points, with their joining lines and planes, form a figure of structure 12 (·, 4, 4) 16 (3, ·, 2) 8 (6, 4, ·), arising from two of three desmic tetrads (Vol. II, p. 213). The eight planes of this figure enable us to construct eight circles meeting four circles, given in a plane, all at equal angles.

Ex. 6. Regarding two circles in a plane as projections of plane sections of a quadric, the condition that the two circles should cut at a given angle is that the range, of four points, consisting of the two poles of the corresponding plane sections of the quadric, and the two points where the join of these meets the quadric, should be related to a given range. Prove that a circle cutting two given circles at given angles cuts any circle coaxial with the given circles at a constant angle, and touches two fixed circles. Construct a circle cutting three given circles at given angles (Steiner, *Ges. Werke*, I, p. 21, 1826).

Ex. 7. If three circles have three tangents, each touching two of the circles, which meet in a point, they have three other tangents, also each touching two of the circles, which likewise meet in a point.

Let α, β, γ be three plane sections of a quadric, U a quadric cone containing β and γ, V a quadric cone containing γ and α, and W a quadric cone containing α and β. The pair of cones V, W, as both containing α, have two common tangent planes; so W and U have two common tangent planes, and U and V have two common tangent planes. There is thus a quadric, say Σ, of which U, V, W are all enveloping cones (Vol. III, p. 10). Any section of the original quadric touching β and γ lies in a plane touching one of the two cones containing β and γ; if this cone be U, this plane touches the quadric Σ. Consider three sections of the original quadric whose planes meet in a line, l, the planes of these sections touching, respectively, the cones U, V, W; these planes, therefore, all touch the quadric Σ. Of this quadric, then, the line l is a generator. Let m be any generator of Σ which meets l; this lies in a particular tangent plane of every enveloping cone of Σ. There is therefore a plane through m touching the sections β, γ; another plane through m touching the sections γ, α; and another plane through m touching the sections α, β.

Regard, now, three circles in a plane as projections of three sections of a quadric, the centre of projection being a point, H, of this quadric. A common tangent line of two of the circles is the projection of a section of the quadric, touching the two given sections, by a plane which passes through H. Three concurrent tangents, each of two of the given circles, are thus projections of three sections of the quadric by planes which intersect in a line, l, passing through H. By taking for m the other generator of the quadric Σ which passes through H, the theorem above stated is obtained.

Ex. 8. Given three circles in a plane, construct three other circles each to touch two of the given circles, each of the constructed circles, moreover, touching the other two. For the case when the given circles are lines, the problem was put by Malfatti, *Mem. d. Soc. Ital. d. Sc.*, Modena, 1803, Vol. x. For this case, and the general one, a solution was enunciated, without proof, by Steiner, *Ges. Werke*, I, pp. 35–39. Steiner's construction was justified by Hart, *Quarterly J.*, I, 1857, p. 219. An analytical investigation of the corresponding theorem for sections of a quadric was given by Cayley, *Papers*, II, p. 57 (1852).

The theorem is as follows: Let α, β, γ be three plane sections of a quadric, whose planes meet in O, the poles of these planes being P, Q, R, respectively; let U be the vertex of a cone containing the sections β, γ, the vertex of a cone containing γ and α being V, and of a cone containing α and β being W, these points U, V, W being in a line in the plane PQR. Next, let θ, ϕ, ψ be, respectively, the sections by the polar planes of U, V, W; the planes of these will meet in a line passing through O, and the sections will have two points in common; let I be one of these common points. The section θ, for example, contains the two points common to the sections β, γ, and the plane of θ contains the vertex, other than U, of a cone containing the sections β, γ. Now, let λ be a section touching ϕ, ψ and α, the point of contact with α being A; similarly, let μ be a section touching ψ, ϑ, and touching β, in the point B; and let ν be a section touching ϑ, ϕ, and touching γ, in the point C. The sections λ, μ, ν, being touched in pairs by sections, θ, ϕ, ψ, whose planes meet in the line OI, are also (Ex. 7) touched in pairs by sections, say θ', ϕ', ψ', whose planes meet in another line OI'; it can in fact be shewn, the sections λ, μ, ν being properly taken, that there is a section, with plane passing through O, which passes through A, and touches μ and ν; let this be θ'; and that there is a section, ϕ', through O and B, touching ν and λ, and a section, ψ', through O and C, touching λ and μ. There is then a section touching β, γ, and ϕ', ψ', say ξ; also a section, η, touching γ, α and ψ', θ'; and a section, ζ, touching α, β and θ', ϕ'. Further, the sections η, ζ touch one another, touching θ' at the same point; similarly ζ and ξ touch one another, and ξ and η touch one another. The sections ξ, η, ζ are those to be constructed.

Ex. 9. We may easily investigate the equations of the sections touching the three given sections α, β, γ. Let the planes of these be, respectively, $x = 0$, $y = 0$, $z = 0$, the polar plane of the point, O, in which these meet, being $t_1 = 0$, so that the equation of the quadric is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = t_1^2.$$

Let $A = bc - f^2$, $F = gh - af$, etc., these being the minors of a, f, etc. in the determinant of the quadratic form, Δ, whose value is

$$abc + 2fgh - af^2 - bg^2 - ch^2.$$

The pole, P, of the plane α has the coordinates $(A, H, G, 0)$, the poles of Q and R being, respectively, $(H, B, F, 0)$ and $(G, F, C, 0)$. The equation of a quadric cone containing the sections β, γ is of the form $(a, b, c, f, g, h \between x, y, z)^2 + 2\lambda yz = t_1^2$, and λ is such that the three derivatives of the left side all vanish at the vertex, $(x', y', z', 0)$, of this cone. With $\lambda = \Delta\,[(BC)^{\frac{1}{2}} - F]^{-1}$, these are satisfied by taking

$$(x', y', z') = B^{-\frac{1}{2}}(H, B, F) - C^{-\frac{1}{2}}(G, F, C),$$

that is, $x' = B^{-\frac{1}{2}}.H - C^{-\frac{1}{2}}.G$, $y' = B^{-\frac{1}{2}}.B - C^{-\frac{1}{2}}.F$, etc. Herein the signs of $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$ are arbitrary, and $(BC)^{\frac{1}{2}}$ means $B^{\frac{1}{2}}.C^{\frac{1}{2}}$, etc. If, for the sake of brevity, we use X to denote the set of four coordinates $A^{-\frac{1}{2}}(A, H, G, 0)$, with, similarly, Y and Z for the sets $B^{-\frac{1}{2}}(H, B, F, 0)$, $C^{-\frac{1}{2}}(G, F, C, 0)$, then three cones, each containing a pair of the sections α, β, γ, with vertices on a line, l, are those whose vertices have coordinates, respectively, $Y - Z$, $Z - X$, $X - Y$. It is then easy to verify that the plane whose equation is

$$(ax + hy + gz)\,A^{\frac{1}{2}} + (hx + by + fz)\,B^{\frac{1}{2}} + (gx + fy + cz)\,C^{\frac{1}{2}} = t_1 M^{\frac{1}{2}}$$

contains this line l, and touches the three sections α, β, γ, provided M satisfy

$$M = aA + bB + cC + 2f(BC)^{\frac{1}{2}} + 2g(CA)^{\frac{1}{2}} + 2h(AB)^{\frac{1}{2}} - \Delta;$$

for the condition that two sections of the quadric touch one another is that the enveloping cone from the pole of the plane of either section should contain the pole of the plane of the other section.

The possible signs of $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, $M^{\frac{1}{2}}$ give the eight sections touching α, β, γ.

Ex. 10. Through any one of the sections α, in Ex. 9, pass two of the cones used in the construction. These cones have then another plane section; shew that this lies on the plane whose equation is

$$y\left[(CA)^{\frac{1}{2}} - G\right] = z\left[(AB)^{\frac{1}{2}} - H\right].$$

The three such planes thus meet in a line, passing through the point O in which the planes α, β, γ intersect.

Ex. 11. If from the vertex $(x', y', z', 0)$, above taken, of a cone containing the sections β, γ, a cone be drawn to contain the section α, prove that this cone meets the quadric again in a section lying on a plane, through O, whose equation is

$$\left[(BC)^{\frac{1}{2}} - F\right]x + \left(GB^{\frac{1}{2}} - HC^{\frac{1}{2}}\right)\left(B^{-\frac{1}{2}}y - C^{-\frac{1}{2}}z\right) = 0.$$

Ex. 12. Prove that the polar planes of the vertices of the three cones used in Ex. 9, which are on the line l, are respectively

$$B^{-\frac{1}{2}}y - C^{-\frac{1}{2}}z = 0, \quad C^{-\frac{1}{2}}z - A^{-\frac{1}{2}}x = 0, \quad A^{-\frac{1}{2}}x - B^{-\frac{1}{2}}y = 0.$$

These are the sections θ, ϕ, ψ of the enunciation in Ex. 8; the point I is then on the line $A^{-\frac{1}{2}}x = B^{-\frac{1}{2}}y = C^{-\frac{1}{2}}z$.

Ex. 13. Shew that if we introduce λ, μ, ν such that

$$\cos\lambda = (bc)^{-\frac{1}{2}}f, \quad \sin\lambda = (bc)^{-\frac{1}{2}}A^{\frac{1}{2}}, \text{ etc.},$$

and take $s = \frac{1}{2}(\lambda + \mu + \nu)$, the section above found, touching α, β, γ, lies in the plane

$$xa^{\frac{1}{2}}\cos(s - \lambda) + yb^{\frac{1}{2}}\cos(s - \mu) + zc^{\frac{1}{2}}\cos(s - \nu) - t_1 = 0.$$

Herein the signs of $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$ are arbitrary, and $(bc)^{\frac{1}{2}}$ means $b^{\frac{1}{2}}.c^{\frac{1}{2}}$; a change of $a^{\frac{1}{2}}$ into $-a^{\frac{1}{2}}$, for example, leads to the substitution of $\pi + \mu$, $\pi + \nu$, $\pi + s$ respectively for μ, ν and s. Four of the eight tangent sections are obtained by substituting $\pm\lambda$, $\pm\mu$, $\pm\nu$, respectively, for λ, μ, ν, retaining the signs of $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$; the other four are then obtained by changing $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$ into $-a^{\frac{1}{2}}$, $-b^{\frac{1}{2}}$, $-c^{\frac{1}{2}}$, in the first four.

Ex. 14. With the substitution of Ex. 13, the plane of Ex. 11 becomes

$$xa^{\frac{1}{2}} + \left(yb^{\frac{1}{2}}\sin\nu - zc^{\frac{1}{2}}\sin\mu\right)\sin(\mu - \nu) = 0.$$

Ex. 15. Taking p, q, r such that

$$\tfrac{1}{2}p = 1 - F(BC)^{-\frac{1}{2}}, \quad \tfrac{1}{2}q = 1 - G(CA)^{-\frac{1}{2}}, \quad \tfrac{1}{2}r = 1 - H(AB)^{-\frac{1}{2}},$$

prove that a section touching the sections by $x=0$, $zC^{-\frac{1}{2}}-xA^{-\frac{1}{2}}=0$, $xA^{-\frac{1}{2}}-yB^{-\frac{1}{2}}=0$, namely the section named λ in the enunciation of Ex. 8, lies in the polar plane of the point (ξ, η, ζ, τ), where $\xi=A^{\frac{1}{2}}$, $\eta=B^{\frac{1}{2}}(1-r^{\frac{1}{2}})$, $\zeta=C^{\frac{1}{2}}(1+q^{\frac{1}{2}})$, and τ is given by

$$\tau^2=(a, b, c, f, g, h \between \xi, \eta, \zeta)^2-\Delta.$$

If we put $-F(BC)^{-\frac{1}{2}}=\cos\lambda'$ (as in Vol. II, p. 205), we have $p^{\frac{1}{2}}=2\cos\frac{1}{2}\lambda'$, etc.

The Hart circles of three circles when the four have a common orthogonal circle. We consider, as before, three sections α, β, γ, of a quadric, whose planes meet in O. Let U be the vertex of one of the two quadric cones containing the sections β and γ. The polar plane of U, as has been remarked, contains the line of intersection of the planes of β and γ; in fact, if these sections meet in A and A', the tangent planes of the quadric, at A and A', touch these sections, and pass through U. If we consider, also, the quadric cone obtained by joining U to the points of the section α, this cone will meet the quadric again in a plane section, say α'; thence, the line in which the plane of α' meets the plane of α, lying equally in the polar plane of U, will meet the line of intersection of the planes β, γ; in other words, the plane of α' passes through the point O in which the planes of α, β, γ intersect. But also the two cones of vertex U, namely $(U;\beta,\gamma)$ and $(U;\alpha,\alpha')$, have four common tangent planes; and these will touch all the four sections α, β, γ, α'.

Thus α' is a section which, added to α, β, γ, gives four sections which have four common tangent sections. Thus, on projection of the quadric, the section α' gives a circle, orthogonal to the common orthogonal circle of α, β, γ; such that the four are all touched by four circles; these likewise have a common orthogonal circle, whose centre is the projection of U. The circle given by α' is a particular kind of Hart circle, of those belonging to α, β, γ.

Since any chord of the quadric passing through O is divided harmonically by O and the polar plane of O, it is clear that the eight sections, which touch the sections α, β, γ, consist of four pairs, such that the planes of a pair intersect on the polar plane of O. Also, it is clear that there are six such sets of four sections as those found above, touching β, γ and α, α'; for there are two cones containing every two sections such as β and γ. Each of these sets consists, then, of two of the four pairs into which the eight tangent sections of α, β, γ can be divided.

The equation of the plane of the section α' has been found above, in Exx. 11 and 14. Four planes touching the sections α, β, γ which all pass through the point U, whose coordinates are

$$(x', y', z', 0)=B^{-\frac{1}{2}}(H, B, F, 0)-C^{-\frac{1}{2}}(G, F, C, 0),$$

are found from

$$(ax+hy+gz)A^{\frac{1}{2}}+(hx+by+fz)B^{\frac{1}{2}}+(gx+fy+cz)C^{\frac{1}{2}}=t_1M$$

by giving to $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$ the signs $(+,+,+)$, $(+,-,-)$, $(-,+,+)$, $(-,-,-)$, respectively.

The Hart circles of three circles in general. Given, as before, three sections α, β, γ of a quadric, we now investigate a fourth section δ, whose plane does not pass through the intersection of the planes, α, β, γ, such that α, β, γ, δ are all touched by four other sections of the quadric.

It will appear that the necessary and sufficient condition that four sections of a quadric, by planes which do not meet in a point, should all be touched by four other planes, is very simple. It may add to the interest of the work to state at once what this condition is. Let the planes of the four sections, α, β, γ, δ meet in threes in the points O, X, Y, Z, the sections β, γ meeting OX in A and A', the sections γ, α meeting OY in B and B', the sections α, β meeting OZ in C and C', the sections α, δ meeting YZ in L and L', the sections β, δ meeting ZX in M and M', and the sections γ, δ meeting XY in N and N'. Through each of the six joins of O, X, Y, Z draw the two planes to the points in which the quadric meets the opposite join; as, for instance, draw through OX the planes OXL and OXL'. Thereby twelve planes are found. In general these twelve planes touch another quadric (Vol. III, p. 54, Ex. 16). It may happen, however, that this quadric degenerates into two points, the planes meeting, in sets of six, in these two points. In such case, choosing the notation properly, we may suppose that the planes OXL, OYM, OZN, YZA, ZXB, XYC meet in a point, say S, the other six planes, OXL', ..., YZA', ..., meeting in a point S'. Then the line AL, as lying in the planes OXL, YZA, passes through S, and so on; namely, the three lines AL, BM, CN meet, in the point S. Conversely, if these three lines meet in a point, say S, the six planes OXL, ..., XYC are easily seen to meet in S; and then, similarly, the three lines $A'L'$, $B'M'$, $C'N'$ also meet in a point, say S'. The condition that this be so may be expressed otherwise: Beginning, as before, with the general case, let points, say A_1 and A_1', be taken on OX, so as to be the respective harmonic conjugates of A and A' in regard to O and X; and let a pair of points be similarly defined on each of the six joins of O, X, Y, Z. It is then easy to see that the twelve new points so obtained lie upon another quadric; we may call this the *harmonic conjugate* of the original quadric in regard to the tetrad O, X, Y, Z. In the particular case when AL, BM, CN meet in a point, say S, the harmonically conjugate quadric breaks up into two planes; and conversely. These two planes are, in fact, the polars of the points S and S', in regard to the original quadric.

What we prove is: If the harmonically conjugate quadric of the original quadric, in regard to O, X, Y, Z, break up into two planes, then the sections of the original quadric by the four planes XYZ, OYZ, OZX, OXY are all touched by four planes. We afterwards prove, conversely, that, for such tangency, it is necessary that the harmonically conjugate quadric should so degenerate.

As before, let $x = 0$, $y = 0$, $z = 0$ be the planes of the sections α, β, γ, the equation of the quadric being

$$(a, b, c, f, g, h \between x, y, z)^2 = t_1^2,$$

where $t_1 = 0$ is the polar plane of $(0, 0, 0, 1)$. For $t_1 = 0$ we can substitute a plane, $t = 0$, in eight ways, so that the condition referred to, that AL, BM, CN meet in a point, is satisfied. For, taking arbitrary signs for the radicals $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$, and denoting $b^{\frac{1}{2}}c^{\frac{1}{2}}$ by $(bc)^{\frac{1}{2}}$, etc., let

$$u = \tfrac{1}{2}\left[f + (bc)^{\frac{1}{2}}\right], \quad v = \tfrac{1}{2}\left[g + (ca)^{\frac{1}{2}}\right], \quad w = \tfrac{1}{2}\left[h + (ab)^{\frac{1}{2}}\right],$$

and $\quad l = (uvw)^{\frac{1}{2}} u^{-1}, \quad m = (uvw)^{\frac{1}{2}} v^{-1}, \quad n = (uvw)^{\frac{1}{2}} w^{-1},$

so that $mn = u$, $nl = v$, $lm = w$. As the product of $(bc)^{\frac{1}{2}}$, $(ca)^{\frac{1}{2}}$, $(ab)^{\frac{1}{2}}$ is without ambiguity there are thus eight possible sets of values for l, m, n. Putting $t = lx + my + nz - t_1$, the equation of the quadric referred to $x = 0$, $y = 0$, $z = 0$, $t = 0$, is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = (lx + my + nz - t)^2;$$

this meets OX in the points A, A' given by $(l \pm a^{\frac{1}{2}})x = t$. Taking for X, Y, Z the points where OX, OY, OZ meet $t = 0$, the points, L, L', where the quadric meets YZ, are given by

$$(m^2 - b)y^2 + 2(mn - f)yz + (n^2 - c)z^2 = 0;$$

since $f = 2mn - (bc)^{\frac{1}{2}}$, these points are given by

$$(m \pm b^{\frac{1}{2}})y = (n \pm c^{\frac{1}{2}})z.$$

Hence, with suitable choice of notation, the lines AL, BM, CN meet in the point, S, given by $(l - a^{\frac{1}{2}})x = (m - b^{\frac{1}{2}})y = (n - c^{\frac{1}{2}})z = t$, and the lines $A'L'$, $B'M'$, $C'N'$ meet in the point, S', given by $(l + a^{\frac{1}{2}})x = \ldots = t$. Conversely, it can be shewn that these intersections require the plane $t = 0$ to be one of the eight we have taken. In general, the harmonically conjugate quadric of that given by the equation $(a, b, c, d, f, g, h, p, q, r \between x, y, z, t)^2 = 0$ has an equation obtained from this by change of f, g, h, p, q, r, respectively, into $-f$, $-g$, $-h$, $-p$, $-q$, $-r$. In the present case, since $f = 2mn - (bc)^{\frac{1}{2}}$, $g = 2nl - (ca)^{\frac{1}{2}}$, $h = 2lm - (ab)^{\frac{1}{2}}$, the harmonically conjugate quadric is found to be

$$(xa^{\frac{1}{2}} + yb^{\frac{1}{2}} + zc^{\frac{1}{2}})^2 - (lx + my + nz + t)^2 = 0.$$

The point S, or $(l-a^{\frac{1}{2}})x = \ldots = t$, is then easily verified to be the pole of the plane $(l-a^{\frac{1}{2}})x+(m-b^{\frac{1}{2}})y+(n-c^{\frac{1}{2}})z+t=0$, in regard to the original quadric. This plane we denote by σ, the plane which is the polar of S' being denoted by σ'.

To prove, now, that the sections α, β, γ, and the section, say δ, of the quadric, by $t=0$, are all touched by four planes, consider the section by the plane

$$xa^{\frac{1}{2}}\cos(s-\lambda)+yb^{\frac{1}{2}}\cos(s-\mu)+zc^{\frac{1}{2}}\cos(s-\nu)-t_1=0,$$

above found (p. 70, Ex. 13) as touching the sections α, β, γ; here $\cos\lambda=(bc)^{-\frac{1}{2}}f$, etc., and $s=\frac{1}{2}(\lambda+\mu+\nu)$. The cone joining O to the section of the quadric by this plane is given by the equation

$$ax^2+\ldots+cz^2+2yz\,(bc)^{\frac{1}{2}}\cos\lambda+\ldots+2xy\,(ab)^{\frac{1}{2}}\cos\nu-[xa^{\frac{1}{2}}\cos(s-\lambda)+\ldots+zc^{\frac{1}{2}}\cos(s-\nu)]^2=0.$$

This is easily found to be the same as is obtained by rationalising the equation

$$[xa^{\frac{1}{2}}\sin(s-\lambda)]^{\frac{1}{2}}+[yb^{\frac{1}{2}}\sin(s-\mu)]^{\frac{1}{2}}+[zc^{\frac{1}{2}}\sin(s-\nu)]^{\frac{1}{2}}=0.$$

Putting p, q, r as abbreviations for $\sin(s-\lambda)$, $\sin(s-\mu)$, $\sin(s-\nu)$, respectively, any generator of the cone is given, in terms of a parameter, θ, by

$$(xa^{\frac{1}{2}}p)^{\frac{1}{2}}:(yb^{\frac{1}{2}}q)^{\frac{1}{2}}:(zc^{\frac{1}{2}}r)^{\frac{1}{2}}=-\theta:\theta-1:1;$$

in particular, one generator is given by

$$xa^{\frac{1}{2}}:yb^{\frac{1}{2}}:zc^{\frac{1}{2}}=p(q-r)^2:q(r-p)^2:r(p-q)^2,$$

and the tangent plane of the cone along this generator has the equation

$$xa^{\frac{1}{2}}(q-r)^{-1}+yb^{\frac{1}{2}}(r-p)^{-1}+zc^{\frac{1}{2}}(p-q)^{-1}=0 \quad \ldots(\text{A}).$$

But, in virtue of $f=(bc)^{\frac{1}{2}}\cos\lambda$, and $2u=f+(bc)^{\frac{1}{2}}$, we have $u=(bc)^{\frac{1}{2}}\cos^2\frac{1}{2}\lambda$; and, thence, by $l=(uvw)^{\frac{1}{2}}u^{-1}$, etc., we may suppose the plane $t=0$ to be that given by

$$xa^{\frac{1}{2}}\frac{\cos\frac{1}{2}\mu\cos\frac{1}{2}\nu}{\cos\frac{1}{2}\lambda}+yb^{\frac{1}{2}}\frac{\cos\frac{1}{2}\nu\cos\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu}+zc^{\frac{1}{2}}\frac{\cos\frac{1}{2}\lambda\cos\frac{1}{2}\mu}{\cos\frac{1}{2}\nu}-t_1=0 \quad \ldots\ldots\ldots(\text{H}).$$

There is ambiguity here in the sign of t_1; we can however suppose that the plane

$$xa^{\frac{1}{2}}\cos(s-\lambda)+yb^{\frac{1}{2}}\cos(s-\mu)+zc^{\frac{1}{2}}\cos(s-\nu)-t_1=0 \quad \ldots(\text{I}),$$

above taken from among the tangent planes of the sections α, β, γ, is chosen to have the same sign for t_1. Then the planes, (I), (H), meet in a line which lies in the plane

$$xa^{\frac{1}{2}}\left[\cos(s-\lambda)-\frac{\cos\frac{1}{2}\mu\cos\frac{1}{2}\nu}{\cos\frac{1}{2}\lambda}\right]$$
$$+\ldots+zc^{\frac{1}{2}}\left[\cos(s-\nu)-\frac{\cos\frac{1}{2}\lambda\cos\frac{1}{2}\mu}{\cos\frac{1}{2}\nu}\right]=0;$$

this, however, is the same, as we easily see, as the tangent plane (A), found above, of the cone which joins O to the section of the quadric by the plane (I). As the line of intersection of the planes (I), (H) is thus a tangent line of this cone, and is, therefore, a tangent line of the section of the quadric by the plane (I), it is a tangent line of the quadric; and, as the plane (H) contains this line, the section of the quadric by the plane (H) has this line for a tangent. The sections of the quadric by the planes (I) and (H) thus touch one another; or the plane (I), beside touching the sections α, β, γ, touches also the section, δ, by the plane (H). This section δ is one of the Hart sections to be associated with α, β, γ. For a definite plane (H) we have four possible planes (I), obtainable, namely, by the changes of λ, μ, ν respectively into λ, $\pm\mu$, $\pm\nu$; all these touch the four sections α, β, γ, δ.

In all there are eight possibilities for the plane (H). There are four in which, in place of $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$, λ, μ, ν, we have: (1), these; (2), $-a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$, λ, $\pi+\mu$, $\pi+\nu$, respectively; (3), $a^{\frac{1}{2}}$, $-b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$, $\pi+\lambda$, μ, $\pi+\nu$, respectively; and (4), $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $-c^{\frac{1}{2}}$, $\pi+\lambda$, $\pi+\mu$, ν. Then there are four others obtainable from these by changing the signs of all of $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$. Thus, given three plane sections of a quadric, it is possible, in eight ways, to add to these another section, by a plane not passing through the common point of the three given planes, so that the aggregate four sections shall have four common tangent planes. Of this result another proof is contained in the work given below to shew that the conditions we have here verified to be sufficient, are also necessary.

Ex. 1. Make an explicit table of the equations of the eight planes (H), and, in each case, of the four planes touching the sections determined by these.

Ex. 2. The planes ABC, $A'B'C'$, where A, ..., C' are as above, may be taken to be

$$(l-a^{\frac{1}{2}})x+(m-b^{\frac{1}{2}})y+(n-c^{\frac{1}{2}})z-t=0,$$

and
$$(l+a^{\frac{1}{2}})x+(m+b^{\frac{1}{2}})y+(n+c^{\frac{1}{2}})z-t=0,$$

respectively. The line of intersection of these planes is the intersection of the polar plane $t_1=0$ with the plane $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$. On the section of the quadric by the plane (I) we have two triads of points: (1), the points of contact of this section with $x=0$, $y=0$, $z=0$; if the points of the section be given, as above, in terms of a parameter θ, by the fact that $(a^{\frac{1}{2}}px)^{\frac{1}{2}}$, $(b^{\frac{1}{2}}qy)^{\frac{1}{2}}$, $(c^{\frac{1}{2}}rz)^{\frac{1}{2}}$ are in the ratios of $-\theta$, $\theta-1$, 1, then these points of contact are given by $\theta=0, 1, \infty$, respectively; (2), the point of contact of the section (I) with the plane $t=0$, taken with the two points in which the section (I) is met by the above plane $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$. The first point of this triad (2) is given, we have seen, by the value $-p(q-r)/r(p-q)$ of the parameter θ; the parameters for the other two points, in which the section (I) is met by the plane we have named, are the roots of the equation

$$p^{-1}\theta^2+q^{-1}(\theta-1)^2+r^{-1}=0.$$

Hence it is easily seen that the points of the first triad are apolar with the points of the second triad, in the sense explained in Vol. II, p. 114. For the condition that θ, θ_1, θ_2 should be apolar with $0, 1, \infty$ is at once seen to be that $\theta=(\theta_1\theta_2-\theta_1-\theta_2)/(1-\theta_1-\theta_2)$; and, when θ_1, θ_2 are the roots of the quadratic equation just given, this leads to $\theta=-p(q-r)/r(p-q)$. Hence we may say: Upon the section (I), touching the three sections α, β, γ, the point of contact with the Hart section (H) is the apolar complement of the two points in which (I) is met by the plane $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$, with reference to the three points of contact of (I) with α, β, γ. The plane $a^{\frac{1}{2}}x+\text{etc.}=0$ is the harmonic conjugate of the point O, in which the planes α, β, γ meet, with respect to the planes ABC and $A'B'C'$; it will receive a better definition below (Ex. 5).

Ex. 3. As before, denote by S the point of intersection of the lines AL, BM, CN, and by S' the point of intersection of the lines $A'L'$, $B'M'$, $C'N'$. Also, denote by σ, σ' the respective polar planes of S and S'. Further, denote the planes ABC, $A'B'C'$ by μ, μ', respectively, and the plane of the section (H) by ϖ; let G, G' and H be the poles of the planes μ, μ' and ϖ, respectively. As we have seen, the equations of the planes σ, μ and ϖ are, respectively,

$$(l-a^{\frac{1}{2}})x+\ldots+t=0,\quad (l-a^{\frac{1}{2}})x+\ldots-t=0,\quad t=0.$$

It can be shewn:

(a), that the planes μ, μ' meet in a line lying on the polar plane of O; the plane, $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$, which joins O to this line, contains also the line of intersection of the planes σ, σ'.

(b), that the two quadric cones which contain the sections by the planes μ and ϖ have their vertices at S, and at the point where OS' meets the plane σ. The former is clear because AL, BM, CN meet in S. For the latter, since O, the point of intersection of AA', BB', CC', is the vertex of one of the cones containing the sections μ, μ', and S' is the vertex of one of the cones containing the sections μ', ϖ, therefore the line OS' contains the vertex of one of the cones containing the sections μ, ϖ, by what we have seen above; and OS' does not contain S. The vertex of the second cone containing μ and ϖ is therefore on OS'. That this second vertex lies on the polar plane, σ, of the vertex, S, of the former cone, is a general property. And the planes μ, ϖ meet on σ, which is harmonically separated from S by these two planes.

(c), that the lines OS, OS' lie in a plane containing the lines OG, OH, and are harmonic conjugates in regard to these. This follows from (b), because the vertices of the two cones containing the sections μ, ϖ lie on the line GH, and are harmonically conjugate in regard to G and H. To point out the meaning of this result, consider the quadric as a sphere of which O is the centre; then the Hart circle of a spherical triangle, ABC, meets BC in L, L'; CA in M, M'; AB in N, N', such that the arcs AL, BM, CN meet in a point S, and the arcs AL', BM', CN' in a point S', these points, S and S', being the centres of similitude of the circle, (μ), circumscribed to ABC, and the Hart circle (ϖ).

It may also be remarked that, from any point, P, and four given points O, X, Y, Z, we can define a plane thus: Let OP, XP, YP, ZP meet the planes XYZ, OYZ, OZX, OXY, respectively, in O', X', Y', Z'. The planes $O'Y'Z'$, $O'Z'X'$, $O'X'Y'$, $X'Y'Z'$, by their intersections with the planes OYZ, OZX, OXY, XYZ, respectively, determine four lines. These four lines lie in the plane to be defined. If, relatively to the points O, X, Y, Z the point P be (ξ, η, ζ, τ), the plane is $x\xi^{-1}+y\eta^{-1}+z\zeta^{-1}+t\tau^{-1}=0$. This being understood, it is clear that the point S, and its polar plane σ in regard to the quadric, are in this relation with respect to the points O, X, Y, Z, determined by the planes of the three original sections α, β, γ and of the Hart section. It is easy to see that in general there are eight points whose polar planes, in regard to a given quadric, are also derivable by this construction, from four given points.

Ex. 4. There are, we have said, eight sections, ϖ, each touched by four of the common tangent planes of the sections α, β, γ. These eight sections fall into four pairs, the planes of a pair intersecting on the polar plane of O (where the planes of α, β, γ meet), being harmonic conjugates in regard to this plane and O. For the pair associated as above with the two planes ABC, $A'B'C'$ the lines OS, OS' are the same. Another pair is associated with the planes $AB'C'$, $A'BC$; a third pair with the planes $BC'A'$, $B'CA$, and a fourth pair with the planes $CA'B'$, $C'AB$.

And it may be noticed that the sections of the quadric by the planes ABC, $AB'C'$, $BC'A'$, $CA'B'$ are all touched by four planes, as follows from the fact that the lines AA', BB', CC' are concurrent. So also the sections of the quadric by the planes $A'B'C'$, $A'BC$, $B'CA$, $C'AB$ are all touched by four planes.

Ex. 5. Given two points, A, B, of a quadric, and a point O, we can define, for the moment, as *mid-axis* of AB in respect to O, the line joining O to the pole, in regard to the section OAB, of the line AB. Then, given three points, A, B, C, of the quadric, let d, e, f denote the mid-axes, in respect to O, respectively of BC, CA and AB. Let p be the line of intersection of the planes ef and BOC; similarly let q and r be the lines (fd, COA), (de, AOB), respectively. It can be shewn that the lines p, q, r lie in a plane meeting the plane ABC on the polar plane of O.

If the quadric be $(a, b, c, f, g, h \between x, y, z)^2 = t_1^2$, where O is $(0, 0, 0, 1)$, and A, B, C are, respectively, $(1, 0, 0, a^{\frac{1}{2}})$, $(0, 1, 0, b^{\frac{1}{2}})$, $(0, 0, 1, c^{\frac{1}{2}})$, the mid-axis, d, of BC in respect to O is $yb^{\frac{1}{2}} = zc^{\frac{1}{2}}$, the plane ef is $-xa^{\frac{1}{2}} + yb^{\frac{1}{2}} + zc^{\frac{1}{2}} = 0$, and the plane pqr is

$$xa^{\frac{1}{2}} + yb^{\frac{1}{2}} + zc^{\frac{1}{2}} = 0.$$

It is easy to shew, further, that the planes (d, OA), (e, OB), (f, OC) meet in a line. This we may call the *centroid axis* of ABC in respect to O.

Consider, next, the plane joining OA to the pole of the plane BOC, and the two other similar planes. The planes are easily seen to meet in a line, which we may call the *altitude axis* of ABC in respect to O.

It can then be proved that the line OH joining O to the pole, H, of the Hart section, whose equation has been written

$$lx + my + nz - t_1 = 0,$$

lies in the plane of the centroid axis and the altitude axis.

Ex. 6. Now consider the particular case of the preceding work which arises when the point O, in which the planes of the sections α, β, γ meet, is on the quadric, so that the points A', B', C' are also at O. Then it is easily seen that also the plane σ', the polar plane of S', and one of the planes of the harmonically conjugate quadric, is the tangent plane of the original quadric at the point O. The intersections of this plane with the lines YZ, ZX, XY of the plane $t = 0$, which were previously denoted by L_1', M_1', N_1', are the harmonic conjugates of L', M', N', respectively, in regard to Y and Z, Z and X, X and Y, the points L', M', N' being points of the Hart section of the quadric, lying respectively in YZ, ZX and XY. Hence we have the following result: Consider three sections of a quadric, α, β and γ, which intersect in a point O of the quadric. Let the lines of intersection of the pairs of planes of β, γ, of γ, α and of α, β, be respectively OX, OY, OZ. In the plane YOZ take the line, l, which is the fourth harmonic, in regard to OY, OZ, of the line in which the tangent plane of the quadric at O meets the plane YOZ. Let the line l cut the section α again in L'. Construct, similarly, the points M' and N' on the sections β and γ, respectively. Let ϖ be the section of the quadric by the plane $L'M'N'$. It can be shewn that there are four plane sections of the quadric which touch the sections α, β, γ. These all touch the section ϖ.

The point of contact with ϖ, of any one of these four tangent sections, say i, can be constructed thus: Take, upon i, its three points of contact with the sections α, β, γ; take also the two intersections of i with the tangent plane of the quadric at O. Then the point of contact of i with ϖ is the apolar complement of these two intersections in respect to the three former points of i.

If we project from O on to an arbitrary plane, the section ϖ becomes the Feuerbach circle of a triangle. This touches the inscribed circle of the triangle (as also the escribed circles) at a point determined, as above, by the points of contact of the inscribed circle with the sides of the triangle and the two Absolute points on the inscribed circle. (See Vol. II, p. 114. This particular case of the general theorem of the text had been found, unknown to the writer, by F. Morley, *American Bull.* I, 1895, pp. 116-124.)

Another construction for the plane of the section ϖ may be referred to. We have mentioned above (p. 77) a construction for the plane whose equation is $x\xi^{-1}+y\eta^{-1}+z\zeta^{-1}+t\tau^{-1}=0$, from the point (ξ, η, ζ, τ); let this plane be called the *polar* of (ξ, η, ζ, τ) in regard to the tetrad consisting of the intersections of the planes $x=0, y=0$, $z=0$, $t=0$. It can be shewn that if three lines be drawn through a given point, O, of a quadric, there exist three unique points, say P, Q, R, one upon each of the lines drawn through O, of which every two are conjugate in regard to the quadric. Let the three lines be the intersections, in pairs, of the planes of the sections α, β, γ. Let S be the pole of the plane PQR in regard to the quadric. Then the plane ϖ is the polar of O in regard to the tetrad P, Q, R, S.

If the point O, relatively to the tetrad P, Q, R, S, be of coordinates $(1, 1, 1, 1)$, the equation of the quadric referred to this tetrad may be supposed to be $ax^2+by^2+cz^2-(a+b+c)t^2=0$. If now, to refer the quadric to the four points consisting of O and the three points in which OP, OQ, OR are met by the polar of O in regard to P, Q, R, S, we put $X=x-t$, $Y=y-t$, $Z=z-t$, $T=x+y+z+t$, the equation is found to take the form

$$aX^2+bY^2+cZ^2-(b+c)YZ-(c+a)ZX-(a+b)XY$$
$$+(aX+bY+cZ)T=0,$$

from which it can be verified that $T=0$ is the plane of the section ϖ arising from the sections $X=0$, $Y=0$, $Z=0$. The harmonically conjugate quadric, in regard to the last tetrad, is

$$(aX+bY+cZ)(X+Y+Z-T)=0,$$

consisting of the tangent plane at O and the plane $t=0$, or PQR.

Ex. 7. If the sections α, β, γ be in the planes $x=0$, $y=0$, $z=0$, meeting on the quadric, the plane of the Hart section ϖ being $t=0$, the equation of the quadric is of the form

$$ax^2+by^2+cz^2-(b+c)yz-(c+a)zx-(a+b)xy+2(x+y+z)t=0,$$

the harmonically conjugate quadric being the aggregate of the planes $x+y+z=0$, $ax+by+cz-2t=0$.

The four planes touching the sections by $x=0$, $y=0$, $z=0$, $t=0$ are of the form

$$x(l-m)(l-n)+y(m-n)(m-l)+z(n-l)(n-m)+t=0,$$

where $\quad -2l^2=b+c, \quad -2m^2=c+a, \quad -2n^2=a+b.$

Further, if we introduce a_1, and the coordinates X, Y, Z, by the definitions

$$a_1^{-1}+a^{-1}+b^{-1}+c^{-1}=0, \qquad ax+a_1X=0,$$
$$y=Y+b^{-1}ax, \qquad z=Z+c^{-1}ax,$$

the quadric has the equation

$$a_1X^2+bY^2+cZ^2-(b+c)YZ-(c+a_1)ZX-(a_1+b)XY$$
$$+2(X+Y+Z)t=0,$$

which is of the same form as the original. For the original equation the points A, L on OX, YZ, of our general theory, are $(a^{-1}, 0, 0, -\frac{1}{2})$ and $(0, b^{-1}, c^{-1}, 0)$, and the lines AL, BM, CN meet in the point $(a^{-1}, b^{-1}, c^{-1}, -\frac{1}{2})$, or S, whose polar in regard to the quadric is the plane $ax+by+cz-2t=0$. But the transformation here given shews that the Hart section, in the plane $t=0$, belongs equally to three other sets of three sections of the quadric, one of these sets being those in the planes $X=0$, $Y=0$, $Z=0$.

Ex. 8. The reader is probably aware that in the nomenclature usual in metrical geometry, the *radius* of the Feuerbach circle of a triad of points in a plane is one-half that of the circle containing these points. This result has also been generalised to the *radius* of the Hart circle arising for a triad of points of a sphere joined by arcs of great circles (Salmon, *Geometry of three dimensions*, 1882, p. 232). We now prove a theorem for sections of a quadric which includes these results.

For this purpose we first define the *radius* of any section of a quadric, in regard to any point not lying on the quadric. It is easy to prove for a *conic*, that if two lines intersect in a point O, not on the conic, and H be one of the points in which one of these lines meets the conic, and the tangent of the conic at H meet the other line in P, then the angle between the lines, measured in regard to the conic, say ρ, is related to the interval OP, also measured in regard to the conic, say δ, by the equation $\sin\rho\sin\delta = \pm 1$, or its equivalent, $\tan\rho = \pm i\sec\delta$ (cf. Vol. II, p. 210, Ex. 6). Now, let O be any point not on the quadric, let ϖ be any section of the quadric lying in a plane whose pole is P, and let H be any point of the section ϖ. By considering the conic in which the plane POH meets the conic, we see that if ρ is the angle between the lines OP, OH, measured in regard to this conic, and δ is the interval OP, measured in regard to the quadric, then $\tan\rho = \pm i\sec\delta$; thus ρ is independent of the position of H on the section ϖ. It is this ρ which is here called the radius of the section ϖ, in regard to the point O. Evidently the relation between O and P is symmetrical.

Now let ϖ and μ be any two sections of the quadric, whose radii, as thus defined, are, respectively, ρ and R; let the plane, joining the line of intersection, l, of the planes ϖ and μ, to the point O, be called ω; and the plane joining l to the vertex of either one of the cones containing the sections ϖ, μ be called σ. If the equations of these planes be written respectively $(\varpi) = 0$, $(\mu) = 0$, $(\omega) = 0$, $(\sigma) = 0$, it is easy to shew that, with a proper constant, c, we have

$$(\sigma) = (\mu) \pm c\,(\varpi), \quad (\omega) = (\mu)\tan R + c\,(\varpi)\tan\rho.$$

This can be applied to the case when, as before, ϖ is the Hart section of three sections, α, β, γ, of the quadric, whose planes meet in a point O not lying on the quadric. For μ we take the section by the plane ABC. The vertex of one of the cones containing the sections μ, ϖ is the point S, whose polar plane in regard to the quadric is the plane σ; it has been seen (Ex. 3) that the vertex of the other cone containing the sections μ, ϖ is on the plane σ (and on OS'). Further, we have seen that (μ), (ϖ), (σ), (ω) are, respectively,

$$(l-a^{\frac{1}{2}})x+\ldots-t=0, \quad t=0, \quad (l-a^{\frac{1}{2}})x+\ldots+t=0, \quad (l-a^{\frac{1}{2}})x+\ldots=0;$$

whence $(\sigma) = (\mu) + 2\,(\varpi)$, $(\omega) = (\mu) + (\varpi)$. Comparing this with the equations above we have $c = \pm 2$, $c\tan\rho = \tan R$. Hence we have the result $\tan R = \pm 2\tan\rho$, which we desired to obtain.

Another proof may be obtained by shewing that $\Delta\tan^2 R = U$, $\Delta\tan^2\rho = V$, where U, V are given respectively (cf. p. 73) by

$$U = Aa + Bb + Cc + 2F\,(bc)^{\frac{1}{2}} + 2G\,(ca)^{\frac{1}{2}} + 2H\,(ab)^{\frac{1}{2}} - \Delta,$$
$$V = Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm - \Delta,$$

where $A = bc - f^2$, $F = gh - af$, etc., as before, and Δ is the discriminant $abc + 2fgh - af^2 - bg^2 - ch^2$. In virtue of $2mn = f + (bc)^{\frac{1}{2}}$, etc., it can be shewn that

$$4V = U = 2\,[(bc)^{\frac{1}{2}} - f]\,[(ca)^{\frac{1}{2}} - g]\,[(ab)^{\frac{1}{2}} - h].$$

Ex. 9. By projection of sections of a quadric, prove that, if three circles in a plane intersect in pairs respectively in A, A'; B, B' and C, C', then the circles ABC, $A'B'C'$, $AB'C'$, $A'BC$, which have a common orthogonal circle and are, in pairs, inverses of one another, are all touched by four other circles which have a common orthogonal circle; also that the circles ABC, $AB'C'$, $BC'A'$, $CA'B'$ are all touched by four other circles. (A. Larmor, *Proc. Lond. Math. Soc.*, XXIII, 1892, p. 149.)

In what has preceded we have shewn that eight sections of the quadric exist which touch three given plane sections of general position, α, β, γ. We have then shewn that there are eight ways in which we can associate with these sections α, β, γ a section, ϖ, such that four of the tangent sections of α, β, γ also touch ϖ—account being now taken only of the cases in which the planes of $\alpha, \beta, \gamma, \varpi$ do not meet in a point. We have shewn that a sufficient condition for this is that another quadric, which we have called the harmonically conjugate quadric, should break up into two planes.

We consider now the converse question *whether this sufficient condition is also necessary*, by examining the most general case in which the sections of the original quadric by the planes $x = 0$, $y = 0$, $z = 0$, $t = 0$ are all touched by four other planes.

The quadric being, in the first instance, referred to the planes of three sections α, β, γ and the polar plane of their point of intersection, let its equation be $(a_1, b_1, c_1, f_1, g_1, h_1 \,)\!(\, x_1, y_1, z_1)^2 = t_1^2$; let a fourth plane be $t = 0$, where $t = t_1 - lx_1 - my_1 - nz_1$; we shall suppose that no one of l, m, n is zero, and put $x = lx_1$, $y = my_1$, $z = nz_1$. The equation of the quadric then takes a form, $\phi(x, y, z, t) = 0$, in which $\phi = (a, b, c, d, f, g, h, u, v, w \,)\!(\, x, y, z, t)^2$, where $u = v = w = d = 1$. The condition that two plane sections of the quadric should touch one another is that, of the cones enveloping the quadric along these sections, the vertex of either should lie on the other. But the cone enveloping the quadric along the section by $t = 0$ has the equation $D\phi(x, y, z, t) = \Delta t^2$, where Δ is the symmetrical discriminantal determinant of ϕ, of four rows and columns, and D is the minor of d therein; and the enveloping cones for the sections by $x = 0$, $y = 0$, $z = 0$ have similar equations. The condition that the four sections of the quadric by the planes $x = 0$, $y = 0$, $z = 0$, $t = 0$ should have a common tangent plane is that the four enveloping cones should have a point in common; this point, which will be the pole of the common tangent plane, will thus necessarily be one of the eight points

$$\pm xA^{-\frac{1}{2}} = \pm yB^{-\frac{1}{2}} = \pm zC^{-\frac{1}{2}} = tD^{-\frac{1}{2}},$$

and the condition is that, with properly chosen signs, we should have

$$\phi(\pm A^{\frac{1}{2}}, \pm B^{\frac{1}{2}}, \pm C^{\frac{1}{2}}, D^{\frac{1}{2}}) = \Delta.$$

If the four sections by $x=0$, $y=0$, $z=0$, $t=0$ are to have four common tangent planes, this last equation must be satisfied for each of four sets of signs of the radicals. These four conditions must be satisfied in virtue of proper choice of the three quantities, l, m, n, without limitation of the values of the original coefficients a_1, b_1, c_1, f_1, g_1, h_1. We suppose that the sections of the quadric by $x=0, y=0, z=0, t=0$ do not degenerate, in either case, into a pair of lines; and so we exclude the possibilities

$$A=0,\ B=0,\ C=0,\ D=0.$$

By taking a particular case we can infer that if the equation $\phi(\pm A^{\frac{1}{2}}, \ldots)=\Delta$ is satisfied by four sets of signs of $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, $D^{\frac{1}{2}}$, then three of these sets must be derivable from one of them by change in the signs of two of the three $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$. We take $f_1=0$, $g_1=0$, $h_1=0$, leading to $f=1$, $g=1$, $h=1$. Then it is easy to verify that

$$\Delta=(a+1)A+(b+1)B+(c+1)C-2D,$$

and, if we put ξ, η, ζ, τ respectively for $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, and $-D^{\frac{1}{2}}$, the equation $\phi(\pm A^{\frac{1}{2}}, \ldots)=\Delta$ becomes

$$(\xi+\eta+\zeta-\tau)^2=2(\xi^2+\eta^2+\zeta^2-\tau^2).$$

We shew that if this equation, and three others chosen from

$$(\pm\xi\pm\eta\pm\zeta-\tau)^2=2(\xi^2+\eta^2+\zeta^2-\tau^2),$$

are all satisfied by the same set of values for ξ, η, ζ, τ, then the three others are those in which the linear functions on the left side are $\xi-\eta-\zeta-\tau$, $-\xi+\eta-\zeta-\tau$, $-\xi-\eta+\zeta-\tau$. The equation $(\pm\xi\pm\eta\pm\zeta-\tau)^2=2(\xi^2+\eta^2+\zeta^2-\tau^2)$ represents, if ξ, η, ζ, τ be regarded as coordinates, the enveloping cone drawn to the quadric $\xi^2+\eta^2+\zeta^2-\tau^2=0$ from the point $(\pm 1, \pm 1, \pm 1, 1)$; there are eight such cones; it is easy to find all the intersections of every three of these cones; then it appears that, excluding intersections on $\xi=0$, $\eta=0$, $\zeta=0$, $\tau=0$, there are only two sets of four cones which have four common points, the vertices of one of these sets being the points $(1, -1, -1, 1)$, $(-1, 1, -1, 1)$, $(-1, -1, 1, 1)$, $(1, 1, 1, 1)$, the vertices of the other set being $(-1, 1, 1, 1)$, $(1, -1, 1, 1)$, $(1, 1, -1, 1)$, $(1, 1, 1, -1)$. This proves the statement made.

This result being assumed, the equation $\phi(\pm A^{\frac{1}{2}}, \ldots)=\Delta$, which, writing $(BC)^{\frac{1}{2}}$ for $B^{\frac{1}{2}}.C^{\frac{1}{2}}$, etc., is

$$aA+bB+cC+dD+2[f(BC)^{\frac{1}{2}}+u(AD)^{\frac{1}{2}}]$$
$$+2[g(CA)^{\frac{1}{2}}+v(BD)^{\frac{1}{2}}]+2[h(AB)^{\frac{1}{2}}+w(CD)^{\frac{1}{2}}]-\Delta=0$$

leads, by combination of the forms it takes when the signs of two of $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$ are changed, to the four equations

$$aA+bB+cC+dD-\Delta=0, \qquad f(BC)^{\frac{1}{2}}+u(AD)^{\frac{1}{2}}=0,$$
$$g(CA)^{\frac{1}{2}}+v(BD)^{\frac{1}{2}}=0, \qquad h(AB)^{\frac{1}{2}}+w(CD)^{\frac{1}{2}}=0.$$

The last three of these equations require that A, B, C, D should be in the ratios of fvw, gwu, huv, fgh, to one another, that is the ratios of the values which A, B, C, D would have if, in the determinant Δ, we had $a=0$, $b=0$, $c=0$, $d=0$. Conversely, if proper signs be given to $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, $D^{\frac{1}{2}}$, the last three of these equations are a consequence of the equations $A=\sigma fvw$, $B=\sigma gwu$, $C=\sigma huv$, $D=\sigma fgh$. We require to shew that if, beside these, we also have $aA+bB+cC+dD-\Delta=0$, then the harmonically conjugate quadric, whose equation is obtained from that of the original by the change of the signs of f, g, h, u, v, w, breaks up into two planes. It will be found that this involves $\sigma=4$.

To carry through the algebra, suppose $u=v=w=d=1$, which is quite general; and, for brevity, put

$$\xi=1-a,\ \eta=1-b,\ \zeta=1-c,\ p=1-f^2,\ q=1-g^2,\ r=1-h^2,$$

and also

$$\alpha=f+g+h, \quad \beta=gh+hf+fg, \quad \gamma=fgh, \quad \epsilon=\alpha-\gamma,$$
$$\lambda=\epsilon(f+f^{-1}), \quad \mu=\epsilon(g+g^{-1}), \quad \nu=\epsilon(h+h^{-1}).$$

Then it is found that $A=\eta\zeta-(1-f)^2$, etc., and the equation $aA+bB+cC+dD-\Delta=0$ is the same as

$$-3\xi\eta\zeta+2\Sigma\eta\zeta-\Sigma p\xi-2\Sigma f^2+2\Sigma gh=0.$$

Hence, the four equations consisting of this and the three

$$A/fvw=\ldots=D/fgh,$$

are found, introducing a quantity θ, to be the same as the aggregate

$$\theta=\eta\zeta\epsilon f^{-1}-\lambda=\zeta\xi\epsilon g^{-1}-\mu=\xi\eta\epsilon h^{-1}-\nu=\xi\eta\zeta+\Sigma p\xi-4,$$
$$2\epsilon\Sigma p\xi=\theta(\alpha-3\gamma)+6\epsilon-2\epsilon\beta.$$

These lead to

$$(\theta+\lambda)(\theta+\mu)(\theta+\nu)=\tfrac{1}{4}\epsilon\gamma^{-1}[\theta(\alpha+\gamma)+2\epsilon+2\epsilon\beta]^2;$$

it is necessary to solve this equation for θ, and then, putting

$$\rho=\tfrac{1}{2}\epsilon^{-1}[\theta(\alpha+\gamma)+2\epsilon+2\epsilon\beta],$$

together with

$$\xi=\rho\epsilon/f(\theta+\lambda), \quad \eta=\rho\epsilon/g(\theta+\mu), \quad \zeta=\rho\epsilon/h(\theta+\nu),$$

to test whether the equation

$$\Sigma p\xi=\theta-\rho+4$$

is satisfied. If this proves to be so, the original equations can be satisfied.

In fact, the three roots of the cubic equation are $\theta = 2\epsilon$, $\theta = -2\epsilon$, $\theta = \epsilon\gamma^{-1}(\frac{1}{4}\epsilon^2 - \beta)$; these lead, respectively, to the three values for ξ,

$$\xi = (1+g)(1+h)(1+f)^{-1}, \quad \xi = (1-g)(1-h)(1-f)^{-1},$$

$$\xi = 4\rho\gamma f^{-1}(\epsilon - 2f)^{-2},$$

with corresponding values of η and ζ. The first and second of these roots lead to verification of the equation $\Sigma p\xi = \theta - \rho + 4$, but the second leads to $A=0$, $B=0$, $C=0$, which we exclude. The third root leads to verification of the required equation only if f, g, h be such as to satisfy

$$\Sigma gh(1-f^2)(\epsilon-2g)^2(\epsilon-2h)^2$$
$$= 16[\gamma + \tfrac{1}{8}\epsilon^2(\alpha+\gamma) - \tfrac{1}{2}\epsilon\beta][3\gamma + \tfrac{1}{8}\epsilon^2(\alpha - 3\gamma) - \tfrac{1}{2}\epsilon\beta];$$

this however is not generally true, as we may see, for example, by taking $f=1$, $g=2$, $h=3$, which lead to a negative value for the left side, and a positive value for the right side.

There remains thus only the possibility of the first root, $\theta = 2\epsilon$. We compute then the harmonically conjugate quadric, and find that it consists of the two planes expressed, if $m = (1+f)(1+g)(1+h)$, by

$$m[x(1+f)^{-1} + y(1+g)^{-1} + z(1+h)^{-1}]^2 = (x+y+z-t)^2.$$

Further, retaining $u = v = w = d = 1$, the four planes touching the sections of the quadric by $x=0$, $y=0$, $z=0$, $t=0$, are the polars of the points $(f^{\frac{1}{2}}, g^{\frac{1}{2}}, h^{\frac{1}{2}}, -f^{\frac{1}{2}}g^{\frac{1}{2}}h^{\frac{1}{2}})$, in which all possible signs are taken for $f^{\frac{1}{2}}, g^{\frac{1}{2}}, h^{\frac{1}{2}}$. This completes the proof that the conditions found sufficient in the early part of this chapter are also necessary.

Ex. 1. In terms of four quantities a, b, c, d let $f = bc - ad$, $g = ca - bd$, $h = ab - cd$, $p = bc + ad$, $q = ca + bd$, $r = ab + cd$; also

$$P = -a+b+c+d, \qquad Q = a-b+c+d,$$
$$R = a+b-c+d, \qquad S = a+b+c-d.$$

Prove that, if the quadric have the equation

$$p^{-1}qrx^2 + q^{-1}rpy^2 + r^{-1}pqz^2 + 2fyz + 2gzx + 2hxy = t_1^2,$$

where $t_1 = 0$ is the polar plane of $(0, 0, 0, 1)$, then one Hart section for the three sections $x=0$, $y=0$, $z=0$, or α, β, γ, is given by $ax + by + cz - t_1 = 0$; also, that the four planes touching α, β, γ and this Hart section are the polars of the points $(-1, 1, 1, P)$, $(1, -1, 1, Q)$, $(1, 1, -1, R)$, $(1, 1, 1, S)$. Prove also that the change of a, b, c, d, respectively, into $-d, c, b, -a$, leaves the equation of the quadric unaltered. This change consists of the interchange of b and c, taken with the interchange, and simultaneous change of sign, of a and d. Similarly, the equation of the quadric is unaltered by the change of a, b, c, d respectively into c, $-d$, a, $-b$; or into b, a, $-d$, $-c$. The equation is also unaltered by the change of sign of t_1. Hence obtain all the eight Hart sections, and, in each case, the four planes touching this and the sections α, β, γ. The form of the equation of the quadric here taken was suggested by Dr G. T. Bennett.

Ex. 2. If the quadric be $fyz + gzx + hxy + t(ux + vy + wz) = 0$, the sections by $x = 0$, $y = 0$, $z = 0$, $t = 0$ are touched by the plane $l^{-1}x + m^{-1}y + n^{-1}z + p^{-1}t = 0$, where

$$l = \epsilon_1 (fu)^{\frac{1}{2}}, \quad m = \epsilon_2 (gu)^{\frac{1}{2}}, \quad n = \epsilon_3 (hu)^{\frac{1}{2}},$$

in which $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, $\epsilon_3 = \epsilon_1 \epsilon_2$, and $p = lmn/uvw$, provided that $l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0$. (Cf. Vol. III, p. 145.)

Ex. 3. For the quadric $(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0$, the pole of a plane touching the three sections by $x = 0, y = 0, z = 0$, is of coordinates

$$A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}}, \quad d^{-1}[M^{\frac{1}{2}} - uA^{\frac{1}{2}} - vB^{\frac{1}{2}} - wC^{\frac{1}{2}}],$$

where M is such that

$$d^{-1}[(uA^{\frac{1}{2}} + vB^{\frac{1}{2}} + wC^{\frac{1}{2}})^2 - M] = (a, b, c, f, g, h \between A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}})^2 - \Delta.$$

Ex. 4. As the condition that two plane sections of a quadric should touch one another is symmetrical, there are symmetrical relations between the planes of four sections which are all touched by four other planes, and these latter planes. It is of interest then to construct the relations of the two tetrads of planes. The following result, in this regard, is due to Study, *Math. Annal.*, XLIX, 1897, pp. 497 ff.

As before, let the planes joining the points O, X, Y, Z meet the quadric in sections all touched by four other planes; let the points of the quadric on the lines OX, OY, OZ, YZ, ZX, XY be, respectively, A, A'; B, B'; C, C'; L, L'; M, M'; N, N', the notation being such that the lines AL, BM, CN meet in a point, S, and $A'L'$, $B'M'$, $C'N'$ in a point, S'. Let the planes $BMNC$, $CNLA$, $ALMB$, $B'M'N'C'$, $C'N'L'A'$, $A'L'M'B'$ be denoted, respectively, by θ, ϕ, ψ, θ', ϕ', ψ'; let the points in which the line (ϕ, ψ') meets the quadric be denoted by P_{23}, Q_{23}, those in which the line (ϕ', ψ) meets the quadric being denoted by P_{32}, Q_{32}; a similar notation being used for the other cases, namely P_{31}, Q_{31} on the line (ψ, θ'), P_{13}, Q_{13} on (ψ', θ), P_{12}, Q_{12} on (θ, ϕ'), P_{21}, Q_{21} on (θ', ϕ). Consider now four planes touching the sections by the planes XYZ, OYZ, etc.; take the intersections in threes of these four planes, and the polar planes of these four points of intersection; let these polar planes be ξ, η, ζ, τ, so that these intersect in threes in the poles of the four tangent planes.

Then the twelve points P_{ij}, Q_{ij}, R_{ij} described above are in fact the intersections of the quadric with the lines of intersection of the pairs of the planes ξ, η, ζ, τ. With a proper choice of notation the line (η, ζ) is the same as the line (P_{23}, P_{32}), and the line (ξ, τ) the same as the line (Q_{23}, Q_{32}), and so on; the sets of six of these points lying on the planes being, respectively, given by

$$\xi(Q_{23}, Q_{32}, P_{31}, P_{13}, P_{12}, P_{21});\ \eta(Q_{31}, Q_{13}, P_{12}, P_{21}, P_{23}, P_{32});$$
$$\zeta(Q_{12}, Q_{21}, P_{23}, P_{32}, P_{31}, P_{13});\ \tau(Q_{23}, Q_{32}, Q_{31}, Q_{13}, Q_{12}, Q_{21}).$$

As we have not distinguished geometrically between the points P_{23}, Q_{23} of the line (ϕ, ψ'), etc., each pair of opposite lines of intersection of the planes ξ, η, ζ, τ is capable of two determinations; there are therefore eight possibilities for this set of four planes, as we have seen.

These statements can be verified easily. We can suppose the quadric to have the equation

$$x^2(1-a^2)+y^2(1-b^2)+z^2(1-c^2)+2yz(bc-1)+2zx(ca-1)$$
$$+2xy(ab-1)+2(x+y+z)t+t^2=0,$$

the harmonically conjugate quadric being then

$$(x+y+z-t)^2-(ax+by+cz)^2=0.$$

Then, with $f=bc-1$, $g=ca-1$, $h=ab-1$, if we put $x_1=xf^{-\frac{1}{2}}$, $y_1=yg^{-\frac{1}{2}}$, $z_1=zh^{-\frac{1}{2}}$, $t_1=t(fgh)^{-\frac{1}{2}}$, the poles of four planes which touch the sections by $x=0$, $y=0$, $z=0$, $t=0$, have, by what we have seen, for coordinates x_1, y_1, z_1, t_1, respectively, $(-1, 1, 1, 1)$, $(1, -1, 1, 1)$, $(1, 1, -1, 1)$, $(1, 1, 1, -1)$; thus the planes ξ, η, ζ, τ have, respectively, the equations

$$-x_1+y_1+z_1+t_1=0,\quad x_1-y_1+z_1+t_1=0,\quad x_1+y_1-z_1+t_1=0$$

and $x_1+y_1+z_1-t_1=0$. The points A, A', L, L' are given, respectively, by

$$y=0=z,\ x(1-a)+t=0;\ y=0=z,\ x(1+a)+t=0;$$
$$x=0=t,\ y(1-b)=z(1-c);\ x=0=t,\ y(1+b)=z(1+c),$$

and so on; and the plane $BMNC$, or θ, is given by

$$-x(1-a)+y(1-b)+z(1-c)+t=0,$$

and so on. Now put

$$\rho_{23}=-(g^{\frac{1}{2}}-h^{\frac{1}{2}})a^{-1}f^{-\frac{1}{2}},\quad \sigma_{23}=-(g^{\frac{1}{2}}+h^{\frac{1}{2}})a^{-1}f^{-\frac{1}{2}},$$

with similar notations; then we easily verify that the coordinates, (x_1, y_1, z_1, t_1), of the twelve points P_{ij}, etc., are given, respectively, by the scheme

$$P_{23}(\rho_{23}, 1, 1, -\rho_{23}),\ Q_{23}(\sigma_{23}, 1, -1, \sigma_{23}),\ P_{32}(-\rho_{23}, 1, 1, \rho_{23}),\ Q_{32}(\sigma_{23}, -1, 1, \sigma_{23}),$$
$$P_{31}(1, \rho_{31}, 1, -\rho_{31}),\ Q_{31}(-1, \sigma_{31}, 1, \sigma_{31}),\ P_{13}(1, -\rho_{31}, 1, \rho_{31}),\ Q_{13}(1, \sigma_{31}, -1, \sigma_{31}),$$
$$P_{12}(1, 1, \rho_{12}, -\rho_{12}),\ Q_{12}(1, -1, \sigma_{12}, \sigma_{12}),\ P_{21}(1, 1, -\rho_{12}, \rho_{12}),\ Q_{21}(-1, 1, \sigma_{12}, \sigma_{12}).$$

From these the statements made easily follow. It can be verified also that the pole of the plane θ lies on the line $\phi'\psi'$, etc.

Ex. 5. With the equation of the quadric taken in Ex. 4, prove that the harmonically conjugate quadric of the given one, in regard to the four planes touching the sections by $x=0$, $y=0$, $z=0$, $t=0$, consists of the two planes which are the polars, in regard to the original quadric, of the points whose coordinates (x, y, z, t) are

$$f[a(m+\sigma)+1],\quad g[b(m+\sigma)+1],\quad h[c(m+\sigma)+1],\quad -fgh,$$

where $m=\frac{1}{2}(abc-a-b-c)$, $\sigma^2=m^2-1$. The discriminant, Δ, of the equation of the original quadric is $16\sigma^2$.

Ex. 6. As before, let the sections of a quadric by four planes OYZ, OZX, OXY, XYZ be touched by four planes. Let the intervals OX, OY, OZ, measured in regard to the quadric, be denoted by $i\alpha$, $i\beta$, $i\gamma$, and the similar intervals YZ, ZX, XY be denoted by $i\alpha'$, $i\beta'$, $i\gamma'$ (cf. Vol. II, pp. 179, 205). Similarly, let the angular intervals of the pairs of planes meeting, respectively, in OX, OY, OZ, be denoted by (A), (B), (C), and those between the plane XYZ and the planes OYZ, OZX, OXY, respectively, by (A'), (B'), (C'). Taking ϵ such that

$$2\tanh\epsilon=\tanh\alpha\tanh\beta\tanh\gamma-\tanh\alpha-\tanh\beta-\tanh\gamma,$$

prove that, with suitable regard to the ambiguities involved in the definitions of these intervals,

$$\alpha'=i\pi+\beta-\gamma,\quad \beta'=i\pi+\gamma-\alpha,\quad \gamma'=i\pi+\alpha-\beta,$$

$$\tan(A)=\sinh\alpha/\cosh(\epsilon+\alpha), \tan(A')=\sinh(\beta-\gamma)/\cosh(\epsilon+\beta+\gamma),$$

and so on. These lead to

$$(A')=(B)-(C),\quad (B')=(C)-(A),\quad (C')=(A)-(B).$$

Prove that if three circles in a plane, α, β, γ, intersect in pairs at angles (A), (B), (C), and ϖ be the circle which intersects them at angles $(B)-(C)$, $(C)-(A)$, $(A)-(B)$, respectively, then a circle can be drawn to touch α, β, γ, ϖ. (Lachlan's *Modern Geometry*, 1893, p. 250.)

CHAPTER III

THE PLANE QUARTIC CURVE WITH TWO DOUBLE POINTS

In this chapter we obtain some properties of a plane curve by projection of a curve which lies in space of three dimensions. The plane curve is one which meets an arbitrary line in four points, and has two double points, or points where the curve crosses itself. The curve in space is the curve of intersection of two quadric surfaces. The matter is dealt with in more detail than is required by its difficulty, because the theory is a model for the subsequent theory of the Cyclide, a quartic surface in three dimensions, regarded as the projection of the intersection of two quadrics of fourfold space. (Chap. VI, below.)

The generation of the curve. The four principal circles. Consider two quadrics which have a common self-polar tetrad. They intersect in a quartic curve, through which there pass four quadric cones. Let V be the vertex of one of these cones, and Ω be one of the quadrics; the curve may be defined by the intersection of the cone and the quadric. Taking upon Ω an arbitrary point, U, we project the curve from U upon an arbitrary plane, ϖ. As each of the generators at U, of the quadric Ω, meets the cone of vertex V in two points, we obtain in ϖ a quartic curve having a double point at each of the points, say I and J, in which these generators meet ϖ. Conversely, any plane quartic curve having double points at the intersections of the line $z=0$ with the two lines $x \pm iy = 0$ has an equation of the form

$$(x^2+y^2)^2 + (x^2+y^2)\,z\,(lx+my+nz)$$
$$+ z^2\,(ax^2+by^2+cz^2+2fyz+2gzx+2hxy) = 0,$$

or say $(x^2+y^2)^2+(x^2+y^2)zP+z^2S=0$, as we may assume. This curve is the projection, from the point $(0, 0, 0, 1)$, of the intersection of the two quadrics $zt=x^2+y^2$, $t^2+tP+S=0$.

Now consider a section of the quadric Ω by the polar plane of V, in regard to Ω. We call this a *principal* section, and its projection upon ϖ a *principal circle*; the centre of this circle is the projection of V (above, p. 7). There are four quadric cones through the curve on Ω; there are thus four principal circles in the plane ϖ; the polar plane of V, in regard to Ω, passes through the vertices of the other three cones, the four vertices forming a self-polar tetrad. Thus (above, p. 8) every principal circle cuts the other three principal circles at right angles. If V_1, V_2, V_3 be the vertices of the other three cones, the line VV_1 is the polar line of V_2V_3 in regard to Ω, and the planes UVV_1, UV_2V_3 are conjugate, and are thus harmonic in regard to the generators at U. Thus the centres of the four principal circles, in the plane ϖ, are such that the join of any two of them is at right angles, in regard to I, J, to the join of the other two; or, any one of these centres is the orthocentre of the other three.

The curve as an envelope of circles. The director conic. Next, consider a section of Ω by a tangent plane of the cone of vertex V. It projects into a circle cutting at right angles the principal circle corresponding to V. The locus of the poles, in regard to Ω, of the tangent planes of the cone (V), is a conic, say v, lying on the polar plane of V, this being the polar reciprocal of the cone (V), in regard to Ω. This conic, v, projects into a conic in the plane ϖ, say σ. The circles, obtained by projecting the sections of Ω by the tangent planes of the cone (V), thus have their centres on a conic, σ, beside cutting at right angles the corresponding principal circle. We may call the conic σ a *director* conic. Every one of the circles, arising from the tangent planes of (V), has in fact two points of contact with the quartic curve of the plane ϖ. For, if H' be an intersection, of the quartic curve on Ω, with the section of Ω by a tangent plane of (V), then H' is on (V), as being on the quartic curve, and is therefore at a point where the tangent plane touches (V); the line of intersection of the tangent plane of Ω at H' with the tangent plane of (V) at this point, is thus both the tangent line of the quartic curve and of the section of Ω by the tangent plane of (V). These two curves thus touch at this point, and the curves in ϖ obtained by the projection of these equally touch. The line VH' meets the quadric Ω in a further point, which also projects into a point of contact of the plane quartic curve with the same circle. The total intersections of the circle with the plane quartic curve count therefore for eight, being two at each point of contact, and two at each of the double points I, J.

Ex. 1. The facts will perhaps be clearer if we consider the equations of a conic and a circle, in a plane, say, respectively, $a^{-2}x^2+b^{-2}y^2=z^2$, and $(x-hz)^2+(y-kz)^2=r^2z^2$. A circle, with centre at the point $(a\cos\theta, b\sin\theta, 1)$, of the former, which cuts the latter at right angles, is given by the equation (cf. Vol. II, pp. 110, 111)

$$(x-a\cos\theta\,.\,z)^2+(y-b\sin\theta\,.\,z)^2$$
$$=[(h-a\cos\theta)^2+(k-b\sin\theta)^2-r^2]\,z^2,$$

or $P\cos\theta+Q\sin\theta=R$, where $P=2az\,(x-hz)$, $Q=2bz\,(y-kz)$, $R=x^2+y^2-z^2\,(h^2+k^2-r^2)$.

This circle then touches the curve expressed by $P^2+Q^2=R^2$, as is easy to verify by combining the equations of the circle and curve, namely at the points where the circle is met by the line given by $P\sin\theta - Q\cos\theta = 0$. The equation of the curve is

$$[x^2+y^2-z^2(h^2+k^2-r^2)]^2-4z^2[a^2(x-hz)^2+b^2(y-kz)^2]=0,$$

and the points of contact are given by

$$a(x-hz)=\lambda z\cos\theta,\quad b(y-kz)=\lambda z\sin\theta,$$

where λ is either root of the equation

$$\lambda^2(a^{-2}\cos^2\theta+b^{-2}\sin^2\theta)+2\lambda(a^{-1}h\cos\theta+b^{-1}k\sin\theta-1)+r^2=0.$$

Cassini's oval is obtained when $h=k=0$, $r^2=a^2+b^2$.

Ex. 2. Prove that the four points of contact, with the quartic curve in ϖ, of any two of the tangent circles which have their centres on the same director conic, lie upon a circle, likewise cutting the associated principal circle at right angles.

The four director conics are confocal. The quartic curve in the plane ϖ is thus the envelope of the circles described, with centres on the conic σ, to cut the corresponding principal circle at right angles. (More generally, it appeared, Vol. III, p. 200, that a general quartic curve in a plane may be regarded as the envelope of a system of conics, in various ways. In that general case, each of the conics had four points of contact with the quartic curve; in the present case, the circles each touch the quartic curve in two points, beside passing through the double points.) It is clear that there are three other ways in which the quartic curve in the plane ϖ may be regarded as the envelope of a system of circles, each system consisting of circles cutting a principal circle at right angles, with their centres on a certain conic. Let these other conics, the projections from U of the conics which are the polar reciprocals, in regard to Ω, of the four quadric cones containing the quartic curve on Ω, be called σ_1, σ_2, σ_3. It can be shewn that the four conics σ, σ_1, σ_2, σ_3 have four common tangents, passing in pairs through the points I and J; or, as we may say, that these four conics are confocals. To prove this, it is only necessary to shew that there are four planes through U, two through each of the generators of Ω at U, which touch the four conics in the principal planes, of which σ, σ_1, σ_2, σ_3 are the projections; or, reciprocating in regard to Ω, that there are two points, on each of the generators of Ω at U, which lie on all the four cones such as (V). The two points in which a generator at U meets the cone (V), lying as they do on the quartic curve upon Ω, lie, however, equally on the other three cones which contain this curve. The four points in question are thus the intersections of the generators of Ω at U with the cone (V); and the four common tangents of the conics σ, σ_1, σ_2, σ_3 are the intersections with ϖ of the tangent planes of Ω at these four points, these tangent planes being the polars of their points of contact. Hence it is clear also that the common tangent lines of the conics σ, σ_1, σ_2, σ_3 are, in pairs, the tangent lines, of the quartic curve in ϖ, at the points I and J. The *foci* of the confocals σ, σ_1, σ_2, σ_3 are the intersections, other than at I and J, of their four common tangent lines. They are, then, the intersections, with ϖ, of the lines of intersection, other than the generators of Ω at U, of pairs of the tangent planes of Ω, at the four points where the cone (V) is met by the generators of Ω at U. These lines of intersection are polar lines, in regard to Ω, of joins of pairs of these four points.

The four director conics σ, σ_1, σ_2, σ_3 are the projections, from U, of four conics, v, v_1, v_2, v_3, which are the polar reciprocals, in regard to Ω, of the four cones (V), (V_1), (V_2), (V_3), with the property that a common point of any two of these cones lies on the others. The conics v, v_1, v_2, v_3 are thus such that a common tangent plane of any two of them is equally a tangent plane of the other two, and they form such a set as those called the focal conics of a set of confocal quadrics (Vol. III, p. 94). It is a familiar fact that of the common tangent planes of such a system of conics there are four which pass through any point (such as U).

The *centre* of the conic σ, namely the pole of the line IJ in regard thereto, is the projection from U, of the pole, in regard to the conic v, (in the principal plane; of which σ is the projection), of the line in which this plane is met by the tangent plane of Ω at U. As this principal plane is the polar plane of V, this line of meeting is the polar line of UV in regard to Ω; and the conic v is the polar reciprocal of the cone (V). The point in the plane of v, which projects into the centre of the conic σ, is thus the pole, in regard to Ω, of the polar plane of U taken in regard to the cone (V). If the polar plane of U, in regard to the cone (V), meet the tangent plane of Ω, at U, in the line l, the pole, in regard to Ω, of this polar plane, lies on the line, λ, which is the polar line of l in regard to Ω, and this passes through U. The centre of the conic σ is the intersection of λ with the plane ϖ. In fact the polar planes of a point, in regard to all quadrics having a quartic curve in common, intersect in a line. Thus the line l, and also the line λ, equally arise from any of the cones (V_1), (V_2), (V_3), or the conics σ, σ_1, σ_2, σ_3 have, as we know, a common centre. It is easy to see also that the foci lie in pairs on lines through this centre; as we also know.

The sixteen foci of the quartic curve. Are intersections of a principal circle and a director conic. In general for an algebraic curve in a plane, considered in relation to two Absolute points, I and J, a point which is the intersection of a tangent of the curve drawn from I, with a tangent of the curve drawn from J, is called a *focus* of the curve. For a curve of class m, that is, of which m tangents pass through a general arbitrary point of the plane, there will then be m^2 foci; unless the curve contain one or other of the points I, J. For a quartic curve with two double points it can be shewn, and will appear below, that four tangents can be drawn to the curve from either double point, other than the two tangents of the curve at each of these points. There will therefore be sixteen foci. We prove that these are the intersections of the principal circles, each with its corresponding director conic.

We may first verify that such an intersection is a focus, namely that the lines which join such a point to I and J are both tangents of the plane quartic curve. Such a point is the projection of a point, say F, in which a principal section of the quadric Ω, say by the polar plane of V, is met by the conic, v, in this plane, which is the polar reciprocal, in regard to Ω, of the cone (V). The tangent plane of Ω at the point F, being the polar of F in regard to Ω, is a plane passing through F which touches the cone (V); namely, F is the point of contact with Ω of a common tangent plane of the cone (V), and the enveloping cone of Ω drawn from V. The two generating lines of Ω at this point F, lying in the tangent plane of Ω at this point, will both touch the cone (V), and the points where these touch this cone, as they lie both on Ω and on (V), are points of the quartic curve on Ω; moreover, as these lines touch (V), they equally touch this quartic curve, at the points where they touch (V). These generating lines thus project, from the point U of Ω, into two lines which are tangents of the plane quartic curve in ϖ. These generating lines, however, intersect the generators of Ω at U. Thus they project into lines passing, one through I and one through J. This shews that an intersection of a principal circle with a director conic is a focus of the plane quartic curve, as has been said. It may be called a point-circle of the enveloping system.

To shew that all the foci of the plane quartic curve are thus obtained, it will be sufficient to shew that there are four tangents of the curve which pass through either I or J. Now a tangent through I, of the plane curve, is the intersection with ϖ of a plane through the generator UI, of Ω, which contains also a point where it touches the quartic curve on Ω, this being the point of Ω which projects into the point of contact on the plane curve; the line UI is a chord of the quartic curve on Ω, meeting this curve in the two points where it meets the cone (V). It will be enough then to shew that, through this chord of the quartic curve on Ω, there can be drawn four planes of which each, in place of having two other points of intersection with the curve, has a point of contact with it, that is, contains a tangent line of the curve. Let such a plane through UI touch the curve in H; as the tangent line of the curve at H meets UI, and touches Ω at H, it is a generator of Ω; as this tangent line touches the cone (V) at H, a tangent plane of (V) can be drawn through it, which, as containing the generator of Ω constituted by this tangent line, is also a tangent plane of Ω. Thus the points such as H arise from common tangent planes of the cone (V) and the quadric Ω, which are four in number. They have already been considered; and we see that the foci already found are all that exist. There are (Vol. III, p. 69) four generating lines of Ω, of either system, which touch the quartic curve on Ω. A plane through one of these eight lines, which touches one of the quadric cones containing the curve, will contain another of the lines; and will touch Ω. The sixteen foci are the projections of the intersections of these eight generators. Incidentally we see that to a general cubic curve in a plane there can be drawn four tangents from any point of the curve, forming a pencil related to the four from any other point.

The equation of the general tangent plane of the cone (V), as for a conic, may be taken in the form $\theta^2 P + 2\theta Q + R = 0$, where $P = 0$, $R = 0$ are two arbitrary tangent planes, and $Q = 0$ is the plane containing the two generators along which these touch the cone. In particular, we may take, for $P = 0$, $R = 0$, two tangent planes of (V) which also touch Ω. There will then be two values of θ, beside $\theta = 0$ and $\theta = \infty$, for which the tangent plane of the cone also touches Ω. If one of these be θ_1, and the corresponding plane be $T = 0$, we have an identity of the form $\theta_1{}^2 P + 2\theta_1 Q + R = \rho^2 T$, where ρ is a constant. If, in this identity, we suppose the coordinates to be those of a point on the cone (V), on which $Q = (RP)^{\frac{1}{2}}$, we infer $\theta_1 P^{\frac{1}{2}} + R^{\frac{1}{2}} = \rho T^{\frac{1}{2}}$, an equation true for points of (V), in which each of $P = 0$, $R = 0$, $T = 0$ represents a plane touching both the cone (V) and the quadric Ω.

If we suppose the quadric Ω referred to four points of which one is the point from which the figure is projected on to the plane ϖ, taken as $(0, 0, 0, 1)$, another is the point $(0, 0, 1, 0)$, and the two others are the intersections of the generators at the former point with the generators at the latter, we may suppose the equation to be $x^2+y^2-2zt=0$. Now, we easily verify the identity

$$(xz'-x'z)^2+(yz'-y'z)^2+2zz'(xx'+yy'-zt'-z't)=z^2(x'^2+y'^2-2z't')+z'^2(x^2+y^2-2zt);$$

wherefore, if $T'=0$, $T''=0$ be the equations of the tangent planes at two points of the quadric, (x', y', z', t') and (x'', y'', z'', t''), we may suppose, at an intersection of the tangent planes with the quadric, that $T'/T''=(D')^2/(D'')^2$, where

$$(D')^2=[(xz'-x'z)^2+(yz'-y'z)^2]/z^2z'^2.$$

Thus (cf. Vol. II, p. 184), we can infer from what is proved above that *the distances of any point of the quartic curve in the plane ϖ, from three of its foci lying on the same principal circle, are connected by a linear equation.*

Construction of the plane quartic curve given four foci

lying on a circle. If the Absolute points, I, J, and also four points, be prescribed, the six points lying on a conic (circle), two of the plane quartic curves, with double points at I and J, can be found, having the four points as foci, to pass through a further given point. And these cut one another at right angles at this point; that is, the tangents of the curves at this point meet the line IJ in points harmonically conjugate in regard to I and J. That a finite number of such curves is to be expected appears by counting constants: That a plane curve should have a given point as double point requires three linear conditions for the coefficients in its equation; that the quartic should have a given focus, or should touch two given lines, requires two conditions, and in the present case there is a reduction because the four foci are not independent points. The number of prescribed conditions for a quartic curve through a given point is thus $6+7+1$, or fourteen; while the number of available coefficients in the equation of a plane curve of order n, or $\frac{1}{2}n(n+3)$, is also fourteen when $n=4$.

Let P, Q, R, S denote the given foci, and O the given point, all in the plane ϖ. Take an arbitrary point U, not in this plane ϖ, and let Ω be a quadric containing the lines UI, UJ (other than the two planes IUJ, ϖ). Let the lines UP, UQ, UR, US, UO meet this quadric respectively in the points P', Q', R', S', O'. The section of Ω by the plane P', Q', R' projects from U, on to ϖ, into the circle PQR, which, by hypothesis, contains S; thus the plane $P'Q'R'$ contains S'. The plane $P'Q'R'S'$ is to be a principal plane, in our previous notation. Let its pole, in regard to Ω, be V. The tangent planes of Ω at P', Q', R', S' pass through V; it is possible to describe a quadric cone, with vertex at V, to touch these four planes, and have the line VO' for generator. In fact, two such cones can be described, and their tangent planes along VO' are harmonic conjugates in regard to the two tangent planes drawn to Ω from VO' (Vol. II, p. 25). Either of these cones meets Ω in such a quartic curve as we have considered; and the tangent line of this curve at O' lies on the tangent plane of the cone at O', which contains the line $O'V$, being the intersection of this plane with the tangent plane of Ω at O'; also, the two tangent planes of the quadric Ω, which can be drawn through $O'V$, contain the two generators of Ω at O'. There is thus, on the tangent plane of Ω at O', a harmonic pencil consisting of the two generators of Ω at O', and the two tangent lines of the two quartic curves on Ω obtained by the two cones (V); for we have seen that the tangent planes to Ω from $O'V$ are harmonic in regard to the two tangent planes of these cones along $O'V$. If now the curves on Ω are projected into two quartic curves on ϖ, this pencil becomes that formed by the tangents at O, of two plane quartics through O, and the lines OI, OJ.

We have thus constructed two quartic curves in ϖ, as desired. Conversely, consider any quartic curve, in the plane ϖ, that may be possible, with I and J as double points, having the same four given foci, and passing through O. Taking U as before, and the quadric Ω, we can shew that this curve arises by the construction we have made. Let the given points P, Q, R, S, O give, as before, the points P', Q', R', S', O', of Ω; project the quartic curve supposed, from U, on to the quadric. The curve on Ω, so obtained, will meet each of the lines UI, UJ in two points other than U; it will touch the two generators of Ω which intersect in P', and also the pairs of generators at Q', R', S'; and it will pass through O'. As this curve lies on Ω it will meet the tangent plane at P' only on the generators at P'; thus, as it touches each of these, its intersections with this plane count for four in number; and similarly at Q', R', S'. The curve is thus of the fourth order, and does not pass through U; it meets every generator of Ω in two points. Thus it lies on another quadric, beside Ω; and therefore lies on a quadric cone, say (W). A tangent line of the curve must thus lie in a tangent plane of this cone (W). In particular, the generators of Ω at P' must touch (W), and equally those at Q', R', S'. We may thus conclude that (W) may be taken to be the same as (V), and thus recover the preceding construction.

Inversion of the curve into itself. We have explained, in Chapter I (above, p. 12), the process of inversion, in regard to the quadric Ω; whereby, being given a point L, and its polar plane, λ, in regard to Ω, we pass from any point, P, to the point P', of the line LP, such that P, P' are harmonic conjugates in regard to L and the point (LP, λ). To a quadric cone there arises, as its inverse, another quadric cone, intersecting the plane λ in the same conic, with vertex the inverse of the vertex of the first cone.

Consider then the curve, on Ω, which lies on the cone (V). The curve which is the inverse of this, in regard to L, is the locus of the second intersections, with Ω, of the lines joining L to the points of the first curve; and this inverse curve lies also on a quadric cone, (V'), obtained by inversion from (V). The tangent planes, and generators, of these two cones, meet on the polar plane, λ, of L, in regard to Ω; and their vertices, V and V', are harmonic inverses in regard to L and λ. By projection from a point, U, of Ω, the curves (Ω, V) and (Ω, V') become two quartic curves in ϖ, both having I and J for double points, though in general with different tangents; and these curves are inverses of one another in regard to the point of ϖ obtained by projection of L, that is, in regard to the circle obtained by projection of the section (Ω, λ). If, in particular, L be taken at V, the two quartic curves on Ω coincide; thus the plane quartic inverts into itself in regard to any one of the principal circles.

Another particular case is when L is so taken that the line UL meets the original quartic on Ω; then the inverse quartic on Ω passes through U, and projects into a cubic curve on ϖ, having each of I and J as a simple point. The original quartic curve in ϖ can thus be inverted into such a cubic curve, namely by taking the centre of inversion on the curve itself. Conversely, the properties of the quartic curve in ϖ are obtainable by inverting a cubic curve which passes through the Absolute points I, J, with respect to any circle in its plane. And such a cubic curve is obtained by projecting a quartic curve on Ω, obtained by the intersection of this with another quadric, from a point U of this curve, the points I, J being the intersections of the generators at U with the plane, ϖ, on which we project.

Ex. 1. We may consider what are the properties of a cubic curve, in a plane ϖ, passing through two given Absolute points, I, J, of this plane, which arise as particular cases of those obtained above for a quartic curve having I, J for double points. There will be four families of circles (conics through I and J), each touching the cubic curve in two points. The circles of one family will have their centres on a parabola, and will all cut a fixed circle at right angles. The four parabolas will have a common focus, the intersection of the tangent lines of the cubic curve at I and J; they will also have a common axis, at right angles (in regard to I and J) to the tangent line of the cubic curve at its third intersection with the line IJ. The four intersections of a parabola with its associated principal circle are foci of the cubic curve.

Ex. 2. To obtain a formulation by equations, we may take as points of reference for coordinates x, y, z, t; first, the centre of projection U, on Ω, as point $(0, 0, 0, 1)$; by hypothesis this point lies on the cone (V); then, the points where the cone (V) is again met by the generators of Ω at U, as points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$; and last, as point $(0, 0, 1, 0)$, the intersection of the generators of Ω at these two points. The equation of Ω is then capable of the form $xy - zt = 0$; and the cone (V) may be regarded as the envelope, for varying θ, of a plane with the equation

$$\theta^2 (x + y + \sigma z) + 2\theta (lx + my) + l^2 x + m^2 y + \rho z + t = 0,$$

the plane $lx + my = 0$ being that which joins the point $(0, 0, 1, 0)$ to the generator of the cone (V) which passes through $(0, 0, 0, 1)$. The coefficients l^2, m^2 are chosen so that the cone contains the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$. By eliminating t, between the equation of the cone and $xy - zt = 0$, we thus find, for the equation of the cubic curve,

$$xy (x + y + zD) + z^2 (Px + Qy + Rz) = 0,$$

where $P = \rho + \sigma l^2$, $Q = \rho + \sigma m^2$, $R = \rho\sigma$, $D = \sigma + (l - m)^2$. This curve touches $x=0$ and $y=0$ on $z=0$, and contains the point $x+y=0$, $z=0$. One family of enveloping circles is given, with varying ϕ, by

$$xy+xz(\phi+l)^2+yz(\phi+m)^2+z^2(\phi^2\sigma+\rho)=0;$$

the centres of these circles lie on the parabola

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}+(l-m)(-z)^{\frac{1}{2}}=0,$$

and they cut at right angles the circle given by

$$xy+xz\sigma n+yz\sigma(1-n)-z^2(\sigma lm+\rho)=0,$$

where $n=l/(l-m)$.

If, instead of ρ, σ, l, m, we regard P, Q, R, D as given, we find, for σ,

$$(\sigma^2-\sigma D+P+Q)^2=4(PQ-RD),$$

and the equations above given then determine ρ, l^2, m^2, and lm.

Ex. 3. The cubic curve just considered arises by taking the centre of projection, U, to lie on the quartic curve on Ω. Another particular case, for the plane curve obtained by projection from U, arises by taking U on one of the principal sections, say, on the polar plane of V, so that V is on the tangent plane at U, of the quadric. In this case the plane quartic curve in ϖ has four foci lying on a line; the conic in ϖ, which is in general the locus of centres of a family of enveloping circles, is in this case a line coinciding with the circle to which these enveloping circles are orthogonal. Taking

$$L=lx+my+nz,\quad C=l\xi+m\eta,\quad L_r=l_rx+m_ry+n_rz,\quad C_r=l_r\xi+m_r\eta,$$

for $r=1, 2$, where ξ, η as well as l, m, n, l_r, m_r, n_r, are constants, such a quartic curve is given by

$$(C_1xy-zL_1)^2=(Cxy-zL)(C_2xy-zL_2),$$

and is the envelope of circles

$$\theta^2(Cxy-zL)+2\theta(C_1xy-zL_1)+C_2xy-zL_2=0,$$

whose centres lie on the line $2(x\xi+y\eta)-z=0$.

Ex. 4. If the centre of projection, U, of the quartic curve on Ω, be taken at one of the four points of a principal section which, in general, project into foci of the plane quartic, namely at a point of contact with Ω of a tangent plane of the cone (V), so that the cone (V) touches the tangent plane at U, of the quadric, another particular case is obtained; then, the intersections of the cone (V), with the generators of Ω at U, coincide in pairs, in two points which lie on the generator of contact of (V) with the tangent plane of Ω at U. The quartic curve in ϖ then has, at each of I and J, a double point with coincident tangents, namely a cusp; and, of one set of four foci of the curve lying on a circle in general, one focus disappears, and the other three lie on a line. The quartic curve is then the curve known as a *Cartesian*. Its equation can be found from that in Ex. 3 by supposing $l_2 = 0$, $m_2 = 0$, $n_2 = 1$, so that $C_2 = 0$ and $L_2 = z$. The *Limaçon*, and, thence, the *Cardioid*, are special cases.

Angles of intersection of enveloping circles of the plane quartic which belong to different families. Returning now to the general case, we prove that, if we take two families of enveloping circles of the quartic curve in ϖ, the angles which a varying circle of one of these families makes with two fixed circles of the other family, have a constant sum or difference (Jessop, *Quarterly Journal of Math.*, XXIII, 1889, p. 375). The angles are measured in regard to the points I, J (as in Vol. II, p. 167).

The angle, between two circles obtained as projection of two plane sections of the quadric Ω, depending on the relation of the tangents of these sections, at one of their points of intersection, to the generators of Ω at this point, is equal to the angle between the planes of the sections, measured in regard to the quadric (Vol. II, p. 193); it is therefore equal to the interval between the poles of the planes of the section, measured by the quadric. The result to be obtained is therefore this: If we take upon the conic v, which is the polar reciprocal of the cone (V), a variable point, P, and take upon the conic v_1, which is the polar reciprocal of the cone (V_1), two fixed points, A, B, the sum, or difference, of the intervals PA, PB, measured in regard to the quadric Ω, is independent of P. As the conics v, v_1, v_2, v_3 are such a system as the focal conics of a system of confocal quadrics, this result has already been proved, Vol. III, p. 96. We return to this point of view below.

The proposition is however a descriptive one. Suppose that we have four planes, α, β, γ, δ, which we take in order; take the four lines of intersection of consecutive pairs of these, say (α, β), (β, γ), (γ, δ), (δ, α). Consider the intersections of these four lines with a given quadric, say, respectively, P, P'; Q, Q'; R, R'; S, S'. It may happen that four of these points, one on each line, lie on a plane; say P, Q, R, S lie on a plane, θ. It is then also the case that the other four points, P', Q', R', S', lie on a plane, say ϕ; for, since the eight points lie on the three quadrics consisting of, (*a*) the given quadric, (*b*) the degenerate quadric consisting of the plane-pair (α, γ), (*c*) the degenerate quadric consisting of the plane-pair (β, δ), and these three quadrics are evidently such that the third does not contain the complete intersection of the first two, it follows (Vol. III, pp. 148, 154) that the quadric consisting of the plane θ and the plane $P'Q'R'$, which contains seven of the eight points, likewise contains the eighth point S', which we suppose not to lie on the plane θ.

This being understood, the descriptive theorem above referred to is that, if (V_1) be a quadric cone passing through the curve of intersection of a quadric Ω and a quadric cone (V), and α, γ be any two tangent planes of the cone (V), while β, δ are any two tangent planes of the cone (V_1), and we consider, as here, the four lines of intersection of the pair of planes α, γ, each with both the planes β, δ, then the eight points, in which these meet the quadric Ω, lie, in two sets of four, upon two planes, say θ and ϕ. The lines of contact with the cone (V), of the two planes α, γ, lie on a plane, say ϵ; likewise the lines of contact of β, δ with (V_1) lie on a plane, say ζ; the planes θ, ϕ intersect in the line of intersection of ϵ and ζ, and are harmonic in regard to these. In order to make this proposition clear we may give at once a very simple proof with the symbols: Denoting by $\alpha = 0$ the equation of the plane α, and so on, the equation of the cone (V) may be supposed to be $\alpha\gamma - \epsilon^2 = 0$; the equation of the cone (V_1) may similarly be supposed to be $\beta\delta - \zeta^2 = 0$. Wherefore the equation of the quadric Ω, which passes through the curve of intersection of these two cones, may be supposed to be $\alpha\gamma - \epsilon^2 - m^2(\beta\delta - \zeta^2) = 0$, where m is a proper constant; this is the same as $\alpha\gamma - m^2\beta\delta = (\epsilon - m\zeta)(\epsilon + m\zeta)$. This equation shews that a point of Ω which is on the line (α, β) lies on one of the two planes $\epsilon - m\zeta = 0$, $\epsilon + m\zeta = 0$, the same being true of points of Ω on any of the lines (β, γ), (γ, δ), (δ, α). This proves the statement made.

Now consider the relation of this descriptive proposition with the metrical theorem above given for the angles of intersection of enveloping circles, of different families, of the plane quartic curve. Let α, β, γ, δ be four planes such that the four lines (α, β), (β, γ), (γ, δ), (δ, α) meet the quadric Ω in, respectively, P, P'; Q, Q'; R, R'; S, S', with the property that P, Q, R, S lie on a plane, θ, and P', Q', R', S' lie on a plane, ϕ; let the angle between the planes α, β, measured in regard to the quadric, that is in regard to the tangent planes of the quadric drawn from the line (α, β), be denoted by $[\alpha, \beta]$; as has been remarked, this is the same as the angle between the tangent lines at a point of intersection of the circles into which the sections by α, β project, measured in regard to I and J. We can then shew that $[\alpha, \beta] + [\gamma, \delta]$ is equal to $[\beta, \gamma] + [\delta, \alpha]$. It is to be understood that the statement is cast in this form in deference to traditional metrical geometry; the symbol $[\alpha, \beta]$ is ambiguous in sign, and by additive multiples of π, unless more particularly defined; the geometrical theorem, which is quite free from ambiguity, is one in regard to the symbols, once before called cross-ratios, such as are defined in Vol. II, p. 166, arising from the pairs of planes. See Ex. 4 below. This being borne in mind, the proof of the equation follows at once from the remark that, if λ, μ, ν be three planes meeting in a point, O, of the quadric,

Ω, the three angles $[\mu, \nu]$, $[\nu, \lambda]$, $[\lambda, \mu]$ are together equal to π, or one of these angles, properly taken, is equal to the sum of the other two; this is only the statement that if the lines of intersection of the three planes, with the tangent plane at O, be represented in terms of the generators, i, j, at this point, by $i + lj$, $i + mj$, $i + nj$, then $(m/n)(n/l)(l/m) = 1$. Using this remark, and considering the two points, P, R, where the plane θ meets the lines (α, β) and (γ, δ), we have

$$\pi = [\theta, \alpha] + [\alpha, \beta] + [\beta, \theta], \quad \pi = [\theta, \gamma] + [\gamma, \delta] + [\delta, \theta],$$

and hence

$$[\alpha, \beta] + [\gamma, \delta] = 2\pi - [\theta, \alpha] - [\theta, \beta] - [\theta, \gamma] - [\theta, \delta],$$

where we have replaced $[\beta, \theta]$ by $[\theta, \beta]$, and $[\delta, \theta]$ by $[\theta, \delta]$. The same symmetrical form arises for $[\beta, \gamma] + [\alpha, \delta]$.

Ex. 1. For circles in a plane there follows, from what has been said, that, if four circles, $\alpha, \beta, \gamma, \delta$, be such that α, β meet in P, P'; while β, γ meet in Q, Q'; γ, δ meet in R, R' and δ, α meet in S, S', and, if P, Q, R, S lie on a circle, then also P', Q', R', S' lie on a circle (Vol. II, p. 72).

Ex. 2. Also that, in this case, the two opposite angles $[\alpha, \beta]$, $[\gamma, \delta]$ have the same sum as the two opposite angles $[\beta, \gamma]$, $[\alpha, \delta]$. Conversely, this theorem of angles is sufficient to ensure that the points P, Q, R, S lie on a circle.

This simple theorem of metrical geometry may well be regarded as fundamental. In Todhunter-Leathem, *Spherical Trigonometry*, 1901, p. 132, there is quoted from Lexell, *Acta Petropolitana*, 1782, the theorem "If, on a sphere, the corners of a spherical quadrilateral lie on a small circle, the sums of its pairs of opposite angles are equal." See also McF. Orr, *Trans. Camb. Phil. Soc.*, XVI, 1897, p. 95.

Ex. 3. Also it is true that, if the sum of the angles of intersection of the pairs of three circles be π, the circles meet in a point.

The condition that the sum of three angles, α, β, γ, should be π, is the vanishing of the determinantal discriminant of the form

$$x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma.$$

It is easy to prove that the angle, in regard to the quadric $x^2 + y^2 + z^2 + t^2 = 0$, between two planes, of equations

$$a_r x + b_r y + c_r z + d_r t = 0 \ (r = 1, 2),$$

has, for cosine, $\sigma_{12}/\sigma_1\sigma_2$, where $\sigma_{12} = a_1 a_2 + \ldots + d_1 d_2$ and

$$\sigma_r^2 = a_r^2 + \ldots + d_r^2.$$

Using this result for every two of three planes, the signs attached to $\sigma_1, \sigma_2, \sigma_3$ being the same throughout, it is easy to deduce that the point of intersection of the three planes lies on the quadric.

Ex. 4. Supposing that the planes above denoted by α, γ, β, δ, θ, ϕ have, respectively, the equations $x=0$, $y=0$, $z=0$, $t=0$, $\xi=0$, $\eta=0$, where there will be two identities, say

$$x+y+z+t+\xi+\eta=0, \quad bx+ay+dz+ct+g\xi+f\eta=0,$$

the equation of the quadric, Ω, will be of the form

$$Pxy+Qzt+R\xi\eta=0.$$

Let the roots of the quadratic equation in λ,

$$P^{-1}(\lambda-a)(\lambda-b)+Q^{-1}(\lambda-c)(\lambda-d)+R^{-1}(\lambda-f)(\lambda-g)=0,$$

be denoted by λ_1, λ_2, and define twelve quantities a_1, b_1, ..., g_2 by the equations

$$Pa_r=a-\lambda_r,\quad Pb_r=b-\lambda_r,\quad Qc_r=c-\lambda_r,\quad Qd_r=d-\lambda_r,$$
$$Rf_r=f-\lambda_r,\quad Rg_r=g-\lambda_r,$$

for $r=1, 2$. Speaking of $x=0, y=0$ as opposite planes, and of $z=0$, $t=0$ as opposite planes, and of $\xi=0$, $\eta=0$ as opposite planes, there will, among the twenty points of intersection of three of these six planes, be eight points through which no two opposite planes pass; these are the points previously denoted by P, Q, ..., R', S'. The equations of the generators of the quadric at any one of these eight points can be written down at once; for instance, at the point $x=0$, $z=0$, $\xi=0$, these generators are given by

$$xa_1^{-1}=zc_1^{-1}=\xi f_1^{-1},\quad xa_2^{-1}=zc_2^{-1}=\xi f_2^{-1};$$

and at the point $x=0$, $z=0$, $\eta=0$ by similar equations, obtained by writing ηg_1^{-1} and ηg_2^{-1} respectively for ξf_1^{-1} and ξf_2^{-1}; and so for all.

Further, considering first, for example, the two planes $x=0, z=0$, the tangent planes of the quadric through their line of intersection are given, by what we have just said, by $xa_1^{-1}=zc_1^{-1}$, $xa_2^{-1}=zc_2^{-1}$. Thus the angle, ω, between the planes $x=0$, $z=0$, measured in regard to the quadric, is given by $e^{2i\omega}=a_1a_2^{-1}/c_1c_2^{-1}$. In general, if we define angles, α, β, γ, δ, θ, ϕ, by means of

$$e^{2i\alpha}=a_1a_2^{-1},\quad e^{2i\beta}=b_1b_2^{-1},\quad e^{2i\gamma}=c_1c_2^{-1},\quad e^{2i\delta}=d_1d_2^{-1},$$
$$e^{2i\theta}=f_1f_2^{-1},\quad e^{2i\phi}=g_1g_2^{-1},$$

then any two of the six planes, $x=0$, $y=0$, ..., $\eta=0$, other than two opposite planes, make with one another an angle, relatively to the quadric, given by the difference of the two corresponding angles chosen from α, ..., ϕ. This is the general form of the theorem, for the angles between enveloping circles of different families of the plane quartic curve, from which we started.

Ex. 5. In Ex. 4, denoting $P^{-1}+Q^{-1}+R^{-1}$ by A,
also $\quad P^{-1}ab+Q^{-1}cd+R^{-1}fg$ by C,
and $\quad P^{-1}(a+b)+Q^{-1}(c+d)+R^{-1}(f+g)$ by B,
and $\quad (B^2-4AC)A^{-1}$ by H,

prove that the condition that a plane, whose equation is written $lx + my + nz + pt + u\xi + v\eta = 0$, should touch the quadric Ω is

$$H(P^{-1}lm + Q^{-1}np + R^{-1}uv) + U_1U_2 = 0,$$

where $\quad U_r = a_rl + b_rm + c_rn + d_rp + f_ru + g_rv \quad (r = 1, 2).$

Ex. 6. The preceding results in regard to the angles between planes lead to results as to the intervals between points, as has been indicated. We indicate some of these, referring, for a less summary account, to a note, *Camb. Phil. Proc.*, xx, 1920, pp. 122–130.

Let two quadrics, V, W, intersect in plane curves; they thus touch one another in two points. It is possible then to find another quadric, U, touching V at all the points of a plane section, and also touching W at all the points of a plane section; the planes of contact necessarily pass through the points of contact of V and W, and are harmonic in regard to the planes in which V and W intersect. Two such quadrics, U, having, we may say, *ring* contact with V and W, can be drawn through an arbitrary point.

Prove that, if AT be any tangent to the quadric V, drawn from any point A, and touching V in T, the interval AT, measured in regard to the quadric U, depends only on A. Deduce that, if A, B, C, D be four points such that the four joins AB, BC, CD, DA all touch V, the sum of the intervals AB, BC, CD, DA, measured in regard to the quadric U, and suitably interpreted, is zero.

Next, let K be any quadric, other than U, which likewise has ring contact with both V and W. Then prove that, if from any point, P, of the quadric K, tangents PX, PY be drawn respectively to V and W, touching these in X and Y, the sum, or difference, of the intervals PX, PY, measured in regard to U, is independent of the position of P upon K.

This result includes many others (cf. Vol. II, p. 211, Ex. 8; Vol. III, p. 96, Ex. 2).

Ex. 7. If $u^2 = b - c$, $v^2 = a - c$, $w^2 = a - b$, the four lines, joining each of the points of coordinates $(w \cosh \alpha,\ 0,\ u \sinh \alpha,\ 1)$, $(w \cosh \gamma,\ 0,\ u \sinh \gamma,\ 1)$, to both the points $(v \cos \beta,\ u \sin \beta,\ 0,\ 1)$, $(v \cos \delta,\ u \sin \delta,\ 0,\ 1)$, touch the two enveloping cones, of the quadric $x^2(a-\rho)^{-1} + y^2(b-\rho)^{-1} + z^2(c-\rho)^{-1} = t^2$, whose vertices have the tangential equation $P^2(b-\rho)^{-1} + Q^2(c-\rho)^{-1} = 0$, where

$$P \sinh \tfrac{1}{2}(\alpha - \gamma) = lw \cosh \tfrac{1}{2}(\alpha + \gamma) + nu \sinh \tfrac{1}{2}(\alpha + \gamma) - p \cosh \tfrac{1}{2}(\alpha - \gamma),$$
$$Q \sin \tfrac{1}{2}(\beta - \delta) = lv \cos \tfrac{1}{2}(\beta + \delta) + mu \sin \tfrac{1}{2}(\beta + \delta) - p \cos \tfrac{1}{2}(\beta - \delta).$$

Ex. 8. From the section of a quadric with a cone of order n, having an r-ple generator, obtain a plane curve of order $2n - r$, with one r-ple point, and two $(n-r)$-ple points, I, J. This curve has $2(n-r)(n+r-1)$ foci, lying on a circle, and is its own inverse in regard to this. (In particular, the invariant of a plane cubic curve is thus clear.)

CHAPTER IV

A PARTICULAR FIGURE IN SPACE OF FOUR DIMENSIONS

We consider now a figure, in space of four dimensions, which is remarkable in that the numbers of the elements which it contains, and the mutual relations of these elements, when properly interpreted, are the same as arise for the figure of the twenty-seven lines of a cubic surface. The figure contains twenty-seven points, corresponding to the twenty-seven lines of the cubic surface. In space of four dimensions, four independent points determine a space of three dimensions; for such a space we shall, in this chapter, use the word *solid*, so that the figure will contain a certain number of points, lines, planes and solids. The twenty-seven points of the figure lie, in sixes, in seventy-two solids, corresponding to the rows of the thirty-six double-sixes of lines of the cubic surface; every one of these rows is in fact determined by four non-intersecting lines lying therein (Vol. III, p. 166). The figure contains two hundred and sixteen lines, each joining two points of the figure; these correspond to two lines of the cubic surface which do not intersect one another (*ibid.*). In a similar way, three lines of the cubic surface of which no two intersect correspond to a plane of the figure; there are 720 such planes. With the notation employed in Vol. II, p. 212, the elements being points, lines, planes and solids, the structure of the figure is represented by the symbol

$$27\,(\,.\,,16,80,16)\,216\,(2,\,.\,,10,5)\,720\,(3,3,\,.\,,2)\,72\,(6,15,20,\,.\,)\,;$$

this means, for example, that sixteen lines, eighty planes and sixteen solids pass through every one of the twenty-seven points; and that every one of the two hundred and sixteen lines contains two points, and lies in ten planes and in five solids; and so on. It should be said that the existence of the figure does not depend on the Propositions of Incidence alone, but requires also the truth of Pappus' Theorem.

In order to simplify the description, the points are represented by the same letters as are used for the lines of a cubic surface, $a_1, a_2, \ldots a_6, b_1, b_2, \ldots b_6, c_{12}, c_{23}, \ldots$. The solids are denoted by capitals, O, P_{123}, P'_{123}, O', P_{12}, etc., where, in P_{123}, P'_{123}, the order of the three suffixes is indifferent, but P_{12}, with two suffixes, is to be distinguished from P_{21}. Each of the numbers occurring in the suffixes is one of the six, 1, 2, ... 6. The relation of the points and the seventy-two solids is, in fact, that represented by the symbols

$O\ (a_1, a_2, \ldots a_6)$, (one case); $P_{123}\ (a_1, a_2, a_3, c_{56}, c_{64}, c_{45})$, (twenty cases);
$O'\ (b_1, b_2, \ldots b_6)$, (one case); $P'_{123}\ (b_1, b_2, b_3, c_{56}, c_{64}, c_{45})$, (twenty cases);
$P_{12}\ (a_1, b_1, c_{23}, c_{24}, c_{25}, c_{26})$, (fifteen cases);
$P_{21}\ (a_2, b_2, c_{13}, c_{14}, c_{15}, c_{16})$, (fifteen cases);

these evidently correspond to the rows of the double-sixes of lines for a cubic surface (Vol. III, p. 161). It will be unnecessary to have a notation for all the lines of the figure; but the six lines containing the pairs of points (a_1, b_1), (a_2, b_2), ... (a_6, b_6) will be denoted, respectively, by $n_1, n_2, \ldots n_6$. A set of five points which do not lie in a solid will be described as a *simplex*; it will be found that there are two hundred and sixteen simplexes in the figure. And, for clearness, it may be stated that a simplex corresponds to five non-intersecting lines of the cubic surface which have two transversals; five non-intersecting lines of the cubic surface which have only one common transversal are represented in the figure by five points belonging to the same solid.

The case here considered, for its interest in connexion with the lines of a cubic surface, may serve as an introduction to a wide literature. In a remarkable paper, "Circles, Spheres and Linear Complexes," *Trans. Camb. Phil. Soc.*, XVI, 1898, pp. 181–188, Mr J. H. Grace, having in mind chains of theorems given by Clifford (*Math. Papers*, 1882, pp. 51, 52), considers theorems in regard to spheres passing in sixes through points. Dr W. Burnside (*Proc. Camb. Phil. Soc.*, XV, 1909, pp. 71–75) remarks that, if the space of the spheres be inverted from a point not lying therein, theorems are thence deducible for points lying on a sphere in Euclidean space of four dimensions; he remarks, however, that the condition of lying on a sphere, or quadric, is not necessary for the figure; and proves the theorems to which we presently proceed. Professor P. H. Schoute, "On the relation between the vertices of a definite six-dimensional polytope and the lines of a cubic surface," *R. Ac. of Sc. of Amsterdam*, 24 Sep. 1910, pp. 375–383, gives, after Mr E. L. Elte, the coordinates of the twenty-seven vertices of a regular figure lying on a sphere in Euclidean space of six dimensions. Such a figure is capable of a group of rotations. For relations with the theory of groups see Burnside, "Groups of rational linear substitutions of finite order, etc.," *Proc. Lond. Math. Soc.*, X, 1911, pp. 300–308 (and, *ibid.*, XI, 1912, pp. 295–299). For the group of the lines of a cubic surface, see also Burnside, *Theory of Groups* (Cambridge, 1911, Second Ed.), pp. 485–488; and *Proc. Roy. Soc.*, LXXVII (1906), p. 182. Also Jordan, *Traité des Substitutions* (1870), pp. 316–319; Burkhardt, *Math. Annal.*, XXXVIII (1891), p. 185, and XLI (1893), p. 320, where the group is represented by linear equations, derived from the theory of hyperelliptic functions. Also L. E. Dickson, *Linear Groups* (1901), Chap. XIV.

We proceed now to shew how the figure can be constructed. We shall require two lemmas which, for clearness, may be given first:

(I) In four dimensions, let $a_1, a_2, \ldots a_6$ be six given points lying in a solid, which we denote by O. Through every three of these points let another definite solid pass; this other solid, through a_1, a_2, a_3, for example, will be denoted by P_{123}. If we consider four of the six given points there will then be such a solid through every three of these points, beside O; consider, for instance, the solids, P_{234}, P_{314}, P_{124}, P_{123}, passing through the threes of the four points a_1, a_2, a_3, a_4; these determine, by their intersection, a fifth point, not lying in O. This we denote by c_{56}, or c_{65}, the suffixes 5, 6 being those not arising in the construction. And so in general. The five points, $a_1, a_2, a_3, a_4, c_{56}$, constitute a simplex; the solids of this simplex, each containing four of these points, are, respectively, P_{234}, P_{314}, P_{124}, P_{123} and O. In all, there are twenty solids such as P_{123}, and fifteen points such as c_{56}; these points, which do not lie in O, belong to simplexes whose other vertices are in O. The solid P_{123} will be used, not only in the construction of c_{56}, but also, when we consider the points a_1, a_2, a_3, a_5, in the construction of c_{64}; and, similarly, in the construction of c_{45}. Thus P_{123} contains the six points $a_1, a_2, a_3, c_{56}, c_{64}, c_{45}$. If we now consider five of the six given points, say a_1, a_2, a_3, a_4, a_5, then, of the five simplexes arising by considering the fours of these five points, there are four simplexes each having a_1 as one vertex, the completing vertices being, respectively, c_{62}, c_{63}, c_{64}, c_{65}. These completing vertices themselves determine a solid. It can be shewn that this solid contains the point a_1. Denoting this solid by P_{16} (not the same as P_{61}), and using a similar notation in general, it can be shewn that the five solids $P_{12}, P_{13}, P_{14}, P_{15}, P_{16}$ have a line in common, which therefore passes through a_1. This line we denote by n_1. These five solids are those which arise by considering the fives from a_1, a_2, ..., a_6 which, in turn, do not include a_2, a_3, a_4, a_5, a_6; just as P_{16} arose from the five other than a_6.

To prove this result, notice that every one of the simplexes used, in obtaining the points c_{rs} here involved, has one vertex at a_1, and has four solids, including O, passing through a_1. Take an arbitrary solid not passing through a_1, say U, and let ϖ be the plane in which this meets the solid O. The five lines joining a_1, respectively, to a_2, a_3, ... a_6, which are lines of O, meet the solid U in points, lying in the plane ϖ; let these points be called, respectively, B_2, ... B_6. The solid P_{123}, containing a_1, a_2, a_3, meets the solid U in a plane, say ϖ_{23}, which passes through B_2 and B_3. Thus, in the solid U, we have a plane, ϖ, and, therein, five points B_2, ... B_6; and through the join of every two of these points is drawn a plane, lying in U, that through B_2 and B_3 being ϖ_{23}. The point of intersection of the planes ϖ_{23}, ϖ_{34}, ϖ_{42}, which arise from the points B_2, B_3, B_4, may appropriately be named C_{56}; for it lies in the solids $P_{123}, P_{134}, P_{142}$, which contain a_1 and c_{56}, and is thus the point where the line of intersection of these solids, joining a_1 to c_{56}, meets the solid U. By considering every three of the four points B_2, B_3, B_4, B_5, and the three planes, ϖ_{rs}, through the joins of these three points, we obtain four points, such as C_{56}, namely $C_{62}, C_{63}, C_{64}, C_{65}$. It has been shewn above (p. 18) that these four points lie in a plane, which will then be in the solid U. We denote this plane by Π_6. These four points are on the lines joining a_1 to $c_{62}, c_{63}, c_{64}, c_{65}$, respectively; thus, the solid containing $c_{62}, \ldots c_{65}$ also contains a_1, as we desired to prove, this being the solid containing a_1 and the plane Π_6. We denote this solid by P_{16}. It has, however, also been shewn above (p. 29) that the five planes, $\Pi_2, \Pi_3, \ldots \Pi_6$, obtained by considering every four of the five points $B_2, B_3, \ldots B_6$, meet in a point, in the solid U. Wherefore, the five solids $P_{12}, P_{13}, \ldots P_{16}$ meet in a line, passing through a_1, as we also desired to prove. This is the line we denoted by n_1.

(II) The second lemma we require deals with the relations of the solids $P_{12}, P_{13}, \ldots P_{16}$, which intersect in a line, n_1, passing through the point a_1, and the solids $P_{21}, P_{23}, \ldots P_{26}$, which, similarly, intersect in a line, n_2, passing through the point a_2. We build up the result we require from the beginning. Suppose we have two lines, n, n', in space of four dimensions, and three solids passing through each of these; namely, the solids P, Q, R passing through the line n, and the solids P', Q', R' passing through the line n'. The solids P, P' meet in a plane, say α; similarly, Q, Q' meet in a plane, β, and R, R' in a plane, γ. Let A be the point of intersection of the planes β, γ, or the point common to the solids Q, Q', R, R'; similarly, let B be (γ, α), or (R, R', P, P'), and C be (α, β), or (P, P', Q, Q'). Through the plane ABC can be drawn a range (∞^1, a pencil) of solids; if H be the intersection with the line n of one of these solids, and H' the intersection of the same solid with n', the range of various positions of H on n is clearly related to that of the resulting positions of H' on n'. In particular, let the solid P, through n, meet the line n' in U', and the solid P', through n', meet the line n in U, so that U, U' are the intersections of the plane α respectively with n and n'; similarly, let the plane β meet n and n' in V and V', respectively, and the plane γ meet n and n' in W and W', respectively. The line UU', in the plane α, meets the line BC, which lies in this plane; thus U and U' are in a particular solid containing the plane ABC, that, namely, containing the plane of the lines UU', BC, and the point A. So V, V' are in a solid containing the plane ABC, as are W and W'. Wherefore, for any position of H, and its corresponding point H', the range U, V, W, H is related to the range U', V', W', H'.

Now let S be a fourth solid through the line n, meeting the line n' in T', and S' a fourth solid through the line n', meeting the line n in T, and let δ be the plane (S, S'); also let A' be the point

(α, δ), or (P, P', S, S'); and, similarly, B' be the point (β, δ), or (Q, Q', S, S'), and C' the point (γ, δ), or (R, R', S, S'). As, then, we considered the points A, B, C, arising by taking the solids P, Q, R with the solids P', Q', R', so we can consider the points A, B', C', arising by taking the solids S, Q, R with the solids S', Q', R'; thus we can infer that if a solid through A, B', C' meet the line n in H, and the line n' in H', the range V, W, T, H is related to the range V', W', T', H'. For these points H, H' to be the same as those previously so denoted, since we had U, V, W, H related to U', V', W', H', it is necessary, as a condition for the solids S, S', that U, V, W, T should be related to U', V', W', T'. Conversely, suppose this is so; let H be taken arbitrarily on n; then a point, H', can be taken on n' so that the range U', V', W', T', H' is related to U, V, W, T, H; and these points H, H' will be such that the five points A, B, C, H, H' lie in a solid, and also the five points A, B', C', H, H' lie in a solid. Therefore, similarly, the five points B, C', A', H, H' lie in a solid, and so do C, A', B', H, H'. Thus we have proved that, if solids P, Q, R, S, through the line n, respectively, meet the line n' in U', V', W', T', and solids P', Q', R', S', through the line n', meet n in a related range U, V, W, T; if, further, $\alpha, \beta, \gamma, \delta$ be the planes (P, P'), (Q, Q'), (R, R'), (S, S'), respectively, and, respectively, A, B, C, A', B', C' be the points (β, γ), (γ, α), (α, β), (α, δ), (β, δ), (γ, δ); and if H, H' be any two points, respectively, on n, n', such that the range U, V, W, T, H is related to the range U', V', W', T', H'; then each of the four sets of five points (A, B, C, H, H'), (A, B', C', H, H'), (B, C', A', H, H'), (C, A', B', H, H') lies in a solid.

We now make application of the results of these lemmas. Take, for the lines n, n' of lemma II, the lines previously denoted, in lemma I, by n_1 and n_2. Take, for the solids P, Q, R, S, respectively, those denoted by $P_{13}, P_{14}, P_{15}, P_{16}$; and, for the solids P', Q', R', S', respectively, those denoted by $P_{23}, P_{24}, P_{25}, P_{26}$. Then, for example, the solids P_{13}, P_{23}, respectively, contain the points $(a_1, c_{32}, c_{34}, c_{35}, c_{36})$ and $(a_2, c_{31}, c_{34}, c_{35}, c_{36})$. Thus we have the respective intersections

$$(P_{14}, P_{15}, P_{24}, P_{25}) = c_{45},\ (P_{15}, P_{13}, P_{25}, P_{23}) = c_{53},\ (P_{13}, P_{14}, P_{23}, P_{24}) = c_{34},$$
$$(P_{13}, P_{16}, P_{23}, P_{26}) = c_{36},\ (P_{14}, P_{16}, P_{24}, P_{26}) = c_{46},\ (P_{15}, P_{16}, P_{25}, P_{26}) = c_{56}.$$

Therefore the points which, in lemma II, were called A, B, C and A', B', C', are, respectively, c_{45}, c_{53}, c_{34} and c_{36}, c_{46}, c_{56}. These points A, B, C, however, lie in the solid P_{126}, which meets the lines n_1 and n_2, respectively, in the points a_1 and a_2; and these points A, B', C', or c_{45}, c_{46}, c_{56}, lie in the solid P_{123}, which meets the lines n_1 and n_2, respectively, in the same points a_1 and a_2. Thus, by lemma II, we see that the solids $P_{13}, P_{14}, P_{15}, P_{16}$ meet the line n_2 in a range which is related to that in which the line n_1 is met by the solids

P_{23}, P_{24}, P_{25}, P_{26}. And in these related ranges a_2 and a_1 are corresponding points.

But we can also infer that, if any other solid be drawn through A, B, C, or c_{45}, c_{53}, c_{34}, meeting n_1, n_2, respectively, in the points b_1 and b_2, then b_2, b_1 will also be corresponding points of these two related ranges, respectively. Let such a solid be denoted by P'_{126}. Further, we can infer that the five points b_1, b_2, A, B', C', that is, b_1, b_2, c_{45}, c_{46}, c_{56}, lie on a solid, say P'_{123}; also, that the five points b_1, b_2, B, C', A', or b_1, b_2, c_{53}, c_{56}, c_{36}, lie on a solid, say P'_{124}; and that the five points b_1, b_2, C, A', B', or b_1, b_2, c_{34}, c_{36}, c_{46}, lie on a solid, say P'_{125}. Fully to justify the notation, however, it must be proved that the order of the suffixes here used is indifferent. Consider, for example, P'_{126}, containing b_1, b_2, c_{34}, c_{45}, c_{53}. Let the point in which this solid meets the line n_6, whose definition is analogous to that of n_1 and n_2, be denoted by b_6. Taking together the lines n_1, n_6, there are, through the former, the solids P_{12}, P_{13}, P_{14}, P_{15}; and, through the latter, the solids P_{62}, P_{63}, P_{64}, P_{65}; also we have the intersections represented by

$$(P_{13}, P_{14}, P_{63}, P_{64}) = c_{34},\ (P_{14}, P_{15}, P_{64}, P_{65}) = c_{45},\ (P_{13}, P_{15}, P_{63}, P_{65}) = c_{53}.$$

Thus, as before, we can define b_6 directly from b_1, and then obtain b_2 by a solid which, with parity of notation, would be denoted by P'_{162}. This is then the same as P'_{126}. And so on.

We have thus justified the propriety of all the symbols obtainable, by change of the suffixes, from

$$P_{123}\,(a_1, a_2, a_3, c_{56}, c_{64}, c_{45}),\quad P_{12}\,(a_1, b_1, c_{23}, c_{24}, c_{25}, c_{26}),$$
$$P'_{123}\,(b_1, b_2, b_3, c_{56}, c_{64}, c_{45}).$$

Finally, we prove that the six points b_1, b_2, ... b_6 lie in a solid. For this, consider, for instance, the five points b_1, c_{62}, c_{63}, c_{64}, c_{65}, which lie in the solid P_{16}. With every four of these we can form a simplex, whose completing vertices lie, by lemma I, in a solid which contains the point b_1. The simplexes in question, with the solids containing the sets of four vertices of these, are, in fact,

$$\begin{array}{ll}
b_1, c_{63}, c_{64}, c_{65} & b_1, c_{62}, c_{64}, c_{65} \\
P_{26}, P'_{123}, P'_{124}, P'_{125}, P_{16}; & P_{36}, P'_{123}, P'_{134}, P'_{135}, P_{16}; \\
b_1, c_{62}, c_{63}, c_{65} & b_1, c_{62}, c_{63}, c_{64} \\
P_{46}, P'_{124}, P'_{134}, P'_{145}, P_{16}; & P_{56}, P'_{125}, P'_{135}, P'_{145}, P_{16},
\end{array}$$

where the completing vertices, not written, are, respectively, b_2, b_3, b_4, b_5. Thus it appears that the five points b_1, b_2, b_3, b_4, b_5 lie in a solid. By a similar argument it appears that b_1, b_2, b_3, b_4 and b_6 lie in a solid. This is then the same as before. We can then complete the symbols used, by writing, also, $O'\,(b_1, b_2, \ldots b_6)$.

The statement of the structure of the figure, which has been given, can now be justified. The 27 points of the figure are those which have been named. The 216 lines of the figure consist of the joins of the pairs of the six points lying in each of the 72 solids which have been obtained; this would give 72 . 15 lines; but each line lies in five of the solids, so that there are 72 . 3, or 216 lines. In fact, the notation has involved that the line n_1, joining the points a_1 and b_1, lies in the five solids $P_{12}, \ldots P_{16}$. That any other line also lies in five of the solids may be verified; for instance, the line joining the points b_1 and b_2 lies in the solids O', P'_{123}, P'_{124}, ... P'_{126}; or, again, the line joining the points a_1 and c_{56} lies in the solids P_{123}, P_{124}, P_{134}, P_{15}, P_{16}. The 720 planes of the figure are, similarly, those defined by three points lying in the same solid of the figure. But it will be seen, on examination, that every such plane lies in two of the solids, so that there are $72 . 20 \div 2$, or 720 such planes. For example, the plane defined by a_1, a_2, a_3 lies in the solids P_{123} and P'_{123}; or again, the plane defined by a_1, c_{64}, c_{65} lies in P_{123} and also in P_{16}. The other numbers of the structure symbol can also be justified. For instance, that a line lies in ten planes: the line joining a_1 and b_1 lies in the ten planes containing a_1, b_1, c_{rs}, where r, s are any two of the numbers 2, 3, ... 6. Or, again, the line joining a_1 and a_2 is in the planes joining these two points to a_3, a_4, a_5, a_6, c_{ij}, where i, j are any two of the numbers 3, 4, 5, 6.

That a point lies in sixteen solids arises from $72 . 6 \div 27 = 16$; that a point lies in 80 planes similarly arises from $720 . 3 \div 27 = 80$; and that there are sixteen lines through a point can be similarly shewn, the point a_1, for example, lying in the lines joining this point to a_2, a_3, ... a_6, b_1, c_{rs}, where r, s are any two of the numbers 2, ... 6.

It is also of interest to consider the possible simplexes, of which there are 216. Of these, 40 exist with any one of the points as one vertex, 10 exist with two of the points, joined by a line of the figure, as two vertices, 3 exist with three of the points, lying in a plane of the figure, as three vertices, and 1 exists with any four of the points, lying in a solid of the figure, as four vertices. In fact, taking any four of the points which lie in one of the 72 solids, there will, through every three of these points, be another solid; and the four solids so obtained give, by their intersection, the fifth vertex of the simplex. Taking three of the points of a solid, there are three other points in this solid of which each may make a fourth vertex of a simplex, taken with the first three; the three completing vertices of the three simplexes so obtained lie in the other solid which contains the first three points. Or, the number is obtained by remarking that each of the 216 simplexes has ten sets of three vertices, while there are in all 720 planes in the figure; since

$$216 \,.\, 10 \div 720 = 3.$$

Again, each of the simplexes has ten edges, while there are 216 lines in the figure. Thus there are $216 \,.\, 10 \div 216$, or ten, simplexes with two vertices common. Also, the 216 simplexes give $216 \,.\, 5$ points, while there are 27 points in the figure; thus there are $216 \,.\, 5 \div 27$, or 40, simplexes with vertex at any point of the figure. There is in fact a correspondence between the simplexes and the lines of the figure. For example, the simplex a_1, a_2, a_3, a_4, c_{56} corresponds to the line joining the points b_5, b_6, the correspondence consisting in the fact that no one of the ten lines joining b_5, b_6 to the vertices of the simplex is a line of the figure.

The figure, we see, depends on the assignment of 4 points to determine the solid O, requiring 16 constants; then on two other points therein, requiring 6 more; then on another constant to determine each of the 20 solids such as P_{123}; and then, finally, on another constant to determine b_1. In all upon $16 + 6 + 21$, or 43, constants.

Ex. 1. The points A, B, C, D lie in one plane, ϖ, in space of four dimensions, and the points A', B', C', D' in another plane, ϖ'; these planes intersect in the point O. The condition that the range (or pencil) of four solids, joining the plane ϖ' to A, B, C, D, respectively, should be related to the range of four solids joining the plane ϖ to A', B', C', D', respectively, is that the range (flat pencil) of lines joining O to A, B, C, D should be related to the range of lines joining O to A', B', C', D'.

Ex. 2. Consider four general planes, 1, 2, 3, 4, in space of four dimensions. Let the points of intersection of the plane 4 with the planes 1, 2, 3 be denoted, respectively, by A, B, C, the points of intersection of the planes (2, 3), (3, 1), (1, 2) being, respectively, A', B', C'. Thus, in terms of the six points, the planes 1, 2, 3, 4 are, respectively, (A, B', C'), (B, C', A'), (C, A', B'), (A, B, C). It is a known fact (Vol. I, p. 92; below, pp. 118, 120) that all lines meeting the four given planes meet another plane. Prove, in fact, that, if A, B, C, A', B', C' be symbols for the six points, subject to

$$A + B + C + A' + B' + C' = 0,$$

then all lines meeting the four given planes meet also the plane which contains the four points whose symbols are $A + B' + C'$, $B + C' + A'$, $C + A' + B'$, $A + B + C$.

It can be shewn that the unique line which can be drawn, from the point $xA + yB + zC$, to meet the first three planes, contains the point

$$x^{-1}(A + B' + C') + y^{-1}(B + C' + A') + z^{-1}(C + A' + B').$$

In threefold space, given two arbitrary lines and two arbitrary planes, let the lines meet the first plane in B, C, and meet the second plane in B', C'. Let $B'C'$ meet the first plane in A, and BC meet the second plane in A'. It can be proved that the points $xA+yB+zC$, $x^{-1}A'+y^{-1}B'+z^{-1}C'$ are on a transversal of the two given lines.

Ex. 3. The figure dual to that considered in the text is of considerable interest. This will consist of seventy-two points and twenty-seven solids. Starting with a point O, and six solids, a_1, a_2, ... a_6, passing through this, every three of these solids meet in a line. On each of these lines is taken a point, such as P_{123}; there are twenty such points. The four points P_{234}, P_{314}, P_{124}, P_{123} determine a solid, c_{56}; and so on. The four solids c_{61}, c_{62}, c_{63}, c_{64} meet in a point lying in the solid a_1, namely the point P_{16}. The five points P_{12}, P_{13}, P_{14}, P_{15}, P_{16} lie in a plane, which is in the solid a_1. This plane is n_1. If N_{12} denote the common point of the planes n_1, n_2 (cf. Ex. 1, above), the two flat pencils of lines $N_{12}(P_{12}, \ldots, P_{16})$, $N_{12}(P_{21}, \ldots, P_{26})$ are related to one another. And so on.

Ex. 4. If in Ex. 3, the point O, and the assumed twenty points such as P_{123}, lie on a quadric threefold of the fourfold space, then all the derived points, $72-21=51$, in number, lie on this quadric. By projection, of the sections of this quadric threefold by the twenty-seven solids which arise, from a point of this quadric, on to an arbitrary solid, we derive a figure consisting of 27 spheres meeting in sixes in 72 points. We may initiate the figure with six arbitrary spheres having a common point; every three of these have then another common point. This is the point of view of Mr Grace's paper referred to (*loc. cit.*, p. 182).

CHAPTER V

A FIGURE OF FIFTEEN LINES AND POINTS, IN SPACE OF FOUR DIMENSIONS; AND ASSOCIATED LOCI

The figure of fifteen lines and points. Let a, b, c, d be four lines of general position, in space of four dimensions. Any two of these will then define a threefold space. For a threefold space we shall here use the name *solid*. There will then be six such solids, say, $[A]$, or $[a, d]$; $[B]$, or $[b, d]$; $[C]$, or $[c, d]$; $[A']$, or $[b, c]$; $[B']$, or $[c, a]$; $[C']$, or $[a, b]$. Conversely, if six such solids be given, of which no three have a plane in common, they define four lines, each the intersection of three of these solids, namely a as the intersection of $[A], [B'], [C']$, b as the intersection of $[B], [C'], [A']$, c from $[C], [A'], [B']$, and d from $[A], [B], [C]$. The three lines b, c, d have a common transversal, which we denote by a'; this is the line common to the solids $[A'], [B], [C]$. Similarly, let b', c', d' denote, respectively, the common transversals of c, a, d; of a, b, d; and of a, b, c. There are, then, twelve points of intersection of the eight lines now obtained, which we denote as follows:

$A = (b', c)$, $B = (c', a)$, $C = (a', b)$; $A' = (b, c')$, $B' = (c, a')$, $C' = (a, b')$,
$P = (a, d')$, $Q = (b, d')$, $R = (c, d')$; $P' = (a', d)$, $Q' = (b', d)$, $R' = (c', d)$.

The point A, as containing a point of the line c, is in the solid $[b, c]$; as containing a point of the line b', which lies wholly in the solid $[a, d]$, this point A is also in the solid $[a, d]$. Similarly, the point A' is in both the solids $[b, c]$ and $[a, d]$; and each of the points P, P' is in both these solids $[b, c]$, $[a, d]$. Denote the plane common to these solids by λ. The lines AA' and PP', both lying in this plane, have a common point; denote this point by L. By similar reasoning, the plane, say μ, common to the solids $[c, a]$ and $[b, d]$, contains both the lines BB' and QQ'; let the point of intersection of these lines be called M. Finally, the plane, say ν, common to the solids $[a, b]$ and $[c, d]$, contains the lines CC' and RR'; let N be the common point of these lines.

We can now prove that the three points L, M, N are in line, namely, in the line which is common to the three solids $[a, a']$, $[b, b']$, $[c, c']$. For, consider the point L: this point lies on the line PP', joining a point of the line a to a point of the line a'; thus L lies in the solid $[a, a']$; again, L is on the line AA', joining a point of the line b to a point of the line b', so that L is in the solid $[b, b']$; the line AA', however, equally joins a point of the line c to a point of the line c', so that L is also in the solid $[c, c']$. Thus L is in the three solids $[a, a']$, $[b, b']$, $[c, c']$. By similar reasoning M and N are also in these three solids. Thus L, M, N are in a line. This line we denote by e. See the diagram given as frontispiece to the Volume. But further, the point L, as it lies on the line PP', joining a point of the line d to a point of the line d', lies also in the solid $[d, d']$; and, similarly, M and N are also in this solid. Wherefore, the four solids $[a, a']$, $[b, b']$, $[c, c']$, $[d, d']$ meet in a line; and, as was previously remarked (Vol. I, p. 92), the proof of this depends on the Propositions of Incidence only. We denote the lines $AA'L$, $BB'M$, $CC'N$, respectively, by l, m, n, and the lines $PP'L$, $QQ'M$, $RR'N$, respectively, by p, q, r. The figure now contains fifteen points and fifteen lines, three points lying on each line, and three lines passing through each point. In each case, taking any one of the lines, and the six lines, other than this, which pass, in couples, through the three points lying on this line, the eight remaining lines consist of four non-intersecting lines and of the four transversals, each meeting three of these lines; the figure is entirely symmetrical. We shall speak of the set, a, b, c, d, e, of five lines, as being *associated*; any four of these may be regarded as primary and the fifth determined from them, as e was determined from a, b, c, d. But equally, e is determinable from a', b', c', d', just as e was determined from a, b, c, d, and the five lines a', b', c', d', e are also associated. In all, the figure contains $15 . 2 \div 5$, or six, sets of associated lines, every line of the fifteen being common to two of these sets. As is easily seen, these sets are given by the adjoined scheme, which may be read either in rows or columns.

	I	II	III	IV	V	VI
I	.	e	d	a	b	c
II	e	.	d'	a'	b'	c'
III	d	d'	.	l	m	n
IV	a	a'	l	.	r	q
V	b	b'	m	r	.	p
VI	c	c'	n	q	p	.

This scheme suggests another notation for the lines, which will be found to be very useful. Namely, the line of which the symbol here occurs in the r-th row and the s-th column may be denoted either by rs or by sr. For instance, the line e is then 12 or 21, and p is 56 or 65. In this notation, *two lines* of the fifteen *intersect if, and only if, their duad symbols have no number in common*, as may be verified easily. Thus the three lines which meet in any point have duad symbols which, together, employ all the six numbers 1, 2, ..., 6; for instance, the three lines 12, 34, 56 meet in a point. Such a set of three duads was called by Sylvester a syntheme; the fifteen points of the figure correspond to the fifteen synthemes which are possible. As another illustration of the notation, no two of the three lines 23, 31, 12 intersect, but each is met by all the three lines 56, 64, 45; thus these six lines are generators of a quadric surface, and lie in a solid. There are, then, *ten* such solids, each corresponding to one of the ways of dividing the six numbers into two triads; for instance, the solid we have considered corresponds to the triads 123, 456, and may be denoted, appropriately, by either of these triads; and so for the other solids. These ten solids are, in fact, as is seen at once, the ten previously denoted by $[b, c]$, ..., $[a, d]$, ..., $[e, a]$, ..., each defined by a pair of the five associated lines a, b, c, d, e; each contains, likewise, one pair from every set of five associated lines. We shall call these solids the ten *singular* solids. Each of the fifteen lines lies in four of these solids; for instance the line 12 lies in the four solids 123 (or 456), 124 (or 356), 125 (or 346), 126 (or 345). Further, through any one of the fifteen points where three of the lines intersect, there pass six of the singular solids, two of these intersecting in the plane of any two of these three lines. For instance, through the point 12.34.56 there pass the solids 134 (or 256) and 156 (or 234), beside the four which meet in the line 12; the former two solids contain both the lines 34, 56, but not the line 12.

Proof of the incidences with the help of the symbols. If desired, the incidences of the figure may be proved very briefly by use of symbols. If the symbols of the points A, B, C, A', B', C' be represented by the same letters, these symbols, since the six points are in space of four dimensions, must be connected by a syzygy, which, by proper choice of the symbols, may be supposed to be $A+B+C+A'+B'+C'=0$. Thereafter, no further multiplication of these symbols by an algebraic symbol is legitimate, save by one the same for all (cf. Vol. I, p. 71). We suppose the six symbols not to be connected by any further syzygy, the six points not being in a space of three dimensions. The symbols of the points P, Q, R, which lie, respectively, on the lines BC', CA', AB', are then each expressible linearly by the symbols of the two points on whose join it lies, in such forms as $P=mB+nC'$. The points P, Q, R are, however, in line, and their symbols are connected by a syzygy; by the expressions for P, Q, R, this becomes a syzygy for the six symbols A, B, ..., C', which must then agree with the fundamental syzygy for these. Hence, by absorption of proper algebraic multipliers in the symbols P, Q, R, we have, as is easily seen, the results

$$P+B+C'=0,\quad Q+C+A'=0,\quad R+A+B'=0.$$

By a similar argument, the symbols for the points P', Q', R' may be taken so that

$$P'+B'+C=0,\quad Q'+C'+A=0,\quad R'+A'+B=0.$$

These, however, lead to $P+P'=A+A'$, expressing that there is a point of the line PP' which is also a point of the line AA'. The symbol, L, of this point, may then be taken so that $L+A+A'=0$. The points of intersection of BB' and QQ', and of CC' and RR', are similarly established, with symbols M and N such that $M+B+B'=0$ and $N+C+C'=0$. These equations, however, in virtue of the fundamental syzygy, lead to $L+M+N=0$; this proves that the points L, M, N lie on a line, e.

If, instead of the six points A, B, ..., C', we introduce other six points, F, G, ..., H', with respective symbols

$$F=\tfrac{1}{2}(B+C-A),\quad G=\tfrac{1}{2}(C+A-B),\quad H=\tfrac{1}{2}(A+B-C),$$
$$F'=\tfrac{1}{2}(B'+C'-A'),\ G'=\tfrac{1}{2}(C'+A'-B'),\ H'=\tfrac{1}{2}(A'+B'-C'),$$

so that $F+G+H+F'+G'+H'=0$, then it is seen at once that the symbols of the original fifteen points of the figure are the fifteen sums of twos of the six symbols F, G, ..., H'. This is equivalent to saying that, if, in one of the fifteen possible ways, the six points F, G, ..., H' be divided into three pairs, and the joins of the points of the pairs be taken, then the common transversal of the three joins is one of the fifteen lines of the figure, and the points where the transversal meets the joins are three of the original fifteen points of the figure. Or again, any one of the original fifteen points, say $F+F'$, is the intersection, of the join of the two points F and F', with the solid defined by the four remaining points, G, H, G', H'. The three lines of the figure through $F+F'$ are the transversals, drawn in this solid (G, H, G', H'), to meet one of the three pairs of opposite joins of these four points, G, H, G', H'. This shews that the six points F, G, ... are entirely symmetrical in regard to the figure of fifteen points and lines, and, though they do not belong to it, may be used to construct the figure. Essentially, only one such figure exists (Vol. I, p. 152).

The figure of six general points in space of four dimensions was investigated by Mr H. W. Richmond, *Quart. J. of Math.* XXXI, 1900, pp. 125—160; *Math. Annal.* LIII, 1900, pp. 161—176. To the present writer the figure of fifteen points and lines was independently suggested, as arising from four lines, by the problem of a double-six of lines (*Roy. Soc. Proc.* LXXXIV, 1911, p. 599; *Proc. Camb. Phil. Soc.* XX, 1920, p. 133). The locus of the third order, associated with the figure, to which reference will be made below, was considered by Segre (*Atti...Torino,* XXII, 1887, p. 547, and *Memorie...Torino,* XXXIX, 1889, pp. 3—48). It has been remarked above (p. 36) that spheres and circles in threefold space may be regarded as derived from points and lines in fourfold space. Thus the theory of the figure of fifteen lines, in fourfold space, may be stated in terms of circles in threefold space. From this point of view an ample introduction, which includes the two number notation for the lines, and the symmetrical equation for the cubic locus studied by Segre, is given by Stéphanos, *Compt. Rend.* XCIII, 1881, p. 634 (and p. 578). For him, five associated lines have their (ten) coordinates linearly connected (p. 120, below). It is shewn below (Chap. VII) that the original fifteen lines of the figure may be regarded as arising by transformation from the fifteen joining lines of six points in space of *five* dimensions.

Ex. 1. Any two of five associated lines determine a solid; this meets each of the three other lines in a point. Prove that these three points are in line.

Ex. 2. Any six points, in space of four dimensions, taken in order, form a skew hexagon (say $AB'CA'BC'$); of this hexagon there will be two sets each of three alternate sides (BC', CA', AB' and $B'C$, $C'A$, $A'B$); and there will be three diagonals (AA', BB', CC'). Prove that the two transversals, each of one set of alternate sides, form, with the three diagonals, a set of five associated lines.

Ex. 3. In space of four dimensions, the three lines which are the transversals, of one set of alternate sides, of the other set of alternate sides, and of the diagonals, of a skew hexagon, lie in a solid. If we take the two hexagons $AB'CA'BC'$ and $AB'BA'CC'$, of which the latter is obtained from the former by the interchange of B and C, the three transversals so obtained, from one of these hexagons, all meet the three transversals so obtained from the other hexagon, and the two solids are the same. (Cf. Ex. 13, p. 144, below.)

Ex. 4. Referring to the notation used above, let λ, μ, ν denote, as before, respectively, the planes $LAA'PP'$, $MBB'QQ'$, $NCC'RR'$. Then the point of intersection of the planes μ, ν is the point of symbol $F-F'$ (or $A'+B+C$); and so on. Thus the plane λ contains the points $G-G'$ and $H-H'$. Further, the solid (λ, e), defined by the plane λ and the line e meeting this plane, is the solid defined by the four points G, H, G', H'; and so on. Now, in space of four dimensions, given a point, O, and a solid, Π, we can define a transformation, a *harmonic inversion*, from a point, P, to a point P', by taking the point, M, where OP meets Π, and then taking the point, P', on the line OP, which is the harmonic conjugate of P in regard to O and M. Shew that, if this transformation be employed, with O as the point (μ, ν), or $F-F'$, and Π as the solid (λ, e), then the points F, F' are interchanged, but each of the points G, H, G', H' is unaltered. The figure of fifteen points and lines is thus unaltered, in its aggregate, by this harmonic inversion. In fact, the four lines BQ, $B'Q'$, CR', $C'R$ meet in the point (μ, ν); and the four points which are the harmonic conjugates of the point (μ, ν) in regard, respectively, to the pairs B, Q; B', Q'; C, R'; C', R, all lie in the solid (λ, e).

The plane λ is the intersection of the solids $[b, c]$, $[a, d]$; as has been remarked. Cf. Ex. 21, p. 148, below.

Ex. 5. In space of four dimensions, the dual of a line is a plane. Consider, briefly, the dual of what precedes, arising when we begin with four arbitrary planes, of general position, say α, β, γ, δ. Every two of these planes have a point in common; let the six points so obtained be denoted as follows:

$$A=(\alpha,\delta),\ \ B=(\beta,\delta),\ \ C=(\gamma,\delta),$$
$$A'=(\beta,\gamma),\ \ B'=(\gamma,\alpha),\ \ C'=(\alpha,\beta);$$

being in space of four dimensions, these six points will not be independent. Conversely, if the six points be supposed to be given, arbitrarily, the four planes are determined, each by three of these, as follows:

$$\alpha=(A,B',C'),\ \ \beta=(B,C',A'),\ \ \gamma=(C,A',B'),\ \ \delta=(A,B,C).$$

The dual of the meeting of two lines in a point is the lying of two planes in a solid, or the meeting of these two planes in a line (instead of a point merely). Thus any three of the planes are met, each in a line, by another plane. In fact, each of the four planes

$$\alpha'=(A',B,C),\ \ \beta'=(B',C,A),\ \ \gamma'=(C',A,B),\ \ \delta'=(A',B',C')$$

meets three of the given planes in a line. For instance, the plane δ' meets α in the line $B'C'$, meets β in $C'A'$, and meets γ in $A'B'$; and so on. As the dual of what is proved above, the four points (α,α'), (β,β'), (γ,γ'), (δ,δ') lie in a plane; we may also prove this anew, by a method not the dual of that used above. Recall first, in space of three dimensions, that, if three lines be given, of which no two intersect, and, from a variable point, P, of one of these lines, the transversal be drawn to the other two lines, and thereon the point, Q, be taken, which is harmonically separated from P by the two lines, then the locus of Q is a fourth line. In space of four dimensions, if two planes, ξ, η, be given, intersecting in a point O, and H be any point upon a line, l, which does not meet ξ nor η, the two solids (ξ, H), (η, H) meet in a plane, passing through O; this plane, as lying in the solid (ξ, H), meets ξ in a line; and, likewise, meets η in a line. Thus lines can be drawn, through the point H, meeting the planes ξ, η; let such a line meet ξ and η, respectively, in X and Y, and let K be the harmonic conjugate of H in regard to X and Y. Then the locus of K, as H varies on l, is a plane passing through O. This we see by projecting the figure, from O, on to any threefold space. With this in mind, consider, in the figure described above, the three lines AA', BB', CC', and the common transversal of these, say l, meeting AA', BB', CC', respectively, in L, M, N. Let L', on AA', be the harmonic conjugate of L in regard to A and A'; similarly, let M', N', respectively on BB' and CC', be the harmonic conjugates of M and N. Then, it follows from what has been said that the plane $L'M'N'$ contains the point common to the two planes ABC and $A'B'C'$; and, for the same reason, that this plane contains the point common to the two planes $A'BC$ and $AB'C'$; and so on. Thus the four points (α,α'), (β,β'), (γ,γ'), (δ,δ') all lie in the plane $L'M'N'$, which is thus the plane *associated with* the planes α, β, γ, δ; or associated with the planes α', β', γ', δ'.

If we use the symbols A, B, ... for the points A, B, ..., choosing them so that the necessary syzygy for these six symbols is $A+B+C+A'+B'+C'=0$, we see, for instance, that the symbol of the point (δ, δ') is $A+B+C$ (or its equivalent, $A'+B'+C'$). Thus the theorem we have proved is that the four points whose symbols are

$$A+B'+C',\ B+C'+A',\ C+A'+B',\ A+B+C$$

lie in one plane; as follows also because these four symbols have a vanishing sum. The second and third of these symbols have, for sum, $A-A'$, which is equally the sum of the first and fourth symbols; the symbols of L, M, N are, respectively, $A+A'$, $B+B'$, $C+C'$. Thus the symbols of L', M', N' are, respectively, $A-A'$, $B-B'$, $C-C'$; and each of these points is the intersection of two of the six joins of the four points (α, α'), (β, β'), (γ, γ'), (δ, δ').

Ex. 6. We have considered a process of harmonic inversion, in space of four dimensions, in which a given point and a given solid were fundamental; we have also considered a process of harmonic inversion in which two given planes were fundamental. Suppose now that we are given a fixed line, l, and a fixed plane, ϖ, not intersecting the line l. The plane joining l to any point, P, meets the plane ϖ in a point. Thus a line, say p, can be drawn from any point P to meet the given line l and also the given plane ϖ; thereon the point, Q, can be taken which is the harmonic conjugate of P in regard to the points where p meets l and ϖ. This defines a transformation from P to Q (or conversely).

If X, Y, Z be points of the plane ϖ, and T, U be points of the line l, it can easily be shewn that, with coordinates relative to these five points, the transformation is expressed by the equations

$$x'=x,\ y'=y,\ z'=z,\ t'=-t,\ u'=-u.$$

This transformation is evidently the combination of two such as are considered in Ex. 4, each defined with reference to a fixed point and a fixed solid.

In Ex. 5, the planes δ, δ' correspond to one another by such a transformation as that considered here, in which the fixed elements are the line l and the plane $L'M'N'$ (as do the planes α, α', etc.). Similarly, in the original figure of the text, the lines d, d' correspond to one another by such a transformation, of which the fixed elements are the line e, and the plane which meets in a line the three planes λ, μ, ν (i.e. the planes $LAA'PP'$, etc.; see Ex. 4); and the lines a, a' correspond in the same transformation, as do b, b' and c, c'. The plane meeting, in a line, each of λ, μ, ν, is the plane defined by the three points $A-A'$, $B-B'$, $C-C'$ (or by the three points $F-F'$, $G-G'$, $H-H'$). Cf. Ex. 21, p. 148, below.

Ex. 7. The figure of fifteen points and fifteen lines can be separated (in fact in ten ways) into two parts, of which one part consists of two triads of points together with the nine lines joining the points of one triad to the points of the other triad, and the other part consists of two triads of lines together with nine points in which the lines of one triad intersect the lines of the other triad. Of the lines and points named, no one is common to the two parts, but the two parts together exhaust the whole figure. The nine lines of the first part can be arranged in three sets, of three each, such that the transversal line of a set is one line of a triad of lines of the second part, and the set forms, with the other two lines of this triad, an associated system of five lines. The other triad of the second part is similarly obtainable from another arrangement of the nine lines of the first part.

Ex. 8. An illustration of the transformation considered in Ex. 6 is as follows:—Let A, A', B, B', C, C' be six points of a rational quartic curve in space of four dimensions; let L, M, N be the points where the chords AA', BB', CC', of this curve, are met by their common transversal, l; let L', M', N' be, respectively, the harmonic conjugates of L, M, N in regard to A, A'; to B, B'; and to C, C'; and denote the plane $L'M'N'$ by ϖ. By applying the transformation of Ex. 6, with l and ϖ as fundamental elements, the given quartic curve gives rise to another quartic curve having, likewise, AA', BB', CC' as chords. Prove that any plane, containing one of these chords, which meets one of the two curves, also meets the other curve (the second point of meeting not corresponding to the first, however, by the transformation in question). (*Proc. Camb. Phil. Soc.* XXI, 1923, p. 684.)

The associated line of a set of five is met by all planes meeting the first four lines. We return now to our original point of view, in which we suppose four non-intersecting lines of general position to be given, say a, b, c, d, and prove that all planes which meet a, b, c, d also meet the associated line, e. It will appear that there are ∞^2 such planes. It will also appear that, of such planes, there are two which pass through an arbitrary point, O, of general position. For some positions of O these two planes coincide; the locus of such points O is of great importance. For some positions of O the number of these planes is infinite; for instance when O is on the associated line e. The proof does not follow from the Propositions of Incidence alone, but requires Pappus' theorem.

It will add to clearness to interpolate here some remarks in regard to a construct occurring in space of four dimensions, which we shall call a *quadric point-cone*; and in regard to an associated conception, that of a *quadric line-cone*. In space of three dimensions the dual of a conic, considered as the envelope of its tangent lines, which lie in a plane, is the quadric cone, or conical sheet, considered as consisting of its generating lines, which pass through its vertex. In space of four dimensions, lying in a particular solid in this space, we may have an ordinary quadric surface. This may be considered to consist of its two sets of generating lines, of which any line of either set meets all the lines of the other set. The dual conception, in the space of four dimensions, will be that of two sets of planes, all passing through a point, say V, of which any plane of either set meets every plane of the other set in a line, passing through V, while two planes of the same set meet only in V. For the original figure, of a quadric surface lying in a solid of the fourfold space, there pass, through every general line in this solid, two planes each containing two lines of the quadric surface; this line also contains two points at each of which two of these four generators meet. Thus, in the dual figure, an arbitrary plane through V contains two lines, passing through V, each of which is the intersection of a plane of one set with a plane of the other set; and there are two solids containing this plane, each of which contains two of the four planes so arising. It is the aggregate of points lying in these two sets of planes which we call a quadric *point-cone*, the point V being the *vertex*. In the original case, of the quadric surface, two generating lines which intersect determine a point of the surface, and a tangent plane; so, in the case of the point-cone, two generating planes which meet in a line lie in a solid, called the *tangent solid* of the point-cone, the line of intersection of the planes being the *line of contact* of this tangent solid with the point-cone. There are ∞^2 tangent solids, each passing, with its line of contact, through the vertex. It is clear that the generating planes of the point-cone meet an arbitrary solid in the generating lines of a quadric surface lying in that solid. Conversely, a point-cone is generated by the planes which join the generating lines of a quadric surface to a point, not lying in the solid in which the quadric surface lies.

We may equally consider the dual, in space of four dimensions, of the tangent lines of a conic. This will consist of ∞^1 planes passing through a line, say l, two of these planes lying in an arbitrary solid which contains the line l. It is the aggregate of the points of these planes which we call a quadric *line-cone*, l being the *axis*. A solid through the axis may contain only one generating plane of the quadric line-cone; the solid is then said to touch the line-cone, at every point of this plane; of such *tangent-solids* there are ∞^1. The lines of section of the generating planes of the line-cone, with an arbitrary solid, are the generating lines of a quadric conical sheet, lying in this solid, whose vertex is on the axis of the line-cone. Conversely, a line-cone consists of the planes joining a line to the points of a conic whose plane does not meet the line.

The general homogeneous quadratic function, of the five coordinates which are appropriate for space of four dimensions, can be written, in infinitely many ways, as a sum of squares of at most five independent linear functions of the coordinates (cf. Vol. III, p. 15). When this sum consists of only four squares it represents a point-cone, when equated to zero; the line-cone is obtained when the number of these squares is three; if the number is two the line-cone degenerates into two solids meeting in a plane.

Any solid, passing through the vertex of a quadric point-cone, meets this in a conical sheet. The solid meets the two generating planes of the point-cone which lie in any tangent solid, each in a line, these two lines being generating lines of the conical sheet. If the solid contain the line of intersection of the two generating planes, then its plane of intersection with the tangent solid, which touches the point-cone at the points of this line, is the tangent plane of the conical sheet at the points of this line. This is clear by remarking that the tangent solid of the point-cone is that containing the

vertex and a tangent plane of the quadric surface in which the point-cone meets an arbitrary solid. Similarly if a solid be drawn through the axis of a line-cone, this meets the line-cone in two planes which coincide when the solid is a tangent solid. Further, the tangent solids of a point-cone which pass through any point, not lying on the cone, are the tangent solids of a line-cone, whose axis joins the point to the vertex of the point-cone.

Resuming our discussion, let, as before, a, b, c, d be four non-intersecting lines, in space of four dimensions, of which the transversals of threes are a', b', c', d'; and let e be the associated line, common to the four solids $[a, a']$, $[b, b']$, $[c, c']$, $[d, d']$. Let E be any point of the line e; then, as d' is the transversal of a, b, c, the plane Ed' meets these lines; and this plane meets the line d, because e is in the solid $[d, d']$. Thus, by similar reasoning, any one of the four planes Ea', Eb', Ec', Ed' meets, in a line through E, every one of the four planes Ea, Eb, Ec, Ed. Therefore, these planes meet an arbitrary solid in two sets of four lines, belonging, respectively, to the two sets of generators of a quadric surface. Thence, through E can be drawn an infinity of planes each meeting all the lines a, b, c, d, these being the planes, of one set, of a quadric point-cone, of vertex E; as well as an infinity of planes each meeting all the lines a', b', c', d'. Conversely, as a line meeting three generators of the same system of a quadric surface, equally meets all the generators of that system, it follows that a plane which meets e and meets any three of the lines a, b, c, d, likewise meets the fourth. We have seen, however, that the five lines a, b, c, d, e are symmetrical; thus any plane meeting a, b, c, d also meets e. The argument, as assuming the last quoted property of a quadric surface, depends on Pappus' theorem.

Next, let O be any point of general position. If any point, P_4, be taken on the line d, a plane can be drawn through O and P_4 to meet the lines a, b, this being the plane through OP_4 which contains the transversal of the three lines OP_4, a, b. If this transversal meet a, b in P_1 and P_2, respectively, this plane can also be described as the plane through O and P_1 which meets b and d; for the line P_2P_4 is the transversal of the three lines OP_1, b, d. Consider the line in which the solid $[a, b]$ is met by the plane Od; as P_4 varies on d, the lines OP_4 and P_1P_2 meet in a point of this line. Thus the ranges (P_1), (P_4) are related. We can similarly construct a plane through O and P_1 which shall meet the lines c and d; if this plane meet d in P_4', the ranges (P_1), (P_4') are related, as P_1 varies on a. Thus the ranges (P_4), (P_4'), on d, are related. There are, therefore, in general, two positions of P_1, on a, for which P_4 and P_4' coincide. When this is so, there is a plane through O and P_1 which meets b, c and d. There are then, in general, two planes through O which meet a, b, c, d. But these may coincide for particular positions of O. Or, if three planes can be drawn through O to meet a, b, c, d, then an infinite number of such planes is possible, and a plane drawn through O to meet three of a, b, c, d necessarily meets the fourth. We have seen that this is the case when O is on the line e; thus, also, an infinite number of planes meeting a, b, c, d, e passes through any point of any one of these five lines.

The two planes through a general position of O which meet a, b, c, d, also meet e, as we have proved; let them meet e in U and T. Then, as we have seen, considering the point-cone of vertex U, containing the planes through U which meet a, b, c, d, and those which meet a', b', c', d', there will be, through OU, a plane meeting a', b', c', d'. A plane meeting a', b', c', d' can, similarly, be drawn through OT. These are then the two planes which, by a similar argument, can be drawn through O to meet a', b', c', d'.

The six systems of planes. There are six systems of planes, those of any system being the planes which meet the five lines of one of the six associated systems of lines. A plane of a system is determined, as the argument shews, by its intersections with two of the five associated lines; thus the aggregate of the planes of each system is ∞^2. Through a general point, O, can be drawn two planes of each system; taking two of the systems, if α, β be the planes of one of these through this point, and α', β' be the planes of the other system, there are two points, T, U, upon the line common to the two associated systems of lines, from which the planes are defined, such that OT is the line of intersection of the plane α with one of α', β', say with β', while, similarly, the planes β, α' meet in the line OU. We shall denote the systems of planes by the numbers 1, 2, ..., 6, these being the same as those marking the systems of associated lines, in the scheme given above (p. 114).

It follows that, if O be such that the planes α, β, of one system, drawn through O, coincide with one another, then the planes, α', β', of any other system, drawn through O, likewise coincide. In this case the planes of the various systems which pass through O are six in number, one of each system. One of these planes, say that meeting the lines a, b, c, d, e, will be met in five lines, passing through O, by the other five planes, these lines passing to the points where a, b, c, d, e meet this plane. But, two planes meeting in a line define a solid; and another plane, which meets each of these in a line, lies in this solid. Thus, for such a position of O as now contemplated, the six planes through O lie in a solid. It will be found immediately that there are ∞^3 such positions of O, of which four lie on an arbitrary line; we shall be much concerned with the locus so determined.

The various systems of planes are also of importance in connexion with the singular solids which have been described. Each of these solids can be generated by planes of any one of the six systems, these forming an axial pencil of planes whose axis is one of the six lines lying in the solid. More precisely, taking the singular solid previously denoted by 123, or 456, this is generated by planes of the system 1, all passing through the line 23; also by planes of system 2, all passing through the line 31; and so on; and, finally, by planes of the system 6, all passing through the line 45. For instance, if a plane in the solid be drawn through the line 23, this plane, beside meeting the other five lines which lie in this solid, meets the remaining lines, of the original fifteen lines, which meet the line 23; the plane thus meets the lines 12, 13, 56, 64, 45 and 14, 15, 16; as it meets, therefore, the lines 12, 13, 14, 15, 16, it is a plane of system 1. This argument is of general character. It follows also, from this, that the planes joining any point, O, of the singular solid, to the six lines lying in the solid, are of the six systems, respectively; it will be found that every point of the singular solid is such that the two planes of any system, drawn through this point, coincide with one another.

The planes of the figure deduced with the help of the symbols. The results in regard to the planes can also be obtained with great simplicity by use of the symbols. Thereby, too, we can define with greater precision the locus of points, O, through which only one plane of each system can be drawn.

A plane meeting the lines a, b, c, not containing any one of these lines, can evidently be defined by three points, one on each of the lines, with symbols, respectively, of the forms

$$yB + zC', \quad zC + xA', \quad xA + \eta B'.$$

For this plane to meet the line d, there must be a syzygy

$$\lambda(yB + zC') + \mu(zC + xA') + \nu(xA + \eta B') + \rho(B' + C) + \sigma(C' + A) = 0,$$

connecting these three points with the two points, P', Q', of the line d. As this must be equivalent to the fundamental relation between A, B, ..., C', it is easily seen that $\eta = y$. Thus, any general plane meeting a, b, c, d must contain three points, respectively on a, b, c, of symbols $yB + zC'$, $zC + xA'$, $xA + yB'$. Conversely, the plane of these points contains the point whose symbol is

$$x(yB + zC') + y(zC + xA') + z(xA + yB'),$$

or, save for sign, $yzP' + zxQ' + xyR'$; and this is a point of the line d. This plane also contains the point whose symbol is

$$(yB + zC') + (zC + xA') + (xA + yB'),$$

or, save for sign, $xL + yM + zN$; and this is a point of the line c. In virtue of $L + M + N = 0$, this point depends only on the ratio of any two differences of x, y, z.

Now let O be any general point. We can suppose it to have a symbol

$$O = \xi A + \eta B + \zeta C + \xi' A' + \eta' B' + \zeta' C',$$

where, in virtue of the fundamental syzygy, only the ratios of the differences of ξ, η, ζ, ξ', η', ζ' are determinate. That this point should lie on the plane containing the three points $yB + zC'$, $zC + xA'$, $xA + yB'$ requires an identity, in regard to A, B, C, A', B', C', of the form

$$O = \lambda\,(yB + zC') + \mu\,(zC + xA') + \nu\,(xA + yB') + \rho S,$$

where S denotes the sum of A, B, ..., C'. Using this to express ξ, η, ..., ζ', we can deduce the three equations

$$x^{-1}(\xi' - \xi) + y^{-1}(\eta' - \eta) + z^{-1}(\zeta' - \zeta) = 0,$$

$$x(\eta' - \zeta) + y(\zeta' - \xi) + z(\xi' - \eta) = 0,$$

$$(\eta - \zeta')(\eta' - \zeta) + \sigma^{-1}(\zeta - \xi')(\zeta' - \xi) - (1 + \sigma)^{-1}(\xi - \eta')(\xi' - \eta) = 0,$$

where σ denotes $(z - x)/(y - z)$. The first two of these equations determine two sets of ratios of x, y, z when ξ, η, ..., ζ' are given; that is, they determine the two planes meeting the lines a, b, c, d, e which pass through O. These planes meet the line e in the two points $\sigma_1 L - M$, $\sigma_2 L - M$, where σ_1, σ_2 are the two roots of the third equation. We easily find, also, that

$$(\eta - \zeta')(\eta' - \zeta) = (\zeta - \xi')(\zeta' - \xi)/\sigma_1\sigma_2 = (\xi - \eta')(\xi' - \eta)/(1 + \sigma_1)(1 + \sigma_2);$$

when the two points of the line e are given, these are two conditions for O which will be of interest below.

When the point O lies on the line e, so that $\xi' - \xi = 0 = \eta' - \eta = \zeta' - \zeta$, the first equation is satisfied identically; the second equation has then the solution $x = \xi + \theta$, $y = \eta + \theta$, $z = \zeta + \theta$, where θ is arbitrary. Thus an infinite number of planes meeting a, b, c, d, e can be drawn through a point of e, as was seen. This is also so when O is on any one of a, b, c, d.

The equation of an associated quartic locus. If we put p, q, r, p', q', r', respectively, for $\eta - \zeta'$, $\zeta - \xi'$, $\xi - \eta'$, $\eta' - \zeta$, $\zeta' - \xi$, $\xi' - \eta$, it is easy to see that the third (quadratic) equation, above, gives equal roots for σ provided

$$(pp')^{\frac{1}{2}} + (qq')^{\frac{1}{2}} + (rr')^{\frac{1}{2}} = 0;$$

this is then the condition for points O from which the two planes of any system coincide with one another; and may be regarded as the equation of the locus of such points. The equation, when rationalised, is evidently of the fourth order, and there are four such points upon an arbitrary line.

We may with advantage reach this result in another way, defining O by means of one of the planes through it which meet the lines a, b, c, d, e, and the position of O in this plane. This is done by writing the symbol of the point O in the form

$$\lambda (yB + zC') + \mu (zC + xA') + \nu (xA + yB').$$

The other plane, through the point O, which meets a, b, c, d, e, can then be shewn to be that containing the points $y_1B + z_1C'$, $z_1C + x_1A'$, $x_1A + y_1B'$, where

$$x_1 = (\mu - \nu)(\mu z - \nu y)^{-1},\ y_1 = (\nu - \lambda)(\nu x - \lambda z)^{-1},\ z_1 = (\lambda - \mu)(\lambda y - \mu x)^{-1}.$$

The conditions that this other plane should coincide with the first plane, which are $x_1/x = y_1/y = z_1/z$, reduce to one condition, and lead, effectively, to

$$\lambda = x (t - x)^{-1},\ \mu = y (t - y)^{-1},\ \nu = z (t - z)^{-1},$$

where t is arbitrary. These values express that the point O lies on a conic in the plane of the points $yB + zC'$, $zC + xA'$, $xA + yB'$, of which the points are given by varying t. This conic is, in fact, that passing through the points in which the plane is met by the lines e, d, a, b, c; for these points are given, respectively, by the values 0, ∞, x, y, z of t, as is easily seen. If the symbol of the general point of this conic be written $\xi A + \eta B + \zeta C + \xi' A' + \eta' B' + \zeta' C'$, we see that, with $x' = t - x$, $y' = t - y$, $z' = t - z$, we have

$$\xi = 1/yz',\ \eta = 1/zx',\ \zeta = 1/xy',\ \xi' = 1/y'z,\ \eta' = 1/z'x,\ \zeta' = 1/x'y.$$

These values satisfy the irrational equation above given for the locus of the point O. Conversely, they lead to the equations

$$1 - x^{-1} t = 2qr (pp' - qq' - rr')^{-1},\ 1 - y^{-1} t = 2rp (qq' - rr' - pp')^{-1},$$
$$1 - z^{-1} t = 2pq (rr' - pp' - qq')^{-1}.$$

Thus the locus is of the kind called *rational*, being in (1, 1) birational correspondence with the points of a threefold space, in which x, y, z, t are coordinates. Through any point of the locus there pass six conics lying on the locus, each conic lying in a plane of one of the six systems, and meeting the five associated lines of that system; in the representation here given, one of these conics corresponds to a definite set of values for the ratios of x, y, z, the various points of the conic corresponding to different values of t. It will appear incidentally below that such parameters, replacing x, y, z, t, can be taken for each of the six conics. It is easy to shew that the four points, in which the locus is met by any line lying in the plane of one of the conics, consist of the two points in which the line meets the conic, each taken doubly. Thus all the conics, and the fifteen fundamental lines also, may be spoken of as being *double* upon the locus. We shall denote the locus by Σ.

Ex. It is easy to represent the points of the six conics that pass through any point, O, of this locus; incidentally there arises thus a proof that the six points, on one of these conics, constituted by O and the points where the conic is met by the associated lines, form a range related to that of the corresponding points on any other conic, if the points be taken in proper order. A geometrical proof of this is given below. More precisely, taking the scheme given above (p. 114), of the six sets of five associated lines, let each entry be supposed, now, to mean the point where the line, given by that entry, meets the proper plane of the six drawn through O; and imagine the point O supplied as the diagonal element of every row of this scheme. Then the points of the six planes through O form related ranges, upon their respective conics, when taken in the order given by this scheme. This can be shewn by choosing a proper parameter for each of the six conics, and shewing that the six points in question arise, on all the conics, for the same values of the parameter. See, also, p. 131, below.

To put the results in concise form, let a, b, c be three parameters, and use a', b', c', respectively, for $1-a$, $1-b$, $1-c$. Also, use the symbols P, Q, R, P', Q', R', respectively, for A/bc', B/ca', C/ab', $A'/b'c$, $B'/c'a$, $C'/a'b$, where A, B, C, A', B', C' are the symbols previously used for the six fundamental points. We have seen that any point of the locus Σ is given by a symbol $A/yz' + \ldots + C'/x'y$, where $x' = t-x$, $y' = t-y$, $z' = t-z$; in particular, suppose that O arises for the values $a, b, c, 1$, of x, y, z, t, respectively. If, in the symbol $A/yz' + \ldots + C'/x'y$, say Ω, we replace x, y, z by a, b, c, respectively, the points upon the conic, through O, which meets the lines e, d, a, b, c, are given by varying t in the symbol

$$(t-a)^{-1} a' (Q+R') + (t-b)^{-1} b' (R+P') + (t-c)^{-1} c' (P+Q'),$$

as we have seen; the six points consisting of O and the points upon the lines e, d, a, b, c arise for the values $t = 1, 0, \infty, a, b, c$, respectively. If, in the general symbol Ω, we replace x, y, z, t, respectively, by $t-a$, $t-b$, $t-c$, $t-1$, another conic is given by varying t in the symbol thence arising, which is

$$(t-a)^{-1} a (Q'+R) + (t-b)^{-1} b (R'+P) + (t-c)^{-1} c (P'+Q);$$

this conic meets the line e, goes through the point O, and meets the lines d', a', b', c'. These occur, respectively, also for the values $t = 1, 0, \infty, a, b, c$. Again, if, in the same general symbol Ω, we replace x, y, z, t, respectively, by $a(t-a)^{-1}$, $b(t-b)^{-1}$, $c(t-c)^{-1}$, $(t-1)^{-1}$, the symbol thence arising, namely

$$(t-a)^{-1} (P+P') + (t-b)^{-1} (Q+Q') + (t-c)^{-1} (R+R'),$$

gives another conic as t varies; this meets the lines d, d', goes through the point O, and meets the lines l, m, n, these occurring for the same respective values, $t = 1, 0, \infty, a, b, c$. And if, in the same general symbol, we replace x, y, z, t, respectively, by $t, b, c, 1$, the symbol thence arising,

$$P + P' + (1-t)^{-1} a'(Q + R') + t^{-1} a(Q' + R),$$

likewise represents the points of a conic as t varies; this conic meets the lines a, a', l, goes through the point O, and meets the lines r, q, for the same respective values, $1, 0, \infty, a, b, c$ of t. Similarly if, in the same general symbol, we replace x, y, z, t, respectively, by a, t, c, 1, the symbol thence arising,

$$t^{-1} b(R' + P) + Q + Q' + (1-t)^{-1} b'(R + P'),$$

represents the points of a conic, meeting the lines b, b', m, r, passing through the point O, and meeting the line p, for the same respective values of t. Finally, if, in the same general symbol, we replace x, y, z, t, respectively, by a, b, t, 1, there is a conic, of points given by

$$(1-t)^{-1} c'(P + Q') + t^{-1} c(P' + Q) + R + R',$$

which meets the lines c, c', n, q, p, and goes through the point O, for the same respective values of t.

The tangent solid of the quartic locus Σ. When a locus, in space of four dimensions, is given by a single equation, in coordinates X, Y, Z, T, U, say $F(X, Y, Z, T, U) = 0$, it is easy to prove that the lines drawn through a point $(X_0, Y_0, Z_0, T_0, U_0)$, of this locus, which meet the locus in two coincident points here, generate a solid, called the *tangent solid*, whose equation is $DF_0 = 0$, where D denotes the operator $X\partial/\partial X_0 + \ldots + U\partial/\partial U_0$, and F_0 is F, with $X_0, \ldots$ substituted for $X, \ldots$. The lines through this point which meet the locus in three coincident points here, lie on a conical sheet, obtainable as the intersection of the tangent solid with the (∞^3) quadric given by $D^2F_0 = 0$; while the lines, through this point, meeting the locus in four coincident points here, are the six which are the intersection of this conical sheet with $D^3F_0 = 0$. When, as in what precedes, we use coordinates $\xi, \eta, \ldots, \zeta'$, six in number, of which only the differences are effective, the operator D, it can be seen, is replaced by $\xi\partial/\partial\xi_0 + \ldots + \zeta'\partial/\partial\zeta_0'$. Hence it may be directly computed, by forming the rational equation for Σ from $[(\eta - \zeta')(\eta' - \zeta)]^{\frac{1}{2}} + \ldots = 0$, that the tangent solid of the locus Σ, at the point O, or $A/bc' + \ldots + C'/a'b$, has the equation

$$bc'\xi + ca'\eta + ab'\zeta - b'c\xi' - c'a\eta' - a'b\zeta' = 0.$$

This equation, however, is easily obtained otherwise, by first proving that any line, in the plane of any one of the six conics of Σ passing through O, discussed above, meets the locus in two coincident points at each of its two intersections with the conic, and thence inferring that the tangent solid of the locus, at any point of one of these conics, contains the plane of this conic. For, the general solid, through the plane containing the three points $bB + cC'$, $cC + aA'$, $aA + bB'$, has the equation

$$b(t-c)\xi + c(t-a)\eta + a(t-b)\zeta - c(t-b)\xi' - a(t-c)\eta' - b(t-a)\zeta' = 0,$$

in which t is arbitrary, as is quite clear. The value of t for which this passes through the point O is $t = 1$; and the equation then agrees with the above. It is verified at once, moreover, that the tangent solid, as given by this equation, contains the six conics passing through the point O; it is the solid in which the six planes of these conics were proved to lie. In passing, it may be remarked that the tangent solid contains the tangent lines of these conics at the point O, these being the lines through this point which meet the locus Σ in four coincident points there; these six lines lie on a quadric cone; it will be proved that the six planes touch a quadric cone.

Ex. If we put $u = bc'$, $v = ca'$, $w = ab'$, $u' = -b'c$, $v' = -c'a$, $w' = -a'b$, the equation of the tangent solid is given by

$$u\xi + v\eta + w\zeta + u'\xi' + v'\eta' + w'\zeta' = 0,$$

with the two identities

$$u + v + w + u' + v' + w' = 0, \quad uvw + u'v'w' = 0.$$

Conversely, the point of contact depends on the values a, b, c expressed by

$$a = -(v' + w)/(u' + u), \quad b = -(w' + u)/(v' + v), \quad c = -(u' + v)/(w' + w).$$

The equations of the singular solids. The tangent solid at a general point of the locus Σ, which, as we have seen, is capable of the equation $y(t-z)\xi + \ldots - (t-x)y\zeta' = 0$, or

$$yz(\xi - \xi') + zx(\eta - \eta') + xy(\zeta - \zeta')$$
$$+ tx(\eta' - \zeta) + ty(\zeta' - \xi) + tz(\xi' - \eta) = 0,$$

meets the locus Σ in a quartic surface. There are, however, ten tangent solids for which this quartic surface reduces to a repeated quadric surface; these are in fact the ten singular solids discussed above. They arise for the respective values of x, y, z, t given by

$$(0, y, z, 0), \quad (x, 0, z, 0), \quad (x, y, 0, 0), \quad (1, 1, 1, 0),$$
$$(0, 1, 1, 1), \quad (1, 0, 1, 1), \quad (1, 1, 0, 1),$$
$$(x, 0, 0, t), \quad (0, y, 0, t), \quad (0, 0, z, t),$$

and have the respective equations

$$\xi - \xi' = 0, \quad \eta - \eta' = 0, \quad \zeta - \zeta' = 0, \quad \xi + \eta + \zeta - \xi' - \eta' - \zeta' = 0,$$
$$\eta - \zeta' = 0, \quad \zeta - \xi' = 0, \quad \xi - \eta' = 0,$$
$$\eta' - \zeta = 0, \quad \zeta' - \xi = 0, \quad \xi' - \eta = 0.$$

For instance, the solid represented by the equation $\eta - \zeta' = 0$ meets the locus Σ in the repeated quadric surface given by

$$[(\zeta - \xi')(\zeta' - \xi) - (\xi - \eta')(\xi' - \eta)]^2 = 0,$$

as follows from the irrational equation of the locus. But the equation is capable of other irrational forms; for example, two of these are

$$[(\xi - \xi')(\eta - \zeta')]^{\frac{1}{2}} + [(\eta - \eta')(\zeta - \xi')]^{\frac{1}{2}} + [(\zeta - \zeta')(\xi - \eta')]^{\frac{1}{2}} = 0,$$

$$[(\xi + \eta + \zeta - \xi' - \eta' - \zeta')(\xi' - \xi)]^{\frac{1}{2}} + [(\zeta - \xi')(\xi' - \eta)]^{\frac{1}{2}} + [(\zeta' - \xi)(\xi - \eta')]^{\frac{1}{2}} = 0,$$

as may be verified by the parametric forms which have been given for the general point of the locus. These shew that the ten particular solids do each meet the locus in a repeated quadric surface.

That these particular solids agree with the singular solids as previously defined, may then be verified by shewing that these quadric surfaces contain the lines, of the original fifteen, which are characteristic of the singular solids. It appears that the solids, arranged as above (p. 129), are in fact the singular solids given by the respective notations

124, or 356, 125, or 364, 126, or 345, 123, or 456
234, or 156 235, or 146 236, or 145
134, or 256 135, or 246 136, or 245.

Beside the six lines, from the original fifteen, characteristic of any one of the singular solids, there are, upon the quadric surface lying in this solid, infinitely many other lines, which, therefore, also lie upon the locus Σ. The fifteen original lines are distinguished from these by the fact that they are double lines of the locus; every line drawn through a point of one of the fifteen lines has two of its four intersections with Σ coincident at this point.

The intersections of a tangent solid of Σ with the fifteen fundamental lines. We consider the tangent solid of the locus Σ at a point O, and the fifteen points in which this solid is met by the fifteen fundamental lines, denoting the intersection with the line 12 by (12), and so on. We thus have, in all, sixteen points in the tangent solid. We also have sixteen planes in the tangent solid, constituted by the six planes, of the diverse systems, which pass through O, and the ten planes of intersection of the singular solids with the tangent solid. Each of the six planes, in the tangent solid, through O, contains six of the points, namely O and five points on the associated lines which meet this plane. Each of the ten planes, in which the tangent solid is met by the singular solids, likewise contains the six points lying on the lines which characterise the singular solid. And six of the planes pass through every one of the sixteen points. This has been seen for the point O. For the point (12), for example, these six planes are the intersections of the tangent solid with the singular solids 123, 124, 125, 126, all of which contain the line 12, together with the two planes, of systems 1 and 2, which lie in the tangent solid, through O: for these two planes meet the line 12 in the same point, which must lie in the tangent solid at O. But it can be shewn further that the six points, on any one of the sixteen planes, lie on a conic, and that the six planes, through any one of the sixteen points, touch a quadric cone; and also, that the ranges of six points on the conics, and the sets of six tangent planes of the cones, are thirty-two related sets, if the elements be taken in proper order. With the indices 1, 2, ..., 6, this order can be specified by a symbolical rule which is of great simplicity.

Before indicating the proof of these statements we make two remarks: (1) If two triads of points upon a conic be considered, each with its three joining lines, the six joins touch another conic; and, among the tangents of this other conic, these joins are a set related each to its opposite point on its own triad, the points being regarded as belonging to a range of the original conic (Vol. II, p. 29, Ex. 7); (2) The line which joins any two of the sixteen points of the tangent solid, is also the intersection of two of the sixteen planes lying in this solid. For instance, consider the line joining the points (12), (13). This lies in the plane of system 1 through O, and also lies in the plane (123), in which the solid 123 (or 456) meets the tangent solid. Or, again, consider the line joining the points (12), (34). This lies in both the singular solids 125 (or 346) and 126 (or 345), and, therefore, in the planes, (125) and (126), in which these solids meet the tangent solid.

The proof of the statements made depends on the following circumstance: let the six points, in one of the sixteen planes, be divided into two triads, in any one of the ten possible ways; then, among the ten remaining points, which do not lie in this plane, there is one which is such that the six planes through this point contain the six joining lines of the points of the two triads. To prove this, take, first, the six points (23), (31), (12), (56), (64), (45), lying in the plane (123), arising from the singular solid 123 (or 456). These six points lie on a conic because the six lines in the singular solid lie on a quadric surface. The join of (31) and (12) lies on the plane of system 1 through O; and so on. Thus the six planes, of the systems 1, 2, 3, 4, 5, 6, through O, touch a quadric cone; and further, as tangent planes of this cone, they are related, in this order, respectively to the points of the conic, in the order taken. Take, next, the same six points, but arranged as the two triads (31), (12), (56) and (64), (45), (23). The six joins of these then lie on the six planes passing through the point (14). For instance, the join of (12), (56) lies on the solid 124 (or 356), which contains the line 14, as well as the lines 12 and 56; in the same way, the join of (56) and (31) lies on the solid 314 (or 256), which contains the line 14, as well as the lines 56 and 31; and likewise, the join of (31), (12) lies on the plane of system 1 through O, which also contains the point (14). The other triad may be similarly considered. Then, first, we have the result that, the six points (31), (12), (56), (64), (45), (23), in the range of points of the conic on which they lie, are related, respectively, to the six planes, through the point (14), obtained in order by, the solids 124, 314, the plane of system 1 through O, the solids 145, 146, and the plane of system 4 through O, these planes being considered in the series of tangent planes of the quadric cone which they touch. But, second, rearranging the points, and using the result found before, we see that the planes through O, of systems 1, 2, ..., 6 are, respectively, related to the six planes (4), (124), (134), (1), (145), (146), through the point (14). And we notice that the notations for these latter six planes are obtainable by a symbolical multiplication, of the symbol 14, respectively with 1, 2, ..., 6. Now consider, finally, the points in the plane of system 1 through O, taken in the two triads O, (12), (13) and (14), (15), (16). We shew that the joins of these triads lie on the six planes through the point (23). In fact, the three joins (12), (13); (13), O; O, (12) lie, respectively, on the solid 123, the plane 3 through O, and the plane 2 through O; while the three joins (15), (16); (16), (14); (14), (15) lie, respectively, on the solids 234, 235, 236. Assuming that it has been proved as above, that the six planes through (23), so arising, touch a quadric cone, and are related, respectively, as planes of this cone, to the planes through O of systems 1, 2, ..., 6 (these being the numbers obtained by symbolical multiplication, with 23, of the notations for the six planes), it follows that the points O, (12), (13), (14), (15), (16), of plane 1 through O, lie on a conic, and are related to the planes through O, respectively, in the order taken.

We can thus take the six points of any one of the sixteen planes, as also the six planes through any one of the sixteen points, in such an order that they are associated, respectively, with the numbers 1, 2, ..., 6. The association is obtained, in the former case, by symbolical multiplication of the symbol of the plane, respectively, with 1, 2, ..., 6; and, in the latter case, of the symbol of the point. Thus, for example, the number 1 is associated with the point (12) lying in the plane 2 through O, and is associated with the point (23) of the plane (123), in which the tangent solid is met by the singular solid 123 (or 456); or again, the number 1 is associated with the plane 3 through O, regarded as a plane through the point (13), and is associated with the plane (134), through the point (34). This association enables us to make the sixteen points (as also the sixteen planes) correspond in pairs, in any one of fifteen ways. For instance, take the two numbers 1, 2. From any point, we may pass, with the number 1, to a particular plane through this point; and then, with the number 2, we may pass from this plane to a particular one of the six points which it contains. The second point is evidently obtained from the first point by symbolical multiplication with the numbers 1 and 2, in either order; for instance, if the first point be (45), the second is (36).

What has been said suggests the arrangement of the sixteen points in such a scheme as that adjoined. This is typical of ten such schemes, of which the other nine are obtainable from this by interchange of any one of the numbers 1, 2, 3 with any one of 4, 5, 6. If, in this scheme, we take any element in the last row, or in the last column, other than O, the other six elements in the same row and column, as this chosen, are the points lying in one of the six planes through O. But, the points, given by the elements of the scheme, which are in the same row and column as any of the other ten elements, are those in one of the ten planes arising from the singular solids. Further, considering the elements of the scheme as consisting of the four tetrads which are in the four rows, the points given by any two of these tetrads are at once seen to be Moebius tetrads, inscribed to one another (Vol. I, pp. 61, 91). For instance, taking the first and second rows, the point (14) lies in the plane of the points (25), (26), (31), since these four points are in the singular solid 256 (or 314); similarly, the point (25) lies in the same plane with (14), (16), (23), these four points being in the singular solid 146 (or 235). The same relation holds for the four tetrads given by the columns of the scheme. Moreover, there is a quadric surface in regard to which the four tetrads given by the four rows are all self-polar; and, taking the four lines joining elements in the same column, in the first two rows of the scheme, and, likewise, the four lines joining elements in the same column, in the last two rows, these eight lines have two common transversals, which are generators of this quadric. By taking the four rows in two pairs, in this way, three sets each of two generators are obtained. These are all of the same system, of generators of the quadric, and the two of any set are harmonic conjugates in regard to those of either of the other two sets (cf. Vol. III, pp. 68, 138, 143).

(14),	(15),	(16),	(23)
(24),	(25),	(26),	(31)
(34),	(35),	(36),	(12)
(56),	(64),	(45),	O

The intersections of a tangent solid of Σ with the planes of the six systems. In space of four dimensions, a plane meets a solid in a line. Thus the ∞^2 planes of any one of the six systems, which have been considered, give rise to ∞^2 lines in the tangent solid of Σ at a particular point, O. As these planes all meet three lines, not lying in the tangent solid, whose transversal does not lie in this solid (and this in various ways), we infer that the ∞^2 lines, obtained in the tangent solid, belong to (at least one) tetrahedral complex of lines in this solid (p. 32, above). Through any general point of the tangent solid there pass, we have seen, two planes of any one of the six systems, giving rise to two lines of the system considered. We may expect, therefore, that the lines of this system belong also to a linear (as well as to a tetrahedral) complex. This we now prove, in a direct way, which also shews that the six linear complexes so arising are mutually conjugate or apolar (cf. p. 42, above). General results in regard to quadratic congruences of lines are given below (Chap. VII); the sixteen points and planes we have considered, in what precedes, then arise as *singular* points and planes. The proof we now give is, in part, algebraic; another direct proof, of a synthetic kind, arises below (p. 155), from a dual point of view.

Consider any particular line, *of the second system*, obtained by the intersection of the tangent solid with the plane of the second system containing the three points $\eta B' + \zeta C$, $\zeta C' + \xi A$, $\xi A' + \eta B$. We first prove that any line of the first system which meets this line, also meets another determinate line of the second system. Through any point, say P, of the given line of the second system, there pass, we have remarked, two lines of the first system; we are to prove that both these meet another line of the second system, whose determination is independent of the position of P upon the first line. The plane of the second system, (ξ, η, ζ), referred to, meets the line e, or 12, in a particular point, say T; through this point there pass ∞^1 other planes of the second system, meeting the tangent solid, at O, in the generating lines, of one system, of a quadric surface, as we have seen. Through T there pass, also, ∞^1 planes of the first system, giving rise to the other system of generators of the quadric surface spoken of; one of these passes through P; it meets all lines of the second system which lie on planes through T. There is through P another line of the first system; this line, in fact, is not on a plane of the first system passing through T, and does not lie on the quadric surface, in the tangent solid at O, obtained by planes through T. This line meets this quadric surface in another point, say P'. Through P' there passes a generator of the quadric surface, lying on a plane of the second system through T; this is of the same system as the generator, (ξ, η, ζ), on which P lies. We shew that this second generator, through P', depends only on ξ, η, ζ, and not on the position of P upon the generator (ξ, η, ζ). This second generator is then met by both the lines, obtained by planes of the first system, through all points, P, of the generator (ξ, η, ζ).

For, the point common to the plane defined by the points $\eta B' + \zeta C$, $\zeta C' + \xi A$, $\xi A' + \eta B$, and the plane, of the first system, defined by the points $yB + zC'$, $zC + xA'$, $xA + yB'$, is the point whose symbol is

$$x^{-1}\xi(\eta B' + \zeta C) + y^{-1}\eta(\zeta C' + \xi A) + z^{-1}\zeta(\xi A' + \eta B),$$

its coordinates, relative to A, B, C, A', B', C', being

$$y^{-1}\eta\xi,\ z^{-1}\zeta\eta,\ x^{-1}\xi\zeta,\ z^{-1}\zeta\xi,\ x^{-1}\xi\eta,\ y^{-1}\eta\zeta;$$

the condition that this point lies on the tangent solid at O, whose equation is $bc'X + \ldots = b'cX' + \ldots$, where $a' = 1 - a$, etc., is

$$\xi^{-1}a'x(bz - cy) + \eta^{-1}b'y(cx - az) + \zeta^{-1}c'z(ay - bx) = 0.$$

This assumes that the two planes do not meet in a line; that is, do not meet the line e in the same point; for which the condition is

$$\xi(y - z) + \eta(z - x) + \zeta(x - y) = 0.$$

In order to prove what we have stated it is thus necessary and sufficient to prove that, if these two conditions are satisfied by one set of values of ξ, η, ζ, then they are satisfied by another set of values, which depend on ξ, η, ζ only, and are independent of x, y, z. Such another set, ξ_1, η_1, ζ_1, is, in fact, given, in terms of the former set, by

$$\xi_1(\eta - \zeta) = a'(\eta c' - \zeta b'),\quad \eta_1(\zeta - \xi) = b'(\zeta a' - \xi c'),$$
$$\zeta_1(\xi - \eta) = c'(\xi b' - \eta a').$$

For the former condition is equivalent, with proper values of λ, μ, to the three

$$\xi^{-1}a' = \lambda + \mu a x^{-1},\quad \eta^{-1}b' = \lambda + \mu b y^{-1},\quad \zeta^{-1}c' = \lambda + \mu c z^{-1},$$

leading to $\eta^{-1}b' - \zeta^{-1}c' = \mu(by^{-1} - cz^{-1})$, etc. Thus, what we wish to verify as to ξ_1, η_1, ζ_1 satisfying the first condition, is clear, this being that the sum of three such quantities as

$$x(bz - cy)(\eta^{-1} - \zeta^{-1})(\eta^{-1}b' - \zeta^{-1}c')^{-1}$$

vanishes. The latter condition, for proper ρ, σ, is equivalent to the three equations $\xi = \rho x + \sigma$, etc.; what we wish to verify for this is thus the obvious fact that the sum of $\xi_1(\eta - \zeta)$, $\eta_1(\zeta - \xi)$ and $\zeta_1(\xi - \eta)$ vanishes.

It can now be further verified that, if we form the same functions of ξ_1, η_1, ζ_1 that ξ_1, η_1, ζ_1 are of ξ, η, ζ, we shall obtain the quantities ξ, η, ζ, save for a common factor. These are then obtained from ξ_1, η_1, ζ_1 just as ξ_1, η_1, ζ_1 are obtained from ξ, η, ζ.

Whence, considering, first, only those lines in the tangent solid at O, that lie upon planes of the second system which pass through a particular point of the line e, these being the generators, of one system, of a particular quadric surface, we have shewn that all lines, obtained by intersection of the tangent solid with planes of the first kind, meet this quadric surface on generators which are pairs of an involution of these generators, or, are themselves generators of this quadric, of the opposite system. From this it follows that these lines of the first kind belong to a linear complex of lines, in the tangent solid at O (Vol. III, p. 67). By parity of reasoning, considering now all the lines, in the tangent solid, obtained by planes of the second kind, these will also belong to a linear complex. But further, as we have shewn that all lines of the first linear complex, which meet any particular line of the second linear complex, likewise meet another line of the second linear complex, it follows that the pair of associated lines of the second complex are polar lines of one another in regard to the first complex. Thus either complex consists of pairs of lines which are polar lines of one another in regard to the other complex. Whence it follows that the two complexes are apolar, or conjugate (Vol. III, p. 65. Above, p. 42). A similar argument applies to any two of the six systems of lines in the tangent solid. The fact that any two of the linear complexes are conjugate involves the fact, remarked above (p. 133), that we can associate together two of the sixteen points of the tangent solid, in one of fifteen ways; the converse is also true.

Ex. 1. Taking any two points, P, Q, given, relatively to $A, B, \dots, C'$, respectively by

$$P = X_1A + \dots + Z_1'C', \text{ and } Q = X_2A + \dots + Z_2'C',$$

and putting

$$S_1 = yz(X_1 - X_1') + zx(Y_1 - Y_1') + xy(Z_1 - Z_1'),$$
$$U_1 = x(Y_1' - Z_1) + y(Z_1' - X_1) + z(X_1' - Y_1),$$

with a corresponding meaning for S_2, U_2, prove that the planes of the first kind, joining three points $yB + zC'$, $zC + xA'$, $xA + yB'$, which meet the line joining the two points P, Q, are given by the values of x, y, z which satisfy the equation $S_1U_2 - S_2U_1 = 0$. Regarded as representing a locus in the plane (x, y, z), this equation represents a cubic curve passing through the four points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$. When the two points P, Q are both in the tangent solid at O, this point being given, as before, by the coordinates $(bc')^{-1}$, $(ca')^{-1}$, etc., the equations $S_1 + U_1 = 0$, $S_2 + U_2 = 0$ are both satisfied by $(x, y, z) = (a, b, c)$, and the cubic curve contains the fifth point (a, b, c). When the two points P, Q, not necessarily lying in the tangent solid at O, are in a plane of the second kind, being, respectively,

$$p_1(\eta B' + \zeta C) + q_1(\zeta C' + \xi A) + r_1(\xi A' + \eta B), \text{ and } p_2(\eta B' + \zeta C) + \text{etc.},$$

the cubic locus breaks into two, becoming

$$[\xi uyz + \eta vzx + \zeta wxy][x(\eta - \zeta) + y(\zeta - \xi) + z(\xi - \eta)] = 0,$$

where $u = q_1r_2 - q_2r_1$, $v = r_1p_2 - r_2p_1$, etc. If the two points satisfy both conditions, being on a line of the second kind in the tangent solid at O, the ratios of u, v, w are the same as of $a(\eta c' - \zeta b')$, $b(\zeta a' - \xi c')$, $c(\xi b' - \eta a')$. Both the ratios of ξu, ηv, ζw, and the ratios of $\eta - \zeta$, $\zeta - \xi$, $\xi - \eta$, are unaltered by replacing ξ, η, ζ by ξ_1, η_1, ζ_1, where $\xi_1(\eta - \zeta) = a'(\eta c' - \zeta b')$, etc., as above.

Ex. 2. If ξ, η, ζ, ξ', η', ζ' be the coordinates of any point relatively to $A, B, C, \ldots$, then independent coordinates, for a point in the tangent solid at O, may be taken to be X, Y, Z, T, where

$$T = (c - a)(ab'\zeta - a'b\zeta') - (a - b)(ca'\eta - c'a\eta'),$$

and $X = \xi - \xi'$, $Y = \eta - \eta'$, $Z = \zeta - \zeta'$. The tetrad of reference for these coordinates consists of the four points where the tangent solid is met by the four lines 34, 35, 36, 12 (or l, m, n, e). If then the line coordinates of any line in the tangent solid be defined by writing its equations in the forms

$$l'T + mZ - nY = 0, \quad l'X + m'Y + n'Z = 0,$$

and we put

$$P = x(b - c) + y(c - a) + z(a - b),$$
$$Q = yza(b - c) + zxb(c - a) + xyc(a - b),$$

it can be shewn that the plane of the first system, containing the points $yB + zC'$, $zC + xA'$, $xA + yB'$, meets the tangent solid at O in the line whose coordinates, l, m, n, l', m', n', are, respectively,

$$a'(b - c)xQ, \quad b'(c - a)yQ, \quad c'(a - b)zQ, \quad yzP, \quad zxP, \quad xyP.$$

These coordinates satisfy the equation

$$l/a' + m/b' + n/c' - l'a(b - c) - m'b(c - a) - n'c(a - b) = 0,$$

expressing that the line belongs to a linear complex, beside the equations $ll' + mm' + nn' = 0$, $ll'/a' + mm'/b' + nn'/c' = 0$, corresponding to a tetrahedral complex.

Similarly, a plane of the second kind, joining the points $y_1B' + z_1C$, $z_1C' + x_1A$, $x_1A' + y_1B$, if

$$P_1 = x_1(b - c) + \text{etc.}, \text{ and } Q_1 = y_1z_1a'(b - c) + \text{etc.},$$

gives a line whose coordinates are

$$a(b - c)x_1Q_1, \quad b(c - a)y_1Q_1, \quad c(a - b)z_1Q_1, \quad y_1z_1P_1, \quad z_1x_1P_1, \quad x_1y_1P_1.$$

These satisfy the equation

$$l/a + m/b + n/c - l'a'(b-c) - m'b'(c-a) - n'c'(a-b) = 0,$$

beside $ll' +$ etc. $= 0$ and $ll'/a +$ etc. $= 0$. Evidently the two linear complexes thus obtained are conjugate to one another.

Ex. 3. With X, Y, Z, T as in Ex. 2, the points of the quadric surface, in the tangent solid at O, obtained by planes of the second kind drawn through the point $\xi(A+A') + \eta(B+B') + \zeta(C+C')$, are expressible, in terms of the ratios of three parameters, x, y, z, in the forms

$$X = (y-z)(Q\nu - R\mu), \quad Y = (z-x)(R\lambda - P\nu), \quad Z = (x-y)(P\mu - Q\lambda),$$

$$T = \lambda(b-c)QR + \mu(c-a)RP + \nu(a-b)PQ,$$

where

$$P = bc'y - b'cz, \quad Q = ca'z - c'ax, \quad R = ab'x - a'by,$$

$$\lambda = \eta - \zeta, \quad \mu = \zeta - \xi, \quad \nu = \xi - \eta.$$

The plane sections of this quadric surface correspond to conics, in a plane (x, y, z), which pass through the two points $(1, 1, 1)$, $(a^{-1}a', b^{-1}b', c^{-1}c')$. The generators of the quadric surface, on the planes of the second kind, correspond to the lines in this plane which pass through the second of these points. The planes of the first and second kinds, defined, respectively, by

$$(yB+zC',\ zC+xA',\ xA+yB') \text{ and } (\eta B'+\zeta C,\ \zeta C'+\xi A,\ \xi A'+\eta B),$$

meet on the tangent solid at O if

$$x^{-1}\xi = a^{-1}a', \quad y^{-1}\eta = b^{-1}b', \quad z^{-1}\zeta = c^{-1}c'.$$

Ex. 4. If, instead of X, Y, Z, T in Ex. 2, we use $X_1 = X/a'$, $Y_1 = Y/b'$, $Z_1 = Z/c'$, $T_1 = T/a'b'c'$, prove that the intersection of the locus Σ with the tangent solid at O has the equation $P^{\frac{1}{2}} + Q^{\frac{1}{2}} + R^{\frac{1}{2}} = 0$

where $\quad P = a(b-c)X_1[-a'T_1/a + (c-a)Z_1 - (a-b)Y_1]$,

with similar values for Q and R. This equation represents a quartic surface. It has sixteen double points, namely at O, and at the points where the tangent solid is met by the fifteen fundamental lines. These double points lie, in sets of six, in sixteen planes, each of which touches the surface at all the points of a conic. The lines in which the tangent solid is met by the planes of any one of the six systems, belonging both to a quadratic (tetrahedral) complex, and to a linear complex, are said to form a quadratic *congruence*. As any line in one of these planes meets the locus Σ in only two points, each a pair of coincident points, the lines of the quadratic congruence are double tangents of the quartic surface. As the two planes of any system, which pass through a point of the locus Σ, coincide with one another, the points of the quartic surface are those points, in the tangent solid at O, for which the two lines of the quadratic congruence, generally possible through any point, coincide with one another. Any tangent plane of the quartic surface meets it in a plane quartic curve, to which six tangent lines can be drawn from the (double) point of contact of the plane. These are the six bitangents of the quartic surface which are possible through any point of it, arising from the six planes of the various systems through any point of Σ. See, also, p. 142, below.

The quartic surface is generally called *Kummer's quartic surface.* We return to the consideration of it below (Chap. VII), from the point of view of space of five dimensions; and it will be proved that any quadratic congruence of lines lies in forty tetrahedral complexes.

Ex. 5. A set of six linear complexes of lines, in space of three dimensions, of which every two are conjugate, can be constructed from six arbitrary points, of which no four lie in a plane. It has been said (Vol. III, p. 68, Ex. 26) that, if five points, A, B, C, D, E, be taken, in threefold space, in a particular order, there is a polar system in which each of these points has, for its polar plane, the plane through this point which contains the two contiguous points; so that the five sides of the pentagon are lines of the associated linear complex. In this polar system, the polar line of the joining line of two alternate points of this pentagon, for example of the join AD, is the transversal, drawn from the intermediate point, E, to the two sides of the pentagon, AB and CD, which are contiguous to the side, BC, opposite to E. Hence, if A, P, Q, B, C, D be any six points of the threefold space, and two polar systems be determined, respectively by the two ordered pentagons A, P, Q, B, C and A, Q, P, B, D (wherein the four points A, P, Q, B are the same, but P, Q occur in reverse orders), then these polar systems, or the associated linear complexes, are conjugate to one another. For, the line AP, which is self-polar in the first system, has, for polar in the second system, the transversal drawn from Q to PB and AD; while this transversal, drawn from Q in the plane PQB, is likewise self-polar in the first system. Namely, the linear complex, associated with the first polar system, contains two lines which are polars of one another in the second polar system. This is a sufficient condition for the two associated linear complexes to be conjugate (cf. p. 42, above, Ex. 1).

These preliminaries being clear, take, in the threefold space, six points, A, B, C, P, Q, R. We can then define six polar systems respectively by the ordered pentagons

$$B, R, Q, C, A; \quad C, P, R, A, B; \quad A, Q, P, B, C;$$
$$B, Q, R, C, P; \quad C, R, P, A, Q; \quad A, P, Q, B, R;$$

these we denote, respectively, by α, β, γ and α', β', γ', and prove that every two of them are conjugate. For this it is sufficient to prove that the four pairs, (β, γ), (β', γ'), (α, α') and (α, β'), are conjugate. But, β, γ are determined, respectively, by the two pentagons A, B, C, P, R and A, C, B, P, Q; β' and γ' are determined by the two pentagons R, P, A, Q, C and R, A, P, Q, B; α and α' are determined by the two pentagons B, R, Q, C, A and B, Q, R, C, P; while the two systems α, β' are determined, respectively, by the two pentagons R, Q, C, A, B and R, C, Q, A, P. In every case there are four common points to the two pentagons, of which two come in reverse orders. This proves the statement, by what is said above.

Recurring to the scheme given above (p. 133), it will be found that we may take, for A, B, C, a set of three points in one of the rows of this scheme, then, for P, a point of another row, in the same column as A; for Q, a point of another row, in the same column as B; and for R, the point of the remaining row which is in the same column as C.

Ex. 6. In the preceding Example, if the planes BPC, CQA, ARB meet in D, and, referred to A, B, C, D, the points P, Q, R be, respectively, $(0, q_1, r_1, k_1)$, $(p_2, 0, r_2, k_2)$, $(p_3, q_3, 0, k_3)$, and we take θ_1, θ_2, θ_3 so that

$$q_3 r_2 \theta_1 = p_2 k_3 - p_3 k_2, \quad r_1 p_3 \theta_2 = q_3 k_1 - q_1 k_3, \quad p_2 q_1 \theta_3 = r_1 k_2 - r_2 k_1,$$

prove that the six linear complexes considered have the respective equations

$$l - \theta_1 l' = 0, \quad m - \theta_2 m' = 0, \quad n - \theta_3 n' = 0,$$

and $$k_1 \theta_1^{-1} L + r_1 M - q_1 N = 0, \quad -r_2 L + k_2 \theta_2^{-1} M + p_2 N = 0,$$

$$q_3 L - p_3 M + k_3 \theta_3^{-1} N = 0,$$

where $$L = l + \theta_1 l', \quad M = m + \theta_2 m', \quad N = n + \theta_3 n'.$$

It is easy to see that any two of these are conjugate.

Without loss of generality, we may replace the coordinates, x, y, z, t, of a point, respectively, by lx, my, nz, t. Thereby, for θ_1, θ_2, θ_3, we obtain, respectively, $mn\theta_1/l$, $nl\theta_2/m$, $lm\theta_3/n$. Thus we can choose l, m, n so that θ_1, θ_2, θ_3 have any assigned numerical values.

Ex. 7. Prove that, in the figure of fifteen lines, which we have considered, in space of four dimensions, there are sixty pairs of non-intersecting lines, and forty-five pairs of intersecting lines. Also, that there are eighty triads of non-intersecting lines, namely sixty triads with symbols such as 12, 13, 14, and twenty triads with symbols such as 23, 31, 12. These eighty triads correspond to the eighty *tetrads* of points represented by such schemes as that given above (p. 133).

Ex. 8. Prove that if, in the tangent solid of the locus Σ at the point O, we take coordinates given by $X = \eta - \zeta'$, $Y = \zeta - \xi'$,

$Z = \xi - \eta'$, $T = bc(\xi - \xi') + ca(\eta - \eta') + ab(\zeta - \zeta')$, for which the points of reference are the intersections of the tangent solid with the lines 14, 15, 16, 23 (or a, b, c, d'), then the line coordinates, of the line in which the plane $(yB + zC', zC + xA', xA + yB')$ meets the tangent solid, are

$$(b - c)\,UV'W', \quad (c - a)\,VW'U', \quad (a - b)\,WU'V',$$
$$-a'U'VW, \quad -b'V'WU, \quad -c'W'UV,$$

where $U = y - z$, etc., and $U' = yc - zb$, etc. These satisfy the equations

$$ll' + mm' + nn' = 0, \quad ll'/a' + mm'/b' + nn'/c' = 0,$$

and the equation

$$l + m + n - a(b - c)\,l'/a' - b(c - a)\,m'/b' - c(a - b)\,n'/c' = 0.$$

The corresponding results may be obtained when the coordinates

$$X = \eta' - \zeta, \quad Y = \zeta' - \xi, \quad Z = \xi' - \eta,$$

and $$T = b'c'(\xi - \xi') + c'a'(\eta - \eta') + a'b'(\zeta - \zeta')$$

are taken, for which the reference points are the intersections of the tangent solid with the lines 24, 25, 26, 31 (or a', b', c', d). Or when the coordinates

$$X = c\eta - b\zeta' + c'\eta' - b'\zeta, \quad Y = a\zeta - c\xi' + a'\zeta' - c'\xi, \text{ etc.},$$
$$T = \xi - \xi' + \eta - \eta' + \zeta - \zeta'$$

are taken, for which the reference points are the intersections of the tangent solid with the lines 56, 64, 45 (or p, q, r), and the point O. See the following Example.

Ex. 9. Take a set of four tetrads of points, in space of three dimensions, of which every two are mutually inscribed, consisting of the tetrad of reference and three tetrads $(0, c_r, -b_r, a_r)$, $(-c_r, 0, a_r, b_r)$, $(b_r, -a_r, 0, c_r)$, $(a_r, b_r, c_r, 0)$, for $r = 1, 2, 3$, where $a_r a_s + b_r b_s + c_r c_s = 0$. Such a set is given by the rows of the scheme above (p. 133). Let $Q = a_1 a_2 a_3 yz + b_1 b_2 b_3 zx + c_1 c_2 c_3 xy$, and $P_1 = a_1 x + b_1 y + c_1 z$. Consider any one of the ∞^2 lines given, for varying x, y, z, by the line coordinates

$$xQ, \quad yQ, \quad zQ, \quad a_2 a_3 yzP_1, \quad b_2 b_3 zxP_1, \quad c_2 c_3 xyP_1;$$

these are such that

$$ll'/a_2 a_3 = mm'/b_2 b_3 = nn'/c_2 c_3,$$

and $$(l - l')\,a_1 + (m - m')\,b_1 + (n - n')\,c_1 = 0.$$

The line can be joined to the points of any one of the four tetrads, so forming four axial pencils each of four planes. Prove that these four axial pencils are related to one another.

If x, y, z be regarded as coordinates in a plane, the line coordinates, equated to zero, represent cubic curves with five points in common. This is a general result (cf. Chap. VII, below).

Ex. 10. Consider any solid, Π, passing through the line e, in the figure of fifteen lines, in space of four dimensions, considered above. The eight lines, of the fifteen fundamental lines, which do not meet the line e, namely $a, b, c, d, a', b', c', d'$, will meet this solid in points which we may denote by $(a), (b), \ldots$. The plane of the three points (a), (b), (c), in this solid, will meet the line e; this is a plane of the first system, and meets the line d, in (d). Similarly, the points (a'), (b'), (c'), (d') are in a plane of the second system. Again, the points $(b), (c), (d'), (a')$, lying, respectively, on the lines 15, 16, 23, 24, are in the singular solid 234 (or 156), and are in the plane in which this solid meets Π. Similarly for $(c), (a), (d'), (b')$ and $(a), (b), (d'), (c')$. Thus the two tetrads of points, $(a), (b), (c), (d')$ and $(a'), (b'), (c'), (d)$, form a pair of mutually inscribed Moebius tetrads.

Using, as before, coordinates $\xi, \eta, \ldots, \zeta'$ referred to $A, B, \ldots, C'$, put

$$X = \eta - \zeta', \quad Y = \zeta - \xi', \quad Z = \xi - \eta',$$

$$P = b\xi + c\eta + a\zeta - c\xi' - a\eta' - b\zeta',$$

$$T = bc(\xi - \xi') + ca(\eta - \eta') + ab(\zeta - \zeta').$$

The equation $tP - T = 0$, for varying t, is the general solid containing the points $bB + cC'$, $cC + aA'$, $aA + bB'$; the solids $X = 0$, $Y = 0$, $Z = 0$ contain, respectively, the lines b, c, d', a'; c, a, d', b'; a, b, d', c'. It is easy to verify such identities as

$$a[P + (c - a)Z - (a - b)Y] = T - (a - b)(a - c)(\xi - \xi');$$

thus the locus Σ, in the space of four dimensions, has the equation (see the irrational form, p. 130; also pp. 138, 158)

$$[a(b - c)X[P - a^{-1}T + (c - a)Z - (a - b)Y]]^{\frac{1}{2}} + \text{etc.} = 0.$$

If, herein, we replace T by P, the equation represents the Kummer surface, in which the locus Σ is met by the tangent solid at the point (a, b, c), the equation of this tangent solid being $T - P = 0$. The reference points for the coordinates X, Y, Z, P are (a), (b), (c), (d'). Any solid containing the line e may be represented by $T = 0$. The surface in which the locus Σ is met by this solid, Π, in terms of the coordinates X, Y, Z, P, is therefore given by putting, in this equation for Σ, zero in place of T. The surface then arising is one studied by Plücker (*Geometrie des Raumes*, 1868, pp. 163 ff.), under the name of the *Meridian Surface*. It is the locus of the conics touched by the lines of a quadratic complex which meet an arbitrary line; and is the envelope of the cones generated by lines of this complex which meet this line. It will be seen (Chap. VII, below) that the lines of a quadratic complex which belong to a linear complex (or, in particular, meet a line), belong to a tetrahedral complex (to forty such). Thus Plücker's meridian surface may be regarded as generated from the tetrahedral complex $ll'/A = mm'/B = nn'/C$, where $A + B + C = 0$, by lines meeting the *particular* line (l, m, n, l', m', n'); its equation is then of the form

$$\{Ax(l't + mz - ny)\}^{\frac{1}{2}} + \text{etc.} = 0,$$

as we easily verify, and of many other forms. The surface is of the fourth order, and rational (Vol. III, p. 102, Ex. 9), as is, in fact, every quartic surface in threefold space which has a double line. If, in this equation, we take p, q, r so that $l' = qr$, $m' = rp$, $n' = pq$; put $x = pX$, $y = qY$, $z = rZ$, $t = P$; and also take a, b, c so that

$$a(Bnq - Cmr) = Amn, \quad b = a - n/r, \quad c = a + m/q,$$

it is easily seen that this form reduces to that obtained above.

The surface has the eight points (a), (b), ... for double points; these lie in pairs, (a) and (a'), (b) and (b'), etc., upon four lines, lying on the surface, which meet the line e; and there is a corresponding dual theorem. The complete investigation need not be given here. (Cf., for example, the writer's *Multiply Periodic Functions*, 1907, p. 159.)

Determine the equations of the congruences of lines in which the solid Π is met by the planes of the systems 3, 4, 5, 6. The meridian surface is evidently the locus of points from which the two lines, of any one of these congruences, are coincident; namely, is the *focal* surface of the congruence. (Cf. Chap. VII, below.)

Ex. 11. Six arbitrary lines of general position, in space of four dimensions, are met by five planes. These form an associated system of planes (as defined above, p. 118).

For, taking four of the lines, a, b, c, d, as in the preceding work, consider the planes joining three points, $yB + zC'$, $zC + xA'$, $xA + yB'$, (which therefore meet a, b, c, d), which are also such as to meet two other arbitrary lines. The condition for such a line to meet a further arbitrary line, we have seen (Ex. 1, p. 136, above), is that (x, y, z), regarded as coordinates in a plane, should be the coordinates of a point of a cubic curve passing through four given points. Two plane cubic curves have nine common points. There are, then, five other sets of values of (x, y, z), and, therefore, five planes, which meet two arbitrary lines in addition to meeting the four independent lines a, b, c, d. And, as any cubic curve, which passes through eight of the nine intersections, passes through the ninth (Vol. III, p. 156, Ex. 6), four of these planes determine the fifth. (Cf. Segre, ***Rend. d. Palermo***, II, 1888, pp. 45—52.)

As is indicated in Ex. 1, p. 136 above, the theorem has exceptions when the six given lines are not of general positions.

Ex. 12. Given four arbitrary planes, in space of four dimensions, these are met, by a solid, in four lines. These four lines will lie on a quadric surface, in the solid, if this solid is any solid passing through a certain fifth plane, *associated* with the given four. Let the four given planes be α, β, γ, δ; and the four planes each meeting three of these in a line be α', β', γ', δ', the common associated plane of these two sets being ϵ. Then the solid, drawn through the plane ϵ, meets α', ..., δ' in four generators of the same quadric surface, of the opposite system. These meet the four generators in the planes α, ..., δ, respectively, at the points (α, α'), ..., (δ, δ'), which are on the plane ϵ, on the conic in which the plane ϵ meets the quadric surface. Thus the first four generators, as lines of their system of generators, are *related with* the second four generators, as lines of their system.

Dually, if a, b, c, d be four general lines, and a', b', c', d' the transversals of threes of these, and O be a point such that the planes Oa, Ob, Oc, Od belong to a quadric point-cone (and, therefore, on a certain line, e), of which the planes Oa', Ob', Oc', Od' are planes of the other system, then the first four planes meet any plane, through O, which meets a, b, c, d, in a flat pencil of lines which is related to the pencil in which the second four planes meet any plane, through O, which meets a', b', c', d'.

Ex. 13. Let three points, A, B, C, in space of four dimensions, be joined, by lines, each to one of three other points, A', B', C'. In this way there arise six triads of joining lines, according to the order in which the points A', B', C' are taken. An arbitrary solid, by its intersections with the lines of a triad, determines a plane. It may be shewn that the six planes so arising meet in a point, and touch a quadric cone.

The triads of joining lines may also be described as the two sets of alternate sides, and the set of diagonals, of the hexagon $AB'CA'BC'$, together with the three other similar triads from the hexagon $AB'BA'CC'$ (which is obtained from the former hexagon by the interchange of B and C).

Let the given points be of symbols A, B, C, etc.; and let ξ, η, ζ, etc. be coordinates relative to these points; the equation of the solid may be written

$$a^{-1}\xi + b^{-1}\eta + c^{-1}\zeta = a_1^{-1}\xi' + b_1^{-1}\eta' + c_1^{-1}\zeta'.$$

The points in which this solid meets the joins of A, B, C, respectively, to B', C', A', for example, are $aA + b_1B'$, $bB + c_1C'$, $cC + a_1A'$. The plane of these points passes through the point, say O, whose symbol is $aA + bB + cC + a_1A' + b_1B' + c_1C'$; as, similarly, do all the six planes in question. Next, consider the points, taken in order, whose symbols are

$$bB + a_1A',\ aA + b_1B',\ cC + a_1A',\ bB + b_1B',\ aA + a_1A',\ cC + b_1B'.$$

These form a hexagon of which the diagonals meet in the point

$a_1A' - b_1B'$; thus any side of the hexagon is met, not only by the two contiguous sides, but also by the opposite side. Wherefore, the six sides of this hexagon are generators of a quadric surface. The solid in which this surface lies may be defined by three alternate vertices of the hexagon, together with the point of intersection of the diagonals, and this space is identical with the solid by which the six planes are constructed. The planes joining O to the sides of this hexagon are, in fact, these six planes, which, therefore, touch a quadric cone, as stated.

If the six given points be in a fivefold space, and the six planes be similarly defined by a fourfold, with equation

$$a^{-1}\xi + \ldots = a_1^{-1}\xi' + \ldots,$$

these six planes lie upon the quadric $a^{-2}\xi^2 + \ldots = a_1^{-2}\xi'^2 + \ldots$, and pass through the point where this is touched by the given fourfold.

Ex. 14. The theorem, for fourfold space, proved in the preceding Example, includes the result, given above, that the six planes, of the various systems, which can be drawn through a point of the quartic locus Σ, touch a quadric cone; as is easy to see. Now let an arbitrary plane, ϖ, be drawn in one of the singular solids, say 123 (or 456). Through the two points where this plane meets a pair of lines, of one of the two triads of non-intersecting lines lying in this singular solid, say the two lines 12, 13, there can be drawn a definite plane of one of the six systems, in this case of system 1. As only a single tangent solid of the locus Σ contains the plane ϖ, because this lies in the singular solid, we infer that the six planes obtained as described, meet in a point and touch a quadric cone.

Ex. 15. We may relate the points of three planes, in space of four dimensions, by taking four arbitrary points in one of these planes, and making correspond to these, respectively, four points in each of the other two planes (Vol. I, p. 148). Then we may consider the aggregate of planes containing three corresponding points, one in each plane. A particular case is that when the three sets of four corresponding points are on four lines, each meeting the three planes. On the first plane, α, let the four points be U, V, W, K; on the second plane, α', let the points corresponding thereto be U', V', W', K'; on the third plane, α'', let the corresponding points be U'', V'', W'', K''—so that U, U', U'' are in line, as also V, V', V''; W, W', W'' and K, K', K''. Each three of these four lines has a transversal. It can be shewn, if H, H', H'' be any three corresponding points respectively lying in the related planes $\alpha, \alpha', \alpha''$, that the plane $HH'H''$ meets the four transversals so arising.

We may identify the four lines, $UU'U''$, etc., which meet the three given planes, respectively with the lines a, b, c, d of the figure discussed above, in the text; then the points U, V, W, K, and the general point, H, of the plane α, may be taken to have symbols, respectively,

$$yB+zC',\ zC+xA',\ xA+yB',\ yz(B'+C)+zx(C'+A)+xy(A'+B)$$

and $$\lambda x(yB+zC')+\mu y(zC+xA')+\nu z(xA+yB');$$

the points U, V, W, K arise from the last point, H, for the particular values $(\lambda, \mu, \nu)=(1, 0, 0)$; $(0, 1, 0)$; $(0, 0, 1)$ and $(1, 1, 1)$. The point, H', in the plane α', may then be taken to be, similarly, of symbol $\lambda x'(y'B+z'C')+$ etc., where λ, μ, ν are the same as for H; and the point, H'', in the plane α'', may be represented by $\lambda x''(y''B+z''C')+$ etc. The general point of the plane $HH'H''$ has, then, a symbol

$$\xi(\nu B'+\mu C)+\eta(\lambda C'+\nu A)+\zeta(\mu A'+\lambda B),$$

where ξ, η, ζ are of the respective forms $\xi=pyz+p'y'z'+p''y''z''$, $\eta=pzx+p'z'x'+$ etc., and $\zeta=pxy+$ etc. The ratios of p, p', p'' identify any individual point of this plane; in particular we obtain the points of intersection of this plane with the four transversals, spoken of, by choosing p, p', p'' so that, for these four lines, respectively, $(\xi, \eta, \zeta)=(1, 0, 0)$; $(0, 1, 0)$; $(0, 0, 1)$; $(1, 1, 1)$.

Ex. 16. With the notation of the text, the common point of the two planes of the first kind, containing, respectively, the points $yB+zC'$, $zC+xA'$, $xA+yB'$ and y_1B+z_1C', z_1C+x_1A', etc., is

$$\frac{y_1-z_1}{yz_1-y_1z}(yB+zC')+\frac{z_1-x_1}{zx_1-z_1x}(zC+xA')+\frac{x_1-y_1}{xy_1-x_1y}(xA+yB').$$

The point of intersection of the former plane with the plane of the second system containing the points; P, or $\eta B'+\zeta C$; Q, or $\zeta C'+\xi A$; R, or $\xi A'+\eta B$, provided these two planes do not meet in a line, is

$$(y\zeta)^{-1}A+(z\xi)^{-1}B+(x\eta)^{-1}C+(\eta z)^{-1}A'+(\zeta x)^{-1}B'+(\xi y)^{-1}C',$$

which is $x^{-1}\xi P+y^{-1}\eta Q+z^{-1}\zeta R$. The other plane of the first system through this point is that given by x', y', z', where

$$x'(\eta-\zeta)=x(y\zeta-z\eta),\ y'(\zeta-\xi)=y(z\xi-x\zeta),\ \text{etc.};$$

these are such that

$$x/x'=\theta x\eta\zeta+\phi,\ y/y'=\theta y\zeta\xi+\phi,\ z/z'=\theta z\xi\eta+\phi,$$

where θ, ϕ are symmetrical quantities.

Ex. 17. Three planes, α, β, γ, of the first three systems, respectively, are given by the triads of points

$$(yB+zC',\ zC+xA',\ xA+yB'),\ (y_1C+z_1B',\ z_1A+x_1C',\ x_1B+y_1A'),$$
$$(y_2A+z_2A',\ z_2B+x_2B',\ x_2C+y_2C');$$

prove that the conditions for the pairs of planes (α, β), (α, γ), (β, γ) to meet in lines are, respectively,

$$x_1^{-1}(y-z)+\text{etc.}=0,\quad x^{-1}(y_2-z_2)+\text{etc.}=0,\quad x_2^{-1}(y_1-z_1)+\text{etc.}=0,$$

and that these equations are not consistent, in general. Cf. Ex. 23.

Ex. 18. With coordinates, $\xi, \eta, \dots, \zeta'$, referred to the six points $A, B, \dots, C'$, as in the text, prove that the equation

$$(\eta-\zeta')(\eta'-\zeta)+\sigma^{-1}(\zeta-\xi')(\zeta'-\xi)-(1+\sigma)^{-1}(\xi-\eta')(\xi'-\eta)=0$$

represents the quadric point-cone, drawn from the point $\sigma L - M$, of the line e, whose planes are those of the first and second kinds, meeting, respectively, the lines a, b, c, d and a', b', c', d'. Prove, also, that every tangent solid of this cone is a tangent solid of the locus Σ.

Ex. 19. Consider the rational quartic curve, in space of four dimensions, whose points have coordinates, (x, y, z, t, u), of the forms $(\theta^4, \theta^3, \theta^2, \theta, 1)$; the equations of a plane meeting this curve in three points, a so-called *trisecant* plane, are of the forms

$$Ax+By+Cz+Dt=0,\quad Ay+Bz+Ct+Du=0.$$

If L, M be any two points of the space, the trisecant planes of the curve, which pass through the point $\sigma L - M$, lie upon a quadric point-cone, whose equation is of the form $U + 2\sigma V + \sigma^2 W = 0$. Determine the other system of planes of this cone; and shew that the cone becomes a line-cone for three values of σ.

Prove that the trisecant planes of the quartic curve which meet a line are projected, from an arbitrary point of the curve, by the tangent solids of a quadric point-cone. Deduce, for a cubic curve in space of three dimensions, that, if three tetrads of points be taken on the curve, the twelve planes, each containing three points of one tetrad, all touch a quadric surface; further, that the two tangent planes of this quadric surface, drawn from an arbitrary chord of the cubic curve, determine a further tetrad of tangent planes of this quadric surface (of which these are two); and also (as was remarked to the writer by Mr Vaidyanathaswamy), that there are three chords of the cubic curve which are generators of the quadric surface. (Cf. Vol. III, Ex. 8, p. 135; and p. 120, above.)

Ex. 20. The tangent solid of the locus Σ, whose equation, with coordinates referred to the points $A, B, \dots C'$, was $u\xi + \dots + u'\xi' + \dots = 0$, subject to $u + v + \dots + w' = 0$, $uvw + u'v'w' = 0$, may be referred to the symmetrical points $F, G, \dots$, previously used (p. 116 above). For this, let the symbol

$$\xi A + \dots + \zeta' C',\ \text{or}\ \xi(G+H)+\eta(H+F)+\dots+\zeta'(F'+G'),$$

be written $xF + \dots + z'H'$, so that $x = \eta + \zeta, \dots, z' = \xi' + \eta'$. The equation of the tangent solid,

$$u\xi+\dots=0,\ \text{or}\ u(y+z-x)+\dots+u'(y'+z'-x')+\dots=0,$$

becomes $Ux + \ldots + W'z' = 0$, with $u = V + W, \ldots, w' = U' + V'$. The conditions for the coefficients then become

$$U + V + \ldots + W' = 0,$$

$$(V + W)(W + U)(U + V) + (V' + W')(W' + U')(U' + V') = 0.$$

For any p, q, r, however, we have

$$(p + q + r)^3 = p^3 + q^3 + r^3 + 3(q + r)(r + p)(p + q).$$

Thus, the second condition for the tangent solid, the *tangential equation* of the locus Σ, becomes

$$U^3 + V^3 + W^3 + U'^3 + V'^3 + W'^3 = 0,$$

where $U + \ldots + W' = 0$.

It may be proved, further, that the ten singular solids have equations $Ux + \ldots + W'z' = 0$, where three of $U, V, \ldots, W'$ are 1, and three are -1.

Ex. 21. It has been shewn (p. 117 above) that the figure of fifteen points and fifteen lines, with the associated planes and solids, is changed into itself by fifteen involutory transformations. There exists also a quadric (∞^3), in regard to which the figure is its own polar reciprocal. In particular, the polar solid of the point $F - F'$, in regard to this quadric, is the solid defined by the points G, H, G', H'. Thus, the fifteen points such as $F - F'$ are the complete intersection of the polar solids, in regard to this quadric, of the six points $F, G, \ldots, H'$, any four of these solids intersecting in one of these fifteen points; also, the plane of the points $F - F'$, $G - G'$, $H - H'$ is the polar plane, in regard to this quadric, of the line, e, containing the points $F + F'$, $G + G'$, $H + H'$, the polar solids of the points of this line all containing this plane. With coordinates $x, y, \ldots$, referred to $F, G, \ldots$, as above, the equation of this quadric is

$$(x + y + z + x' + y' + z')^2 - 6(x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2) = 0;$$

or, if $3X$ denote $2x + 2x' - y - y' - z - z'$, with similar forms for $3Y$, $3Z$, the equation is also

$$(x - x')^2 + (y - y')^2 + (z - z')^2 + X^2 + Y^2 + Z^2 = 0,$$

which is explicitly a function only of the differences of the coordinates. The condition that a solid, expressed by

$$Ux + \ldots + W'z' = 0,$$

should touch this quadric is

$$U^2 + V^2 + W^2 + U'^2 + V'^2 + W'^2 = 0,$$

subject to $U + \ldots + W' = 0$. In terms of coordinates $\xi, \eta, \ldots$, referred to $A, B, \ldots$, if $p = \eta - \zeta'$, $q = \zeta - \xi'$, $r = \xi - \eta'$, $p' = \eta' - \zeta$, etc., and $3P = q + q' - r - r'$, etc., the equation is

$$(p - p')^2 + (q - q')^2 + (r - r')^2 + P^2 + Q^2 + R^2 = 0.$$

Ex. 22. The reciprocity of the preceding Example is a particular case of a theorem already many times referred to (cf. Vol. II, p. 218; Veronese, *Math. Annal.* XIX, 1882, p. 161). The twenty-one points, consisting of the six F, G, etc., and the fifteen $F-F'$, $F-G'$, etc., may be regarded as consisting of two *simplexes*, each of five points, which are in perspective with one another from a centre, (which is also one of the twenty-one points), together with the ten points, lying in a solid, in which joining lines, of corresponding pairs of points of the two simplexes, intersect one another. These twenty-one points may be obtained either, by projection of the intersections of seven fourfolds, lying in a space of five dimensions which contains the fourfold space of the given figure; or, as the intersections, with this fourfold space, of the joins of seven points of this fivefold space. Taking the latter point of view, the fourteen linearly independent (∞^4) quadrics, which can be drawn through the seven points of the fivefold space, meet the fourfold space of the given figure in fourteen (∞^3) quadrics. These are all outpolar to the quadric in regard to which the specified polarity has place, and determine this quadric. (Cf. p. 10, above, Ex. 1.)

Ex. 23. Take any point O, of general position, not lying on the locus Σ. Through this point there pass fifteen planes, each joining the point to one of the fifteen lines of the figure. There also pass through this point two planes of each of the six systems, which we may denote, respectively, by $(1, 1')$, $(2, 2')$, ..., $(6, 6')$. Each of the two planes of the first system meets one of the planes of the second system in a line, intersecting the line 12; we can choose the notation so that the planes $1, 2'$ meet in a line, and also the planes $1', 2$ meet in a line. And, then, so that the planes $1, 3'$ meet in a line, and also the planes $1', 3$. Then we can prove that the planes $2, 3'$ meet in a line, and also $2', 3$. For, if it were the planes 2, 3 which meet in a line, the planes $1', 2, 3$ would be in a solid; and, then, the lines 23, 31, 12, each of which meets a pair of these planes, would be in this solid; this would thus be the singular solid 123 (or 456). But this solid contains the point O; and a general point O does not lie in any of the ten singular solids. Thus it is not the planes 2, 3, but the planes $2, 3'$, and, therefore, also, the planes $2', 3$, which meet in a line. This argument being repeated, we can infer that, with proper notation, no two of the six planes 1, 2, ..., 6 meet in a line, and no two of the six planes $1', 2', \ldots, 6'$; but the plane r (for $r = 1, 2, \ldots, 6$) meets in a line every one of the planes $1', 2', \ldots, 6'$ other than r'. Now, let the symbols 1, 2, ... denote, not the planes through the point O, but the conics in which these planes meet the locus Σ. Then, for instance, the conics $1, 2'$ have a point in common, where the line of intersection of the planes $1, 2'$ meets the line 12. We have, therefore, twenty-seven loci on the locus Σ, the fifteen lines and the twelve conics; and we easily see that, upon each of these twenty-seven loci, there are five points, through each of which two other of the twenty-seven pass. For instance, on the line 12 there are, first, the three points through which pass, respectively, the three pairs of lines (34, 56), (35, 64), (36, 45), and there are the two points through which pass, respectively, the two pairs of conics $(1, 2')$ and $(1', 2)$. Or again, on the conic $1'$ there are the five points in which this is met by the lines 12, 13, ..., 16; and through these there pass, respectively, also the conics 2, 3, 4, 5, 6. The total number of such intersections of three of the twenty-seven loci is, therefore, forty-five.

Now, let Π be an arbitrary solid, not containing O. Allow the planes, through O, containing the twenty-seven loci, to meet this solid. We obtain twenty-seven lines in Π, meeting in threes in forty-five points. Transferring the notation to these lines, the lines $1, 2'$, in the solid Π, meet in a point, as do the lines $1', 2$; and the join of these points is the line 12. Also, two lines in Π, denoted by pq and rs, meet if the symbols p, q, r, s are all different, but not otherwise. The figure in the solid Π is, in fact, the *dual* of that of the twenty-seven lines of a cubic surface, explained in Vol. III, p. 160. And, as in that case, it may be shewn that there are thirty-six ways of choosing a set of twelve of the lines, in the solid Π, to play the parts here played by $1, 1', \ldots, 6, 6'$, the other fifteen being then constructed from these. If we consider, in the solid Π, beside the lines, the planes in which this solid is met by the tangent solids, of the locus Σ, which pass through O, we shall easily see that these are an aggregate dual to that of the points of a cubic surface. Three tangent solids of the locus Σ pass through an arbitrary plane of the fourfold space; in particular, three through a plane which contains O. Thus, in Π, three planes of the aggregate pass through an arbitrary line.

We may deal with the matter algebraically. If the point O be $aA + \ldots + a'A' + \ldots$, so that (a, b, c, a', b', c') are its coordinates referred to $A, B, \ldots, C'$, the tangent solid of the locus Σ, whose equation is $y(t-z)\xi + \ldots = (t-y)z\xi' + \ldots$ (p. 128, above), contains the point O provided $t = S/P$, where

$$S = yz(a'-a) + zx(b'-b) + xy(c'-c),$$
$$P = x(b'-c) + y(c'-a) + z(a'-b).$$

This relation connecting x, y, z, t may be regarded as representing a quadric surface in a space (x, y, z, t). To every point of this quadric surface corresponds a tangent solid, of the locus Σ, passing through O; and, therefore, also, a plane of the cubic aggregate in the solid Π. By means of this equation, the coefficients of $\xi, \eta, \ldots$, in the equation of the tangent solid of Σ, take the forms $y(S-zP)$, $z(S-xP)$, ... When equated to zero, these represent cubic curves, in the plane (x, y, z), with six common points. (Cf. Vol. III, p. 189.)

More geometrically, we may shew that the planes, of the aggregate in Π, are the dual of the points of a cubic surface, by shewing that they meet three fixed planes in corresponding points of three related plane systems (cf. Vol. III, p. 185; and Ex. 15, above). That this is so follows from the figure in the space of four dimensions, because a variable plane of one system meets three fixed planes of another system in such related plane systems (cf. Ex. 16, above); while a tangent solid of the locus Σ, at a point of such a plane, contains this plane.

The dual of the figure which has been considered. We have already referred briefly to the dual of the figure, here considered, which arises when we begin with four planes, instead of four lines (Ex. 5, p. 117, above). There will be fifteen planes, and six systems of lines, each line meeting five associated planes. Instead of the aggregate of the tangent solids of the quartic locus Σ, each containing planes of all the six systems, meeting in its point of contact, there will be a locus of points, which we shall denote by S; this will be expressible by an equation $\xi\eta\zeta=\xi'\eta'\zeta'$, where $\xi+\eta+\zeta=\xi'+\eta'+\zeta'$, there being three points of the locus upon an arbitrary line. The six systems of lines will lie entirely upon this cubic locus S, a line of each system passing through every ordinary point of the locus. The locus S will have ten double points, corresponding to the singular solids, such that any line through one of these points meets the locus only in one further point. All the points of any one of the fifteen planes will be points of the locus. As is indicated in Ex. 20 above, the equation of the locus S can be written in symmetrical form by taking coordinates $X, Y, \ldots$ such that $\xi=Y+Z$, $\eta=Z+X$, $\zeta=X+Y$, $\xi'=-(Y'+Z')$, $\eta'=-(Z'+X')$, $\zeta'=-(X'+Y')$. These lead to

$$X+Y+Z+X'+Y'+Z'=0,$$

and the equation is

$$X^3+Y^3+Z^3+X'^3+Y'^3+Z'^3=0.$$

A plane lying entirely on the locus is then represented by $X+X'=0$, $Y+Y'=0$, which involve $Z+Z'=0$; and there are fifteen such planes, each corresponding to a mode of dividing the six coordinates into three pairs. The double points of the locus are those given by $X^2=Y^2=\ldots=Z'^2$, and are the points of which three of the coordinates $X, Y, \ldots$ are $+1$, and three of them -1. To state the equations of a system of five planes of which no two meet in a line, let the coordinates be denoted in order by $1, 2, \ldots, 6$; let such an equation as $X+Y=0$ be denoted by 12. There are six ways of forming a system of five synthemes, such as 12.34.56, so that these five synthemes contain, in their aggregate, all the fifteen duads such as 12 (cf. Vol. II, p. 221). Such a system of synthemes gives such a system of five planes. When such a system of five planes is obtained, the solids which join an arbitrary point of one such plane, to three others of the planes, meet in a line intersecting four of the planes; this line, therefore, lies entirely on the locus; and it meets the fifth plane.

But, the locus S is obtained in a natural way by considering, in space of three dimensions, all the quadric surfaces which pass through five arbitrary points of general position. If coordinates, in this threefold space, be x, y, z, t; and the five given points, which we denote by A, B, C, D, E, be, respectively, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, then six degenerate quadric surfaces through these five points are $\xi = 0$, $\eta = 0$, ... etc., where

$$\xi = y(t-z),\quad \eta = z(t-x),\quad \zeta = x(t-y),$$
$$\xi' = (t-y)z,\quad \eta' = (t-z)x,\quad \zeta' = (t-x)y.$$

These are subject to $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$, and may be interpreted as coordinates in space of four dimensions. They are also subject to $\xi\eta\zeta = \xi'\eta'\zeta'$, and represent a cubic (∞^3) locus, S, in the fourfold space. Between the general points of this locus, and the points of the threefold space, there is thus set up a correspondence, the reverse equations being

$$t^{-1}x = (\eta' - \zeta)/(\xi - \xi'),\quad t^{-1}y = (\zeta' - \xi)/(\eta - \eta'),\quad t^{-1}z = (\xi' - \eta)/(\zeta - \zeta').$$

In general, any quadric surface through the points A, B, ..., E, with equation of the form $A\xi + B\eta + C\zeta = A'\xi' + B'\eta' + C'\zeta'$, corresponds to the section of the locus S by a solid; this is a cubic surface. Thus, the quartic curve of intersection, of two such quadric surfaces, corresponds to the cubic curve in which the locus S is met by a plane. The three points, beside the five points A, B, ..., E, in which three such quadric surfaces meet, correspond to the three intersections of the locus S with a line, of which two determine the third.

Though the results obtainable can only be the dual of those already obtained for the locus Σ, it is instructive to follow out the correspondence. Of this we give some indications:—There are, through the five fundamental points A, B, C, D, E, of the space of three dimensions, ∞^2 cubic curves, one such curve passing through a point of a plane taken to contain two of the five points. Each of these cubic curves lies on three linearly independent quadric surfaces, which are, then, among those through the five fundamental points; thus, each of these curves corresponds to a line lying on the locus S, of the fourfold space; and, of lines of this system, there is one passing through every general point of S. Again, taking an arbitrary line through one of the points $A, B, \ldots, E$, there are three linearly independent quadric surfaces, containing this line, passing through the other four points; these quadric surfaces correspond to solids in the fourfold space, and they have in common only the arbitrary line first taken. Thus the lines, ∞^2 in aggregate, through one of $A, B, \ldots, E$, correspond to a system of lines lying on the locus S; and, of these, one passes through each general point of S. We thus obtain six systems of lines lying on S. Further, upon any quadric surface through $A, B, \ldots, E$, there are two cubic curves passing through these five points (Vol. III, p. 129); and there are two lines of the quadric surface through any one of these five points. Thus, two of the lines of any system, of the locus S, lie in an arbitrary solid of the fourfold space—just as there are two planes of either system, in the dual figure considered above, which pass through an arbitrary point. But there are particular solids, in the fourfold space, each containing ∞^1 of the lines of S, of each of two systems. For, the degenerate quadric surface, which consists of the plane ABC and a plane through the line DE, contains lines through D and E; and the quadrics of this description are linearly dependent from two of them. They correspond, therefore, to a set of solids, in the fourfold space, having a plane in common. Each of these solids contains ∞^1 lines, of each of two systems, on the locus S, of which any line of one system meets every line of the other; this will be so if the solid meets S in a cubic surface degenerating into a quadric surface and a plane. We thus reach ten planes, lying wholly on the locus S, each corresponding to the aggregate of a plane containing three of the points $A, B, \ldots, E$ taken with the line joining the other two of these five points. Again, a quadric cone with vertex, say, at A, containing the other four points B, C, D, E, contains ∞^1 lines through A, and also contains ∞^1 cubic curves passing through the five points; and any one of these lines has an intersection, not at the fundamental points, with any of these curves; these cones are linearly dependent from two of them. We thus reach five other planes, lying wholly on the locus S in the fourfold space, each corresponding to the aggregate of the joins, of one of the five fundamental points, to the other four. There are then fifteen planes on S. Moreover, these fifteen planes, each taken twice over, consist of sets of five planes all met by the lines of one system on S. For, the condition that a plane, in fourfold space, should be met by a line, is that the two should lie in a solid; such solid corresponds to a quadric surface, in the threefold space, passing through the five fundamental points; and, first, through a cubic curve which contains these points can be drawn a quadric cone with vertex at any one of them; while, next, through any line drawn through the fundamental point A can be passed such a cone, with vertex at A, to contain the other four points, beside four degenerate quadrics, such as that consisting of the plane BCD and the plane defined by AE and this line. Consider, now, the joining line of two of the five fundamental points, in the threefold space. Through this line, and the other three of the five points, there pass four linearly independent quadric surfaces; the corresponding solids in the fourfold space have a point in common. Moreover, three of these quadric surfaces, it is easily seen, have one further point in common, not one of the fundamental points. Thus, the point of the fourfold space, on the locus S, is such that an arbitrary line, through it, meets S in one further point. We thus reach ten double points of the locus S, each corresponding to the join of two of the five fundamental points A, B, ..., E. We can then proceed to prove that each of the fifteen planes lying on S contains four of the double points; that every solid through one of these planes contains an infinite number of lines, of each of two systems of S, has already appeared. In fact, the aggregate of a plane through three of the points A, B, ..., in the threefold space, with the line joining the other two, contains this last line and the joins of the three points; while the joins of one of the points A, B, ..., to the others, are four in number. The corresponding dual property, for the figure studied above, was that a line, say e, lay in four singular solids, $[a, a']$, $[b, b']$, $[c, c']$, $[d, d']$, each point of this line e being the vertex of a quadric point-cone whose planes belonged to two of the six fundamental systems of planes. These singular solids, moreover, could be, respectively, generated, we saw, by axial pencils of planes, of axis e, of the four other systems of planes. In the present figure in fourfold space, for the locus S, each of the four double points, in one of the fifteen planes of S, is the centre of a flat pencil of lines, lying in this plane; these are lines of S of systems different from one another, and different from the two systems remarked as lying in any solid containing the plane. Again, as we likewise see from the dual figure, through each of the ten double points of the locus S, there pass six planes lying on S, from among the fifteen; and each of these planes is the seat of a flat pencil of lines, these belonging to the six systems on S, respectively, the double point being the vertex of all these pencils.

Now, let O be an arbitrary general point of the locus S; we suppose it to correspond to a point $(a, b, c, 1)$ of the threefold space. Take, in the fourfold space, a solid, Π, upon which we project, from O as centre of projection. Every line lying on S gives rise, by intersection of Π with the plane joining O to this line, to a line on Π. We thus have six systems of such lines. Every line through O meets S in two further points; through each of these there passes a single line of any specified one of the six systems of lines of S. Thus, of any one of the six systems of lines in Π, there are two lines which pass through an arbitrary point of Π. In fact, as we know from the dual figure, the lines, of any system, in Π, belong to a quadratic congruence, being common to a linear complex and a quadratic complex. We give now a proof, independent of the preceding (p. 135, above), that all the lines of one of these systems, in Π, which meet a definite line of another system, equally meet a second line of this system; from this it follows, as before, that every two of the linear complexes are conjugate (or apolar). For this we prove, by consideration of the figure in the threefold space, the following theorem:—Let a line, say e, of the locus S, in the fourfold space, be joined to O by a plane, meeting S, further, in a conic, ϵ. From every point of e, and from every point of ϵ, let the unique line of another system, upon S, be drawn; this other system we denote by (a). Then there is a second line, e', of S, of the same system as e, which is met by all the lines (a) drawn through the points of e; and the conic, ϵ', in which the plane Oe' further meets S, is likewise met by all the lines, of the system (a), which can be drawn from the points of the conic ϵ. To see this, let the line e, of S, be that corresponding to a line, l, drawn through the point E, in the threefold space of the points A, B, ..., E. The solids which contain the plane Oe correspond to quadric surfaces in the threefold space, these passing, not only through the five points A, B, ..., E, but also through the point $(a, b, c, 1)$, and through the line l. Any two of these quadric surfaces meet in a cubic curve, which contains the four points A, B, C, D, and the point $(a, b, c, 1)$, and has the line l for chord. This cubic curve, which we denote by θ, corresponds then to the conic ϵ. We may now suppose that the lines (a), of the solid S, which meet the line e, correspond to the lines, in the threefold space, which pass through A and lie in the plane Al (which contains E); and that the lines (a), which meet the conic ϵ, correspond to the generating lines of the quadric cone, of vertex A, which contains the curve θ. What we desire to prove is, then, that there is, upon this cone, another cubic curve, say θ',—likewise containing the points A, B, C, D and the point $(a, b, c, 1)$—such that the chord, l', of this cubic curve θ', which can be drawn through E, lies in the plane Al. For this, consider the quadric cones, with vertex at the point $(a, b, c, 1)$, which contain the points A, B, C, D. Any one of these meets the first cone in a cubic curve containing the five desired points. If the line l meet the curve θ in P and Q, a single cone, of vertex $(a, b, c, 1)$, can be drawn to contain A, B, C, D and any specified point, P', of AP; the intersection, Q', of this cone, with AQ is then definite. Thus the ranges, (P'), (Q'), upon the lines AP, AQ, are related; and the lines $P'Q'$ touch a conic in the plane APQ (or Al), of which EPQ is one tangent line; for the curve θ lies upon one of the cones of vertex $(a, b, c, 1)$. Thus, we can draw through E, in the plane Al, a single line, EP_1Q_1, which is a common chord, first, of the cone which projects the curve θ from A (of which AP, AQ are two generators), and, second, of another cone, vertex $(a, b, c, 1)$, passing through A, B, C, D. Thereby we find the line l', and the curve θ', desired.

The two further points in which the locus S is met by an arbitrary line through O evidently correspond to the two remaining intersections of three quadric surfaces drawn, in the threefold space, through the six points consisting of A, B, ..., E and the point $(a, b, c, 1)$. If these two remaining intersections, of which one determines the other, coincide, it is easy to see that their point of coincidence is the vertex of a quadric cone containing the first six points. The locus of the vertex of such a quadric cone, containing six given points, may be proved to be a quartic surface (see the Examples below). This we shall call a *Weddle surface*. This surface then corresponds to the points of the locus S which, when joined to O, are coincident intersections of S with lines from O. Writing $\alpha = bc'$, $\beta = ca'$, $\gamma = ab'$, $\alpha' = b'c$, $\beta' = c'a$, $\gamma' = a'b$, where $a' = 1 - a$, $b' = 1 - b$, $c' = 1 - c$, we easily see, by substitution of $\xi + \lambda\alpha$, $\eta + \lambda\beta$, etc. for the coordinates in the equation of S, that the points of S concerned lie on the quadric represented by

$$\alpha\eta\zeta + \beta\zeta\xi + \gamma\xi\eta = \alpha'\eta'\zeta' + \beta'\zeta'\xi' + \gamma'\xi'\eta'.$$

This, which we may call the polar quadric of the point O, or $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$, in regard to S, has the same tangent solid at O as the locus S; and the sextic surface in which it meets S has a double point at O. By projection from O, it gives rise to a quartic surface in the solid Π. This quartic surface, which we may denote here by K, can at once be seen, from the properties of the locus S, to be Kummer's quartic surface, with sixteen double points and sixteen tangent planes each touching it at the points of a conic. The plane which joins O to any line of S contains also a conic lying on S, which meets this line in two points; these are clearly upon the intersection of S with the polar quadric of O. Wherefore, every line of S projects, from O, into a line, in the solid Π, which is a double tangent of the surface K. Further, at every general point of the surface K, there are six tangent lines of K which touch this surface again, these being projections of lines of S of the six systems. The polar quadric of O, by its definition, passes through the ten double points of S; and contains, also, the six lines of S, of the various systems, which pass through O. This shews the origin of the sixteen double points of the surface K. Again, the polar quadric of O meets each of the fifteen planes, lying upon S, in a conic; thus, by projection from O, are obtained fifteen conics lying on the surface K, along each of which the surface is touched by a plane. It may be proved that the tangent solid of S, at the point O, gives rise to another such singular tangent plane of K. Further, we saw that six of the planes lying on S pass through every double point of S; thus, for ten of the double points of K, it is true that six singular tangent planes pass through a double point. This is also true for the other six double points of K. For, any one of the six lines of S, passing through O, meets, we have seen, five of the planes (of an associated system) lying on S; and this line meets the tangent solid of S at O. Again, we saw that any one of the planes of S contains four of the double points; and that any solid through this plane also holds ∞^1 lines, of each of two systems, lying on S. Defining such a solid as that which contains O, we see that the fifteen planes of S project, from O, into planes in Π, each containing six double points of K, two of these arising from lines of S which pass through O. The remaining singular tangent plane of K, lying in the tangent solid of S at O, contains double points on the six lines of S which pass through O.

In general, as has been said, there pass, through a point of the solid Π, two lines arising as projections of lines of S of a particular system. But there are, in fact, sixteen points of Π through each of which there pass ∞^1 lines of this system, forming a flat pencil. These are the singular points of the surface K, as is clear from what has been said; and the same points equally arise, with the same characteristic property, for each of the six systems of lines. There are, similarly, sixteen planes, in the solid Π, each of which, instead of containing two lines of a particular system, contains a pencil of such lines; and each of these planes contains, also, a pencil of lines of every system. These six points and planes arise, in fact, from the systems of lines of the space Π. From this point of view they are reconsidered below, in Chap. VII.

We see that the whole theory is analogous to that of the projection, from a point of a cubic surface, of the sextic curve, of this surface, lying on the polar quadric surface of this point, taken in regard to the cubic surface (Vol. III, p. 200). Indeed, the present theory includes the former, if we take a section of the present figure by an arbitrary solid passing through O (cf. Vol. III, p. 205). Thus, also, it appears that the plane quartic curve obtained, in the former theory, by projection of the 'apparent contour' of a cubic surface, which was stated to be a general plane quartic curve, may be obtained as a plane section of the Kummer surface

K; a conclusion which was enunciated by Kummer. But, further, it also appears, comparing the former theory with the present, that six intersections of suitable pairs of bitangents of a plane quartic curve, which lie on a conic (Vol. III, p. 201), may be regarded as the polar points, of the plane of the quartic curve, in regard to six linear complexes of lines which are mutually conjugate (p. 43, above).

Ex. 1. The fifteen planes of the locus S, whose equation is $\xi\eta\zeta = \xi'\eta'\zeta'$, subject to $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$, are the nine expressed by such a pair of equations as $\xi = 0$, $\xi' = 0$; and the six expressed by such a pair of equations as $\xi - \xi' = 0$, $\eta - \eta' = 0$. The ten double points are $(1, 1, 1, 1, 1, 1)$, with the nine of coordinates $(\xi, \eta, \zeta, \xi', \eta', \zeta')$ in which two of ξ, η, ζ are zero, and two of ξ', η', ζ' are zero. If $x' = 1 - x$, etc., the six lines, of the various systems, lying on S, which pass through the point $(yz', zx', xy', y'z, z'x, x'y)$, are those which join this point, respectively, to the points $(z'/z, x'/x, y'/y;\ y'/y, z'/z, x'/x)$, $(y, z, x;\ z, x, y)$, $(z', x', y';\ y', z', x')$, $(yz', 0, y';\ y'z, z', 0)$, $(z', zx', 0;\ 0, z'x, x')$, $(0, x', xy';\ y', 0, x'y)$.

Ex. 2. The equation of the Weddle surface, the locus of the vertex of a quadric cone, in space of three dimensions, which contains six given points, A, B, C, D, E, F, say $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, $(a, b, c, 1)$, is obtainable from

$$\alpha\eta\zeta + \beta\zeta\xi + \gamma\xi\eta = \alpha'\eta'\zeta' + \beta'\zeta'\xi' + \gamma'\xi'\eta'$$

by putting $\alpha = b(1 - c)$, $\alpha' = (1 - b)c$, $\xi = y(t - z)$, $\xi' = (t - y)z$, etc. It is capable of many forms. For instance,

$$(t - x)(y - z)(bcxt + ayz) + \ldots = 0.$$

Or, if we put

$$X/a = \alpha\xi' - \alpha'\xi,\quad Y/b = \beta\eta' - \beta'\eta,\quad Z/c = \gamma\zeta' - \gamma'\zeta,$$
$$-T/abc = a(\xi - \xi') + b(\eta - \eta') + c(\zeta - \zeta'),$$

it is capable of the form $P^{\frac{1}{2}} + Q^{\frac{1}{2}} + R^{\frac{1}{2}} = 0$, where

$$P = a(b - c)X\,[-a'T/a + (c - a)Z - (a - b)Y], \text{ etc.}$$

The surface has a double point at each of the six fundamental points; and contains the cubic curve which can be drawn through these. Any chord of this curve meets the surface in two points which are harmonic conjugates in regard to the end points of the chord. The surface contains the fifteen joining lines of the six fundamental points, and also contains ten other lines, each the intersection of a plane containing three of these points with the plane containing the other three. The coordinates of a point of the cubic curve, through the six points, may be written, in terms of a parameter, θ, as $a(\theta - a)^{-1}$, $b(\theta - b)^{-1}$, $c(\theta - c)^{-1}$, $(\theta - 1)^{-1}$;

thus the respective values $\theta = 1, 0, \infty, a, b, c$ give the points D, E, F, A, B, C. See, also, p. 142, above.

The surface is expressible by those multiply-periodic functions of two variables which arise from the irrationality involved in the square root of a sextic polynomial in one variable (see the writer's *Multiply-Periodic Functions*, Cambridge, 1907).

Ex. 3. In the space of four dimensions, four independent solids passing through the point O, or $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$, are $X = 0$, $Y = 0$, $Z = 0$, $T = 0$, where X, Y, Z, T have the values of Ex. 2. These may be used as coordinates in the solid, Π, upon which we project from O. Putting $p = a'/a$, $q = b'/b$, $r = c'/c$, and considering a line, of one system, of the locus S, given, for proper values of the parameters x, y, z, by the equations $\xi' = yz^{-1}\xi$, $\eta' = zx^{-1}\eta$, $\zeta' = xy^{-1}\zeta$, prove that this projects from O into a line, of the solid Π, whose line coordinates, relatively to X, Y, Z, T, are

$$(b-c)\,U'VW,\quad (c-a)\,V'WU,\quad (a-b)\,W'UV,$$
$$-UV'W',\quad -VW'U',\quad -WU'V',$$

where $U = y - z$, $U' = yr - zq$, etc. The lines of the system, obtained by varying x, y, z, thus belong to the tetrahedral complex $all' + bmm' + cnn' = 0$, and also to the linear complex

$$a'l + b'm + c'n - a(b-c)\,l' - b(c-a)\,m' - c(a-b)\,n' = 0.$$

Ex. 4. Shew that, with proper constant values of A, B, C, the relation connecting u, v, w, p, in order that the quadric surface, in threefold space, expressed by $AuX + BvY + CwZ + pT = 0$, where X, Y, Z, T are the same functions of the coordinates of the space as in Ex. 2, should be a cone, is identical with the equation, in these coordinates, of the Weddle surface, given in Ex. 2.

Ex. 5. The equation $(XX')^{\frac{1}{2}} + (YY')^{\frac{1}{2}} + (ZZ')^{\frac{1}{2}} = 0$, where

$$X + Y + Z + X' + Y' + Z' = 0,$$

is identically satisfied by writing, for X, Y, Z, X', Y', Z', respectively,

$$aa'(b-c),\quad bb'(c-a),\quad cc'(a-b),\quad a'bc(b-c),\quad b'ca(c-a),\quad c'ab(a-b),$$

where $a' = 1 - a$, $b' = 1 - b$, $c' = 1 - c$. The reverse equations are

$$2aYZ = XX' - YY' - ZZ',\quad 2bZX = YY' - ZZ' - XX',$$
$$2cXY = ZZ' - XX' - YY'.$$

The equation of the tangent solid of the locus, in space of four dimensions, represented by the equation, at the point (a, b, c), is then

$$bcX + caY + abZ + aX' + bY' + cZ' = 0.$$

If the values of the parameters a, b, c, at any point common to the locus and this tangent solid, be denoted by $t^{-1}x$, $t^{-1}y$, $t^{-1}z$, these are found to satisfy the equation

$$(t-x)(y-z)(bcxt+ayz)+\text{etc.}=0,$$

identical in form with the equation of the Weddle surface mentioned in Ex. 2.

The locus is identical with that before called Σ, which appeared with the equation $[(\eta_1-\zeta_1')(\eta_1'-\zeta_1)]^{\frac{1}{2}}+\text{etc.}=0$. (Above, p. 126.) This was expressed by parameters, x_1, y_1, z_1, in the forms $\xi_1=(y_1z_1')^{-1}$, etc. Shew that there is complete agreement if $(1-x_1)(1-a)=1$, $(1-y_1)(1-b)=1$, $(1-z_1)(1-c)=1$.

Ex. 6. In space of three dimensions, a general quartic surface with *fifteen* double points is represented by the first two equations of Ex. 5, taken with a single further relation

$$l'X+m'Y+n'Z+lX'+mY'+nZ'=0.$$

By the birational transformation of Ex. 5, this is changed into the quartic surface with *five* double points, which is expressed by the equation

$$(t-x)(y-z)(lyz+l'xt)+\text{etc.}=0.$$

In general, this surface contains the ten joins of the five fundamental points, and ten other lines, each in a plane containing three fundamental points. It becomes the Weddle surface if $ll'=mm'=nn$.

Ex. 7. The locus S, in space of four dimensions, given by $\xi\eta\zeta=\xi'\eta'\zeta'$, $\xi+\eta+\zeta=\xi'+\eta'+\zeta'$, may be obtained by projection of the (∞^3) locus, in space of five dimensions,—the intersection of two quadric line-cones,—which is represented by the equations

$$XX'/aa'=YY'/bb'=ZZ'/cc'.$$

For, putting, herein, $X=a(1+\lambda x)$, $X'=a'(1+\lambda x')$, etc., the elimination of λ leads to the equation

$$(y-z')(z-x')(x-y')+(y'-z)(z'-x)(x'-y)=0.$$

This may be described, briefly, as the condition that the pairs x, x'; y, y'; z, z' should be in involution. A tetrahedral complex of lines, in space of three dimensions, can be represented, in space of five dimensions, by the intersection of two quadric line-cones. (Cf. Segre, *Atti...Torino*, XXII, 1886–7, p. 556. See, also, Chap. VII, below.)

Ex. 8. The Weddle and Kummer surfaces may be obtained, as appears from what precedes, without space of four dimensions, by considering the four quadric surfaces passing through six points in space of three dimensions. A more general theory is that which arises by considering any four quadrics whatever. See the beautiful account given by Reye in three chapters of the *Geometrie der Lage* (for other references see also *Proc. Lond. Math. Soc.* XXI, 1923, p. 121).

CHAPTER VI

A QUARTIC SURFACE IN SPACE OF FOUR DIMENSIONS. THE CYCLIDE

A quartic surface in space of four dimensions. We have shewn in detail, in Chapter III, how the theory of a plane curve of the fourth order, which has two double points, may be deduced from that of a curve, lying in space of three dimensions, defined as the intersection of two quadric surfaces. In a similar way, we may take two quadrics in space of four dimensions, each represented by the vanishing of a quadratic function of five homogeneous variables, and consider the locus of their common points; this will be an aggregate of ∞^2 points, say a *surface*, having four points on any general plane of the fourfold space. If, in analogy with the case previously discussed, this surface be projected, from an arbitrary point of one of the quadrics, on to any threefold space, there arises a quartic surface in this space. This surface passes through a certain conic of the threefold space, which depends only on the centre of projection and the quadric upon which this is taken; this conic is a double curve for the surface obtained. As in the other case, if this definite conic be regarded as the Absolute conic of the threefold space, it is easily proved, from the four dimensional figure, that the quartic surface, in the threefold space, can be generated as the envelope of *spheres*, each touching it in two points, the generation being possible in five different modes. The spheres of one system have their centres on a quadric surface, and cut a fixed sphere at right angles. Further, the five quadric surfaces, which are the loci of the centres of the spheres of the different systems, are confocal, all touching the common tangent planes of any two of them. These common tangent planes all touch the quartic surface, at points of its double conic; this conic, being the Absolute conic, lies on all the spheres. If the projection be made from a point which is common to two quadrics of the fourfold space, the locus obtained, in the space of three dimensions, is a *cubic* surface, on which the Absolute conic is a simple curve. The theory of *inversion*, which we have given previously (above, pp. 12, 96), is likewise applicable here. As inversion in a plane, in regard to a circle, was reduced to projection in space of three dimensions, so inversion in threefold space, in regard to a sphere, is reducible to projection in space of four dimensions. The fact that we can obtain, from the same quartic surface in fourfold space, either a quartic or a cubic surface in threefold space, becomes the fact that a quartic surface, in threefold space, possessing a double conic, can be inverted into a cubic surface passing through this conic. In particular, from the theory of the lines of a cubic surface, we can thus infer that such a quartic surface contains sixteen lines; the lines of the cubic surface which consist of the line lying in the plane of the Absolute conic, and the ten others meeting this, disappear in the inversion.

The theory from this point of view is very analogous to that we have given for the case of two and three dimensions; but the surface of the fourth order in space of four dimensions is in fact simpler than the curve which is the intersection of two quadric surfaces in threefold space. The latter is a curve of which the points are expressible, in single-valued form, only by elliptic functions. But the surface in fourfold space is a rational surface, whose points are in (1, 1) birational correspondence with the points of a plane. It is thus proper to develop the theory of this surface for itself. The theory of a quartic surface, in threefold space, with a double conic, and the theory of a cubic surface, arise then as consequences. This seems, in fact, the simplest approach to the theory of the cubic surface.

The surface in question is capable of an extremely simple definition. Let a, b, c be any three non-intersecting lines of the space of four dimensions, and T, U be two points, all of general position. From one of these points, say T, there can be drawn ∞^1 planes to meet the lines a, b, c; from U can also be drawn ∞^1 planes to meet a, b, c. The surface we consider is the locus of the point common to one of the planes through T and one of the planes through U.

Preliminary algebraic consideration of the surface. We first shew, directly from this definition, with help of the symbols, that the surface is the intersection of two quadric (∞^3) loci, obtain the parametric expression, and prove that the surface contains sixteen lines. This involves some repetition of work that has preceded; moreover the surface appears later (Chap. VII), as the representation, in space of five dimensions, of a quadratic congruence of lines. But perhaps this is not a disadvantage.

Given the lines a, b, c, as stated, let the equations of the solids $[b, c]$, $[c, a]$, $[a, b]$ be written, respectively, $p = 0$, $q = 0$, $r = 0$; and, denoting the line TU by e, let the equations of the solids $[a, e]$, $[b, e]$, $[c, e]$ be, respectively, $P = 0$, $Q = 0$, $R = 0$. We can suppose that, identically, $p + q + r + P + Q + R = 0$. The lines a, b, c, e are then given, respectively, by

$$q = r = P = 0,\quad r = p = Q = 0,\quad p = q = R = 0,\quad P = Q = R = 0.$$

Thus a plane through the point U, to meet the lines a, b, c, may be supposed to meet these, respectively, in points for which p, q, r, P, Q, R are in the ratios of

$$(\alpha, 0, 0;\ 0, -y, z),\quad (0, \beta, 0;\ x, 0, -z),\quad (0, 0, \gamma', -x', y', 0);$$

and here, since the plane contains U, for which $P = Q = R = 0$, we can suppose $x' = x$, $y' = y$, and put $\gamma' = \gamma$. Then $\alpha = y - z$, $\beta = z - x$, $\gamma = x - y$, so that $\alpha + \beta + \gamma = 0$. Hence we find, for the equations of the plane, any two of the three

$$-P + x(\beta^{-1}q - \gamma^{-1}r) = 0,\quad -Q + y(\gamma^{-1}r - \alpha^{-1}p) = 0,$$
$$-R + z(\alpha^{-1}p - \beta^{-1}q) = 0,$$

the plane meeting the line e, in the point U, where p, q, r, P, Q, R are in the ratios of $(\alpha, \beta, \gamma, 0, 0, 0)$. Instead of P, Q, R we may use coordinates, p', q', r', such that

$$p' + P + q + r = q' + Q + r + p = r' + R + p + q = 0,$$

and, therefore, $p + q + r + p' + q' + r' = 0$. The equations of the plane then become two of the three

$$p' + z\beta^{-1}q - y\gamma^{-1}r = 0,\quad q' + x\gamma^{-1}r - z\alpha^{-1}p = 0,$$
$$r' + y\alpha^{-1}p - x\beta^{-1}q = 0,$$

shewing, incidentally, that the plane also meets another line, say d, given by $p' = q' = r' = 0$, $p/x\alpha = q/y\beta = r/z\gamma$. The equations of the plane shew that it lies on the quadric point-cone given by

$$\alpha^{-1}pp' + \beta^{-1}qq' + \gamma^{-1}rr' = 0,$$

whose vertex, U, has, for coordinates p, q, r, p', q', r', the ratios of $\alpha, \beta, \gamma, \alpha, \beta, \gamma$, respectively.

Let the point T be that for which the coordinates are $\alpha', \beta', \gamma', \alpha', \beta', \gamma'$; a plane through this, which meets the lines a, b, c, will be given by equations arising from the former by putting x', y', z', respectively, for x, y, z, where $\alpha' = y' - z'$, $\beta' = z' - x'$, $\gamma' = x' - y'$.

Now, let μ denote any one of $\beta\gamma' - \beta'\gamma$, $\gamma\alpha' - \gamma'\alpha$, $\alpha\beta' - \alpha'\beta$, which are at once seen to be equal to one another; put

$$t = \alpha'\beta\gamma(p + p') + \beta'\gamma\alpha(q + q') + \gamma'\alpha\beta(r + r'),$$
$$u = \alpha\beta'\gamma'(p + p') + \beta\gamma'\alpha'(q + q') + \gamma\alpha'\beta'(r + r'),$$

which give rise to

$$\mu^2(p + p') + \alpha't + \alpha u = 0,\quad \mu^2(q + q') + \beta't + \beta u = 0,$$
$$\mu^2(r + r') + \gamma't + \gamma u = 0.$$

The equations $t = 0$, $u = 0$ represent solids passing, respectively, through the points U, T; in terms of these the equation of the quadric point-cone found above, with vertex at U, becomes

$$\alpha^{-1}(p - p')^2 + \beta^{-1}(q - q')^2 + \gamma^{-1}(r - r')^2 + t^2/\mu^2\alpha\beta\gamma = 0.$$

There is a corresponding point-cone, with vertex at T, whose equation is found from this by putting $\alpha', \beta', \gamma', u$ in place of α, β, γ, t, respectively. The quartic surface under consideration is the intersection of these two point-cones; its equations are then given by

$$pp'/\alpha\alpha' = qq'/\beta\beta' = rr'/\gamma\gamma',$$

and these, we see, are equivalent to

$$[\mu^4(p-p')^2 - (\alpha' t + \alpha u)^2]/\alpha\alpha' = [\mu^4(q-q')^2 - (\beta' t + \beta u)^2]/\beta\beta'$$
$$= [\mu^4(r-r')^2 - (\gamma' t + \gamma u)^2]/\gamma\gamma'.$$

These are equivalent, also, to what are obtained by changing the sign of u throughout.

Thus the surface contains the four lines a, b, c, d, given, respectively, by $q=r=p'=0$, $r=p=q'=0$, $p=q=r'=0$, and $p'=q'=r'=0$. Likewise it contains the four lines, a', b', c', d', given, respectively, by $q'=r'=p=0$, $r'=p'=q=0$, $p'=q'=r=0$, and $p=q=r=0$; we easily see that these are the transversals of triads of a, b, c, d. But the equations shew that the surface contains the sixteen lines given by

$$\mu^2(p-p')\,\theta = \alpha' t + \epsilon\alpha u, \quad \mu^2(q-q')\,\phi = \beta' t + \epsilon\beta u,$$
$$\mu^2(r-r')\,\psi = \gamma' t + \epsilon\gamma u,$$

wherein each of $\theta, \phi, \psi, \epsilon$ is ± 1; the value $\epsilon = 1$ gives the eight lines already named, in virtue of the identities such as

$$\mu^2(p+p') + \alpha' t + \alpha u = 0.$$

The equations also shew that the pentad, of five points, given by the intersections, in fours, of the five solids whose equations are $p-p'=0$, $q-q'=0$, $r-r'=0$, $t=0$, $u=0$, is self-polar in regard to all quadrics containing the quartic surface; and that all the sixteen lines may be obtained, from any one of them, by combination of the five harmonic inversions, in which one of the solids, and the opposite vertex of the *simplex*, are taken as fundamental elements. The solids, $p-p'=0$, $q-q'=0$, $r-r'=0$, and the opposite vertices, are independent of the positions of T and U upon the line e. The four lines consisting of e and the joins of any point, O, of the line e, to these three vertices of the simplex, are conjugate, in pairs, in regard to the quadric point-cone formed by the planes drawn from O to meet the lines, a, b, c.

We have said that the quartic surface is representable rationally upon a plane. The parameters for this expression are those which determine planes lying, respectively, upon two quadric point-cones containing the surface. Taking the ratios of ξ, η, ζ, such that $\xi/p\alpha^{-1} = \eta/q\beta^{-1} = \zeta/r\gamma^{-1}$, the equations

$$pp'/\alpha\alpha' = qq'/\beta\beta' = rr'/\gamma\gamma'$$

give

$$p/\alpha\xi = \ldots = \alpha'\eta\zeta + \beta'\zeta\xi + \gamma'\xi\eta,$$

$$p'/\alpha'\eta\zeta = q'/\beta'\zeta\xi = r'/\gamma'\xi\eta = -(\alpha\xi + \beta\eta + \gamma\zeta);$$

thereby the coordinates, $p, q, \ldots, r'$, of a point of the surface, are expressed by the coordinates, ξ, η, ζ, of a point of a plane; the six curves $p = 0, \ldots, r' = 0$, in this plane, are cubics through the five points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, $(\alpha^{-1}\alpha', \beta^{-1}\beta', \gamma^{-1}\gamma')$. The last two points are the intersections of $\alpha\xi + \beta\eta + \gamma\zeta = 0$, $\alpha'\eta\zeta + \beta'\zeta\xi + \gamma'\xi\eta$. These cubics are subject to $p + \ldots + r' = 0$. Conversely, the theory of the quartic surface may be initiated by considering the cubic curves, of which five are linearly independent, passing through five arbitrary points of a plane. (Cf. Vol. III, p. 189.)

Ex. 1. Shew that an arbitrary plane, drawn through any line of the quartic surface, meets the surface again in one point. As such a plane meets a fixed plane, of general position, in a point, it establishes a representation of the quartic surface upon a plane.

There are five lines of the quartic surface which meet the line through which the planes are drawn. These are represented by five fundamental points of the plane. The other ten lines of the surface become the joins of these five fundamental points. Obtain, on the quartic surface, the representation of a general line of the plane; and, also, of a general cubic curve passing through the five fundamental points of the plane.

Ex. 2. Any two general quadric point-cones, in fourfold space, can be expressed by the equations

$$x^2 + y^2 + z^2 + t^2 = 0, \quad a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 + u^2 = 0.$$

Denoting $(b - c)^{\frac{1}{2}}$, $(c - a)^{\frac{1}{2}}$, $(a - b)^{\frac{1}{2}}$, respectively, by λ, μ, ν, verify that, if we take

$$p = \lambda\mu(\lambda x + \mu y + \nu z), \quad q = c(\mu x - \lambda y + \nu t), \quad r = -b\mu x + a\lambda y - \nu(abc)^{\frac{1}{2}}u,$$

$$p' = \lambda\mu(\lambda x + \mu y - \nu z), \quad q' = c(\mu x - \lambda y - \nu t), \quad r' = -b\mu x + a\lambda y + \nu(abc)^{\frac{1}{2}}u,$$

together with $\alpha = c - a$, $\beta = -c$, $\gamma = a$, $\alpha' = c - b$, $\beta' = -c$, $\gamma' = b$, so that $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 0$, then $p + q + r + p' + q' + r' = 0$, and the two point-cones are expressed, respectively, by

$$pp'/\alpha\alpha' = qq'/\beta\beta' \quad \text{and} \quad pp'/\alpha\alpha' = rr'/\gamma\gamma'.$$

Ex. 3. Given, in space of n dimensions, a simplex consisting of $(n + 1)$ points, and the $(n + 1)$ *primes*, or spaces of $(n - 1)$ dimensions, which contain the sets of n of these points, consider the $(n + 1)$ harmonic inversions, in each of which one of these points and the complementary prime are the fundamental elements. Shew that an arbitrary line, subjected to all possible combinations of these inversions, gives rise to a set of 2^n lines. (Cf. Vol. III, p. 80.)

Ex. 4. We may verify directly from the equations that the quartic surface, in the fourfold space, contains no other line than the sixteen we have found. Any line can be represented by equations of the forms $\rho_1(p-p')=l_1t+m_1u$, $\rho_2(q-q')=l_2t+m_2u$, $\rho_3(r-r')=l_3t+m_3u$; the substitution of the values of $p-p'$, $q-q'$, $r-r'$ found from these, in the equations of the surface, determines all the existing lines.

Ex. 5. Prove that the two lines obtained, respectively, from the equations

$$\theta x=\alpha' t+\epsilon\alpha u,\quad \phi y=\beta' t+\epsilon\beta u,\quad \psi z=\gamma' t+\epsilon\gamma u,$$

by taking $\epsilon=\pm 1$, lie in the plane expressed by

$$(\theta x-\alpha' t)/\alpha=(\phi y-\beta' t)/\beta=(\psi z-\gamma' t)/\gamma;$$

and that this plane meets, in a line, any one of the four planes whose equations are obtained from these by changing the sign either of one, or of all, of θ, ϕ, ψ,—where $\theta^2=\phi^2=\psi^2=1$.

Resumption of the descriptive theory of the quartic surface. The sixteen lines. As before, let a, b, c, e be four lines, of which no two intersect and no three lie in a solid; and let T, U be any two points of e. We consider the surface which is the locus of the intersection of a plane drawn from T to meet a, b, c, with a plane drawn from U to meet these lines. All such planes, we have shewn, will meet another line, d, associated with a, b, c, e. Let a', b', c', d' be the transversals of the threes of the lines a, b, c, d, the line d' meeting a, b, c, and so on. Through any point of a plane, which meets a, b, c, d, there passes also a plane which meets a', b', c', d', and meets e in the same point as does the former plane (p. 123, above). Thus the quartic surface may also be defined by planes through T meeting a, b, c, taken with planes through U meeting a', b', c'; or *vice versa*; or by planes through T and U both meeting a', b', c'.

The planes from T meeting a, b, c intersect an arbitrary solid in generating lines, of one system, of a quadric surface lying in this solid (p. 122, above); another quadric surface is obtained in this solid by planes from U meeting a, b, c. The quartic surface under discussion thus meets an arbitrary solid in a quartic curve, the intersection of two quadric surfaces; and, therefore, meets an arbitrary plane in four points. We thus speak of the surface as being a quartic, or of order four; the order of a locus of r dimensions, lying in a space of n dimensions, being the number of points of this locus which lie in a (planar) manifold of $n-r$ dimensions, of general position, when this number is independent of this manifold. A plane through T (or U), which meets a, b, c, contains, however, not four, but an infinite number of points of the quartic surface, these lying on a conic; for the planes through U (or T)

which meet a, b, c determine a quadric surface in any solid containing this plane.

Every one of the eight lines a, b, c, d, a', b', c', d' lies on the quartic surface, such a line being a common generator of the two quadric surfaces determined by T and U, as above described, in any solid drawn through this line. Any other line lying on the surface must likewise be a common generator of two such quadric surfaces; this line must, therefore, be common to a plane through T meeting a, b, c, and to a plane through U which either meets a, b, c, or meets a', b', c'; or the statement must hold with interchange of T and U. Now, a plane through T, meeting a, b, c (and, therefore, d), cannot have a line in common with a plane through U meeting a, b, c, unless this line be one of a', b', c', d'. For the two planes, having a line in common, determine a solid; and a line which meets both the planes lies in this solid, unless it is a line, meeting the line of intersection of the planes, not lying in either plane. Thus the line e lies in this solid; and, as two of the lines a, b, c, d do not lie in the same solid with e, three of these lines must meet the line of intersection of the two planes. This line is, therefore, one of a', b', c', d'. Similarly, a line lying on the surface can be one of a, b, c, d. There is next the possibility of a line, lying on the surface, which is the intersection of a plane, α, through T, meeting a, b, c, d, with a plane, β, through U, meeting a', b', c', d'. For this case, consider a solid through the former plane, α, and the quadric surface, lying therein, which is determined by the planes through U of the two systems, those meeting a, b, c, d, and those meeting a', b', c', d'. As this plane α contains a generator of this quadric surface, lying on a plane through U meeting a', b', c', d', the plane α also contains a generator lying on a plane through U meeting a, b, c, d. This second generator must then, as we have seen, be one of a', b', c', d'. By a similar argument the plane β must contain one of a, b, c, d. That is, the line in question is the intersection of a plane through T containing one of a', b', c', d' (which, therefore, meets a, b, c, d), with a plane through U containing one of a, b, c, d (which, therefore, meets a', b', c', d'). The solid determined by these two planes, as it contains T and U, contains the line e. The only solids possible, containing e and also one of a', b', c', d' and one of a, b, c, d, are the four containing, respectively, the lines (a, a', e), (b, b', e), (c, c', e) and (d, d', e). Conversely if, for example, the plane Td' meet the line d in H, and the plane Ud meet the line d' in K', it is at once seen that HK' is a line of the quartic surface; and, if TH meet d' in H', and UK' meet d in K, that $H'K$ is another such. The four lines d, d', HK', $H'K$ are the degenerate quartic curve in which the solid (d, d', e) meets the quartic surface.

Three other pairs of lines are similarly obtained by drawing the transversals, from T and U, to a, a', to b, b', and to c, c'. The only remaining possibility, for a line of the quartic surface, of being the intersection of a plane through T meeting a', b', c', d' with a plane through U meeting a, b, c, d, leads to the same four pairs of lines.

Thus, beside the original eight lines, $a, b, \ldots, d'$, there are eight others, two in each of the solids $(a, a'), \ldots, (d, d')$; and these are all. Another geometrical derivation of the lines arises below (p. 170). Now denote, temporarily, the two lines so obtained which meet d, d' by t and t'; and, similarly, those meeting (a, a'), (b, b'), (c, c'), respectively, by (x, x'), (y, y'), (z, z'). As the section of the quartic surface by any solid is the curve of intersection of two quadric surfaces, there cannot be three lines of the surface lying in a plane, or meeting in a point. Thus, for instance, the line x, meeting a' (which meets d), cannot meet d; and the same line x, meeting a, cannot meet d'. The intersection of x with the solid (d, d') must then be on t or t'. The six points in which x, x', y, y', z, z' meet the solid (d, d') must, in the same way, be all on t or t'. It will appear that there are three on each. Thus we see that every one of the sixteen lines is met by five others.

The self-polar pentad for the quartic surface. The quartic surface has appeared as the intersection of a quadric point-cone of vertex T, with a quadric point-cone of vertex U, each generated by planes through its vertex meeting either a, b, c, d or a', b', c', d'. By an appeal to the equations of these cones, it thus appears that the surface lies on ∞^1 quadrics, each a locus of ∞^3 points, of which the equation of any one is of the form $(T) + \lambda(U) = 0$. As in the analogous cases in a plane, or in three dimensions, there are then, in the general case considered, three other quadric point-cones containing the quartic surface; and the vertices of the five cones form a self-polar pentad for all the quadrics containing the quartic surface. It is easy to specify the vertices of these other cones. Using a preceding notation (p. 113, above), let λ, μ, ν denote, respectively, the planes of the pairs of lines (l, p), (m, q), (n, r), of which, for example, the first contains the points A, A', P, P', L; and let X, Y, Z be, respectively, the points of intersection (μ, ν), (ν, λ), (λ, μ). These are the points of which, (cf. Ex. 4, p. 117), the symbols are $F - F'$, $G - G'$, $H - H'$, or $A' + B + C$, $B' + C + A$, $C' + A + B$ (p. 117, above). It can be shewn that the self-polar pentad is formed by these points, X, Y, Z, together with T and U. It will be noticed that X, Y, Z do not depend on the positions of T and U upon the line e. For, it was seen that the points Y, Z are in the solid $[\lambda, e]$; and that, in regard to X and this solid, the two points Q, B, that is the points (b, d') and (c', a), are harmonic conjugates; as are also the points of each of the pairs (R, C'), (R', C), (Q', B') (p. 117, above). The four pairs of points all lie on the quartic surface, by what we have proved; and a solid is determined by four points. Wherefore the solid $[\lambda, e]$, which contains the four points Y, Z, T, U, is the polar solid of the point X in regard to every quadric containing the quartic surface. Similarly for Y and Z. Again, the solid containing X, Y, Z, U is the polar of T in regard to all these quadrics. For, this follows if we shew that, for the quadric point-cone constituted by the planes drawn to meet a, b, c, d, or a', b', c', d', from any point, O, of the line e, the polar solid of the line e is that containing X, Y, Z and O; and, for this, it is sufficient to shew that the plane XYZ contains three points, each of which is a harmonic conjugate of a point of the line e, in regard to a pair of points lying on the quartic surface. In fact, the point which is the harmonic conjugate of the point L, in regard to the points A, A', both of which lie on the quartic surface, lies on the plane XYZ; this point lies, indeed, on the line YZ (p. 117, above); and there is a similar statement for M and N. This discussion shews also that U is the pole of the solid containing X, Y, Z, T. The property of X, Y, Z, T, U is thus established.

It follows hence, from the harmonic property of the polar solid of a point in regard to a quadric, that to any line of the quartic surface there corresponds another, the harmonic inverse of the former in regard to any vertex of the polar pentad and its opposite solid. These two lines intersect on this solid, so that any one of the lines meets five others. All the sixteen lines are obtainable from any one of them by combination of the five possible inversions. As a line in space of four dimensions depends on six parameters, the number of parameters necessary to specify one line and the five points of the polar pentad is $6 + 5 . 4$, or twenty-six; this is the same number as that, $3 . 6 + 2 . 4$, required to specify the three lines a, b, c and the two points T, U.

Again, in any one of the solids of the polar pentad, say that opposite to the vertex U, there will be eight points, each an intersection of two lines of the quartic surface. These pairs of lines lie in eight planes through U, which are planes of the quadric point-cone, of vertex U, containing the quartic surface. The plane of the two lines which intersect in one of these eight points, say E, of the polar solid of U, is easily seen to lie in the tangent solid, at E, of any quadric which contains the quartic surface. Thus, this plane is tangent, at E, to the quartic curve in which the quartic surface meets the polar solid of U. This quartic curve lies on the quadric surface, say ω, in which this polar solid is met by the quadric point-cone, of vertex U, which contains the quartic surface. It is known that there are four generators, of either system, of a quadric surface in threefold space, which touch the quartic curve in which this quadric surface is met by another quadric surface (Vol. III, p. 69). The eight points such as E are thus identified as the points in which generators, of the quadric surface ω, touch the quartic curve in which the quartic surface meets the polar solid of U. For every generator of ω determines a plane of the cone of the vertex U.

Ex. 1. It has been shewn above (p. 119) that the original figure of fifteen points and lines is transformed into itself by the harmonic inversion in which the line e and the plane XYZ are fundamental elements. Thus if, as above, a line from T meet d and d', respectively, in H and H', this line will meet the plane XYZ, and the solid (X, Y, Z, T) will contain the points H, H'; we have seen that two lines of the quartic surface intersect at both these points. The same solid contains three other pairs of intersections, on (a, a'), (b, b'), (c, c'). Similarly for the solid (X, Y, Z, U).

Ex. 2. The plane ν, in a preceding notation, contains the points C, C', R, R', each of which is an intersection of a pair of lines of the quartic surface. Prove that the solid (X, Y, T, U), that is, the solid (ν, e), also contains four intersections, each of a line, such as HK', which meets d and d', with a line meeting another pair (a, a'), or (b, b'), or (c, c').

Precisely, with the symbols, the line joining the points

$$(z-x)^{-1}B-(x-y)^{-1}C' \quad\text{and}\quad (z_1-x_1)^{-1}B'-(x_1-y_1)^{-1}C,$$

meets the line joining the points

$$(x_1-y_1)^{-1}C-(y_1-z_1)^{-1}A' \quad\text{and}\quad (x-y)^{-1}C'-(y-z)^{-1}A,$$

in a point of the solid (ν, e).

Ex. 3. The fact that the sixteen lines of the quartic surface are obtainable from any one of them by repeated harmonic inversion, in respect to the five solids named, suggests a notation for the lines; in terms of which, also, a particular property of the lines can be conveniently stated. If one of the lines be denoted by 0, and those which are derived from this by harmonic inversion in the five solids be denoted by 1, 2, 3, 4, 5, respectively, then it is easy to see that inversion of the line 1, in the second solid, gives the same line as is obtained by inversion of the line 2 in the first solid; so that this line may be denoted by 12 or 21. There will be ten such lines. The notation 123 would then denote, for instance, the line obtained, by inversion in the third solid, from the line 12; it is, however, easy to see that this is the line 45; and so on, every line for which three indices might be used being the same as that denoted by the two complementary indices. The origin of the notation shews, for instance, that the lines 12, 13 both intersect the line 1; while they do not intersect one another. But the lines 12, 34 intersect, for example; and, in general, two lines represented each by two numbers, when the four numbers involved are all different. Consider now the two sets of four lines, 1, 23, 24, 25, and 2, 13, 14, 15; it is easy to see that each set consists of the four transversals of the threes of the other set. By using different numbers, such a pair of sets of four lines can be formed in ten ways. Again, the two sets of four lines, 1, 2, 3, 45 and 23, 31, 12, 0, have the same character, each consisting of the transversals of the threes of the other set; and of such pairs of sets there are also ten. We may say, then, that, from the lines of the surface, there can be formed *twenty double-fours* of lines.

The conics lying on the quartic surface. The quartic surface meets any generating plane of the quadric point-cone, of vertex *U*, which contains the surface, in a conic; as has been remarked. This conic is on the quadric surface in which any solid, drawn through the plane, is met by the quadric point-cone of vertex *T*, which contains the quartic surface. The planes of the former cone, of vertex *U*, are of two systems, two planes of different systems meeting in a line. Thus the two conics, in two generating planes, of the cone, of different systems, have two points in common, where the line of intersection of their planes meets the point-cone, with vertex at *T*, which determines these conics. In particular, the conic, in such a plane through *U*, may consist of two lines of the quartic surface; then these lines will meet the conic lying in any plane of the other system through *U*. The sixteen lines of the surface, we have seen, consist of four pairs lying in four generating planes of the same system, of the cone of vertex *U*, and of four other pairs lying in four generating planes of the other system. Thus a conic of the surface, generated as here, is met by eight lines of the surface. In particular, two lines of the surface, lying in one generating plane of the cone of vertex *U*, are each met by four other lines. Again, through any point, say *O*, of the quartic surface, can be drawn two of the generating planes of the cone of vertex *U*, meeting in a line through *U*. These two planes define a solid, the tangent solid at *O* of the cone. This solid contains the tangent plane at *O* of the quartic surface, which is the intersection of the tangent solids of all the quadrics containing the surface. Thus, the tangent plane of the quartic surface at *O* meets, in a line, each of the two generating planes of the cone, of vertex *U*, which pass through *O*. We assume, on the basis of what has been proved, that the quadric point-cones containing the quartic surface, with vertices at *X*, *Y*, *Z*, *T*, *U*, play entirely symmetrical parts, in general. Then it follows that, through any point, *O*, of the surface, there pass ten conics lying on the surface; two of these, which intersect in another point beside O, lie in planes, of different systems, of the cone of vertex U, and there are four other such pairs. Every one of the sixteen lines of the surface meets five of the ten conics, one of each of the five pairs; and each conic is met by eight of the lines. Further, each conic is met, besides at O, by the tangent plane of the surface at O. We can fix these relations by remarking the consequences which follow when we project from O on to an arbitrary solid:—The tangent plane of the quartic surface at O will meet this solid in a line, say t. Each of the conics through O will project into a line meeting the line t; and the ten lines so found will intersect in pairs. Each of these ten lines will also be met by eight of the sixteen lines which arise by projection of the lines of the quartic surface; and will thus be met by ten lines in all. A line arising by projection of one of these sixteen lines of the quartic surface will be met by five others of these, and will also be met by five of the lines arising by projection of the ten conics. As an arbitrary plane through O meets the quartic surface in three other points, the quartic surface will project into a cubic surface; and this will contain the twenty-seven lines which have been described.

The theory of the quartic surface may be studied in further detail, for its great interest and simplicity. In particular, the systems of curves which lie upon the surface may be examined; either directly, or, as in the case of other rational surfaces, by their representation upon a plane; it appears that every curve is co-residual with an aggregate formed from *six* fundamental curves. The corresponding theory of curves upon a cubic surface in threefold space (cf. Vol. III, pp. 191 ff.) may be deduced from this theory by projection. In this place only indications can be given, of an extensive theory. The most important source of information, with full references to previous literature, is Segre, *Math. Annal.* XXIV, 1884, pp. 313–444. See also Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques*, Paris, 1873, and the Bibliography, to 1872, appended thereto. For an introduction to the theory of the co-residuation of curves upon the surface, reference may be made to C. V. H. Rao, *Proc. Lond. Math. Soc.* XVII, 1919, pp. 272–305.

The cubic surface in space of three dimensions; and the theory of inversion. Regard the quartic surface which we have discussed as the intersection of a quadric point-cone (U), and a particular quadric, Ω. Denote the surface by Γ. We have just considered the projection of Γ, from a point O common to (U) and Ω, upon a solid, say Π; this gave a cubic surface in Π. Through the point O, of Ω, there passes a conical sheet of lines, lying on Ω and on the tangent solid of Ω at O (p. 37, above). Each of these lines meets the cone (U) in a further point, which is, then, also on Γ. The conical sheet of lines meets Π in a conic, lying in the plane in which Π is met by the tangent solid of Ω at O. Denote this conic by ω. The cubic surface in Π, obtained by projection of Γ, contains the conic ω. Every one of the sixteen lines of Γ meets the tangent solid of Ω, at O, in a point, also lying on Ω, as does the line; thus, the sixteen lines of the cubic surface which arise by projection of the sixteen lines of Γ intersect the conic ω. Other ten lines of the cubic surface, arising by projection of the conics of Γ which pass through O, meet the line of the cubic surface, lying in the plane of ω; this arises, as we have seen, from the tangent plane of Γ at O.

In this derivation of the cubic surface, the centre of projection, O, lies on the cone (U), as well as on Ω. We may, however, obtain, on the quadric Ω, another quartic surface, say Γ', exactly similar to Γ, but not passing through O. Take any point, H, not lying on Ω, and the polar solid of H in regard to Ω; to any point, P, of the fourfold space, make correspond the point, P', of the line HP, which is the harmonic conjugate of P, in regard to H and the polar solid of H. In particular, when P is on Ω, the point P' is the second intersection of the line HP with Ω. Then, as in the case of threefold space (p. 13, above), to the quadric point-cone (U) will correspond another quadric point-cone, say (U'), whose generating planes meet those of (U) on the polar solid of H. To the surface Γ will then correspond the intersection of Ω with this cone (U'); this will be a surface, Γ', exactly similar to Γ, but not in general containing O. If the surface Γ' be projected from O, on to the solid Π, we obtain, therein, a surface of the fourth order. This may, then, be obtained, from the cubic surface into which Γ projects, by a process of inversion in this space Π, in regard to a *sphere* lying in this solid (the Absolute conic being ω). The centre of this sphere is the projection, from O, of the point H (pp. 14, 37 ff., above). By the process of harmonic inversion, with H and its polar solid, the sixteen lines of Γ, which lie upon Ω, become lines of Γ'; and these are projected from O into lines of the quartic surface, in Π, into which Γ' projects. Thus the sixteen lines of the cubic surface, in the space Π, invert into lines. But the other eleven lines of the cubic surface, as we see in a similar way, do not invert into lines.

Ex. 1. A cubic surface, in the threefold space of coordinates x, y, z, u, which meets the plane $u=0$ in the conic $x^2+y^2+z^2=0$, and also in the line $lx+my+nz=0$, is expressible by an equation $(x^2+y^2+z^2)(lx+my+nz)-u\phi=0$, where $\phi=0$ is a quadric cone of vertex $(0, 0, 0, 1)$. This surface arises, by projection, from the point, in the fourfold space (x, y, z, u, t), whose coordinates are $(0, 0, 0, 0, 1)$, of the quartic surface which is the intersection of the quadric $x^2+y^2+z^2+tu=0$, and the point-cone

$$t(lx+my+nz)+\phi=0,$$

whose vertex is $(0, 0, 0, 1, 0)$.

Ex. 2. With the notation of the text, let the conic of Γ, lying in a plane through O and T which meets the lines a, b, c, d, be called $(5')$; and the conic, in a plane through OT, which meets a', b', c', d', be called (6). Through OU there is a plane, meeting a, b, c, d, which contains a conic of Γ which we call $(6')$, and a plane, meeting (a', b', c', d'), containing a conic (5). The two sets of six elements, a, b, c, d, (5), (6), and a', b', c', d', $(5')$, $(6')$, project from O into a double-six of lines (cf. Vol. III, p. 160). The conics of Γ in planes from O to the vertex X, of the self-polar pentad, may be called (23) and (14); those similarly arising for Y and Z being, respectively, (31), (24) and (12), (34). The eight other lines of Γ may then be denoted by symbols $(r5)$, $(r6)$, for $r = 1, 2, 3, 4$.

Ex. 3. Prove by projection in fourfold space that, by inversion in threefold space, a line meeting the Absolute conic inverts into another line meeting this conic, and meeting the former line.

The Cyclide, or quartic surface with a double conic, in threefold space. Consider now more particularly the character of the surface, in the threefold space Π, which is obtained by projecting, from the point, O, of the quadric Ω, the intersection of Ω with the quadric point-cone (U), not passing through O. Denote the quartic surface constituting this intersection by Γ; and let the surface in Π, obtained by projection of Γ, be called the *Cyclide*. As Γ is met by an arbitrary plane, of the fourfold space, in four points, the Cyclide is met by an arbitrary line of the threefold space in four points, and is a quartic surface. The lines of the conical sheet, in which Ω is met by its tangent solid at O, each meet the cone (U) in two points. Thus the Cyclide has the conic, ω, in which this conical sheet meets Π, as a double conic. Every line lying on Γ, since it meets the tangent solid at O, and lies itself on Ω, meets the conical sheet; thus the Cyclide has sixteen lines, which all meet the conic ω.

The poles, in regard to Ω, of the tangent solids of the cone (U), are an aggregate of ∞^2 points; they lie in the solid which is the polar of the vertex, U, of the cone (U), in regard to Ω; they describe a quadric surface in this polar solid. This we denote by $Q(U)$. A tangent solid of the cone (U) touches this cone along a line; this line meets Ω in two points. The tangent solid of (U) contains the tangent plane of Γ at each of these two points. The section of Ω, by this tangent solid of (U), projects from O into a quadric surface in Π, containing the conic ω; or, as we shall say, into a *sphere*, the conic ω being regarded as Absolute conic of the space Π. The *centre* of this sphere, being the projection of the pole, in regard to Ω, of the tangent solid of (U), lies on the quadric surface in (U) which is the projection of the quadric $Q(U)$,

(p. 37, above). This sphere, then, touches the Cyclide in two points. The section of Ω by the polar solid of the vertex U, of the point-cone, also projects into a sphere, in Π. This sphere, which depends only on U and O, is cut at right angles by the sphere arising from the chosen tangent solid of (U), because this tangent solid and the polar solid of U are conjugate in regard to Ω (p. 40, above). Thus it appears that the Cyclide is touched, in two points, by an aggregate of spheres, each defined as having its centre on a certain fixed quadric surface, in the space Π, and cutting at right angles a certain fixed sphere of this space. Or, we may regard the matter in a slightly different way: Any plane, in the fourfold space, meets the quadric Ω in a conic. Thus the section of Ω, by a plane, projects from O, on to the solid Π, into a conic having two points of intersection with the Absolute conic, ω, that is, into a *circle*. When the plane, not passing through O, meets Ω in two lines, these become, on projection, two lines meeting the conic ω. If the plane be a generating plane of the cone (U), so consisting wholly of points of this cone, this plane meets the quartic surface Γ in a conic, projecting from O into a circle lying on the Cyclide. Through such a generating plane, of the cone (U), there passes an infinity of tangent solids of (U), each containing another generating plane, of the opposite system. Thus, by what is said above, through the circle on the Cyclide, just obtained, there passes an infinity of spheres, each meeting the Cyclide in another circle. Any such sphere will touch the Cyclide at the two intersections of the two circles which lie upon the sphere. As the tangent solids of (U) which pass through a plane have their poles, taken in regard to Ω, upon the polar line of this plane in regard to Ω, it follows that the spheres, passing through a particular circle which lies on the Cyclide, have their centres on a line. This is then a generator of the quadric surface, in Π, obtained by projection of the quadric surface $Q(U)$. A special case arises when the particular circle consists of two lines lying on the Cyclide.

Ex. 1. In the fourfold space, we may take coordinates, (x, y, z, u, t), for which the point O is $(0, 0, 0, 0, 1)$, the tangent solid of Ω at O being $u = 0$, choosing x, y, z so that the conical sheet of lines of Ω at O is given by $u = 0$, $x^2 + y^2 + z^2 = 0$. Then the equation of Ω is of the form $-ut + x^2 + y^2 + z^2 + 2uP + du^2 = 0$, where P is a homogeneous linear form in x, y, z. If the point U be $(0, 0, 0, 1, 0)$, the equation of the cone (U) is $t^2 + 2tQ + V = 0$, where Q, V are homogeneous in x, y, z, respectively linear and quadratic. The elimination of t between these equations gives the equation of the projection of their intersection, upon the solid $t = 0$; this solid is here taken to contain the point U. If, in the eliminant, we put $x + lu$, $y + mu$, $z + nu$, respectively, for x, y, z, and choose l, m, n

properly, the equation of the projection takes the form

$$(x^2+y^2+z^2)^2+u^2W=0,$$

where W is a homogeneous quadratic form in x, y, z and u. This equation, however, can be written $(x^2+y^2+z^2+\lambda u^2)^2+u^2M=0$, where $M=W-2\lambda(x^2+y^2+z^2)-\lambda^2u^2$. By taking, for λ, a root of a certain quintic equation, the equation $M=0$ represents a quadric cone; thus, for such λ, we can suppose $M=XY-Z^2$, where X, Y, Z are homogeneous linear forms in x, y, z and u. Then the equation of the Cyclide becomes $CD+u^2XY=0$, where C, D are of the forms $x^2+y^2+z^2+\lambda u^2 \pm uZ$. This is satisfied by $C=0$, $X=0$, and by $C=0$, $Y=0$. The equation $C=0$ represents a sphere, and $Z=0$ is any plane through the vertex of the cone $M=0$, of which $X=0$, $Y=0$ are tangent planes. There are, therefore, ∞^2 such spheres.

Ex. 2. The Cyclide can be generated as the locus of a circle which is the intersection of a varying sphere passing through a fixed circle of the surface, with a corresponding varying sphere passing through another fixed circle of the surface.

Ex. 3. The Cyclide, we have seen, is the envelope of a varying sphere, with centre at a varying point, say H, of a fixed quadric surface, Q', the sphere cutting at right angles a fixed sphere, S. The varying sphere touches the Cyclide in two points. These two points of contact are, in fact, the limiting points of the system of coaxial spheres defined by the fixed sphere S, and the tangent plane, at H, of the quadric Q' (cf. Vol. III, p. 74). The Cyclide is thus the locus of the limiting points, so defined by the fixed sphere S and the varying tangent plane of the quadric surface Q'. To prove this, consider the figure in the space of four dimensions. Let L, L' be the two points where a generator of the cone (U) meets Ω; take the polar plane, in regard to Ω, of this line LL'. This plane lies on the polar solid, in regard to Ω, of every point of the line. In particular, it lies in the principal solid which is the polar of the vertex U. The plane touches the quadric surface $Q(U)$. The plane meets Ω in a conic, which we denote by λ; this conic is, therefore, on the section of Ω by the polar solid of U. As the plane is in the tangent solid of Ω at L, the conic λ is on the conical sheet of lines of Ω through L. When we project, from O, this conical sheet becomes a quadric cone, in Π, whose vertex is the projection of L. This cone contains the conic which is the projection of λ; and also, as in preceding cases, contains the conic ω. The cone, therefore, contains the section of a *principal sphere*, S, by a tangent plane of the quadric surface, Q', (arising by projection of $Q(U)$), and contains ω. The vertex of this cone, the projection of L, is, therefore, a limiting point of the coaxial system of spheres defined by the principal sphere, and the tangent plane. By definition, L is on the quartic surface Γ, and projects into one of the points of contact of the Cyclide with the sphere obtained from the section of Ω by the tangent solid, along LL', of the cone (U).

If, in the threefold space, the quadric surface Q' and the sphere S be, respectively,

$$a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 + d^{-1}t^2 = 0,$$

$$x^2 + y^2 + z^2 + 2fxt + 2gyt + 2hzt + kt^2 = 0,$$

the Cyclide is

$$4t^2 \left[a(x+ft)^2 + b(y+gt)^2 + c(z+ht)^2\right] + d(x^2+y^2+z^2-kt^2)^2 = 0.$$

Ex. 4. The planes of the circles, in which the Cyclide is met by the enveloping spheres, are tangent planes of one of five quadric cones. These are often called the *cones of Kummer*.

The five generations of the Cyclide, with confocal quadric surfaces. The Cyclide has, at any point of its double curve, ω, two tangent planes, one touching each sheet. We prove, now, that all these planes touch the quadric surface, Q', of the solid Π, which is obtained by projection, from O, of the quadric $Q(U)$. We may call this quadric surface a *principal quadric* of Π; and call the sphere S, obtained by projection of the section of Ω by the polar solid of the vertex U, a *principal sphere*. Every generating line, of the conical sheet of lines of Ω at O, meets the point-cone (U) in two points, say K and K'. The locus of the point K is the curve, in the tangent solid of Ω at O, which is the intersection of the conical sheet of lines of Ω at O, with the quadric surface in which the cone (U) meets the tangent solid of Ω at O. This curve is of the fourth order. Consider the tangent solid of Ω at the point K. This solid passes through O, and meets the tangent solid of Ω at O in a plane, which touches the conical sheet of lines through O (p. 38, above). This plane, therefore, contains a tangent line of the conic ω. The point K lies on the surface Γ. The tangent plane of Γ at this point,—being the intersection of the tangent solid of (U) at this point with the tangent solid of Ω at this point,—projects, from O, into a tangent plane of the Cyclide. This plane is the intersection of Π with the solid, through O, containing the tangent plane of Γ at K; this solid is no other than the tangent solid of Ω at K. Thus, the tangent plane of Γ, at K, projects into a tangent plane, of the Cyclide, at the point where OK meets the conic ω. The other tangent plane of the Cyclide, at this point, arises, in a similar way, from K'.

Next consider the line KU. It lies on the cone (U), and is the intersection of two generating planes of this cone. Its polar plane, in regard to Ω, is, by definition, a tangent plane of the quadric surface $Q(U)$. This plane also lies in the tangent solid of Ω at K, because the line KU contains the point K. Thus the plane, into which the tangent plane of Γ, at K, projects, is also a tangent plane of the principal quadric Q', of Π. The point of contact is the projection of the point, of the quadric surface $Q(U)$, which is the pole, in regard to Ω, of the tangent solid of the cone (U) along KU.

It is thus shewn that the tangent plane of Γ, at K, projects into a tangent plane, of the Cyclide, at a point of the conic ω, this plane also touching the quadric Q'.

In the general case, there are, beside the cone (U), four other quadric point-cones containing the surface Γ; to each cone corresponds a *principal* solid, the polar of its vertex in regard to Ω, containing the other four vertices. The sections of Ω by these principal solids project into five principal spheres in the solid Π. To each cone will also correspond a quadric surface, such as $Q(U)$, lying in the corresponding principal solid, this being the polar, in regard to Ω, of the aggregate of the tangent solids of the cone; the quadric surfaces, so arising, project into principal quadric surfaces in Π, such as Q'. From the conjugate relation of the vertices of the cones in the fourfold space, it is clear that, in the solid Π, the centres of four of the principal spheres form a self-polar tetrad for the remaining sphere, and for the corresponding principal quadric surface; also, the join of any two of these centres is at right angles (in regard to the conic ω, as Absolute conic) to the plane containing the other three; while, further, any two of the principal spheres cut at right angles (p. 40, above). Likewise, by what we have just proved, the five principal quadrics belong to a confocal system, being all touched by the tangent planes of the Cyclide at the points of the conic ω. This conic is then one of the four focal conics of these confocal quadric surfaces (cf. Vol. III, p. 92). The Cyclide is the envelope of a sphere, with its centre on any one of the principal quadric surfaces, described to meet the corresponding principal sphere at right angles.

Ex. 1. A confocal system of quadric surfaces, in space of three dimensions, has *four* focal conics. Thus, by what has been shewn, the tangent solids of Ω, at the points such as K, should meet the solid Π in planes containing, not only the tangent lines of the conic ω, but also the tangent lines of three other conics. Reciprocating in regard to Ω, this statement is, that the joins of an arbitrary point, P, to the points K, lie on the planes of four quadric *line-cones* (see above, p. 121). In fact, K describes a quartic curve in the tangent solid of Ω at O, and, thus, lies on four quadric conical sheets in this solid. If V be the vertex of one of these, the line PK evidently lies in the plane VPK, which describes a line-cone with VP as axis. This proves the statement in its reciprocal form.

Ex. 2. Let U, T be the vertices of two quadric point-cones (U) and (T), which contain the surface Γ. If a generating line, UP, of (U), be such that the tangent solid of (U), along UP, touches Ω, the plane UTP meets the polar plane of the line TU, in regard to Ω, upon a certain conic. The same conic is obtained from the cone (T) by a similar definition. Denote this conic by γ.

The angle between two enveloping spheres of the Cyclide, of different systems, arising, say, from tangent solids of the cones (U) and (T), respectively, is equal to the interval between these two tangent solids, measured in regard to Ω (p. 39, above); or, equal to the interval between the poles of these solids, in regard to Ω. Let these solids touch (U) and (T) along the lines UK and TL, respectively; and let the planes UTK and TUL meet the polar plane of the line UT, in regard to Ω, in the points K' and L', respectively. The angle in question, between the corresponding enveloping spheres of the Cyclide, is equal to the interval $K'L'$, measured in regard to the conic γ. (Cf. Jessop, *Quartic Surfaces*, 1916, p. 106.)

Ex. 3. It has been seen that the Cyclide may be regarded as the inverse of a cubic surface containing the conic ω; and, conversely, that such a cubic surface is obtained by inversion of the Cyclide when the centre of inversion lies thereon. By taking the vertex of the harmonic inversion, in the space of four dimensions, in regard to Ω, to be at the vertex, U, of one of the point-cones, we see that the Cyclide inverts into itself when the centre of inversion is the centre of any one of the five principal spheres. When the centre of inversion is arbitrary, a Cyclide inverts into another Cyclide.

Ex. 4. For a quartic curve, in threefold space, which is the intersection of a quadric surface with a quadric cone, there are four points of the curve of which the tangent line is a generator of the cone; these are the points of the curve lying on the polar plane of the vertex of the cone, taken in regard to the quadric surface. By supposing the curve to be that considered above (p. 177), the intersection of (U) with the conical sheet of lines of Ω through O, we see that there are four points K which coincide with the associated points, K'. These will lie on the plane which is the intersection of the polar solids of O in regard to all quadrics containing the quartic surface Γ. The tangent solids of Ω at the four points K will meet in the polar line of this plane, in regard to Ω; and this line passes through O. Any such tangent solid contains the tangent plane of Γ at the corresponding point K. Thus, on projection, we see that there are four points of the conic

ω at which the two tangent planes of the Cyclide coincide with one another; and that the four tangent planes at these points have a point common.

Taking, again, the plane which is the intersection of the polar solids of O in regard to all the quadrics containing Γ, we may consider the locus of the polar lines of this plane, in regard to these quadrics. This is a cubic cone, with O as vertex. Hence, for the Cyclide, we can infer a cubic curve containing the following nine points:—the five points obtained from the vertices of the cones such as (U), these being the centres of the principal spheres; the three intersections of the opposite pairs of joining lines, of the four points of the conic ω at which the tangent planes of the Cyclide coincide; and the point of intersection of the tangent planes of the Cyclide at these four points (Segre, *Math. Annal.* XXIV, 1884, p. 330).

Further, considering the intersection, with Γ, of the polar solids of O, in regard to the quadrics containing Γ, and the intersection of such a solid with Ω, we can prove, for the Cyclide, that there are ∞^1 quadric surfaces, each touching the Cyclide along a quartic curve, all passing through the four specified points of the conic ω, there being one of these quartic curves through any point of the Cyclide. Each of these curves is the intersection of the Cyclide with a quadric surface passing through ω, say with a sphere; these spheres have a common centre, the point at which the tangent planes of the Cyclide, at the four specified points of ω, co-intersect (*loc. cit.* p. 337).

For the Cyclide expressed by

$$(x^2+y^2+z^2)^2+(ax^2+by^2+cz^2)\,t^2+2\,(fx+gy+hz)\,t^3+dt^4=0,$$

the tangent planes coincide at any one of the four points $(p, q, r, 0)$, where $p^2=b-c$, $q^2=c-a$, $r^2=a-b$; and the tangent planes at these four points intersect in $(0, 0, 0, 1)$.

Confocal Cyclides. An enveloping sphere of the Cyclide was obtained by projecting the section, of Ω, by a solid which touches one of the point-cones (U). Of such solids, however, there is an infinity (∞^1) which also touch Ω; and the locus of the points of contact of these solids with Ω is an important curve. We denote this curve by q. Consider, in the principal solid which is the polar of the vertex U in regard to Ω, the quadric surface in which this polar solid meets Ω; this quadric surface we denote by $[\Omega]$; and, also, the quadric surface in which this solid meets the cone (U), denoting this quadric surface by $[U]$. The quadric surface $Q(U)$, in this solid, is the polar reciprocal of $[U]$ in regard to $[\Omega]$. A tangent solid of Ω which touches the cone (U), as it passes through the point U, must touch Ω at a point of the quadric surface $[\Omega]$, and must also touch the quadric surface $[U]$. This tangent solid thus meets the principal solid, which is the polar of the point U, in regard to Ω, in a common tangent plane of the quadric surfaces $[\Omega]$, $[U]$. The curve q is the locus of the points of contact, of these common tangent planes, with $[\Omega]$; it is a quartic curve, being the intersection of $[\Omega]$ with $Q(U)$. When we project from O, on to the solid Π, the quadric surface $[\Omega]$ becomes a principal sphere, and the quadric surface $Q(U)$ becomes a principal quadric surface, Q'. Thus the curve q projects into the intersection of these. Such a curve, obtained by projection of q, of which there are five in general, may be called a *focal curve* of the Cyclide. The section of Ω, by a tangent solid of the cone (U) which touches Ω, is a conical sheet of lines, of which the generators meet the generators of the conical sheet of lines of Ω through O (p. 38, above); thus, the projection from O, on to the solid Π, of this section, is a quadric cone passing through the conic ω; the vertex of this cone is a point of the quartic focal curve, q', obtained by projection of q. Such a cone, when ω is regarded as the Absolute conic, is often called a *point-sphere*. The focal curve q' can then be said to be the locus of the vertices of point-spheres whose section with the Cyclide consists of two circles (which touch one another).

It can be shewn that if, upon a sphere, of a threefold space Π, that is, a quadric surface passing through a given conic, ω, of this space, there be given a quartic curve, q', the intersection of the sphere with another quadric surface; and, also in this solid Π, there be given a point, H'; then, three Cyclides can be constructed, containing the point H', all having the curve q' as focal curve. Further, that the tangent planes of any two of these Cyclides, at any point common to both, are at right angles; that is, that these tangent planes meet the plane of the conic ω in lines which are conjugate to one another in regard to this conic. It can then be shewn, further, that only these three Cyclides are possible under the given conditions. The corresponding result for curves in a plane is proved above, p. 95. For, in a fourfold space containing the solid Π, take a point, O, not itself in Π; consider the conical sheet of lines joining O to the given conic ω. A quadric, Ω, can be described containing this conical sheet, this being any quadric whose section, by the solid which is defined by O and the plane of ω, consists of this conical sheet. Next, consider the quadric point-cone which is the aggregate of the lines joining O to the points of the given sphere in Π. This will meet the quadric Ω in a quartic surface; part of this, however, is the quadric surface constituted by the conical sheet $O\omega$; the remaining part of the intersection is then another quadric surface. Thus, assuming that a quadric *surface*, in fourfold space, necessarily lies in a solid, we can regard the given sphere, in Π, as the projection, from O, of the section of Ω by a solid; denote this solid by Φ. The curve q', on the given sphere in Π, is the projection from O of a curve, say q, lying in the section, with Ω, of the solid Φ; this section is a quadric surface, and the curve q, like q', is a quartic curve. Let the pole, in regard to Ω, of the solid Φ, be the point U, supposed not to lie on Ω; also, let the line OH' meet Ω again in the point H. The polar plane, in regard to Ω, of the line UH, will lie in the solid Φ, which is the polar solid of U; denote this polar plane of UH by η. We know that, in the solid Φ, there can be drawn, through the curve q, three quadric surfaces to touch the plane η (Vol. III, p. 120); the polar reciprocals of these quadric surfaces, in regard to Ω, are three quadric point-cones, with vertex at U; these cones touch the tangent solids of Ω at the points of the curve q (which lies on Ω), and contain the line UH. The intersection of Ω, with any one of these three point-cones, projects from O, on to Π, into a Cyclide having the given curve q' as focal curve, and passing through H'. This proves the existence of three such Cyclides.

We next prove that any two of these cut at right angles (in regard to ω, as Absolute conic), at their common point H':—The plane η, polar of the line UH in regard to Ω, is the intersection of the solid Φ with the tangent solid of Ω at H. Thus the plane η contains the conic, in which the solid Φ is intersected by the conical sheet of lines of Ω which pass through H; this conic is also on the quadric surface (Ω, Φ), upon which the curve q lies; denote this conic by h. It is known that the three points of contact with the plane η, of quadric surfaces, in the solid Φ, which contain the curve q and touch η, form a self-polar triad in regard to the conic section of η with any quadric surface, in Φ, containing q; these three points of contact are thus a self-polar triad in regard to the conic h, which is the intersection of the quadric surface (Ω, Φ) with the tangent solid of Ω at H. The polar solids, in regard to Ω, of these three points of contact, are the tangent solids, at H, of the three point-cones, of vertex U, above drawn through the line UH; the polar planes, in regard to Ω, of the lines of the conical sheet, of lines of Ω through H, are the tangent planes of this same conical sheet. Thus, reciprocating in regard to Ω, the tangent planes at the point H, of the three surfaces in which Ω is met by the three point-cones above drawn through UH, are conjugate in pairs in regard to the conical sheet of lines of Ω through H. On projecting from O, since every line of Ω through H meets a line of Ω through O, we obtain three Cyclides through the point H', of which every two have their tangent planes, at H', at right angles in regard to the conic ω, regarded as Absolute conic. By a similar argument every two of these surfaces cut at right angles at every one of their common points.

And, as in the corresponding case of a quartic curve in a plane, it was proved (p. 95, above), that there were only two such curves, so it can be shewn here that there are only three Cyclides subject to the given conditions.

Ex. 1. Let the Cyclide be the projection of the intersection of the quadric Ω given by $x^2+y^2+z^2+t^2+u^2=0$, with a quadric given by $ax^2+by^2+cz^2+dt^2+eu^2=0$. Then the coordinates, x_1, y_1, z_1, t_1, u_1, of a point of Ω whereat the tangent solid touches the point-cone (U), of vertex $(0,0,0,0,1)$, containing the intersection of these quadrics, are such that

$$u_1=0,\quad x_1^2+y_1^2+z_1^2+t_1^2=0,$$
$$(a-e)^{-1}x_1^2+(b-e)^{-1}y_1^2+(c-e)^{-1}z_1^2+(d-e)^{-1}t_1^2=0.$$

Thus the curve q, and the focal curve, q', obtained by projecting this, depend only on the differences of a, b, c, d, e. A system of Cyclides with common focal curves are then those given by the projection of the intersection of

$$x^2+y^2+z^2+t^2+u^2=0,\quad (a+\lambda)^{-1}x^2+(b+\lambda)^{-1}y^2+\ldots+(e+\lambda)^{-1}u^2=0,$$

where λ is arbitrary. The condition that two such Cyclides, given by the values, λ_1, λ_2, of λ, should cut at right angles at the common point, $(x_1, y_1, \ldots, u_1)$, is (p. 39, above) that the tangent solids $(a+\lambda_1)^{-1}xx_1+\ldots=0$, $(a+\lambda_2)^{-1}xx_1+\ldots=0$ should be conjugate in regard to Ω. This condition is

$$(a+\lambda_1)^{-1}(a+\lambda_2)^{-1}x_1^2+\ldots=0;$$

it is a consequence of the two equations

$$(a+\lambda_1)^{-1}x_1^2+\ldots=0,\quad (a+\lambda_2)^{-1}x_1^2+\ldots=0.$$

The condition that a Cyclide, of the confocal system obtained, should pass through an arbitrary point, is a cubic equation for λ.

Ex. 2. Through any point, H, of the quartic surface, Γ, defined as the intersection of the quadric Ω, and another quadric, Ω', a line may be drawn which is the polar, in regard to Ω, of the tangent plane of Γ at H. This line is the locus of the poles, in regard to Ω, of the tangent solids, at H, of all the quadrics (say $\theta\Omega+\Omega'=0$) which contain Γ. The line lies in the tangent solid of Ω at H; and is the polar line, in regard to the conical sheet of lines of Ω through H, of the tangent plane of Γ at H, this plane, and the conical sheet, being both in the tangent solid of Ω at H. By projection of Γ, from a point O, of Ω, into a Cyclide of the solid Π, this line becomes the normal of the Cyclide at the point arising from H; namely, the line which is conjugate to the tangent plane of the Cyclide with respect to the Absolute conic ω. If H be $(x_0, y_0, z_0, t_0, u_0)$, and Ω be $x^2 + \ldots + u^2 = 0$, while Ω' is

$$ax^2 + \ldots + eu^2 = 0,$$

the general point of the line in question is $(\xi, \eta, \zeta, \tau, v)$, where $\xi = (a + \theta) x_0, \ldots, v = (e + \theta) u_0$, the θ being variable. The line, as we see by taking polars in regard to Ω, touches all the quadrics which are the polar reciprocals, in regard to Ω, of the quadrics $\lambda\Omega + \Omega' = 0$; that is, it touches all the quadrics

$$(a + \lambda)^{-1} x^2 + \ldots + (e + \lambda)^{-1} u^2 = 0;$$

except two of these, upon which it lies entirely. These two are those for which λ satisfies the equation

$$(a + \lambda)^{-1} x_0^2 + \ldots + (e + \lambda)^{-1} u_0^2 = 0,$$

which is a quadratic in virtue of

$$x_0^2 + \ldots + u_0^2 = 0, \quad ax_0^2 + \ldots + eu_0^2 = 0.$$

The tangent solid of Ω at H touches every one of these polar reciprocal quadrics, at points of this line. In particular, however, when λ has one of the five values $-a, \ldots, -e$, the line meets the corresponding locus, now become a quadric surface, only once; in the point in which the line meets the principal solid in which this quadric surface lies.

The aggregate of all the points of all such lines, when $(x_0, y_0, \ldots, u_0)$ becomes, in turn, all points of the surface Γ, is a ruled locus of ∞^3 points. By substitution of $x_0 = (a + \lambda)^{-1} \xi$, etc., in the equations $x_0^2 + \ldots + u_0^2 = 0$, $ax_0^2 + \ldots + eu_0^2 = 0$, or by the geometrical property of the lines, we see that the equation of this ruled locus can be formed as the discriminant, in regard to λ, of the equation

$$(a + \lambda)^{-1} \xi^2 + \ldots + (e + \lambda)^{-1} v^2 = 0;$$

so that the locus is of the twelfth order. Thus, on projection, we infer that twelve normals of a Cyclide pass through an arbitrary point. The line of the ∞^3 ruled locus, through a point, $(x_0, \ldots, u_0)$, on the curve in which the surface Γ meets the quadric

$$(a + \lambda)^{-1} \xi^2 + \ldots + (e + \lambda)^{-1} v^2 = 0,$$

lies entirely on this quadric, as already remarked. In particular, for $\lambda = \infty$, the lines of this locus which meet Γ in the curve given by $x_0^2 + \ldots + u_0^2 = 0$, $ax_0^2 + \ldots + eu_0^2 = 0$, $a^2x_0^2 + \ldots + e^2u_0^2 = 0$, lie entirely on the quadric Ω; such lines form a surface, lying on Ω, of order $2 . 12 - 2 . 4$, or sixteen. The lines of this surface are obtainable, also, by remarking, first, that, in the tangent plane of Γ at any point P, there are two lines lying on Ω, the intersection of this tangent plane with the conical sheet of lines of Ω through P;

and noticing, then, that, when P is on the curve whose equations have been given, the tangent plane of Γ touches this conical sheet along a line of points, with coordinates $(a+\lambda)x_0, \dots, (e+\lambda)u_0$. Thus, the tangent plane of Γ, at any point of this curve, is also a tangent plane, at another point, of the surface in which Ω is met by any one of the quadrics $(a+\lambda)^{-1}x^2 + \dots + (e+\lambda)^{-1}u^2 = 0$. For, this tangent plane of Γ lies in the tangent solid of Ω at every point of the line whose points have coordinates $(a+\lambda)x_0, \dots, (e+\lambda)u_0$; and lies in the tangent solid of the quadric

$$(a+\lambda)^{-1}x^2 + \dots + (e+\lambda)^{-1}u^2 = 0,$$

because this, being the tangent solid of Ω at the point $(x_0, \dots, u_0)$ of the curve, touches the line at this point. The curve in question,

$$(x_0^2 + \dots + u_0^2 = 0,\ ax_0^2 + \dots + eu_0^2 = 0,\ a^2x_0^2 + \dots + e^2u_0^2 = 0),$$

lies also on the polar reciprocal of the quadric $x^2 + \dots + u^2 = 0$, in regard to the quadric $ax^2 + \dots + eu^2 = 0$. On projection, the focal curves of the Cyclide appear as lying doubly on the developable surface formed by the tangent planes of the Cyclide at points of the conic ω, these planes touching the confocal Cyclides.

We may, however, go further. Of the ∞^1 lines, forming the surface of order sixteen, spoken of, lying on Ω, there are sixteen lines which lie on Γ. For the line of points of coordinates

$$(a+\lambda)x_0, \dots, (e+\lambda)u_0,$$

will lie on $ax^2 + \dots + eu^2 = 0$, if, beside the three equations for $(x_0, \dots, u_0)$, we also have $a^3x_0^2 + \dots + e^3u_0^2 = 0$. The four equations have sixteen solutions; if we denote the quintic polynomial, in t, whose roots are a, b, c, d, e by $f(t)$, the four equations are satisfied by $x_0^2 f'(a) = y_0^2 f'(b) = \dots = u_0^2 f'(e)$. The curve, of order eight, for which $x_0^2 + \dots = 0$, $ax^2 + \dots = 0$, $a^2x_0^2 + \dots = 0$, is expressed, in terms of a parameter θ, by

$$x = (\theta+a)^{\frac{1}{2}}[f'(a)]^{-\frac{1}{2}}, \dots,\quad u = (\theta+e)^{\frac{1}{2}}[f'(e)]^{-\frac{1}{2}};$$

the points of any one of the sixteen lines of the surface Γ are given by $x = (\lambda+a)[f'(a)]^{-\frac{1}{2}}, \dots,\ u = (\lambda+e)[f'(e)]^{-\frac{1}{2}}$, for varying values of λ. Here $f'(x)$ denotes df/dx.

More generally, in space of n dimensions, in which $x_1, \dots, x_{n+1}$ are the coordinates, we may consider the surface given by the $(n-2)$ equations, $a_1^r x_1^2 + \dots + a^r_{n+1}x^2_{n+1} = 0$, for $r = 0, 1, \dots, (n-3)$. Upon this there is a curve, of order 2^{n-1} and genus $2^{n-2}(n-3)+1$, given by these equations together with

$$a_1^{n-2}x_1^2 + \dots + a^{n-2}_{n+1}x^2_{n+1} = 0,$$

of which the points have coordinates of the forms

$$(\theta + a_1)^{\frac{1}{2}}[f'(a_1)]^{-\frac{1}{2}}, \dots,$$

where $f(t)$ is the polynomial having $a_1, \ldots, a_{n+1}$ for roots. This leads to the consideration of 2^n lines of the surface, each consisting of points of coordinates such as $x_s = (\lambda + a_s)[f'(a_s)]^{-\frac{1}{2}}$. For the case $n = 3$, cf. Vol. III, p. 92. The case $n = 5$ arises in Chap. VII below.

Ex. 3. It is not a difficult matter to obtain *the lines of curvature upon a Cyclide.* We do this by a method which is, in part, applicable to any surface. The question is, however, of special interest here because, as will be seen in Chap. VII (below, p. 231), these lines are in correspondence with the inflexional lines of a Kummer surface.

At any point of a surface, in space of three dimensions, two lines can be drawn, in the tangent plane at this point, which meet the surface in three coincident points there. These are the *inflexional* directions. If the point be $(x_0, \ldots)$, and the surface be $f(x, \ldots) = 0$, the lines are found by combining the tangent plane with the polar quadric of the point, given by $D^2 f_0 = 0$, where D is the operator $x\partial/\partial x_0 + \ldots$, and f_0 the value of f at $(x_0, \ldots)$. If any sphere be drawn to touch the surface at the point, its inflexional directions give two other lines lying in the tangent plane of the surface. The directions of the lines of curvature of the surface, at the point, are along the two lines, in the tangent plane, which are the double rays of the involution determined by the two pairs of inflexional directions which we have spoken of. In fact, the curve of intersection of the sphere with the surface has a double point at the point of contact, whose tangent lines lie in the tangent plane, and form a pair of the involution in question; there are two such spheres for which the curve of intersection has a cusp at the point; the two cuspidal tangents are the directions of the lines of curvature.

We prove these statements by regarding the surface under consideration, in the threefold space, Π, as obtained by projection, upon Π, of the intersection of a quadric (∞^3), Ω, lying in a space of four dimensions, which contains Π, with another (∞^3) locus, Ω', lying in this fourfold space. We shall speak of Ω' also as a quadric, so that the direct application is to the Cyclide. The centre of projection, O, is on Ω; the point of the surface (Ω, Ω'), which we consider, is denoted by P, and its coordinates by $(x_0, y_0, z_0, t_0, u_0)$; the tangent solids of Ω and Ω' at P are denoted by T and T'. We take solids, $\theta T + T'$, passing through the tangent plane, (T, T'), of the surface (Ω, Ω'), at the point P. These solids intersect Ω in surfaces which project, from O, into *spheres* in the space Π, the Absolute conic being the intersection of Π with the conical sheet of lines of Ω through O. There are two lines through P, the intersection of the plane (T, T') with the cone (T, Ω), in which the solid $\theta T + T'$ meets the conical sheet of lines of Ω through P; these lines meet the conical sheet of lines of Ω through O, and project into the generators, or inflexional lines, of the sphere into which the surface $(\theta T + T', \Omega)$ projects, at the point which is the projection of P. These two lines lie in the plane (T, T'), and are independent of θ. The solid $\theta T + T'$ meets the surface (Ω, Ω') in a curve; Ω' being a quadric, this curve, lying on the quadric $\theta\Omega + \Omega'$, lies on the conical sheet $(\theta T + T', \theta\Omega + \Omega')$, the solid $\theta T + T'$ being the tangent solid of $\theta\Omega + \Omega'$; the curve may be given as the intersection of this conical sheet, $(\theta T + T', \theta\Omega + \Omega')$, with the quadric Ω. Thus, the curve has a double point at P, whose tangent lines are the two in which T meets the conical sheet; that is, these tangent lines are the two lines in which the quadric $\theta\Omega + \Omega'$ is met by the plane (T, T'). For different values of θ these form an involution of pairs of lines in this plane, intersecting at P. They project, from O, into the tangent lines of the curve in which the surface, obtained by the projection of (Ω, Ω'), is met by a sphere touching this surface at the projection of P. For the particular value of θ for which the solid $\theta T + T'$ contains the point O, this sphere is replaced by the tangent plane of the surface obtained by projection of (Ω, Ω'), and the tangent lines become the inflexional directions of this surface. For the value $\theta = \infty$, we have the two lines, first considered, in which the plane (T, T') meets Ω, which project into the common generators of all the spheres. For five particular values of θ, the quadric $\theta\Omega + \Omega'$ becomes a cone, and we have, on projection, the tangent lines of a pair of conics of the Cyclide, through the point into which P projects.

There are, also, two values of θ for which the tangent plane (T, T') meets $\theta\Omega + \Omega'$ in two coincident lines, namely, the plane $(T, \theta T + T')$ touches the conical sheet $(\theta T + T', \theta\Omega + \Omega')$. For, we know that the tangent planes of this cone are the intersections, of $\theta T + T'$, with the tangent solids of $\theta\Omega + \Omega'$ at points of this conical sheet (p. 38, above). Thus, for the plane $(T, \theta T + T')$ to touch the conical sheet, it must be identical with a plane $(K, \theta T + T')$, where K is the tangent solid of $\theta\Omega + \Omega'$ at a point, $(x_1, y_1, \ldots)$, of this conical sheet. Now, if Ω, Ω' be, respectively, $x^2 + \ldots + u^2 = 0$, $ax^2 + \ldots + eu^2 = 0$, the condition, that the planes

$$[(a+\theta)\,xx_1 + \ldots = 0,\ (a+\theta)\,xx_0 + \ldots = 0],$$
$$[(a+\theta)\,xx_1 + \ldots = 0,\ xx_0 + \ldots = 0]$$

should be identical, is the existence of equations of the form $(a+\theta)\,x_1 = \rho\,(a+\psi)\,x_0$, for each coordinate, for proper values of ψ and ρ; or, if $x_2 = x_1 - \rho x_0$, etc., that $(a+\theta)x_2 = \rho\,(\psi - \theta)\,x_0$, etc., or, say $(a+\theta)\,x_2 = \sigma x_0$. The condition that $(x_1, \ldots)$, and hence also

$(x_2, \ldots)$, should lie on the conical sheet, requiring

$$(a+\theta)\,x_0x_2 + \ldots = 0, \quad (a+\theta)\,x_2^2 + \ldots = 0,$$

is satisfied if $x_0^2(a+\theta)^{-1} + \ldots = 0$, in virtue of $x_0^2 + \ldots = 0$. As we also have $ax_0^2 + \ldots = 0$, this equation for θ is only a quadratic; denote its roots by λ and μ. Then the curve, in which the surface (Ω, Ω') is met by either of the two solids $\lambda T + T'$, $\mu T + T'$, has a cusp at P. The two cuspidal tangents give, on projection, the cuspidal tangents of two curves in which the Cyclide is met by two spheres. These we define as the directions of the lines of curvature.

The surface (Ω, Ω'), if $f(t)$ be the quintic polynomial whose roots are $a, b, \ldots, e$, has the coordinates of its points $(x, y, \ldots)$ represented by equations $x^2 = (a+\theta)(a+\phi)/f'(a), \ldots, u^2 = (e+\theta)(e+\phi)/f'(e)$; the values of θ, ϕ appropriate to the point $(x, y, \ldots)$ are the roots of the quadratic equation $x^2(a+\theta)^{-1} + \ldots + u^2(e+\theta)^{-1} = 0$. If we pass, along the curve of the surface for which θ is constant, from $(x, y, \ldots)$ to $(x+dx, y+dy, \ldots)$, we have $dx = \frac{1}{2}x\,d\phi\,(a+\phi)^{-1}$, etc.; namely, the tangent line of this curve is the line from $(x, y, \ldots)$ to the point $(x(a+\phi)^{-1}, y(b+\phi)^{-1}, \ldots)$. This tangent line is then, by what we have proved above, the cuspidal tangent of the intersection of (Ω, Ω') with the solid $\phi T + T'$, where T, T' are the tangent solids of Ω and Ω' at the point considered. Thus, the curve along which θ is constant has its tangent line along a direction of curvature, at every point, but the *osculating* solid (and consequent *sphere*) has a parameter ϕ which varies from point to point of this curve.

This result, after what is proved above, is in accord with a theorem, associated with the name of Dupin, that if, in space of three dimensions, three surfaces be such that every two of them cut at right angles, in regard to the Absolute conic, at all their common points, then, at a point common to all the surfaces, the tangent line of the common curve of any two of these surfaces, is the tangent line of a line of curvature on each. In particular, a Cyclide is intersected in a line of curvature by any one of the five principal spheres.

Ex. 4. It was remarked that the line joining a point, $(x, y, \ldots)$, of the surface Γ, or (Ω, Ω'), to a point of coordinates $(a+\psi)\,x$, $(b+\psi)\,y, \ldots$, becomes, on projection, a normal of the Cyclide. This line is, evidently, on projection, at right angles to the line, considered in Ex. 3, joining $(x, y, \ldots)$ to the point of coordinates $(a+\psi)^{-1}x$, $(b+\psi)^{-1}y, \ldots$. Consider two points $(x, y, \ldots)$, $(x+dx, y+dy, \ldots)$, of Γ, on the curve given, for constant θ, by the equation $(a+\theta)^{-1}x^2 + \ldots + (e+\theta)^{-1}u^2 = 0$, these points corresponding to values ϕ, $\phi + d\phi$, of the second parameter, ϕ, determining points of Γ, as in Ex. 3. Corresponding to the first point, take the point of coordinates $(a+\phi)\,x$, $(b+\phi)\,y, \ldots$; this is, in fact, the pole, in regard to Ω, of the tangent solid of $\phi\Omega + \Omega'$ at the point $(x, y, \ldots)$, and projects into the centre of the sphere into which the surface $(\Omega, \phi T + T')$ projects. Corresponding to the second point, take the point of coordinates $(a + \psi)(x + dx)$, $(b + \psi)(y + dy), \ldots$, where $\psi = \phi - \frac{1}{2}d\phi$. Since $dx = \frac{1}{2}x\,d\phi\,(a + \phi)^{-1}$, we have

$$(a + \phi - \tfrac{1}{2}d\phi)(x + dx) = (a + \phi - \tfrac{1}{2}d\phi)\left[1 + \tfrac{1}{2}(a + \phi)^{-1}d\phi\right]x$$
$$= (a + \phi)x - \tfrac{1}{2}d\phi\,dx.$$

Thus, on projection, we have the result often expressed by saying that the normals, at two consecutive points of a line of curvature, intersect one another. These normals, that is, form a developable surface, whose *cuspidal edge* (Vol. III, pp. 131, 132), or *edge of regression*, is the locus of the centres of the osculating spheres.

Ex. 5. By considering when the two cuspidal directions obtained in Ex. 3 can coincide, prove that the lines of the surface Γ lie on the quartic locus, obtained as the discriminant, in regard to θ, of the quadratic equation $(a + \theta)^{-1}x^2 + \ldots = 0$, whose equation is

$$(p_1\Omega_2 + \Omega_3)^2 = 4\Omega_2(p_2\Omega_2 + \Omega_4),$$

where

$$\Omega_r = a^r x^2 + \ldots + e^r u^2, \quad p_1 = a + \ldots + e, \quad p_2 = ab + ac + \ldots + de.$$

Ex. 6. By inversion of a surface in threefold space, a line of curvature is changed into a line of curvature of the inverse surface.

Ex. 7. Consider, in threefold space, whose Absolute conic is ω, a curve, γ, lying upon a surface of this space. The tangent lines of this curve γ meet the plane of the Absolute conic in points, T, describing a curve, τ, of this plane. The normals of the surface, at points of the curve γ, that is, the lines conjugate to the tangent planes in regard to the Absolute conic, meet the Absolute plane in points, N, describing a curve, ν, of this plane. If, for every point of the curve γ, the tangent line of the curve ν, at the point N, passes through the corresponding point, T, of the curve τ, then the curve γ is a line of curvature of the surface. The condition may be expressed briefly by saying that the curve defined by the normals is a curve of pursuit of the curve defined by the tangents.

Ex. 8. Two surfaces, in threefold space, are such as to cut at right angles at every point of their curve of intersection, which is known to be a line of curvature on one of the surfaces. Considering, in the Absolute plane, the curve enveloped by the tangent planes of one of the surfaces, at the points of their common curve, and the like curve enveloped by the tangent planes of the other surface, at the points of this curve, shew that these curves are polar reciprocals of one another in regard to the Absolute conic.

Ex. 9. If a line of curvature of a surface, in threefold space, be a plane curve, prove that the tangent plane of the surface at points of the curve is at a constant inclination to the plane of the curve (with respect to the Absolute conic). Deduce this by proving that, if the normals of a plane curve, defined as lines conjugate to the tangent lines with respect to an Absolute *conic* of the plane, all meet in a point, then the curve is the locus of a point which is at a constant *interval* from the point of intersection of the normals, the interval being defined by the Absolute conic (as in Vol. II, pp. 168 ff.).

Ex. 10. If the tangent plane of a surface, in threefold space, at a point (x, y, z, t), be expressed by $lX + mY + nZ + pT = 0$, an inflexional line of the surface satisfies the differential equation

$$dx\,dl + dy\,dm + dz\,dn + dt\,dp = 0.$$

If l, m, n, l', m', n' be the line coordinates of the normal of the surface, a line of curvature satisfies the differential equation

$$dl\,dl' + dm\,dm' + dn\,dn' = 0.$$

More generally, while the lines of a ruled surface, in space of three dimensions, are represented, in space of five dimensions, by the points of a curve lying on the fundamental quadric Ω (p. 40, above), the lines of a developable surface, or the tangents of a curve, in the threefold space, are represented by a curve on Ω of which all the tangent lines also lie on Ω.

The matters discussed in these Examples arise again below, in Chap. VII (p. 232). For a further generalisation see Darboux, *Théorie...des Surfaces*, Livre VII, p. 485, § 840.

In regard to some particular cases. The Dupin Cyclide. In considering the quartic surface, in the space of four dimensions, as the intersection of two quadrics, we have generally assumed that the two quadrics are general, so that there are five quadric point-cones passing through the surface, whose vertices form a self-polar pentad for the quadrics. In this case the equations of the two quadrics can be supposed to be of the forms, $x^2 + \ldots + u^2 = 0$, and $ax^2 + \ldots + eu^2 = 0$, in which no two of the coefficients $a, b, \ldots, e$ are equal. And we have given the centre of projection, by which we pass to the Cyclide, a quite general position. In this Volume we do not enter into the discussion of the possibilities which can arise in less general cases; Segre (*Math. Annal.* XXIV, p. 440) enumerates more than seventy. We make, however, some remarks, with the aim of indicating the nature of the special cases; and, in particular, on account of its historical and intrinsic interest, we shew how Dupin's Cyclide can be obtained.

(*a*) It may happen that, through a surface, in fourfold space, which is defined by the intersection of two quadrics, which are cones, there passes no quadric which is not a cone. In other words, if $U=0$, $V=0$ be the given quadric cones, it may happen that the discriminantal determinant, of five rows and columns, of the quadric $U+\lambda V=0$, vanishes for all values of λ. To illustrate this possibility, suppose $U=y^2+z^2+u^2-tu$, $V=yz-xu$, each of which, as depending on four linear functions of the coordinates only, belongs to a cone. We then have

$$U+\lambda V=y^2+\lambda yz+z^2+u\,(u-t-\lambda x);$$

for every value of λ, this likewise depends on four linear functions of the coordinates only, and also belongs to a cone.

The projection of the surface (U, V), in this case, from the point $(0,0,0,0,1)$, gives, in the space (x, y, z, t), the surface whose equation is $y^2z^2+z^2x^2+x^2y^2-xyzt=0$. This is *Steiner's quartic surface* (Vol. III, p. 222); its properties can then be deduced by considering the surface (U, V). The cones $U=0$, $V=0$ have a common tangent solid at every point of the line joining their vertices, and the projection is made, on to this solid, from a point on one of the cones.

The Steiner surface may also be obtained by projecting the Veronese surface, in space of five dimensions (p. 52, above), by means of planes passing through a line which does not meet this surface.

(*b*) When, beside cones, there is at least one general quadric containing the surface, say $U=0$, the discriminant of $U+\lambda V$, not vanishing identically in regard to λ, vanishes for, at most, five values of λ. There are then other quadrics which are not cones, beside $U=0$, containing the surface. Let $V=0$ be such a quadric. Then, two cases are possible. Either, there is *not* a point, common to the quadrics $U=0$, $V=0$, at which these have the same, definite, tangent solid; or there is such a point, or several such. We sketch now, incompletely, a proof that, in the former of these cases, the equations of $U=0$, $V=0$ can be supposed to be of the respective forms $x^2+y^2+\ldots+u^2=0$, $ax^2+by^2+\ldots+eu^2=0$, in which no two of the coefficients $a, b, \ldots, e$ are equal. This will then be the general case considered above. In fact, under the hypothesis made, there will be at least one point, not lying on $U=0$, nor on $V=0$, for which the five ratios, respectively of $\partial U/\partial x$, $\partial U/\partial y$, ..., $\partial U/\partial u$ to $\partial V/\partial x$, $\partial V/\partial y$, ..., $\partial V/\partial u$, are equal. If we change the notation so that this point becomes $(0,0,0,0,1)$, and its polar solid, in regard to either $U=0$, $V=0$, which is the same for both, becomes $u=0$, then the equations of the two quadrics take the forms $u^2+\phi=0$, $eu^2+\psi=0$, where ϕ, ψ are quadratic forms in x, y, z, t only. The quadric surfaces, $\phi=0$, $\psi=0$, regarded as lying in the solid $u=0$, will not have a common point at which their tangent planes coincide; for then, the solid joining this plane to the point $(0,0,0,0,1)$

would be a common tangent solid of the two original quadrics, at a common point. And the discriminant of $\lambda\phi + \psi$ is not identically zero. We can therefore apply a similar argument to $\phi = 0$, $\psi = 0$, reducing them to the respective forms $t^2 + \xi = 0$, $dt^2 + \eta = 0$, where ξ, η are quadratic forms in x, y, z, only. And so on. Thereby U, V are reduced, respectively, to forms $x^2 + \ldots + u^2$, $ax^2 + \ldots + eu^2$. Herein, however, we cannot have, for example, $d = e$; since, then, the quadrics $U = 0$, $V = 0$ would have the common tangent solid, $t + iu = 0$, at the common point $(0, 0, 0, 1, i)$.

(*c*) Suppose now that $U = 0$, $V = 0$ have a common point, which we may take to be $(0, 0, 0, 0, 1)$, and, thereat, a definite common tangent solid, which we may take to be $t = 0$. Their equations are then capable of the forms $ut + \phi = 0$, $ut + \psi = 0$, where ϕ, ψ are quadratic forms in x, y, z, t, only. The surface of intersection of the quadrics thus lies on $\phi - \psi = 0$. This is, then, either a point-cone whose vertex is $(0, 0, 0, 0, 1)$, or a line-cone whose axis passes through this point, or a pair of solids, or a single repeated solid. In all these cases, the projection of the quartic surface (U, V), from this point, upon any solid, is a *quadric* surface, of a general or particular kind. Thus the Cyclide arising by projection of the surface (U, V) from a general point, say, of $U = 0$, is one which is the inverse, in the solid upon which we project, of a quadric surface, of general or of particular kind. It is clear that there are many cases.

(*d*) In what has preceded, we have spoken of projecting the quartic surface of fourfold space, into a Cyclide, from a centre of projection lying on a particular quadric which contains the surface; and have used this quadric to define inversion in the space of the Cyclide. In general, as a single quadric of the form $U + \lambda V = 0$ passes through an arbitrary point of the fourfold space, this is equivalent to defining the Cyclide as arising by projection from an arbitrary point of the space. If we do this, however, special cases may arise. For instance, it may happen that the quadric, of the family $U + \lambda V = 0$, which passes through the centre of projection, is a cone. Then it is easy to see that the surface of the fourth order, obtained by projection, in place of having a double conic of general kind, has two double lines, which may coincide. Or, again, it may happen, when the surface in the fourfold space lies on a line-cone, that the tangent solid, at the centre of projection, of the quadric, of the family, which passes through this point, is a tangent solid of this line-cone. In this case, while there may exist a double conic for the quartic surface obtained by projection, it will be such that the tangent planes of this surface coincide at every point of this conic. The double conic then becomes a *cuspidal* conic.

(*e*) For historical and intrinsic reasons, some account may be given of the case in which the quartic surface, in fourfold space, lies on one, and, more particularly, on two quadric line-cones. The former arises when the surface is given by two equations of the forms $x^2+y^2+\ldots+u^2=0$, $ax^2+by^2+eu^2=0$; the latter arises when, further, $a=b$, the second line-cone containing the surface then being $a(z^2+t^2)+(a-e)u^2=0$. The latter surface projects into what has been called *Dupin's Cyclide* (Dupin, *Applications de géométrie et de méchanique*, 1822, p. 200). This surface is in point correspondence with the *Tore* (or *Anchor Ring*); and either can be inverted into the other, or into a quadric cone of revolution. Beautiful stereoscopic diagrams of forms of Dupin's Cyclide are given in Maxwell's *Scientific Papers*, Vol. II, p. 158.

Consider first the case when the quartic surface in fourfold space lies upon only one quadric line-cone. This cone, then, arises in place of two of the five point-cones existing in the general case. The axis of the line-cone meets a general quadric of the family, say the quadric Ω, in two points, say M and N. On projection from a point, O, of Ω, these become two double points of the resulting Cyclide. A tangent solid of the line-cone, which touches this at all points of a plane, meets Ω in a quadric surface; on projection, we have a sphere touching the Cyclide in the points of a circle, passing through the two double points. The centres of such spheres lie on a conic; this is the projection of the conic which is the polar reciprocal of the line-cone, in regard to Ω; the plane of this conic is the projection of the polar plane, in regard to Ω, of the axis of the line-cone. The two conical sheets, of lines lying upon Ω, passing through M and N, respectively, intersect in a conic, lying in the polar plane of the axis of the line-cone; this plane is in the polar solid, in regard to Ω, of every point of this axis. Thus, on projection, there is a circle, lying in the plane which bisects, at right angles, the line joining the two double points of the Cyclide. Every sphere passing through this circle has its centre on this joining line, and the Cyclide inverts into itself in regard to this sphere. In particular, there is one such sphere which reduces, beside the Absolute plane, to a plane; so that there is a plane in regard to which the Cyclide is symmetrical.

It is possible to find points, H, not lying on Ω, such that the points, M', N', in which the lines HM, HN meet Ω again, lie on the tangent solid of Ω at the point O. For, let HO meet Ω again in O'; if M' be a point on the tangent solid at O, the line OM' lies on Ω; the plane, HMO, meeting Ω in the line OM', meets Ω in another line; thus, MO' is on Ω, or O' lies on the tangent solid at M; and conversely. Similarly, if N' be on the tangent solid at O, the point O' lies on the tangent solid at N. Take, then, for O', any point of the conic, on Ω, which is common to the tangent solids at M and N; for H, take any point of the line OO'. This point, H, can be used for the centre of the harmonic inversion, in the fourfold space, which gives rise to inversion, in regard to a sphere, in the threefold space upon which we project. Thereby it is shewn that the Cyclide, obtained by projection of the quartic surface which is the intersection of Ω with the quadric line-cone having MN for axis, can be inverted, in the threefold space, into a Cyclide having two double points on the Absolute conic, the centre of inversion being any point of a certain circle. It is a simpler remark that a point H, not on Ω, can be taken on the line OM; so that the Cyclide can also be inverted into a quadric cone.

Ex. 1. Consider the quartic surface given by the equations $x^2+y^2+\ldots+u^2=0$, $ax^2+by^2+\ldots+eu^2=0$; denote the former quadric by Ω; let O, of coordinates $(x_0, y_0, \ldots u_0)$, be any point of Ω. Also, let A, B, C, D, respectively, denote $a-e$, $b-e$, $c-e$, $d-e$, and A', B', C', E', respectively, denote $a-d$, $b-d$, $c-d$, $e-d$, so that $E'+D=0$. The quartic surface thus lies on the two point-cones

$$Ax^2+By^2+\ldots+Dt^2=0,\quad A'x^2+B'y^2+\ldots+E'u^2=0.$$

We have, identically,

$$(Ax^2+By^2+Cz^2+Dt^2)(Ax_0^2+By_0^2+Cz_0^2+Dt_0^2)$$
$$-(Axx_0+Byy_0+Czz_0+Dtt_0)^2$$

equal to

$$[l'(BC)^{\frac{1}{2}}+l(AD)^{\frac{1}{2}}]^2+[m'(CA)^{\frac{1}{2}}+m(BD)^{\frac{1}{2}}]^2+[n'(AB)^{\frac{1}{2}}+n(CD)^{\frac{1}{2}}]^2,$$

where $l=tx_0-t_0x$, $l'=yz_0-y_0z$, etc. Let this last be denoted by U. We thus infer that, when $(x, y, \ldots t)$ lies on the quartic surface, we have

$$Axx_0+Byy_0+Czz_0+Dtt_0=(-U)^{\frac{1}{2}}.$$

If V denote what U becomes when, for A, B, C, D, are put, respectively, A', B', C', E', and, for l, m, n, are put, respectively, ux_0-u_0x, uy_0-u_0y, uz_0-u_0z, we similarly have, on the quartic surface,

$$A'xx_0+B'yy_0+C'zz_0+E'uu_0=(-V)^{\frac{1}{2}}.$$

Thus, if T denote $xx_0+yy_0+zz_0+tt_0+uu_0$, we have, on the quartic surface,

$$(d-e)T=(-U)^{\frac{1}{2}}-(-V)^{\frac{1}{2}}.$$

In this equation, $T=0$, $l'=0$, $l=0$, etc., $ux_0-u_0x=0$, etc., all represent solids passing through the point O. Thus this equation is, effectively, an irrational form of the equation of the general Cyclide which is obtained by projecting the quartic surface from the point O.

Ex. 2. For the case when the quartic surface is given by a general quadric, Ω, and a *line-cone*, we may proceed as in Ex. 1. We may also proceed as follows: Let $t=0$ be the tangent solid of the quadric Ω at the point from which the projection, to the space of three dimensions, is to be made; this point we now denote by U. Let $u=0$ be the polar solid of U in regard to the line-cone; this solid contains the axis of the cone. Let the point in which $t=0$ meets the axis of the line-cone be Z, and let $z=0$ be the polar solid of Z in regard to Ω; thus $z=0$ is undefined when $t=0$ contains the axis of the line-cone. The solid $z=0$ contains the point U; and contains the polar plane, in regard to Ω, of the axis of the line-cone. We define two other solids, $x=0$, $y=0$, as follows: Consider the plane $u=0$, $z=0$; this meets the quadric Ω in a conic; in this plane take the pole of the line $u=z=t=0$, in regard to this conic, say the point T; pairs of lines drawn from T, conjugate in regard to this conic, meet the line $u=z=t=0$ in pairs of points in involution. The line-cone meets the plane $u=0$, $z=0$ in two lines intersecting in the point in which the axis of the cone, which lies in $u=0$, meets this plane; let this point be $(x_0, y_0, 0, 1, 0)$. We can then choose $x=0$, $y=0$, vanishing at U, so that the solids $x-x_0t=0$, $y-y_0t=0$ contain the axis of the cone, and meet the plane $z=0$, $u=0$ in two lines harmonic in regard to the lines in which the line-cone meets this plane; while, at the same time, the three lines, of the plane $z=0$, $u=0$, given by $x=0$, $y=0$, $t=0$, are a self-polar triad in regard to the conic in which this plane meets Ω.

With this choice of coordinates we may suppose the equations of Ω, and of the line-cone, to be, respectively,

$$x^2+y^2+z^2-r^2t^2-2tu=0,\quad a^2(x-x_0t)^2+b^2(y-y_0t)^2-u^2=0,$$

where a, b, r are certain constants. By elimination of u we have the equation of the Cyclide,

$$(x^2+y^2+z^2-r^2t^2)^2=4t^2\left[a^2(x-x_0t)^2+b^2(y-y_0t)^2\right].$$

This represents a surface, symmetrical in regard to $z=0$, which is the envelope of the sphere

$$(x-ta\cos\theta)^2+(y-tb\sin\theta)^2+z^2=t^2\left[(x_0-a\cos\theta)^2+(y_0-b\sin\theta)^2+r^2-x_0^2-y_0^2\right];$$

the centre of this sphere is $(a\cos\theta, b\sin\theta, 0, 1)$, lying on a certain conic; and the sphere passes through the points

$$[x_0, y_0, \pm(r^2-x_0^2-y_0^2)^{\frac{1}{2}}, 1].$$

The surface inverts into itself, in regard to the sphere,

$$x^2+y^2+z^2-2xx_0t-2yy_0t-2zht+r^2t^2=0,$$

whatever h may be.

If we take λ_1, λ_2, λ_3 as the roots of $f(\lambda)=0$, where

$$f(\lambda)=(a^2-\lambda)^{-1}a^2x_0^2+(b^2-\lambda)^{-1}b^2y_0^2-r^2-\lambda,$$

and put, to define ξ_1, ξ_2, ξ_3, ξ_4,

$$x-x_0t=x_0\sum_{k=1}^{3}(a^2-\lambda_k)^{-1}\lambda_k\xi_k,$$

$$y-y_0t=y_0\sum_{k=1}^{3}(b^2-\lambda_k)^{-1}\lambda_k\xi_k,\quad u=\sum_{k=1}^{3}\lambda_k\xi_k,$$

together with $t=\xi_1+\xi_2+\xi_3+\xi_4$, we find

$$a^2(x-x_0t)^2+b^2(y-y_0t)^2-u^2=\sum_{k=1}^{3}\lambda_k^2f'(\lambda_k)\,\xi_k^2,$$

$$z^2+x^2+y^2-r^2t^2-2tu=z^2+\sum_{k=1}^{3}\lambda_kf'(\lambda_k)\,\xi_k^2+(x_0^2+y_0^2-r^2)\,\xi_4^2,$$

where $f'(\lambda)$ denotes the derivative of $f(\lambda)$.

Ex. 3. The equations when the quartic surface, in the fourfold space, is given by the intersection of *two* line-cones may be obtained as a particular case of Ex. 2. Two of the roots of $f(\lambda)=0$ will, in fact, become equal to b^2, if $y_0=0$ and $a^2x_0^2=(r^2+b^2)(a^2-b^2)$. In general, if a^2, b^2, r^2, $r^2-x_0^2-y_0^2$ are positive real quantities ($a^2>b^2$), the roots of $f(\lambda)=0$ are in the intervals of $-\infty$, 0, b^2, a^2. Suppose the two particular conditions satisfied; introduce μ instead of r, given by $\mu^2=r^2+b^2$, and put $c^2=a^2-b^2$. With $C=(\mu^2-c^2)^{\frac{1}{2}}$, $A=(\mu^2-a^2)^{\frac{1}{2}}$, the reduction to the general formulation is given by putting

$$bCX=a\mu x-\mu^2ct-cu,\quad bACT=acx-\mu c^2t-\mu u,\quad bAU=\mu cx+a(b^2-\mu^2)t-au,$$

which lead to

$$(ax-\mu ct)^2+b^2y^2-u^2=b^2\left[y^2+X^2-A^2T^2\right],$$

$$b^2z^2-(cx-\mu at)^2+(u-b^2t)^2=b^2\left[z^2+C^2T^2-U^2\right],$$

and hence

$$x^2+y^2+z^2+(b^2-\mu^2)t^2-2tu=y^2+z^2+X^2+b^2T^2-U^2.$$

The quartic surface in fourfold space is thus the intersection of two line-cones. The Cyclide obtained by projection has, therefore, four double points; its equation is capable, among others, of the forms

$$\left[x^2+y^2+z^2+(b^2-\mu^2)t^2\right]^2=4t^2\left[(ax-\mu ct)^2+b^2y^2\right],$$

$$\left[x^2+y^2+z^2-(b^2+\mu^2)t^2\right]^2=4t^2\left[(cx-\mu at)^2-b^2z^2\right].$$

A particular irrational form (noticed by Kummer, *Berlin. Monatsber.*, 1863), which may be found as in Ex. 1, is

$$b^2t + [(ax - \mu ct)^2 + b^2y^2]^{\frac{1}{2}} - [(cx - \mu at)^2 - b^2z^2]^{\frac{1}{2}} = 0.$$

The polar reciprocals of the line-cones, with respect to the general quadric, Ω, containing the surface, which is found above, expressed by five squares, are given by $U = 0$, $z = 0$, $a^{-2}x^2 + b^{-2}y^2 = t^2$, and by $X = 0$, $y = 0$, $c^{-2}x^2 - b^{-2}z^2 = t^2$. On projection, to the space (x, y, z, t), we have two conics, in the planes $z = 0, y = 0$, so related that their common tangent planes touch the Absolute conic $t = 0$, $x^2 + y^2 + z^2 = 0$. Reciprocally, this is the statement that, in the fourfold space, the two line-cones meet the solid $t = 0$ in two quadric cones (given by $a^2x^2 + b^2y^2 - u^2 = 0, c^2x^2 - b^2z^2 - u^2 = 0$) whose common curve is on Ω, as well as on $t = 0$. If we denote the irrational form, noticed, by $p + (qr)^{\frac{1}{2}} - (sv)^{\frac{1}{2}} = 0$, the Absolute conic is given by $p = 0$, $qr - sv = 0$; the double points are given by $q = r = p^2 - sv = 0$ and $s = v = p^2 - qr = 0$, that is by

$$ax_0 - \mu ct_0 = y_0 = az_0 \pm b\,(\mu^2 - a^2)^{\frac{1}{2}}\,t_0 = 0$$

and $$cx_1 - \mu at_1 = z_1 = cy_1 \pm b\,(c^2 - \mu^2)^{\frac{1}{2}}\,t_1 = 0,$$

for which $c^{-2}x_0^2 - b^{-2}z_0^2 = t_0^2$, $a^{-2}x_1^2 + b^{-2}y_1^2 = t_1^2$. The Cyclide is obtainable as the envelope of spheres

$$(x - ta\cos\theta)^2 + (y - tb\sin\theta)^2 + z^2$$
$$= t^2\,[(t_0^{-1}x_0 - a\cos\theta)^2 + b^2\sin^2\theta + t_0^{-2}z_0^2],$$

which have their centres on the conic $z = 0$, $a^{-2}x^2 + b^{-2}y^2 = t^2$ and pass through the first pair of double points. It is likewise obtainable as the envelope of spheres, with centres on the conic $y = 0$, $c^{-2}x^2 - b^{-2}z^2 = t^2$, passing through the second pair of double points. Any sphere of the former system touches any sphere of the latter system. It is clear, in fact, in the fourfold space, that, at a common point of the two line-cones defining the surface, the tangent solid of Ω is met in the same plane by the tangent solids of both the line-cones. Thus, for the Cyclide, taking two spheres of either system, the Cyclide can be generated as the envelope of a sphere, with centre on a certain conic, which touches two fixed spheres; or, again, as the envelope of one of the four systems of spheres which touch three given spheres,—a simple result of inversion from a common point of the three spheres. Further, the circles in which the Cyclide is touched by the enveloping spheres of the two systems, are easily seen to be *lines of curvature* (p. 186, above). Dupin's Cyclide may, indeed, be defined by this property of having two systems of lines of curvature which are circles. Also, we notice, any normal of the Cyclide meets the two conics

$(a^{-2}x^2 + b^{-2}y^2 - t^2 = 0,\ z = 0)$, $(c^{-2}x^2 - b^{-2}z^2 - t^2 = 0,\ y = 0)$; and conversely. This Cyclide is the only surface, in threefold space, of which the normals meet two curves; if the surface be a wave-front this is an optical theorem (Maxwell, *Papers*, II, p. 144, 1867). In the case of the general Cyclide, we have defined a line, in the fourfold space (p. 183, above), which, on projection, becomes the normal; in the present case this is the line whose points, for varying ψ, have coordinates

$$\psi x - \mu act,\ (\psi - c^2)y,\ (\psi - a^2)z,\ (\psi - a^2)t + u,$$
$$(\psi - c^2 - \mu)u + \mu c(ax - \mu ct),$$

where (x, y, z, t, u), arising for $\psi = \infty$, is the foot of the normal.

Ex. 4. Prove that the Dupin Cyclide is obtainable by rationalising the equation

$$\psi_1^{\frac{1}{2}} + \psi_2^{\frac{1}{2}} + \psi_3^{\frac{1}{2}} = \mu,$$

where ψ_1, ψ_2, ψ_3 are the three roots of the equation $C(\psi) = 0$, in which

$$C(\psi) = \psi^{-1}x^2 + (\psi - c^2)^{-1}y^2 + (\psi - a^2)^{-1}z^2 - t^2;$$

the equation of the Cyclide so obtained, one factor of the complete result of rationalisation, appears in the form

$$[x^2+y^2+z^2+t^2(c^2+a^2-\mu^2)]^2$$
$$=4t^2[(act-\mu x)^2+x^2(a^2+c^2-\mu^2)+a^2y^2+c^2z^2].$$

This result is obtained by Maxwell from optical considerations. It can be shewn that, if w denote the interval, measured in regard to the quadric surface $C(\theta) = 0$, between two points of a line which touches the two confocal quadric surfaces $C(p) = 0$, $C(q) = 0$, then

$$2w\frac{(\theta - p)(\theta - q)}{\Theta} = \sum_{k=1}^{3}\int\frac{(\psi_k - p)(\psi_k - q)}{(\psi_k - \theta)\Psi_k}\,d\psi_k,$$

where $\Theta^2 = 4\theta(\theta - c^2)(\theta - a^2)(\theta - p)(\theta - q)$, or, say, $\Theta^2 = F(\theta)$, and $\Psi_k^2 = F(\psi_k)$. When $p = c^2$, $q = a^2$, which corresponds to a line meeting the two focal conics, as in Ex. 3, this gives

$$\exp.(2w) = \prod_{k=1}^{3}(\theta^{\frac{1}{2}} + \psi_k^{\frac{1}{2}})/(\theta^{\frac{1}{2}} - \psi_k^{\frac{1}{2}});$$

and if, herein, we suppose θ to increase indefinitely, putting $\theta^{\frac{1}{2}}w = \mu$, this will be found to lead to Maxwell's result quoted. (Cf., also, Darboux, *Théorie...des Surfaces*, Livre IV, pp. 297, 308; and a note, *Proc. Camb. Phil. Soc.*, XX, 1921, p. 129.)

Ex. 5. It was shewn (p. 193) that, in the fourfold space, two given points of the general quadric Ω, can be projected into two other points of Ω lying on the tangent solid at a given point, O, of Ω. It was seen that the centre of such projection is any point (not on Ω), of the conical sheet joining O to the conic in which the polar plane, of the joining line of the two given points, meets Ω. Such a projection changes a line-cone into a line-cone; thus a quartic surface, in the fourfold space, given by the intersection, lying on Ω, of two quadric line-cones, is changed, by such projection, into a quartic surface given by two such line-cones, of which, however, two double points lie on the tangent solid of Ω at O. Such a quartic surface projects, from O, on to any solid, into a *Tore*, or *Anchor Ring*. When this is proved, it will follow that the Dupin Cyclide can be inverted into a Tore. The necessary centre of inversion, in the threefold space, is any point of either of two circles; one of these circles is the intersection of the cones (or *point-spheres*) which join a pair of the double points of the Dupin Cyclide to the Absolute conic.

Suppose, in fact, that Ω contains the intersection of one quadric line-cone, whose axis meets Ω in two points, A and B, with another quadric line-cone, whose axis meets Ω in C and D. Thus CDA is a plane of the line-cone (CD); it can be seen easily that the tangent solid of Ω at A is likewise a tangent solid of the line-cone (CD), touching it along the plane CDA. Thus, the polar plane of the line AB, in regard to Ω, passes through the line CD. Now suppose that A and B are in the tangent solid of Ω at a point O, so that the polar plane of the line AB is the plane OCD. Then, projecting from O, one of the two focal conics of the resulting Cyclide degenerates into the line which is the projection of CD. The other focal conic, the projection of the polar reciprocal, in regard to Ω, of the line-cone (CD), becomes a circle. For, this requires that the polar reciprocal of this line-cone has two points, lying on Ω, which are on the tangent solid at O; or, that there are two tangent planes of this line-cone which touch Ω and pass through O; by what has been said, these are the tangent planes of Ω at A and B. The proof can, then, be completed easily; and it can be shewn, further, that the tangent planes of the Tore, at either of the two double points which lie on the Absolute conic, coincide. Each of these points is itself a coincidence of two of the four points of the Absolute conic, generally existing (p. 180), at which the tangent planes of the Cyclide are the same.

If, in the equations of Ex. 3, we suppose $c = 0$ $(a = b)$, the double points of the Cyclide are given by $t_1 = z_1 = x_1^2 + y_1^2 = 0$, and

$$x_0 = y_0 = z_0 \pm (\mu^2 - a^2)^{\frac{1}{2}}\, t_0 = 0,$$

and the equation of the surface becomes

$$[x^2 + y^2 + z^2 - (\mu^2 - a^2)\, t^2]^2 = 4t^2 a^2 (x^2 + y^2).$$

To obtain such a surface by inversion of the original, we are to take the centre of inversion at any point of the circle of intersection of the cones

$$[cx - \mu at]^2 + [cy \pm b(c^2 - \mu^2)^{\frac{1}{2}} t]^2 + c^2 z^2 = 0,$$

which lies in $y = 0$. Similarly, taking the other pair of double points, a centre of inversion may be taken at any point of the circle $z = 0$, $(ax - \mu ct)^2 + a^2 y^2 + b^2 (\mu^2 - a^2) t^2 = 0$.

By taking the inversion from the point $cx = (\mu a + bC\cos\theta) t$, $cy = bC\sin\theta t$, where $C = (\mu^2 - c^2)^{\frac{1}{2}}$, it is found that the Tore obtained is given by the equation

$$[\xi^2 + \eta^2 + \zeta^2 + R^2 t^2]^2 = m^2 (\xi^2 + \eta^2) t^2,$$

where

$$m^2 = k^4 c^4 C^{-2} (b\mu + aC\cos\theta)^{-2},\ 4R^2 = -A^2 m^2 b^{-2},\ = (a^2 - \mu^2) m^2 b^{-2},$$

and k^2 is arbitrary. If

$$\sigma = \tfrac{1}{2} ck^2/bC,\ \rho^2 = t^2 k^2/(x_1{}^2 + z_1{}^2 + y^2),\ p = \mu a + bC\cos\theta,$$
$$q = \mu b + aC\cos\theta,$$

the actual formulae are

$$x_1 = x - tc^{-1}p, \quad z_1 = z - tc^{-1}bC\sin\theta, \quad \eta = \rho^2 y,$$
$$\xi = \rho^2(x_1\cos\theta + z_1\sin\theta) + \sigma t, \quad \zeta = \rho^2(x_1\sin\theta - z_1\cos\theta) + \sigma t a C q^{-1}\sin\theta.$$

The double points of the Tore are given by $(0, 0, \pm iR, 1)$ and $(1, \pm i, 0, 0)$.

The inversion of a Dupin Cyclide into a Tore is given by Darboux (*Sur une classe remarquable de courbes et de surfaces algébriques*, Paris, 1873, p. 242), with reference to Mannheim (*Nouv. Ann. de Math.*, 1860, p. 67). Darboux operates in threefold space, with spheres, saying (*loc. cit.*, p. 164): *Comme on n'a pas d'espace à quatre dimensions, les méthodes de projection ne s'étendent pas à la géométrie de l'espace.*

Ex. 6. The locus of the pole, for inversion, transforming any two spheres into spheres which are equal, consists of two spheres coaxial with the first, with centres at their centres of similitude. The locus of the pole, for inversion, transforming three spheres, with *directed* radii, into three spheres of equal *and like* radii, is the circle common to the two point-spheres whose centres are the intersections of the orthogonal circle and axis of similitude of the three original spheres (Darboux, *loc. cit.*, pp. 243, 244). With different signs for the three original radii, different axes of similitude arise.

Ex. 7. Determine the character of the curve of intersection of the enveloping cones, to a quadric in fourfold space, drawn from three arbitrary points.

Prove that, in threefold space, the envelope of a sphere touching three given spheres, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, is

$$(\mu_{23}S_1)^{\frac{1}{2}} + (\mu_{31}S_2)^{\frac{1}{2}} + (\mu_{12}S_3)^{\frac{1}{2}} = 0,$$

where, for example, $\mu_{23} = 0$ is the condition of contact of the two latter spheres.

Ex. 8. With the notation of Ex. 3, putting ρ^2 for $(X^2+Y^2+Z^2)^{-1}$, the equations

$$aCx=\rho^2(\mu bX-cAZ)\,t^2+\mu cCt, \quad aCz=\rho^2(cAX+\mu bZ)\,t^2-bACt,$$

with $y = \rho^2 Yt^2$, transform the Dupin Cyclide into the cone of revolution

$$(X - \tfrac{1}{2}ct/Cb)^2 + Y^2 = b^{-2}A^2(Z - \tfrac{1}{2}\mu t/CA)^2.$$

Ex. 9. The Dupin Cyclide may be regarded as the envelope of its tangent planes, and its equation expressed tangentially. If we express tangentially one of the enveloping spheres, whose centre is on a focal conic, the equation of a tangent plane being written $x\xi + y\eta + z\zeta + t\tau = 0$, and find the envelope of this sphere as its centre moves, we find, for the two focal conics, the equations

$$(a\xi - c\omega)^2 + b^2\eta^2 - (\tau + \mu\omega)^2 = 0, \quad (c\xi - a\omega)^2 - b^2\zeta^2 - (\tau + \mu\omega)^2 = 0,$$

where $\omega^2 = \xi^2 + \eta^2 + \zeta^2$. The enveloping spheres, whose centres are at the points of the focal conics which lie on $t = 0$, give four double tangent planes, whose equations are

$$cx - \mu at \pm bz = 0, \quad ax - \mu ct \pm iby = 0.$$

In the fourfold space, we may consider the aggregate of the tangent planes of the quartic surface of intersection of the two quadric line-cones: Considering an arbitrary tangent solid of each of these line-cones, with the plane along which it touches the line-cone, these two planes have a common point; the intersection of the tangent solids is the tangent plane of the quartic surface at this point. After projection, this becomes the statement that any enveloping sphere, of one of the two systems, touches any sphere of the other system. The tangent solids, of the two line-cones, respectively given by

$$(ax-\mu ct)\cos\theta+by\sin\theta-u=0, \quad (cx-\mu at)\cos\phi-ibz\sin\phi-(u-b^2t)=0,$$

meet in a plane lying on the solid $x\xi + y\eta + z\zeta + t\tau = 0$, where

$$\xi=a\cos\theta-c\cos\phi, \quad \eta=b\sin\theta, \quad \zeta=ib\sin\phi, \quad \tau=-\mu(c\cos\theta-a\cos\phi)-b^2,$$

which, with $\omega^2 = \xi^2 + \eta^2 + \zeta^2$, lead to $\omega = c\cos\theta - a\cos\phi$. These satisfy the two equations given above. There are two tangent solids of either line-cone which pass through the centre of projection; these give the four double tangent planes.

Dually, in the fourfold space, we may consider the surface, in the solid $t = 0$, which is the locus of the intersection of this solid with the line joining the two points

$$(a \cos\theta,\ b \sin\theta, 0, 1, \mu c \cos\theta - \mu^2 + b^2),$$
$$(c \cos\phi, 0, -ib \sin\phi, 1, \mu a \cos\phi - \mu^2);$$

these are points of those conics which are the polar reciprocals of the line-cones, in regard to the quadric

$$x^2 + y^2 + z^2 + (b^2 - \mu^2) t^2 - 2tu = 0.$$

This is the line which, on projection, becomes the normal of the Cyclide; it meets $t = 0$ in the point $(\xi, \eta, \zeta, 0, -\tau)$.

Ex. 10. Prove that the Dupin Cyclide is the projection of a quartic surface, in the fourfold space, for which the coordinates of a point are, respectively, the five

$$\mu (a \cos\theta - c \cos\phi) - b^2 \cos\theta \cos\phi,\ b \sin\theta (\mu - a \cos\phi),$$
$$ib \sin\phi (\mu - c \cos\theta),\ c \cos\theta - a \cos\phi,\ b^2 (\mu - a \cos\phi).$$

The first four of these give the coordinates of a point of the Cyclide. The equations of the surface are obtainable by eliminating θ and ϕ.

Prove, also, that the tangent planes of the Cyclide, at any point, $(x_0, y_0, z_0, 0)$, of its double conic, are expressed by

$$(xx_0 + yy_0 + zz_0)^2 - t^2 (a^2 x_0^2 + b^2 y_0^2) = 0;$$

further, that the Cyclide is touched, in two points, by every tangent plane of the cone

$$(\mu x - act)^2 + (\mu^2 - c^2) y^2 + (\mu^2 - a^2) z^2 = 0,$$

the curve of contact, as this plane varies, lying on a sphere. Also, that the tangential equation of the Cyclide is capable of the form

$$[\mu^{-2} \xi^2 + (\mu^2 - c^2)^{-1} \eta^2 + (\mu^2 - a^2)^{-1} \zeta^2 - 2ac\, \xi T + M\tau^2]^2$$
$$- 4\mu^4 T^2 (\xi^2 + \eta^2 + \zeta^2) = 0,$$

where $M = \mu^2 (\mu^2 - a^2)(\mu^2 - c^2)$, $T = (ac\xi + \mu\tau)/M$. Determine all the lines lying upon the Dupin Cyclide.

CHAPTER VII

RELATIONS IN SPACE OF FIVE DIMENSIONS. KUMMER'S SURFACE

Klein's figure in three dimensions, related to a figure in five dimensions. We have studied in some detail, in Chap. v, a figure, in space of four dimensions, containing fifteen lines meeting, in threes, in fifteen points; for convenience we may describe this here as Segre's figure (cf. Stéphanos, *Compt. Rend.*, XCIII, 1881, p. 634; quoted by Segre). We shall shew now that these fifteen lines are in correspondence with the joining lines of six points in space of five dimensions. The transformation utilises formulae arising in the representation of the points, and linear complexes of lines, in space of three dimensions, by use of space of five dimensions; for clearness sake we begin by referring again to this representation.

Let l, m, n, l', m', n' be the coordinates of a line, in space of three dimensions, and put $x = l - l'$, $y = m - m'$, $z = n - n'$, $u = l + l'$, $v = m + m'$, $w = n + n'$. We have seen (p. 46, above) that the lines through a point, (ξ, η, ζ, τ), of the threefold space, are represented, in the fivefold space in which (x, y, z, u, v, w) are coordinates, by the points of a plane whose three equations are $(u, v, w) = D(x, y, z)$, where D, written in explicit form, consists of the elements, of the matrix M given by

$$M = \begin{pmatrix} \tau^2 + \xi^2 - \eta^2 - \zeta^2, & 2(\xi\eta - \zeta\tau) \ , & 2(\zeta\xi + \eta\tau) \\ 2(\xi\eta + \zeta\tau) \ , & \tau^2 + \eta^2 - \zeta^2 - \xi^2, & 2(\eta\zeta - \xi\tau) \\ 2(\zeta\xi - \eta\tau) \ , & 2(\eta\zeta + \xi\tau) \ , & \tau^2 + \zeta^2 - \xi^2 - \eta^2 \end{pmatrix},$$

each element divided by $\xi^2 + \eta^2 + \zeta^2 + \tau^2$. For brevity, the elements of the rows of the matrix M will be denoted, respectively, by l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, and $\xi^2 + \eta^2 + \zeta^2 + \tau^2$ by σ. In the threefold space, the equations $x = 0$, $u = 0$, ... denote six linear complexes, of which every two are conjugate; in the fivefold space these equations define a *hexad*, self-polar in regard to the quadric, Ω, whose equation is $u^2 + v^2 + w^2 = x^2 + y^2 + z^2$. The plane, in the fivefold space, representing the lines through a point of the threefold space, is a plane, of one system, lying on the quadric Ω; and there are planes of Ω of a second system, each representing the lines of a *plane* of the threefold space. For each two of the six linear complexes, or the associated polar systems, of the threefold space, there are two lines which are polars of one another for both systems; if P be any point of the threefold space, and, on the transversal drawn from P to these two common polar lines, P' be the point which is the harmonic conjugate of P, in regard to the polar lines, then P' is the pole, in either focal system, of the polar plane of P in the other focal system (cf. Vol. III, p. 65). In the representation in space of five dimensions, to a point and its polar plane, in regard to the focal system for which the associated linear complex is $u = 0$, there correspond two planes of the quadric Ω, which meet in a line of the fourfold $u = 0$; these are harmonic inverses of one another, in regard to the fourfold $u = 0$ and its pole in regard to Ω (p. 42, above). Thus, the representation, in the fivefold space, of the two points P, P', spoken of, is by two planes of Ω of the same system, either obtainable from the other by the succession of two such harmonic inversions. When we have six linear complexes of which every two are conjugate, there will be fifteen such involutory transformations as that from P to P'; by these, every point of the threefold space gives rise to fifteen others. Likewise, every plane similarly gives rise to fifteen other planes. In the fivefold space there correspond sets of sixteen planes of Ω, of the same system, arising from one of these planes; the linear complexes being $x = 0$, $u = 0$, ..., as above, any point of one of the fifteen derived planes is obtainable, from a point of the primary plane, by change of the sign of *two* of the coordinates of the point; thus, also, any one of the sixteen planes can be regarded as primary. In the threefold space, there is a figure of sixteen points and sixteen planes, arising by taking an arbitrary point and the fifteen derived points, and then the polar planes of all the points in regard to all the six conjugate focal systems. Only sixteen planes arise in this way; each plane contains six of the points, the poles of this plane in the various polar systems; and through each point there pass its six polar planes. The thirty-two planes of the quadric Ω, in the fivefold space, which correspond to this figure, are all derivable from one plane of Ω, by combination of the six processes of harmonic inversion, in which one of the points of the fundamental hexad, and its polar plane in regard to Ω, are fundamental. Further, the six lines in which an arbitrary plane of Ω meets the primary fourfolds, $x = 0$, $u = 0$, ..., touch a conic of this plane. For, in the notation above suggested, these six lines are given, respectively, by

$$x = 0,\quad y = 0,\quad z = 0,\quad l_1x + m_1y + n_1z = 0,\quad \ldots,\quad l_3x + m_3y + n_3z = 0;$$

and they touch the conic in which the plane is met by the cone

$$(l_1l_2l_3x)^{\frac{1}{2}} + (m_1m_2m_3y)^{\frac{1}{2}} + (n_1n_2n_3z)^{\frac{1}{2}} = 0.$$

This conic equally lies on the cone

$$(l_1m_1n_1u)^{\frac{1}{2}} + (l_2m_2n_2v)^{\frac{1}{2}} + (l_3m_3n_3w)^{\frac{1}{2}} = 0.$$

The fact, in the threefold space, which corresponds to this is that, in the figure of sixteen points and planes, the six poles in any plane lie on a conic, and the six planes through any point touch a quadric cone.

Ex. 1. The reverse of the formulae given above, for passing from any point, (ξ, η, ζ, τ), of the threefold space, to a plane,

$$(u, v, w) = D(x, y, z),$$

of Ω, are, if p, q, r, p', q', r', respectively, denote

$$m_3 + n_2,\quad n_1 + l_3,\quad l_2 + m_1,\quad m_3 - n_2,\quad n_1 - l_3,\quad l_2 - m_1,$$

which do not contain l_1, m_2, n_3, these following:

$$\tau^{-1}\xi = q/r' = r/q';\quad \tau^{-1}\eta = r/p' = p/r';\quad \tau^{-1}\zeta = p/q' = q/p'.$$

If $\theta = pp',\ = qq',\ = rr',\ n_2{}^2 = \alpha,\ l_3{}^2 = \beta,\ m_1{}^2 = \gamma$, then

$$p = (\alpha + \theta)^{\frac{1}{2}} + \alpha^{\frac{1}{2}},\quad q = (\beta + \theta)^{\frac{1}{2}} + \beta^{\frac{1}{2}},\quad r = (\gamma + \theta)^{\frac{1}{2}} + \gamma^{\frac{1}{2}}.$$

Ex. 2. In the threefold space, the polar plane,

$$\xi X + \eta Y + \zeta Z + \tau T = 0,$$

of the point (ξ, η, ζ, τ), in regard to the quadric surface

$$X^2 + Y^2 + Z^2 + T^2 = 0,$$

contains the three points $(\tau, -\zeta, \eta, -\xi), (\zeta, \tau, -\xi, -\eta), (-\eta, \xi, \tau, -\zeta)$, of which every two are conjugate to one another in regard to the quadric surface. When the coordinates of the first of these three points are put for ξ, η, ζ, τ, in the matrix M, above, the matrix is unaltered, save for a change of sign of every element in the second and third rows of the matrix. Thus, the plane, in the fivefold space, representing this point, $(\tau, -\zeta, \eta, -\xi)$, is obtained, from the plane representing the point (ξ, η, ζ, τ), by two harmonic inversions in succession, in the fourfolds $v = 0$, $w = 0$, each associated with its pole in regard to Ω. Similarly, the second and third, of the three points named, correspond, respectively, to the pairs of harmonic inversions (w, u) and (u, v). The same plane, $\xi X + \ldots + \tau T = 0$, contains also the three points $(\tau, \zeta, -\eta, -\xi)$, $(-\zeta, \tau, \xi, -\eta)$, $(\eta, -\xi, \tau, -\zeta)$; these are, similarly, a self-polar triad in regard to the quadric surface $X^2 + \ldots + T^2 = 0$, and correspond, respectively, to pairs of harmonic inversions, in the fivefold space, in the pairs of fourfolds (y, z), (z, x), (x, y).

If we take new coordinates (X', Y', Z', T'), given by

$$(X', Y', Z', T') = \begin{pmatrix} \tau, & -\zeta, & \eta, & -\xi \\ \zeta, & \tau, & -\xi, & -\eta \\ -\eta, & \xi, & \tau, & -\zeta \\ \xi, & \eta, & \zeta, & \tau \end{pmatrix} (X, Y, Z, T),$$

the coordinates, (X', Y', Z'), of the six points of the plane $T'=0$, are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3), respectively. These points lie on the conic

$$l_1 m_1 n_1/X' + l_2 m_2 n_2/Y' + l_3 m_3 n_3/Z' = 0,$$

of which the general point is given by

$$X' = l_1 m_1 n_1 \theta, \quad Y' = l_2 m_2 n_2 (1-\theta), \quad Z' = l_3 m_3 n_3 \theta (\theta - 1),$$

namely, for the respective values $\theta = 1$, 0, ∞, $-\mu\nu$, $-\nu\lambda$, $-\lambda\mu$, where $\lambda = l_2/l_3$, $\mu = m_2/m_3$, $\nu = n_2/n_3$. The six points are also given by iH, jH, kH, Hi, Hj, Hk, where $H = i\xi + j\eta + k\zeta + \tau$, the symbols i, j, k being such that $i^2 = -1$, $jk = -kj = i$, etc. (Vol. III, p. 138.)

Ex. 3. Shew that the fifteen points, in the threefold space, can be constructed from the six points $iH, jH, kH, iHi, jHj, kHk$, the notation being as in Ex. 2. These are the six points A, B, C, P, Q, R of Ex. 5, p. 139, above. Determine the coordinates of all the points relatively to the four iH, jH, kH, H.

Ex. 4. In the fivefold space, let the poles, in regard to Ω, of the six primary fourfolds, $x=0, \ldots, w=0$, be divided into three pairs. The joins of the points of a pair meet the quadric Ω in two points, which represent the pair of common polar lines of the two corresponding polar systems of the threefold space. It can be shewn that the three pairs of polar lines so obtained are the pairs of opposite joins of four points of the threefold space; for instance, the tangent fourfolds of Ω at the two points $(1,0,0;\ \pm 1,0,0)$ contain the other two pairs of points similarly arising.

Ex. 5. The formulae suggest the consideration, in the threefold space, of ten quadric surfaces. One of these is expressed by

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = 0;$$

the other nine are obtained by equating to zero the respective elements, l_1, m_1, n_1, etc., of the matrix M. These quadric surfaces arise geometrically because the polar plane of a point, P, in regard to any one of these quadrics, is obtained by first passing from P to a point, P', by means of one of the fifteen involutory transformations explained above, as arising from a pair of the focal systems, and then taking the polar plane of P' in a third focal system. Or, because the generators, of either system, of any one of these quadric surfaces, consist of the lines common to three of the six fundamental linear complexes, the other system of generators being the lines common to the other three complexes. This is clear, either, directly, from the equations of the quadric surfaces, or, from the representation of a point of the threefold space by a plane of Ω in the fivefold space. For instance, from the three equations of such a plane

$$\sigma u = l_1 x + m_1 y + n_1 z, \text{ etc.},$$

any point of the quadric $m_1 = 0$, or $\xi\eta - \zeta\tau = 0$, corresponds to a plane of Ω which contains a point for which $u=0$, $x=0$, $z=0$. These last three equations, however, represent a plane, in the fivefold space, which meets Ω in a conic. The points of the conic thus represent one system of generators of the quadric $m_1=0$. We shall associate the vertices x, y, z, u, v, w of the fundamental hexad in the fivefold space, that is, the points for which these coordinates have, respectively, the values $(1, 0, 0; 0, 0, 0), \ldots, (0, 0, 0; 0, 0, 1)$, with the respective numbers 4, 5, 6, 1, 2, 3. Then, for instance, the plane $u=x=z=0$, containing the points y, v, w, is associated with the triad 235. This plane contains the points of Ω lying on the joins, of pairs of points of the hexad, given by 23, 25, 35; and these points represent three pairs of generators, of the quadric surface $m_1=0$, each of which is a pair of common polar lines for a pair of the six polar systems. The other system of generators, of the quadric surface $m_1=0$, is represented by the conic of Ω which is the polar of the former, lying in the plane $y=v=w=0$; this conic, or the plane in which it lies, is then associated with the triad 164. The quadric surface may then be represented by 235, or 164; and this notation suggests six pairs of generators belonging to the quadric surface. The ten quadrics corresponding to the elements of the matrix M, and to $\sigma=0$, will then, respectively, have the notation of the annexed scheme, as is easily seen in the same way. Now, it is clear that, in the fivefold space, there can be drawn, through the line joining any two of the six fundamental points, four planes, each containing one other of these six points (and, thus, the lines joining this to the two first points). There are, therefore, fifteen sets, each of four of the ten quadric surfaces in the threefold space, such that these four quadrics have two generators in common. Any one of these sets of four quadric surfaces consists of those whose symbols, in the annexed scheme, lie in the same row and column with any one, other than 123 (456), of the sixteen elements of the scheme. For instance, the quadric surfaces so obtainable from the fourth element of the first row all contain the two generators represented by 23; that is, the generators given by the intersections, with Ω, of the line joining the points v, w, in the figure in fivefold space.

$$\left|\begin{array}{cccc} 234(156), & 235(164), & 236(145) & \cdot \\ 314(256), & 315(264), & 316(245) & \cdot \\ 124(356), & 125(364), & 126(345) & \cdot \\ \cdot & \cdot & \cdot & 123(456) \end{array}\right|$$

We may, thus, from the given scheme, obtain another scheme, of four rows and columns, with no entry for the fourth element of the fourth row, by putting, in any place, the symbol for the two generators common to the four quadric surfaces arising, in the way explained, by starting from this place. This new scheme agrees in notation with one given earlier (p. 133, above).

It may be proved that the two generators common to a set of four quadric surfaces are polar lines, each of the other, in regard to each of the six remaining quadric surfaces. Also, the squares of the quadratic functions which, equated to zero, give such a set of four quadric surfaces, are connected by a linear equation. We may also remark that any quadric surface, whatever, may be expressed as a linear function of the ten quadric surfaces; and that every two of these ten are both outpolar and inpolar to one another (cf. Vol. II, p. 149).

The six linear complexes, conjugate in pairs, were studied by F. Klein, *Math. Annal.* II. (1870), p. 198, who remarks the sets of four quadric surfaces having two common generators (*Ges. Math. Abhandl.* I. (1921), p. 63). See also Hudson, *Kummer's Quartic Surface,* 1905, p. 40, etc.

A transformation of Segre's figure, in space of four dimensions, to a figure in five dimensions. The representation, in space of five dimensions, of the figure of six linear complexes in a threefold space, has suggested a notation for the fifteen pairs of common polar lines, of two complexes, for the ten quadric surfaces, and for the fifteen sets of six joins of a tetrad of points (each associated with such a symbol as 12 . 34 . 56). This notation agrees with that previously employed (p. 114, above) for the fifteen lines, for the ten singular solids, and for the fifteen points, arising in Segre's figure. As we saw, the figure of six linear complexes arises in any tangent solid of the locus Σ, discussed in connexion with Segre's figure. We now describe an independent transformation, which, though analogous, is different in essence from the above.

We consider any threefold space, with coordinates ξ, η, ζ, τ. Therein, as we have seen, we can associate, with any point (ξ, η, ζ, τ), fifteen others; the coordinates of these are derivable from ξ, η, ζ, τ by two processes: (1), the change of the signs of two of the coordinates; (2), the interchange of two of the coordinates accompanied by the interchange of the other two. These processes may be applied to the coordinates of any one of the aggregate of sixteen points, and will give the coordinates of the other fifteen, and only these. We thus have a linear group in four (homogeneous) variables. And we have sets of sixteen points of which each set is determined by any one of its points; or, as we may say, we have an *involution* of sets of sixteen points in the threefold space. We now shew that we can determine a set of five functions, all homogeneous polynomials of the same dimension in ξ, η, ζ, τ, with the two properties; (1), that the ratios of these polynomials are the same for all the points of a set of the involution; (2), that these ratios are the same *only* for sixteen points belonging to the same set of the involution. Then we regard these five functions as homogeneous coordinates in a space of four dimensions. Thereby we obtain a correspondence, which is uniquely reversible, between points of this fourfold space, and the sets of the involution in the threefold space. But, the ratios of the five functions, being dependent upon the three ratios of ξ, η, ζ, τ, will be connected by an algebraic relation, which may be taken in a rational form. The correspondence will thus be one between the points of an ∞^3 locus, in the fourfold space, and the unrestricted sets of the involution in the threefold space. For the five functions in question we take the squares of five elements of the matrix M, above (p. 203); namely, recalling the notation used before, $l_1 = \tau^2 + \xi^2 - \eta^2 - \zeta^2$, $n_2 = 2(\eta\zeta - \tau\xi)$, etc., we take the five functions l_1^2, m_2^2, n_3^2, n_2^2, m_3^2. These are easily seen to be unaltered, in their ratios, by passing from any point to any other point of the same set of the involution. The same would not be true of the unsquared functions, l_1, m_2, etc.; for instance the ratio l_1/m_2 is changed in sign by the substitution which replaces ξ, η, ζ, τ, respectively, by τ, ζ, η, ξ. As the ratios of the five chosen functions are unaltered in passing from one point to another of the same set, it follows that when these ratios are given, the ratios of ξ, η, ζ, τ are capable of at least the sixteen values which correspond to the points of such a set of the involution. If we shew algebraically that the ratios of ξ, η, ζ, τ are capable of *only* sixteen values, it will follow that these are those of a set of the involution. Thence will follow the second, characteristic, property of the ratios of the five chosen functions; namely, that they are unaltered *only* for the points of such a set. Now, the expressions of the ratios of ξ, η, ζ, τ, in terms of the five squares l_1^2, m_2^2, n_3^2, n_2^2, m_3^2, are to be found from the equations, previously remarked (p. 205),

$$\tau^{-1}\xi = (n_1 + l_3)/(l_2 - m_1), \quad \tau^{-1}\eta = (l_2 + m_1)/(m_3 - n_2),$$
$$\tau^{-1}\zeta = (m_3 + n_2)/(n_1 - l_3),$$

by putting, beside $\quad m_3 = (m_3^2)^{\frac{1}{2}}, \; n_2 = (n_2^2)^{\frac{1}{2}},$

$$n_1 = (-l_1^2 + m_2^2 + m_3^2)^{\frac{1}{2}}, \quad l_3 = (-l_1^2 + m_2^2 + n_2^2)^{\frac{1}{2}},$$
$$l_2 = (-l_1^2 + m_3^2 + n_3^2)^{\frac{1}{2}}, \quad m_1 = (-l_1^2 + n_2^2 + n_3^2)^{\frac{1}{2}};$$

the ratios $\tau^{-1}\xi$, $\tau^{-1}\eta$, $\tau^{-1}\zeta$ may thus appear to be capable of 2^5, or thirty-two, values, when the ratios of the five squares are given. But the signs are, in fact, not independent; for the product $2m_3n_2 \,.\, n_1l_3 \,.\, l_2m_1$ is expressible by the five squares, in the form

$$2m_3^2n_2^2(m_2^2 + n_3^2) + (m_3^2 + n_2^2)(m_2^2n_3^2 + m_3^2n_2^2 - l_1^2\sigma^2),$$

where $\sigma^2 = -l_1^2 + m_2^2 + n_3^2 + n_2^2 + m_3^2$; as is easily verified.

The representative character of the ratios of the five functions is thus established. To obtain the relation connecting them we introduce the precise notation

$$X=(\tau^2+\eta^2-\zeta^2-\xi^2)^2,\ X'=(\tau^2+\zeta^2-\xi^2-\eta^2)^2,$$
$$Y=4(\eta\zeta-\tau\xi)^2,\ Y'=4(\eta\zeta+\tau\xi)^2,$$
$$Z=-(\tau^2+\xi^2-\eta^2-\zeta^2)^2,\ Z'=-(\tau^2+\xi^2+\eta^2+\zeta^2)^2;$$

then it can be verified at once that

$$X+X'+Y+Y'+Z+Z'=0,\ \text{and}\ (XX')^{\frac{1}{2}}+(YY')^{\frac{1}{2}}+(ZZ')^{\frac{1}{2}}=0.$$

The latter is the equation which may be written $l_1\sigma=m_2n_3-m_3n_2$; it represents the ∞^3 locus, in the space of four dimensions, of which the points are in reversible correspondence with the sets of the involution in the threefold space. By elimination of Z' between these two equations, we have the relation connecting the five chosen functions X, X', Y, Y', Z only.

We may express the matter by speaking of the *invariants* of a linear group, of finite order, in four homogeneous variables. The reader may consult Burnside, *Theory of Groups of Finite Order* (2nd Edit., Cambridge, 1911), p. 359, where, however, the variables are not homogeneous. Also, the present writer's *Multiply-Periodic Functions* (Cambridge, 1907), p. 281. In this, in line 5, the second N should be M; and, in line 22, the words "there being" should read "there not being."

The locus expressed by these equations is none other than the locus Σ previously discussed (pp. 126, 159, above). If we put

$$U=Y+Z+X',\ V=Z+X+Y',$$
$$W=X+Y+Z',\ T=-(X+Y+Z),$$

these corresponding, respectively, to the squares m_1^2, n_1^2, $-l_2^2$, $-l_3^2$, this locus has fifteen double lines, in each of which there vanish four of the ten functions X, Y, Z, X', Y', Z', U, V, W, T. The fifteen sets of four of these are those in the same row and column with the elements, respectively, other than Z', of the adjoined scheme. On comparison with a scheme previously given (p. 133, above), we thus have a notation, by two numbers, for each of the fifteen lines; and also a notation, by three numbers, for each of the ten solids expressed by the equations $X=0, \ldots, T=0$.

$$\left|\begin{array}{cccc} Z, & U, & V & \cdot \\ W, & X, & Y & \cdot \\ T, & Y', & X' & \cdot \\ \cdot & \cdot & \cdot & Z' \end{array}\right|$$

In general, as has been said, to a point of the locus Σ, now found, there corresponds a set of sixteen points of the threefold space (ξ, η, ζ, τ). But the fifteen lines of Σ are exceptional. Each of these corresponds to the points common to four quadric surfaces, of which three are linearly independent; but these four are, in each case, such that they have two lines in common. Thus each of the fifteen lines of the fourfold space corresponds to *two* lines of the threefold space. However, as we have seen above (p. 207), when we represent the lines of the threefold space (ξ, η, ζ, τ) by the points of a quadric Ω, in fivefold space, the representation of the two lines, common to one of the sets of four quadric surfaces, is by two particular points of Ω; namely, the two intersections of Ω with the join of two of the points of the fundamental hexad, self-polar in regard to Ω.

Thus, by passing, through the threefold space, from the fourfold space to the space of five dimensions, every one of the fifteen lines of the Segre figure corresponds to the join of two of the six fundamental points of the fivefold space.

But there is more. The fifteen lines of the Segre figure contain six sets each of five associated lines, (each line being used twice), (p. 114, above). Such a set is that whose notation is *i*1, *i*2, *i*3, *i*4, *i*5, *i*6 (with omission of *ii*). Wherefore, by the number notation we have used for the points of the hexad, in fivefold space, we see that a set of five associated lines, of Segre's figure, is represented by the joins of one point of the hexad to the other five points. All the associated sets are thus accounted for.

Relation of the transformation to the theory of Kummer's surface. We may also apply the transformation by which we have passed from the locus Σ, in the space (X, Y, Z, X', Y', Z'), to the space (ξ, η, ζ, τ), to another locus. We saw that the locus Σ intersects any one of its tangent solids in a Kummer's quartic surface (p. 138, above), having a double point at the point of contact of the tangent solid, and also a double point at each of the fifteen points where the tangent solid is met by the double lines of Σ. The equation of the tangent solid involves the coordinates $X, Y, \ldots$ linearly, and, in the transformation, each of these is a quartic function of the coordinates ξ, η, ζ, τ. Thus the Kummer's surface corresponds to a quartic surface in the space (ξ, η, ζ, τ). Upon this quartic surface there is an involution of sets of sixteen points, each set corresponding to one of the ordinary points of the Kummer surface in the tangent solid of Σ. As for the double points of the Kummer surface, fifteen of them give rise each to two ordinary points of the new surface; for we have seen that each of the double lines of Σ gives rise to two lines of the space (ξ, η, ζ, τ). But, as will be shewn at once, the remaining double point of the Kummer surface, the point of contact of the tangent solid with Σ, gives rise to sixteen points of the new quartic surface, each of which is a double point. In fact, the new surface is also a Kummer surface. This will appear naturally below (p. 215), from another point of view. And, this being so, interesting consequences arise by identifying the space (ξ, η, ζ, τ) with the tangent solid of Σ. For clearness, however, it should be remarked that the two facts, that four of the singular solids in the Segre figure pass through any one of the fifteen lines, and that four of the ten fundamental quadric surfaces, suitably chosen, in the threefold space, have two generators in common, are geometrically distinct. This may be obscured by the adoption of the same number notation for the singular solids and for the quadric surfaces, convenient as this is. If we identify the threefold space (ξ, η, ζ, τ) with a particular tangent solid of the locus Σ, the fifteen sets of six lines, such as 12.34.56, which are the joins of four points of the space (ξ, η, ζ, τ), will be lines which we have not considered in our discussion of Segre's figure in Chapter v.

To see that the point of contact of the tangent solid with Σ gives rise to double points of the surface in the space (ξ, η, ζ, τ), under the transformation discussed, suppose, more generally, that $\phi_1, \phi_2, \phi_3, \phi_4, \phi$ are five polynomials, of the same order, in ξ, η, ζ, τ, which, as regards their ratios, are invariants of a group of substitutions for these (homogeneous) variables ξ, η, ζ, τ; put, respectively, X, Y, Z, T, U for these polynomials. There will exist a rational homogeneous equation $F(X, Y, Z, T, U)=0$, which is an identity in ξ, η, ζ, τ. This equation represents a locus in the fourfold space (X, Y, Z, T, U), of which the tangent solid at any point may be represented by

$$X(\partial F/\partial X)_0+\ldots+U(\partial F/\partial U)_0=0.$$

The intersection of this solid with the locus corresponds to a surface, in the space (ξ, η, ζ, τ), whose equation is obtained from this by putting $\phi_1, \phi_2, \ldots, \phi$, respectively, for $X, Y, \ldots, U$. The conditions for this surface to have a double point at one of the points, say $(\xi_0, \eta_0, \zeta_0, \tau_0)$, which correspond to the point of contact of the tangent solid with the locus $F=0$, are of the form

$$(\partial F/\partial X)_0(\partial\phi_1/\partial\xi)_0+\ldots+(\partial F/\partial U)_0(\partial\phi/\partial\xi)_0=0,$$

wherein for ξ are to be put, in turn, ξ, η, ζ, τ. These four conditions arise, however, by differentiating the identity in ξ, η, ζ, τ, expressed by $F=0$, and then putting ξ_0, η_0, etc., for ξ, η, etc.

Kummer's quartic surface as the locus of singular points of a quadratic complex of lines. A quadratic complex of lines, in threefold space, is an aggregate (∞^3), represented by a single quadratic relation connecting the line coordinates. The lines of such a complex which pass through an arbitrary point are the generators of a quadric cone; those which lie in an arbitrary plane are the tangents of a conic (Vol. III, p. 99). But the quadric cone of lines through a point may break up, for suitable positions of the point, into two planes, the lines through the point then forming two flat pencils in these planes. The points, called *singular points of the complex*, for which this is so, are the points of a Kummer quartic surface. It was from this point of view that the surface was discussed by Kummer (*Berlin. Monatsber.*, 1864, pp. 246, 495; 1865, p. 288; *Berlin. Abhandl.*, 1866, p. 1). Similarly, there is a quartic envelope of planes, wherein the rays of the quadratic complex constitute two flat pencils; but it will appear that this consists of the tangent planes of the former surface. For special forms of the quadratic complex, the locus of singular points may degenerate; for instance, in the case of the tetrahedral complex, the locus breaks up into the four fundamental planes, and the envelope into the four fundamental points. Properties arising in the special cases may, however, suggest more general theorems. For instance, we know (Vol. I, p. 30) that any line meets four planes in a range which is related to that of the four planes joining the line to the four points of intersection of threes of the planes; it will be found that any line meets the general Kummer surface in a range of four points related to that of the four tangent planes which pass through the line.

We suppose the quadratic complex to be represented by a quadratic relation connecting the coordinates of a point in fivefold space; that is, by the intersection of an (∞^4) quadric of this space with the fundamental quadric, Ω, which represents the necessary relation connecting the line coordinates. Further, we suppose the quadratic complex to be so general that the two quadrics can be represented by squares of the same six linear functions of the coordinates. For the fundamental quadric, Ω, and for that, Ω', giving the quadratic complex, we may then suppose the equations to be, respectively,

$$x_1^2 + x_2^2 + \ldots + x_6^2 = 0, \quad k_1x_1^2 + k_2x_2^2 + \ldots + k_6x_6^2 = 0.$$

The lines of the quadratic complex which pass through an arbitrary point of the threefold space, are represented, in the fivefold space, by the conic in which a certain plane, lying on the quadric Ω, meets the quadric Ω'. Similarly, the lines of the quadratic complex which lie in an arbitrary plane, are represented by points, of a conic on Ω', lying in a plane entirely on Ω, this plane being of the opposite system to the former. With a very slight change from the notation used immediately above (p. 203), we may, as before, (p. 46, above) represent a plane lying entirely on Ω by three equations

$$ix_4 = l_1x_1 + m_1x_2 + n_1x_3, \quad ix_5 = l_2x_1 + m_2x_2 + n_2x_3, \quad ix_6 = l_3x_1 + m_3x_2 + n_3x_3,$$

where $l_1^2 + m_1^2 + n_1^2 = 1$, etc. The difference of notation is immaterial in equations in which l_1, m_1, etc. enter homogeneously. Then, the conic in which this plane meets the quadric Ω' corresponds to the equation

$$k_1x_1^2 + k_2x_2^2 + k_3x_3^2 - k_4(l_1x_1 + m_1x_2 + n_1x_3)^2 - k_5(l_2x_1 + m_2x_2 + n_2x_3)^2$$
$$-k_6(l_3x_1 + m_3x_2 + n_3x_3)^2 = 0.$$

The left side of this breaks up into two linear factors provided x_1, x_2, x_3 can be found to satisfy the three equations

$$-k_1x_1 + k_4l_1\,p + k_5l_2\,q + k_6l_3\,r = 0,$$
$$-k_2x_2 + k_4m_1p + k_5m_2q + k_6m_3r = 0,$$
$$-k_3x_3 + k_4n_1\,p + k_5n_2\,q + k_6n_3\,r = 0,$$

where p, q, r denote, respectively, the three linear forms

$$l_1x_1 + m_1x_2 + n_1x_3, \text{ etc.}$$

In virtue of the three such identities as $x_1 = l_1p + l_2q + l_3r$, which follow from the definition of p, q, r, these three equations are the same as

$$\begin{aligned} (14)\,l_1\,p + (15)\,l_2\,q + (16)\,l_3\,r &= 0, \\ (24)\,m_1p + (25)\,m_2q + (26)\,m_3r &= 0, \\ (34)\,n_1\,p + (35)\,n_2\,q + (36)\,n_3\,r &= 0, \end{aligned}$$

where (rs) is put for $k_r - k_s$. The condition for the consistence of the three equations is, therefore,

$$\begin{vmatrix} (14)\,l_1, & (24)\,m_1, & (34)\,n_1 \\ (15)\,l_2, & (25)\,m_2, & (35)\,n_2 \\ (16)\,l_3, & (26)\,m_3, & (36)\,n_3 \end{vmatrix} = 0.$$

In evaluating this determinant, we denote the product $(r4)(s5)(t6)$, where r, s, t are the numbers 1, 2, 3, in some order, simply by rst. The expansion is, then, first,

$$\begin{aligned} 123.l_1m_2n_3 - 132.l_1m_3n_2 + 312.l_2m_3n_1 - 213.l_2m_1n_3 \\ + 231.l_3m_1n_2 - 321.l_3m_2n_1; \end{aligned}$$

herein we substitute, from the properties of the orthogonal determinant $(l_1m_2n_3)$,

$$\begin{aligned} l_2m_3n_1 = m_3\,(l_1n_2 + \epsilon m_3),\ l_2m_1n_3 = n_3\,(l_1m_2 - \epsilon n_3), \\ l_3m_1n_2 = n_2\,(l_1m_3 + \epsilon n_2),\ l_3m_2n_1 = m_2\,(l_1n_3 - \epsilon m_2), \end{aligned}$$

where ϵ is ± 1 according as the determinant $(l_1m_2n_3)$ is ± 1. Then the expansion is

$$\begin{aligned} (123 - 213 - 321)\,l_1m_2n_3 + (-132 + 312 + 231)\,l_1m_3n_2 \\ + \epsilon\,(312.m_3^2 + 213.n_3^2 + 231.n_2^2 + 321.m_2^2), \end{aligned}$$

wherein the coefficients of $l_1m_2n_3$ and $l_1m_3n_2$ are easily verified to be equal, but of opposite sign. Thus, with the omission of a common factor ϵ, after replacing $m_2n_3 - m_3n_2$ by ϵl_1, the whole is

$$321.m_2^2 + 213.n_3^2 + 231.n_2^2 + 312.m_3^2 + (123 - 213 - 321)\,l_1^2.$$

Now consider the coefficients 321, 213, etc., occurring here. Each is a product of three differences of the quantities $k_1, k_2, \ldots, k_6$, and all these enter once into each coefficient. Thus, the ratios of these coefficients are unaltered if we replace $k_1, k_2, \ldots, k_6$ each by the same arbitrary linear function of itself, say k_r by

$$(pk_r + q)/(mk_r + n).$$

In particular, we may replace k_4, k_5, k_6, respectively, by 1, 0, ∞; let the values of k_1, k_2, k_3 then arising be named, respectively,

a, b, c. After this, put a', b', c' for $1-a$, $1-b$, $1-c$, respectively, and $\xi, \eta, \zeta, \xi', \eta', \zeta'$ for bc', ca', ab', $b'c$, $c'a$, $a'b$; so that

$$\xi+\eta+\zeta=\xi'+\eta'+\zeta', \quad \xi\eta\zeta=\xi'\eta'\zeta',$$

and the reverse equations are

$$a(\xi-\xi')=\eta'-\zeta, \quad b(\eta-\eta')=\zeta'-\xi, \quad c(\zeta-\zeta')=\xi'-\eta.$$

Then the expression found becomes

$$\xi m_2^2+\zeta n_3^2+\xi' n_2^2+\eta' m_3^2+(\zeta'-\zeta-\xi)\,l_1^2.$$

The vanishing of this expression is the condition that the plane, $(l_1, m_1, \ldots)$, lying upon Ω, whether it be of the first or second kind of planes of Ω, should meet Ω' in two lines.

We may interpret this result in two ways, in four dimensions or in three; either by regarding the ratios of the squares of l_1, m_2, n_3, m_3, n_2 as coordinates in fourfold space, as above; or, by regarding the plane of Ω as representing a point, (ξ, η, ζ, τ), or a plane, (u, v, w, p), of threefold space.

In the former way, writing, as before, X, Y, Z, X', Y', W, T, respectively, for $m_2^2, n_2^2, -l_1^2, n_3^2, m_3^2, -l_2^2, -l_3^2$, the equation becomes

$$-aW+bX+cY+bcT-caY'-abX'=0.$$

This, however (p. 159, above), is the equation of the tangent solid of the locus expressed by $(XY')^{\frac{1}{2}}+(X'Y)^{\frac{1}{2}}+(WT)^{\frac{1}{2}}=0$, at a point determined by the values of a, b, c; and this locus is no other than the locus, Σ, otherwise expressed by

$$(XX')^{\frac{1}{2}}+(YY')^{\frac{1}{2}}+(ZZ')^{\frac{1}{2}}=0.$$

(Cf. the scheme given above, p. 210.) So that we have verified the statement made (p. 211), that the transformation from the fourfold to the threefold space, changes the section of Σ into a Kummer surface. The values of the parameters a, b, c, in terms of the coordinates, at the point of contact of the tangent solid, which are given in general (p. 159, above) by

$$-2aX'Y'=WT-XY'-X'Y, \quad 2bTX'=XY'-WT-X'Y,$$
$$2cTY'=X'Y-XY'-WT,$$

are easily found to be $a=-\mu\nu$, $b=-\nu\lambda$, $c=-\lambda\mu$, where

$$\lambda=l_2/l_3, \quad \mu=m_2/m_3, \quad \nu=n_2/n_3.$$

Thus we see that the coefficients, in the equation for the quadratic complex, $k_4, k_5, k_6, k_1, k_2, k_3$, if regarded as parameters of points of a conic, lead to a range related to any one of the sixteen ranges previously considered. (Cf. p. 127, above, and Ex. 2, p. 206, above.)

In the latter way, replacing ξ, η, ζ, τ by x, y, z, t, and putting $l_1^2 = (t^2 + x^2 - y^2 - z^2)^2$, etc., the equation found represents a quartic surface in x, y, z, t. On expansion this is of the form

$$2(2\eta-\zeta')(t^2x^2+y^2z^2)+2(2\xi-\zeta')(t^2y^2+z^2x^2)+2(2\zeta-\zeta')(t^2z^2+x^2y^2) \\ +8(\eta'-\xi')xyzt+\zeta'(x^4+y^4+z^4+t^4)=0.$$

The four ratios of the coefficients in this equation are functions of only three, and can be expressed in terms of the ratios of four quantities, p, q, r, s, so that the equation becomes

$$x^4+y^4+z^4+t^4+U(t^2x^2+y^2z^2)+V(t^2y^2+z^2x^2) \\ +W(t^2z^2+x^2y^2)-2Mxyzt=0,$$

with

$$U=(s^2+p^2-q^2-r^2)/(qr-sp),\quad V=(s^2+q^2-r^2-p^2)/(rp-sq), \\ W=(s^2+r^2-p^2-q^2)/(pq-sr),$$

and

$$M=(pqrs)^{\frac{1}{2}}\left[\Pi\,(s+\epsilon_1 p+\epsilon_2 q+\epsilon_1\epsilon_2 r)\right]/(qr-sp)(rp-sq)(pq-sr),$$

the product indicated by the symbol Π being for the four factors in which $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. To see this, remark that the ratios of p, q, r, s are determined when the ratios are determined of the four

$$P,=p-q-r+s,\quad Q,=-p+q-r+s,\quad R,=-p-q+r+s, \\ S,=p+q+r+s;$$

let these be determined so that

$$QR/PS=\eta/(\eta-\zeta'),\quad RP/QS=\xi/(\xi-\zeta'),\quad PQ/RS=\zeta/(\zeta-\zeta');$$

then the coefficients U, V, W are rightly expressed. But these equations, it may be verified, lead to

$$S^{-4}\,\Pi(S+\epsilon_1P+\epsilon_2Q+\epsilon_1\epsilon_2R)=(\zeta')^4(\eta'-\xi')^2/(\eta-\zeta')^2(\xi-\zeta')^2(\zeta-\zeta')^2,$$

of which the left side is $256S^{-4}pqrs$; thus the coefficient M is also expressible as stated.

But the equation, with coefficients expressed in terms of p, q, r, s, represents a surface which can easily be seen to have a double point at the point for which x, y, z, t have the ratios of $p^{\frac{1}{2}}, q^{\frac{1}{2}}, r^{\frac{1}{2}}, s^{\frac{1}{2}}$; and, therefore, also, at the fifteen other points obtainable from this by, (i), change of sign of two of $p^{\frac{1}{2}}$, $q^{\frac{1}{2}}$, $r^{\frac{1}{2}}$, (ii), interchange of two of these accompanied by interchange of $s^{\frac{1}{2}}$ with the remaining one.

The history of this equation (first found by Rosenhain, or Göpel, 1846, 1847, in algebraical investigation of theta functions of two variables, arising from the study of the irrationality involved in the square root of a sextic polynomial) is very interesting, as shewing how gradual the synthesis of a simple conception may be. The reader may compare the writer's *Abel's Theorem* (Cambridge, 1897), pp. 467 and 338—340.

There is, we have seen (cf. p. 46, above), a precisely identical relation in u, v, w, p, representing the quartic envelope of singular *planes* of the quadratic complex. It may be verified by direct algebra that this consists of the tangent planes of the quartic locus; but a geometrical proof occurs below (p. 222). It is easy to see at once that, through any point, (x', y', z', t'), of the locus, there pass six planes of the envelope. For instance, that the plane

$$xt' - yz' + zy' - tx' = 0$$

is a plane of the envelope, follows, if the equation of the envelope is satisfied by $(u, v, w, p) = (t', -z', y', -x')$; and this is true, because the equation of the locus is satisfied by $(x, y, z, t) = (x', y', z', t')$. This plane is the focal plane of (x', y', z', t') in the linear complex whose equation is $l' + l = 0$. The same argument applies to the focal planes in the complexes $l' - l = 0$, $m' \pm m = 0$, $n' \pm n = 0$. When the envelope has been shewn to consist of the tangent planes of the locus, it will thus follow that the six focal planes of any point of the locus are tangent planes of the locus; dually, the six foci of any tangent plane of the locus are points of the locus.

Ex. 1. If, instead of $x_1, x_2, \ldots$, other coordinates, $x_1', x_2', \ldots$, be taken in the fivefold space, such, however, that Ω is expressed by the same form, $x_1'^2 + x_2'^2 + \ldots = 0$, the equations of a definite plane, on Ω, $ix_4 = l_1x_1 + m_1x_2 + n_1x_3$, etc., may become

$$ix_4' = l_1'x_1' + m_1'x_2' + n_1'x_3', \text{ etc.}$$

The point, (x', y', z', t'), of threefold space, determined by such equations as $x'/t' = (n_1' + l_3')/(l_2' - m_1')$, will not be the same as (x, y, z, t), determined by $x/t = (n_1 + l_3)/(l_2 - m_1)$, etc. Prove, in particular, that, if $x_2' = x_3$, $x_3' = x_2$, the other coordinates being unaltered, then $x' = y + z$, $y' = -t - x$, $z' = t - x$, $t' = y - z$.

Ex. 2. A particular case of the above equation arises when the pairs of coefficients (k_1, k_4), (k_2, k_5), (k_3, k_6), interpreted as parameters of points of a range, are those belonging to an involution. Then, as is easy to see, we have $\xi' - \eta' = 0$, and the equation of the Kummer surface, in x, y, z, t, is without the term in $xyzt$. We can, correspondingly, suppose $s = 0$, and the sixteen double points consist of four such sets as

$$(0, r^{\frac{1}{2}}, -q^{\frac{1}{2}}, p^{\frac{1}{2}}),\ (-r^{\frac{1}{2}}, 0, p^{\frac{1}{2}}, q^{\frac{1}{2}}),\ (q^{\frac{1}{2}}, -p^{\frac{1}{2}}, 0, r^{\frac{1}{2}}),\ (p^{\frac{1}{2}}, q^{\frac{1}{2}}, r^{\frac{1}{2}}, 0).$$

The surface then meets each of the planes $x = 0$, $y = 0$, $z = 0$, $t = 0$ in two conics, which intersect in four double points lying in this plane. This surface is the *Tetrahedroid*, or, essentially, it is the *Wave Surface*, of Fresnel.

Of the sixteen double tangent planes of this surface, each touching it along a conic, there are four through each of the four fundamental points, $(1, 0, 0, 0)$, $\ldots$, $(0, 0, 0, 1)$. Those through the last point, for example, contain the four common tangent lines of the two conics in which the surface is met by the fundamental plane $t=0$. The eight points of contact, with these two conics, of these four common tangents, lie, with the four conics in the double tangent planes through (0, 0, 0, 1), upon a quadric surface, in regard to which the four fundamental points are a self-polar tetrad. The four double tangent planes through (0, 0, 0, 1) meet, in pairs, in six lines; each of these lines contains two of the sixteen double points of the surface, harmonic conjugates of one another in regard to the plane $t=0$, and the point (0, 0, 0, 1). The six double points of each of these planes are, then, evidently, in involution on the conic which contains them. Cf. pp. 127, 131.

In fact, as the equation we have given at once shews, the equation of the surface may be formed from that of a general quadric touching the four planes $x=0$, $y=0$, $z=0$, $t=0$, by replacing, therein, x, y, z, t, respectively, by x^2, y^2, z^2, t^2. The four quadric surfaces, such as that described, each containing twelve of the double points, are obtained, by the same substitution, from the polar planes of the four fundamental points in regard to this quadric surface.

The quadratic complex, of which this particular quartic surface is the singular surface, may be taken to be that corresponding to the equations, in the fivefold space, $x_1^2+\ldots+x_6^2=0$, and

$$k_1(x_1^2-x_4^2)+k_2(x_2^2-x_5^2)+k_3(x_3^2-x_6^2)=0;$$

this follows from what is said as to the involution, if we recall the forms of the coefficients, in the Kummer surface, in terms of $k_1, \ldots, k_6$. The equation $k_1(x_1^2-x_4^2)+\ldots=0$, it is easily seen, is that for the complex of lines, in the threefold space, which meet two quadric surfaces in two harmonically conjugate pairs of points.

Kummer's surface in the geometry of space of five dimensions. It has been seen that Kummer's quartic surface is the locus of points for which the cone of lines of a quadratic complex breaks up into two planes. If we represent the lines by points of a quadric, Ω, in fivefold space, and the quadratic complex by the points common to Ω and a further quadric, Ω', the representation of Kummer's surface is by an aggregate of planes lying on Ω, whose conic intersection with Ω' breaks up into two lines; as we have already seen. We may follow the theory through from this point of view, without explicit reference to the lines in threefold space. Thereby we obtain a clearer view of some of the results. But we also, then, make an interesting comparison with the theory of the locus, Γ, the intersection of two quadrics in *fourfold* space, studied in the preceding chapter.

We denote the coordinates in the fivefold space by x, y, z, u, v, w.

We suppose the equations of Ω, Ω', respectively, to be

$$x^2 + \ldots + w^2 = 0, \quad k_1 x^2 + \ldots + k_6 w^2 = 0.$$

We may also denote by Ω'' the quadric whose equation is

$$k_1^2 x^2 + \ldots + k_6^2 w^2 = 0.$$

If a plane, α, lying entirely on Ω, meet Ω' in a conic which breaks up into two lines, it is convenient to have a notation, and a name, for the point of intersection of these two lines. It represents what, in the line theory in the threefold space, is called a *singular line* of the quadratic complex. We shall denote the point by P, and call it the *focus* of the plane α, in regard to Ω'. It is shewn that P lies on Ω'', as well as on Ω and Ω'; and, further, in consequence, that the tangent fourfold of Ω', at P, touches Ω, at a point, Q, lying on the plane α. If $(x_0, y_0, \ldots, w_0)$ be the coordinates of P, the line PQ is the locus of points whose coordinates are of the forms

$$(k_1 + \sigma)\, x_0, \ \ldots, \ (k_6 + \sigma)\, w_0,$$

for different values of σ. This line is then, in part, the generalisation of a line considered, for two quadrics in threefold space (Vol. III, p. 92) and, for two quadrics in fourfold space (p. 183, above). This line is of great importance here, and we may call it the *focal line* of the plane α. If the plane α be of the first kind, of planes lying on Ω, we shew that the plane of Ω of the second kind, drawn through the focal line, is likewise a *singular plane*, meeting Ω' in two lines. This will be seen to give the result, in the threefold space, that the envelope of the singular planes of the quadratic complex, in the threefold space, is the locus of the singular points. The intersections of such focal lines with the fundamental fourfolds $x = 0, \ldots, w = 0$, suggest the consideration of six quadratic *congruences*, associated with the Kummer surface, in a very natural way. Further, it is shewn that the plane, α, which meets Ω' in two lines, equally meets every quadric of equation $k_1' x^2 + \ldots + k_6' w^2 = 0$ in two lines, if k_r' be any (fractional) linear function of k_r, independent of r; and that the focal line is the locus of the foci of the plane in regard to these quadrics.

We shall speak of a fourfold, of the fivefold space, as a *prime*, suggesting this, in general, as a name for an $(n-1)$-fold planar space of a fundamental n-fold space, (for which the word *form* has been used).

Consider a plane, α, lying on Ω, usually meeting Ω' in a proper conic. Let O be any point of this plane; thus O lies on Ω, but is supposed, at first, not to lie on Ω'. Let T be the tangent prime of Ω at the point O, and Π be the polar prime of O in regard to Ω'. The prime T meets Ω in a point-cone, of vertex O, which contains α as one of its planes; this cone meets the solid (T, Π) in a quadric surface, which we denote by ω. The line in which the plane α meets the prime Π is one generator of the quadric surface ω; this line we may denote by l. The solid (T, Π) contains another quadric surface, in which this solid is met by the quadric Ω'; this quadric surface we denote by ω'.

The line l is the polar line of O in regard to the conic (α, Ω'); and meets Ω' in the two points of contact, with this conic, of the two tangent lines drawn from O. But, if the plane α meets Ω' in two lines, these two points of contact coincide. Conversely, if O does not lie on Ω', the intersections of l with Ω' can only coincide when α meets Ω' in two lines. The quartic curve of intersection of the two quadric surfaces, ω and ω', of the threefold space (T, Π), is touched, we know (Vol. III, p. 69), by four generators of ω, of either system of generators (a simple proof of this is by remarking that, on a quadric (∞^4), in fivefold space, the tangent primes of the quadric, at the points of a plane section, meet any other plane in the tangent lines of a conic; this has four tangents common with the conic section of the quadric with this other plane). Therefore, the necessary and sufficient condition that the conic (α, Ω') should be two lines, is that the generator l, or (α, Π), of the quadric surface ω, should be one of these four particular generators. Thus we see that, through a point, O, of Ω, not lying on Ω', there can be drawn four planes, of either of the two systems of planes of Ω, to meet Ω' in two lines. This is the representative of the fact, for a quadratic complex of lines in threefold space, that the locus of the singular points is of the fourth order, and the envelope of the singular planes is of the fourth class.

As was suggested, denote the intersection of the two lines of Ω', in such a plane, α, by P, and call it the focus of this plane in regard to Ω'. The tangent prime of Ω' at P contains these lines, and contains the plane α, which lies on Ω. This prime meets Ω in an (∞^3) quadric, lying in a fourfold space; such a quadric, it is easily proved, cannot contain a plane, unless the quadric be a point-cone with vertex on the plane. Thus the tangent prime of Ω' at P meets Ω in a point-cone, with vertex on the plane, α, of Ω. Therefore, this tangent prime touches Ω, at this vertex; we denote this vertex by Q. Conversely, if the tangent prime of Ω', at a point P, which lies both on Ω and Ω', touch Ω at a point Q, there can be drawn, through the line PQ, a single plane, lying on Ω, of either system of planes of Ω, to meet Ω' in two lines intersecting at P. For, the line QP, lying in the tangent prime of Ω at Q, meets Ω, there, in two coincident points, and meets Ω also at P; this line, therefore, lies on Ω. Thus there can be drawn, through this line, a definite plane of Ω of either system. Such a plane, being in the tangent prime of Ω at Q, that is, in the tangent prime of Ω' at P, meets Ω' in two lines which meet at P. It may be remarked too, considering the polar primes, in regard to Ω', of the ∞^2 points of the plane α, that, among these will be the tangent prime of Ω' at P. This contains the plane α, touching Ω at Q. The ∞^2 polar primes, by their intersection with α, thus give only ∞^1 lines, the lines of the plane passing through P. As suggested, we may call the line PQ the focal line of the plane α; it lies on Ω, and on the tangent prime of Ω' at P; but does not lie in Ω', in general.

With the notation explained above, the point P being $(x_0, \ldots, w_0)$, the condition that the tangent prime of Ω' at P, or

$$k_1 x x_0 + \ldots + k_6 w w_0 = 0,$$

should touch Ω, is $k_1^2 x_0^2 + \ldots + k_6^2 w_0^2 = 0$, the point of contact, Q, being $(k_1 x_0, \ldots, k_6 w_0)$. Thus P lies on a surface common to the three quadrics Ω, Ω', Ω''; and the general point of the focal line PQ has coordinates $(k_1 + \sigma) x_0, \ldots, (k_6 + \sigma) w_0$.

The tangent lines of a Kummer surface at a point. We have seen that in general there are four planes of Ω, of either system, passing through a point of Ω, which meet Ω' in two lines. We have also seen that such a plane, α, contains a focal line, through which there passes a plane of Ω, of the other system, which also meets Ω' in two lines. We consider, now, the four singular planes of Ω, of either system, passing through a general point of this focal line; we shew that these consist of the plane, α, spoken of, in which the line is the focal line, this counting doubly, together with two other planes. Let the point of the focal line be called S. We take the tangent prime, T, of Ω, at the point S, and the polar prime, Π, of S, in regard to Ω'. The tangent prime contains the plane α, and, therefore, contains the focus, P, of the plane; the tangent prime of Ω' at P is the tangent prime of Ω at a point, Q, of the focal line, and also contains the plane α. Thus the polar prime of S, in regard to Ω', likewise contains P. Next, we take the two quadric surfaces in which, respectively, Ω and Ω' meet the solid intersection of T and Π. These two surfaces have a common tangent plane at P. For, to express briefly by the symbols what can also be expressed in words, if the point S be of coordinates

$$(k_1 + \sigma) x_0, \ \ldots, \ (k_6 + \sigma) w_0,$$

where P is $(x_0, \ldots, w_0)$, the two primes T, Π are, respectively,

$$(k_1 x x_0) + \sigma (x x_0) = 0$$

and

$$(k_1^2 x x_0) + \sigma (k_1 x x_0) = 0,$$

and the tangent prime of Ω at P is $(x x_0) = 0$, where $(x x_0)$ denotes $x x_0 + \ldots + w w_0$, etc.; thus, the tangent plane of the former, ω, of the two quadric surfaces, at P, is given by

$$(x x_0) = 0, \quad (k_1 x x_0) = 0, \quad (k_1^2 x x_0) = 0.$$

Again, the tangent prime of Ω' at P is $(k_1 x x_0) = 0$; the tangent plane, at P, of the latter, ω', of the two quadric surfaces, is, therefore, given by the same three equations. This common tangent plane is that of the surface of intersection of the three quadrics Ω, Ω', Ω'', and is independent of the position of S on the focal line (the values $\sigma = 0$, $\sigma = \infty$ not being considered). It is, however, easily proved that, when two quadric surfaces, in threefold space, touch one another, the generating lines, of a particular system, of one of these surfaces, which meet the common curve in two coincident points, consist of the generator of this system, through the point of contact, counting doubly, together with two other generators. This proves what was said, for it was seen that the plane α gives a generator of the quadric ω. It was also seen that the plane, β, of Ω, of the other system from α, which passes through the focal line, meets Ω' in two lines; the argument shews that this also counts doubly, among the four planes of this system which can be drawn through any general point of the focal line.

If we recur to the quadratic complex of lines, in the threefold space, the points of the focal line, PQ, represent lines, in the threefold space, each of which has two coincident intersections with the singular (Kummer) surface, at the same point, this being the point represented by the plane α; that is, the points of the line PQ represent the tangent lines of the singular surface at a point. These lie in the tangent plane of this surface at this point, which is represented by the plane β through the focal line, in the fivefold space. As this, we have seen, is a singular plane, counting doubly, we see that, in the threefold space, *any tangent plane, of the surface which is the locus of singular points, is a plane of the aggregate of singular planes*; as was stated. The point P, of the focal line, whereat the tangent prime of Ω' is equally a tangent prime of Ω, is the point of intersection of the two lines in which the plane α meets Ω', and also of the two lines in which the plane β meets Ω'. Thus, in the threefold space, one of the tangent lines, at any point of the locus of singular points, is the intersection of the two planes into which the quadric cone of complex lines, through the point of contact, breaks up; and this tangent line also contains the two points through one of which passes every complex line in the tangent plane.

The inflexional directions at a point of the Kummer surface. It can be proved from the present point of view (and another proof occurs below) that, in the fivefold space, there are two points of the focal line, PQ, such that, of the four singular planes through the point, three coincide with the plane α. These two points represent the two inflexional tangents, in the threefold space of the quadratic complex of lines, at the point of the Kummer surface. It is interesting to prove this from the present point of view because, as will be seen (p. 231, below), we are then able to put the inflexional curves of the Kummer surface into correspondence with the lines of curvature of a Cyclide. We assume that, when, in threefold space, the curve of intersection of two quadric surfaces, which touch, has a cusp at the point of contact, then, of the generators, of one system, of one of these surfaces, which meet the other surface in two coincident points, three are coincident in the generator of that system which passes through the cusp. The reader may examine the case of the surfaces

$$tz = x^2 - y^2, \quad tz = (a+1)\,x^2 + (b-1)\,y^2 + z\,(lx + my),$$

of which the common curve has a cusp at (0, 0, 0, 1) if a or b be zero. We require then, by what has been said, that the curve of intersection of the two quadric surfaces (T, Π, Ω) and (T, Π, Ω'), lying in the solid (T, Π), in the fivefold space, should have a cusp at P, or $(x_0, y_0, \ldots, w_0)$, where T, Π are defined from a point, S, of the line PQ. Let U, U', U'' denote the tangent primes, respectively of the quadrics Ω, Ω', Ω'', at the point P. Then the tangent prime, T, is $\sigma U + U'$, and the polar prime, Π, is $\sigma U' + U''$. The quadric cone, in the solid (T, Π), with vertex at P, which contains the curve of intersection $(T, \Pi, \Omega, \Omega')$, spoken of, may be given by T, Π and $\sigma\Omega + \Omega'$, this being the intersection of Π with the point-cone $(\sigma U + U', \sigma\Omega + \Omega')$. The two tangent lines of the curve, at P, are the intersections of this cone with U. These coincide if the plane (U, U', U'') touches the cone. This will be so if a point, $(x', y', \ldots)$, exists on the cone, such that the tangent prime of $\sigma\Omega + \Omega'$, at this point, contains P. Namely, with $(k_1 + \sigma)\,x_0 x' + \ldots = 0$, $k_1\,(k_1 + \sigma)\,x_0 x' + \ldots = 0$, $(k_1 + \sigma)\,x'^2 + \ldots = 0$, we must have $(k_1 + \sigma)\,x' x_0 + \ldots = 0$. The equations are satisfied by taking $(k_1 + \sigma)\,x' = x_0, \ldots, (k_6 + \sigma)\,w' = w_0$, provided σ be one of the roots of the equation $(k_1 + \sigma)^{-1}\,x_0^2 + \ldots + (k_6 + \sigma)^{-1}\,w_0^2 = 0$. In virtue of $x_0^2 + \ldots = 0$, $k_1 x_0^2 + \ldots = 0$, $k_1^2 x_0^2 + \ldots = 0$, this is a quadratic equation. If its roots be λ and μ, the two points of the focal line satisfying the conditions are those of coordinates $(k_1 + \lambda)\,x_0, \ldots, (k_6 + \lambda)\,w_0$ and $(k_1 + \mu)\,x_0, \ldots, (k_6 + \mu)\,w_0$.

The associated quadratic congruences. There are also, upon the focal line, six points of great importance, namely those where the line meets the fundamental primes given by $x = 0, \ldots, w = 0$. Consider, in particular, that point for which $w = 0$, whose coordinates are $(k_1 - k_6)\,x_0, \ldots, (k_5 - k_6)\,v_0$, 0. This point we may denote by Q_6. A plane, lying on Ω, which meets Ω' in two lines, will likewise meet the quadric $(k_1 + \theta)\,x^2 + \ldots + (k_6 + \theta)\,w^2 = 0$, or, say, $\theta\Omega + \Omega'$, in two lines; and conversely. And the tangent prime of Ω, at the point of the focal line whose coordinates are

$(k_1 + \theta) x_0, \ldots, (k_6 + \theta) w_0$, will be the tangent prime of the quadric $\theta\Omega + \Omega'$ at the point P, or $(x_0, \ldots, w_0)$. In particular, for $\theta = -k_6$, the quadric $\Omega' - k_6\Omega$ is a point-cone, passing through the intersection of Ω and Ω'; this cone we may denote by K. The tangent prime of Ω at Q_6 is the tangent prime of the cone K at the point P, and, therefore, at all the points of the line joining P to the vertex of this cone. To obtain the planes of Ω meeting Ω' in two lines, which pass through Q_6, we consider the solid intersection of the tangent prime of Ω at Q_6, that is, the tangent prime of the point-cone K at P, with the polar prime of Q_6 in regard to this cone; this polar prime also contains the line joining P to the vertex of the cone K. Then we consider the quadric surfaces which are the intersections of this solid with Ω, and with the cone K. The latter of these quadric surfaces, say ω', consists, however, of two planes. For, the intersection of a quadric point-cone with a prime, Π, passing through its vertex, is a quadric point-cone in this prime, and the intersection of Π with a tangent prime of the former cone touches the latter cone (p. 121, above). Wherefore, the quadric surface ω' is the intersection of a point-cone, lying in the fourfold space Π, with a tangent solid of this. The intersection, we know, consists of two planes, both containing the line joining P to the vertex of this cone. But, the generators of one system, of a quadric ω, in the threefold space, which meet the intersection of this with a pair of planes, ω', in two coincident points, consist of the two generators of ω at the two points where this is met by the line of intersection of the two planes; and each of these counts doubly. Thus we see, for the figure in the space of five dimensions, that the planes of Ω, of the first system, meeting Ω' in two lines, which can be drawn through Q_6, consist of the plane α, containing the line Q_6P, counting doubly, together with another plane also counting doubly. It is clear, however, what this other plane is. For, if the line joining P to the vertex of the cone K meet Ω again in the point P', of coordinates $(x_0, \ldots, v_0, -w_0)$, the point P' is equally on the cone K, on the tangent prime of Ω at Q_6 (which touches the cone at all points of PP'); and P' is also on the polar prime of Q_6 in regard to this cone. The quadric surface ω thus passes through P', as do the two planes constituting the quadric ω'. Thus, the second singular plane, of the first kind, through Q_6, is the plane of Ω, of this kind, which passes through the line Q_6P'. And, there is a plane of the second kind through each of Q_6P and Q_6P', which is likewise a singular plane counting doubly.

In general, an arbitrary plane, lying on Ω, meets the surface expressed by the three equations

$$w = 0,\ x^2 + \ldots + v^2 = 0,\ (k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0,$$

in two points. This surface, however, has the property that the tangent prime of Ω, at any point of this surface, touches the cone K, whose equation is $(k_1 - k_6) x^2 + \ldots + (k_5 - k_6) v^2 = 0$. Thus, the tangent primes of Ω, at the two points where the plane meets this surface, both of which contain this plane of Ω, are two tangent primes of the cone K; they are, therefore, the two tangent primes of K which can be drawn from the solid joining the given plane to the vertex of K. Take the case when, as above, the plane of Ω considered is a plane, α, which meets the cone K in two lines; then, as may be proved, the plane is such that the two tangent primes of the cone through the plane are coincident. We thus infer that the plane α meets the surface in two coincident points.

In the threefold space of the quadratic complex of lines we thus have the following results: Among the tangents at any general point of the singular (Kummer) surface, there are six which touch this surface again (these are the tangent lines from this point to the quartic curve in which the tangent plane cuts the surface). Each of these bitangents, beside belonging to the fundamental quadratic complex, belongs to one of six linear complexes, these complexes being such that any two are conjugate (or apolar). Of the lines of the quadratic congruence given by the quadratic complex and one of the linear complexes, there pass two through an arbitrary point; but each of the bitangents spoken of consists of two coincident lines. The singular surface thus appears as the *focal* surface, or *caustic* surface, of each of six quadratic congruences (cf. for instance, Darboux, *Théorie...des Surfaces*, 2$^{\text{me}}$ Partie, Livre IV). The natural form for the equation of the Kummer surface when regarded as the focal surface of a quadratic congruence is that given above, Ex. 4, p. 138, Chap. V. The theory, in the space of five dimensions, applies equally to both systems of planes of Ω; so that there are also results in the threefold space which are the dual of those we have stated.

The common singular points and planes of the six congruences. In the fivefold space, the surface given by

$$w = 0,\ x^2 + \ldots + v^2 = 0,\ (k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0$$

is the intersection of two (∞^3) quadrics of the fourfold space $w = 0$. This surface has been studied in Chap. VI. It appeared that it contains sixteen lines. Through each of these lines, which lies on Ω, there passes a plane of Ω, of each of two systems. It follows, from what is said above, that each of these two planes lies in infinitely many tangent primes of the cone which we denoted by K; this also appears from a previous remark (p. 48, above), that the two planes of Ω through the line are the intersection of Ω with the polar solid of the line in regard to Ω; the two planes, therefore, lie on the tangent primes of Ω at all the points of the line, and these primes are all tangent primes of the cone K. Thus, there are sixteen planes of the first system of Ω, each of which is a singular plane of special character, having the property of containing, not two but, an infinite number of points of the surface given by

$$w = 0,\ x^2 + \ldots + v^2 = 0,\ (k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0.$$

For greater clearness we may give the equations of these planes. Denote by $f(t)$ the sextic polynomial whose roots are $t = k_1, \ldots, k_6$, and by f_1 the value, for $x = k_1$, of the differential coefficient $\partial f/\partial t$, etc. We use the fact that, for any value of λ, the sum,

$$(k_1 + \lambda)^r/f_1 + \ldots + (k_6 + \lambda)^r/f_6,$$

vanishes, for $r = 0, 1, \ldots, 4$; it is so, in particular, for $\lambda = -k_6$. Denote one of the two square roots of f_r, that is, of $\partial f(k_r)/\partial k_r$, by g_r. When λ, μ vary, the point whose coordinates are

$$g_1^{-1}(k_1 + \lambda)(k_1 + \mu), \ldots, g_6^{-1}(k_6 + \lambda)(k_6 + \mu)$$

describes a plane. This plane meets the prime $w = 0$ in the line which joins the two points

$$[g_1^{-1}(k_1 - k_6)^2, \ldots, g_5^{-1}(k_5 - k_6)^2, 0], [g_1^{-1}(k_1 - k_6), \ldots, g_5^{-1}(k_5 - k_6), 0].$$

This line lies on the quadric $(k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0$. The plane itself lies on the quadric Ω. The tangent primes of Ω, at the points of the line where the plane meets $w = 0$, are the tangent primes of the cone K, or $(k_1 - k_6) x^2 + \ldots + (k_5 - k_6) v^2 = 0$, at the points of the line joining the two points

$$[g_1^{-1}(k_1 - k_6), \ldots, g_5^{-1}(k_5 - k_6), 0], [g_1^{-1}, \ldots, g_5^{-1}, 0].$$

The line joining these two points lies on Ω and on the cone, so that it lies on Ω and Ω'; in fact it lies also on Ω'', the coordinates of every point of the line satisfying the equation

$$k_1^2 x^2 + \ldots + k_5^2 v^2 = 0.$$

By taking all possible combinations of the ambiguous signs in $g_1, \ldots, g_6$, the plane given by the points

$$[g_1^{-1}(k_1 + \lambda)(k_1 + \mu), \ldots, g_6^{-1}(k_6 + \lambda)(k_6 + \mu)]$$

becomes, in turn, thirty-two different planes, sixteen of one system of planes of Ω, and sixteen of the other. If we denote by (pqr) the product $(k_q - k_r)(k_r - k_p)(k_p - k_q)$, the equations of the plane can be shewn to be the three

$$(123)\, g_m x_m = (23m)\, g_1 x + (31m)\, g_2 y + (12m)\, g_3 z,$$

where $m = 4, 5, 6$ and $x_m = u, v, w$. The ratios of the coefficients of the coordinates in these equations are such as are unaffected by any, the same, fractional linear transformation of $k_1, k_2, \ldots, k_6$.

In the threefold space of the quadratic complex of lines, for any one of the six quadratic congruences which have been referred to, there are, therefore, sixteen points (and sixteen planes) through which there pass (in which there lie), not two lines but, a pencil of lines, of the congruence; and these sixteen elements are the same for all the six congruences. These points and planes are the double elements of a Kummer surface. (Cf. Hudson, *Kummer's Quartic Surface*, p. 59.)

The confocal quadrics. It appears from what has been said (p. 214, above) that a plane of Ω, which meets Ω' in two lines, equally meets in two lines any quadric with the equation

$$K_1x^2 + \ldots + K_6w^2 = 0, \text{ where } K_r = (ak_r + b)/(ck_r + d).$$

When $c = 0$, this is the same as $K_r = d^{-1}b + d^{-1}ak_r$; when c is not zero, it is the same as $K_r = A + B(k_r + \theta)^{-1}$, where $\theta = c^{-1}d$, and A, B are independent of the suffix r. It will, therefore, be sufficient to consider quadrics of the form

$$(k_1 + \theta)^{-1}x^2 + \ldots + (k_6 + \theta)^{-1}w^2 = 0.$$

Of such quadrics there are four which pass through an arbitrary general point of Ω; conversely, four arbitrary values of θ determine a set of thirty-two points of Ω. There are also four planes of Ω, through a general point of Ω, which meet Ω' in a pair of lines. We may easily see that these two facts are connected.

For, consider a plane of Ω, say of the *second* system, which meets Ω' in a pair of lines. Let $(x_0, y_0, \ldots, w_0)$ be the focus of this plane, in regard to Ω'. The quadric

$$(k_1 + \sigma)^{-1}x^2 + \ldots + (k_6 + \sigma)^{-1}w^2 = 0$$

contains the point of coordinates $(k_1 + \sigma)x_0, \ldots, (k_6 + \sigma)w_0 = 0$, lying on the focal line of the plane. The tangent prime of this quadric, at this point, is the tangent prime of Ω at the point $(x_0, \ldots, w_0)$, and, thus, contains the plane considered. The quadric, therefore, touches every line of the plane through the point of coordinates $(k_1 + \sigma)x_0, \ldots, (k_6 + \sigma)w_0$. Now, let $(x', \ldots, w')$ be any point of the plane, and $(k_1 + \sigma)^{-1}x^2 + \ldots + (k_6 + \sigma)^{-1}w^2 = 0$ be a quadric through this point. Then, the line, joining $(x', \ldots, w')$ to the point of coordinates $(k_1 + \sigma)x_0, \ldots, (k_6 + \sigma)w_0$, lies entirely on this quadric. Through this line, lying in a plane of Ω of the second system, can be drawn a plane of Ω of the first system; and, this plane, containing one line of the quadric $(k_1 + \sigma)^{-1}x^2 + \text{etc.} = 0$, contains another line of this quadric. Therefore, as has been seen, and will be proved again immediately, this plane is one of the planes of Ω, of the first system, passing through $(x', \ldots, w')$, which cut Ω' in two lines. Whence we see that the four singular planes, of the first system, which can be drawn through the point $(x', \ldots, w')$, meet the focal line, of any singular plane, of the second system, drawn through $(x', \ldots, w')$, in four points of coordinates

$$(k_1 + \sigma_p)\, x_0, \ldots, (k_6 + \sigma_p)\, w_0,$$

where σ_p has the four values belonging to the quadrics,

$$(k_1 + \sigma)^{-1} x^2 + \text{etc.} = 0,$$

which can be drawn through $(x', \ldots, w')$. This result includes also the fact, incidentally referred to above (p. 213), that, in the threefold space, the points, in which a general line meets the Kummer surface, constitute a range which is related to the pencil of four tangent planes of the surface passing through the line. (Cf. also p. 240, below.)

But the quartic equation for θ,

$$(k_1 + \theta)^{-1} x'^2 + \ldots + (k_6 + \theta)^{-1} w'^2 = 0,$$

giving the four quadrics of the system which pass through the point $(x', \ldots, w')$, reduces to a quadratic when $(x', \ldots, w')$ is a point, P, or $(x_0, \ldots, w_0)$, for which, beside $x_0^2 + \text{etc.} = 0$, we have

$$k_1 x_0^2 + \text{etc.} = 0 \text{ and } k_1^2 x_0^2 + \text{etc.} = 0.$$

And, more generally, when, with such values of $(x_0, \ldots, w_0)$, the point $(x', \ldots, w')$ is such that $x' = (k_1 + \sigma)\, x_0, \ldots, w' = (k_6 + \sigma)\, w_0$, the quartic equation reduces to

$$(\theta - \sigma)^2 \left[(k_1 + \theta)^{-1} x_0^2 + \ldots + (k_6 + \theta)^{-1} w_0^2\right] = 0,$$

as we see by writing $(k_1 + \sigma)^2 (k_1 + \theta)^{-1}$ in the form

$$k_1 - \theta + 2\sigma + (\theta - \sigma)^2 (k_1 + \theta)^{-1}, \text{ etc.}$$

The four roots of the quartic equation thus consist of the root σ, repeated, together with the roots of the quadratic

$$(k_1 + \theta)^{-1} x_0^2 + \ldots + (k_6 + \theta)^{-1} w_0^2 = 0.$$

If the roots of this quadratic, which are independent of σ, be denoted by λ and μ, it follows that the two quadrics,

$$(k_1 + \lambda)^{-1} x^2 + \text{etc.} = 0 \text{ and } (k_1 + \mu)^{-1} x^2 + \text{etc.} = 0,$$

contain the line PQ, joining $(x_0, \ldots, w_0)$ and $(k_1 x_0, \ldots, k_6 w_0)$, completely; and, further, that the other two, of the four quadrics, $(k_1 + \theta)^{-1} x^2 + \ldots = 0$, which pass through any point, S, of coordinates $(k_1 + \sigma)\, x_0, \ldots, (k_6 + \sigma)\, w_0$, lying on this line PQ, coincide with one another, both being the quadric $(k_1 + \sigma)^{-1} x^2 + \text{etc.} = 0$. The tangent prime of this quadric, at this point S, is the tangent prime of Ω at P, as remarked above; and this shews, from what has been said, that the plane, α, of the quadric Ω, of the first (or second) system, which passes through the line PS, meets the quadric $(k_1+\sigma)^{-1}x^2+\text{etc.}=0$ in two lines, these intersecting at S. Thus S is the focus of the plane in regard to this quadric, just as P is its focus in regard to Ω'. There are, however, two particular points of the line PS for which three of the quadrics,

$$(k_1+\theta)^{-1}x^2+\text{etc.}=0,$$

passing through the point, coincide with one another; namely, those for which $\sigma=\lambda$ or $\sigma=\mu$. For each of these two points, of the four planes of Ω through the point which meet Ω', and, therefore, any of the quadrics $(k_1+\theta)^{-1}x^2+\text{etc.}=0$, in two lines, there are three which coincide with one another. (Cf. Vol. III, p. 92; and p. 185, Chap. VI, above.)

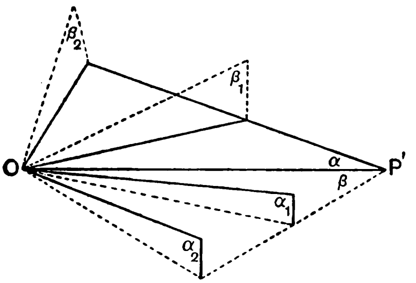
Singular planes through a more particular point. In the preceding we consider planes of Ω, meeting a quadric,

$$(k_1+\theta)^{-1}x^2+\text{etc.}=0,$$

in two lines, which are drawn from a point of Ω. Consider now, briefly, planes of Ω meeting Ω' in two lines, which are drawn from a point, O, which lies on Ω' as well as on Ω. This we excluded above. In this case, any plane of Ω, drawn through O, meets Ω' in a conic passing through O; if this conic break up into two lines, one of these must pass through O. Such a line lies on Ω, and on Ω', and on the tangent primes of Ω and Ω' at O. Conversely, the solid which is the intersection of the tangent primes of Ω and of Ω' at O, is met, by Ω and Ω', in two conical sheets of lines, with vertex at O. These have, in general, four lines in common, through O. Through any one of these four lines there passes a definite plane of Ω, of the first system, (as also one of the second system); this plane meets Ω' in another line, beside this one, intersecting this, say, in the point P; this is the focus, in regard to Ω', of the plane of these two lines. We thus have, as before, four singular planes, of one system of planes of Ω, passing through O.

But, it may happen that the plane, α, of the first system, through one of the four generators common to the two conical sheets, contains another of these four generators. Then the plane of these two generators, which are lines of Ω' meeting in O, lies in the tangent prime of Ω' at O. This prime, then, containing a plane of Ω, is a tangent prime of Ω, at some point, say P', of the plane. Then O is the focus of the plane, in regard to Ω', and P coincides with O. The plane then counts doubly among the four singular planes which can be drawn from O, as passing through two of the four generators. Through the line OP', lying on Ω, can be drawn a plane, β, of the second system of planes of Ω; this plane is on the tangent prime of Ω at P', which is the tangent prime of Ω'

at O. Thus the plane β meets Ω' in two lines which intersect at O; and these, lying on Ω and Ω', are the other two generators, of the four originally proved to pass through O, beside the two which lie in the plane α. Through each of these lines, in the plane β, can be drawn a single plane, of the first system of planes of Ω; say these are α_1 and α_2. In the same way, through each of the two lines lying in the plane α, can be drawn a singular plane, of the second system of planes of Ω; say these are β_1 and β_2. Thus we see that if, of the four lines originally drawn through O, two lie in a plane of the first system, counting doubly among the four singular planes of that system which pass through O, then the other two of these lines lie in a plane of the second system passing through O.



In this figure, the line OP' meets Ω' in two coincident points at O. But it may happen that OP' lies entirely on Ω'. Then it is one of the two lines of Ω' through O lying in the plane α; in which case, the plane of the second system of Ω, say β_1, drawn through this line, coincides with the plane β. But, also, OP' is then one of the two lines of Ω' through O lying in the plane β, and the plane of the first system, say α_1, drawn through this line, coincides with the plane α. In this case, the four singular planes of the first system, passing through O, consist of the plane α, counting three times, together with the plane α_2; and, similarly, the four singular planes of the second system, passing through O, consist of the plane β, counting three times, together with the plane β_2. In general, the solid, which is the intersection of the tangent primes of Ω and Ω' at O, was met by Ω in a conical sheet. In this particular case, this conical sheet consists of the two planes α, β; and the line of intersection, OP', of these planes, is on the conical sheet of lines of Ω', through O, which lie in this solid.

The Inflexional (or Asymptotic) curves of the Kummer Surface. We have considered, in the fivefold space, the surface of points $(x_0, \ldots, w_0)$, for which $x_0^2 + \ldots + w_0^2 = 0, k_1 x_0^2 + \ldots + k_6 w_0^2 = 0$, $k_1^2 x_0^2 + \ldots + k_6^2 w_0^2 = 0$. The points of this surface may be represented, in terms of two parameters, λ and μ, by the equations $x_0 = (k_1 + \lambda)^{\frac{1}{2}} (k_1 + \mu)^{\frac{1}{2}} / [f'(k_1)]^{\frac{1}{2}}$, etc., where $f(t) = (t - k_1) \ldots (t - k_6)$. The parameters belonging to $(x_0, \ldots, w_0)$ are the two roots of the equation $(k_1 + \theta)^{-1} x_0^2 + \ldots + (k_6 + \theta)^{-1} w_0^2 = 0$. Through the point $(x_0, \ldots, w_0)$ there passes a line of which any point has coordinates

$(k_1+\sigma)x_0, \ldots, (k_6+\sigma)w_0$, this line lying entirely on the two quadrics $(k_1+\lambda)^{-1}x^2 + \text{etc.} = 0$, $(k_1+\mu)^{-1}x^2 + \text{etc.} = 0$. The two points of this line, for which $\sigma = \lambda$, $\sigma = \mu$, have been shewn to represent inflexional tangents of the Kummer surface (p. 223). For clearness, we call this line the focal line through $(x_0, \ldots, w_0)$. Consider now the curve, of points $(x_0, \ldots, w_0)$, of the surface given by $x_0^2 + \ldots = 0$, $k_1x_0^2 + \ldots = 0$, $k_1^2x_0^2 + \ldots = 0$, for which μ is constant; draw the focal lines through the points of this curve; and, upon the line drawn through the point (λ) of this curve, take the points of coordinates $(k_1+\lambda)x_0, \ldots, (k_6+\lambda)w_0$. Similarly, upon the focal line drawn through the point $(\lambda + d\lambda)$ of this curve, take the point of which the first coordinate is $(k_1+\lambda+d\lambda)(x_0+dx_0)$; with $dx_0 = \frac{1}{2}x_0(k_1+\lambda)^{-1}d\lambda$, this is the same as

$$(k_1+\lambda)x_0 + 3x_0d\lambda/2 + \tfrac{1}{2}x_0(k_1+\lambda)^{-1}(d\lambda)^2.$$

Thus, neglecting the term in $d\lambda^2$, this point lies on the former focal line. From this we can infer that the focal lines, of the points of the curve in question, along which μ is constant, form a developable surface (cf. Vol. III, p. 132), whose cuspidal edge is the locus, as λ varies, of the point whose coordinates are such as $(k_1+\lambda)x_0$, or $(k_1+\lambda)^{\frac{3}{2}}(k_1+\mu)^{\frac{1}{2}}/[f'(k_1)]^{\frac{1}{2}}$. The points of this cuspidal edge represent, in the original threefold space of the quadratic complex of lines, a sequence of inflexional tangents of the Kummer surface, so taken that, to any one of them there follows, in a limiting sense, another, lying in the same tangent plane of the surface. Thus, the cuspidal edge, in the fivefold space, represents the tangent lines of an inflexional curve of the Kummer surface.

Ex. Shew that this curve, in the threefold space, is of order 16, and of genus 17, when μ is general. There are, however, six particular *principal* inflexional curves, for which μ has one of the values $-k_1, \ldots, -k_6$; these are of order 8, and of genus 5.

It is of interest now to shew that the inflexional curves of the Kummer surface are in correspondence with the lines of curvature of a Cyclide. Consider the two planes of the quadric, Ω, in fivefold space, of the two systems, say α and β, which pass through the focal line joining the points $(x_0, \ldots, w_0)$, $(k_1x_0, \ldots, k_6w_0)$. The solid determined by these planes lies in the tangent prime of Ω at every point of the focal line. This solid meets any fixed prime in a plane, determined by the two lines in which the prime is met by the two planes. Or, this plane may be defined as the intersection of the three primes consisting of: the fixed prime; the tangent prime of Ω at the point, say Q_6, where the focal line meets this prime; and, the tangent prime of Ω at some other specified point of the focal line, say the point P, or $(x_0, \ldots, w_0)$, for which

$x_0^2 + \ldots = 0$, $k_1 x_0^2 + \ldots = 0$, $k_1^2 x_0^2 + \ldots = 0$. It will be sufficient here to take, for the fixed prime, the prime $w = 0$. As P varies, the point Q_6 describes a surface, in $w = 0$; as Q_6 is on Ω, this surface is on Ω; as Q_6 is of coordinates $(k_1 - k_6)\,x_0, \ldots, (k_5 - k_6)\,v_0, 0$, this surface lies also on $(k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0$. Thus, the tangent plane of this surface at Q_6, which is given by $w = 0$, $(k_1 - k_6)\,xx_0 + \ldots + (k_5 - k_6)\,vv_0 = 0$, and $xx_0 + \ldots + vv_0 = 0$, is the plane spoken of, in which the prime $w = 0$ is met by the solid (α, β). We put $X_0 = (k_1 - k_6)\,x_0, \ldots, V_0 = (k_5 - k_6)\,v_0$, and denote the quadrics $(k_1 - k_6)^{-1} x^2 + \ldots = 0$, $x^2 + \ldots = 0$ by Φ and Φ', respectively. The general solid passing through the tangent plane of the surface (w, Φ, Φ'), at the point $(X_0, \ldots, V_0)$, namely, the tangent solid of the quadric $[w, (k_6 + \theta)\,\Phi + \Phi']$, whose equations are $w = 0$, with $(k_1 + \theta)\,xx_0 + \ldots + (k_5 + \theta)\,vv_0 = 0$, is the intersection of $w = 0$ with the tangent prime of Ω at the point of the focal line whose coordinates are $(k_1 + \theta)\,x_0, \ldots, (k_6 + \theta)\,w_0$. This solid meets the surface (w, Φ, Φ') in a quartic curve having a double point at $(X_0, \ldots, V_0)$, and there are two values of θ for which this double point is a cusp (p. 188, Chap. VI, above). These values of θ, as we see at once from what was said in Chap. VI, are given by $(k_1 - k_6)\,x_0^2 (k_1 + \theta)^{-1} + \ldots + (k_5 - k_6)\,v_0^2 (k_5 + \theta)^{-1} = 0$, that is, by $x_0^2 (k_1 + \theta)^{-1} + \ldots + v_0^2 (k_5 + \theta)^{-1} + w_0^2 (k_6 + \theta)^{-1} = 0$. We have seen that there are two points of the focal line with the property that, of the four planes of Ω drawn through each, to meet Ω' in two lines, three are coincident; the tangent prime of Ω at such a point, by its intersection with Ω, Ω' and the polar prime of the point in regard to Ω', gives a quartic curve with cusp at the point P (p. 223, above). It now appears that this same tangent prime, by its intersection with $w = 0$, Ω, and the quadric

$$(k_1 - k_6)^{-1} x^2 + \ldots + (k_5 - k_6)^{-1} v^2 = 0$$

which, together, represent, in fivefold space, a congruence of bitangents of the Kummer surface—gives also a quartic curve with a cusp at the point, Q_6, where the focal line meets $w = 0$. Either of the two particular points of the focal line, in the former case, represents an inflexional tangent of the Kummer surface; in the latter case, as we have seen (p. 186, above), the intersection of Ω with the corresponding tangent prime gives, in $w = 0$, a surface which projects, from any point of Ω in $w = 0$, into a sphere, of an arbitrary threefold space, determining a tangent line of a line of curvature of a Cyclide. We recall, now, the method, previously explained (p. 55, above), by which we pass from a line, of threefold space, to a sphere; this consists in taking the threefold intersection, with an arbitrary prime, of the tangent prime, of the quadric Ω in fivefold space, at the point of this which represents the line, and then projecting the surface in which this threefold meets Ω. We see then that, by this method, the inflexional line of the Kummer surface is transformed into the sphere having stationary contact with the Cyclide along a direction of curvature. Moreover the cuspidal edge spoken of above, whose points are given by $x = (k_1 + \lambda)^{\frac{3}{2}} (k_1 + \mu)^{\frac{1}{2}}/[f''(k_1)]^{\frac{1}{2}}$, etc., with λ variable but μ constant, which represents the inflexional lines of the Kummer surface along an inflexional curve, corresponds to the line of curvature on the Cyclide, represented, also with λ variable but μ constant, by the curve in $w = 0$ whose points are given by

$$x = (k_1 - k_6)(k_1 + \lambda)^{\frac{1}{2}} (k_1 + \mu)^{\frac{1}{2}}/[f'(k_1)]^{\frac{1}{2}}, \text{ etc.}$$

Further, in Lie's transformation, by which the correspondence is established, there are two lines corresponding to a sphere. The second point, in our figure, thus associated with the point

$$x = (k_1 + \lambda)^{\frac{3}{2}} (k_1 + \mu)^{\frac{1}{2}}/[f'(k_1)]^{\frac{1}{2}}, \text{ etc.,}$$

is the point with the same coordinates $x, \ldots, v$, but with a change of sign of w. Thus (p. 224, above), there are two inflexional tangents of the same inflexional curve, of a Kummer surface, corresponding to the same sphere of stationary contact of the line of curvature of the Cyclide; and the points of contact of these are the two points where a bitangent of the Kummer surface touches the surface. The aggregate of these bitangents is a ruled surface in the threefold space, which is of order 8, represented, in the fivefold space, by the curve

$$w = 0,\ x^2 + \ldots = 0,\ (k_1 - k_6)^{-1} x^2 + \ldots = 0,\ (k_1 + \mu)^{-1} x^2 + \ldots = 0.$$

The complete intersection of this ruled surface with the Kummer surface is the inflexional curve, counted twice, which is of order 16 (cf. Klein, *Ges. Math. Abh.*, 1921, pp. 91, 126, 147, etc. Also Hudson, *Kummer's Quartic Surface*, 1905, p. 61, etc.).

The theory is capable of generalisation. By Lie's method of transformation, the tangent lines of a surface (S), in space of three dimensions, correspond to the spheres touching another surface (S'). The tangents at a point, M, of (S), correspond to the spheres touching (S') at a point M'. When the point M describes an inflexional curve of (S), the corresponding point M' describes a line of curvature of (S'). (Lie, *Math. Annal.* v, 1871, pp. 145—256.)

Ex. In the correspondence between a Kummer surface and the Weddle surface which has been obtained above (p. 158), prove that the inflexional directions at any point of either surface correspond to conjugate directions on the other (cf. the writer's *Multiply-Periodic Functions*, 1907, p. 127). In particular, shew that, corresponding to one of the six principal inflexional curves of the

Kummer surface, there is a curve upon the Weddle surface which is of order 7, being the intersection of two cubic curves touching along a generator (*loc. cit.*, p. 322. Also Humbert, *Journ. École Polytechn.*, LXIV, 1894, p. 123).

The rationality of the Quadratic Complex, and of the Quadratic Congruence. We pass now to various theorems, mainly of line geometry, which seem called for by a comparison of some of the results which have been given in this Volume.

The representation, in space of five dimensions, of a quadratic complex of lines of threefold space, is by the (∞^3) locus which is common to two quadrics of this fivefold space. This locus is, in fact, obtainable by considering, in a threefold space, the cubic surfaces which can be drawn through a certain quintic curve. For, consider, in space of five dimensions, any two quadrics of general character. Let U be any point common to both. The lines, through U, lying on both quadrics, lie on the solid intersection of the tangent primes at U, and, therefore, on two conical sheets. We can thus draw four lines through U lying on both quadrics. Let V be any point of one of these lines. The line lies on the tangent primes of both quadrics at V, as well as at U. Denote the tangent primes of the first quadric at U and V, respectively, by $x=0$, $y=0$; and the tangent primes of the second quadric, at these two points respectively, by $z=0$, $t=0$; also, let U, V, beside the zero coordinates x, y, z, t, have the respective coordinates $u=1$, $v=0$ and $u=0$, $v=1$. Then the equations of the two quadrics will be, respectively,

$$\phi + ux + vy = 0, \quad \psi + uz + vt = 0,$$

where ϕ, ψ are homogeneous quadratic polynomials in x, y, z, t. These equations lead to $u(xt-yz) = \psi y - \phi t$, $v(xt-yz) = \phi z - \psi x$. Hence, in terms of parameters ξ, η, ζ, τ, if Φ, Ψ denote the same polynomials in these that ϕ, ψ are in x, y, z, t, the coordinates, x, y, z, t, u, v, of any point of the ∞^3 locus common to the two quadrics, are in the ratios of

$$\xi(\xi\tau-\eta\zeta),\ \eta(\xi\tau-\eta\zeta),\ \zeta(\xi\tau-\eta\zeta),\ \tau(\xi\tau-\eta\zeta),\ \Psi\eta-\Phi\tau,\ \Phi\zeta-\Psi\xi.$$

Regarding ξ, η, ζ, τ as coordinates in a threefold space, each of these six functions, equated to zero, represents a cubic surface. Moreover, the quintic curve which is the intersection, other than the line $\eta=0$, $\tau=0$, of the quadric surface $\xi\tau-\eta\zeta=0$ with the cubic surface $\Psi\eta-\Phi\tau=0$, is the same as the quintic curve which is the intersection, other than the line $\zeta=0$, $\xi=0$, of this quadric, $\xi\tau-\eta\zeta=0$, with the cubic surface $\Phi\zeta-\Psi\xi=0$. The two cubic surfaces have in common, beside this quintic curve, the quartic curve of intersection of the two quadric surfaces $\Phi=0$, $\Psi=0$.

The quintic curve, common to a quadric surface and a cubic surface having a line in common, meets an arbitrary plane drawn through this line, in the two points common to the residual intersections of this plane with the two surfaces—which are, a line and a conic. Thus the quintic curve meets the common line of the two surfaces in three points. It, therefore, meets every generator of the quadric surface, of one system, in three points, and the other generators each in two points, and its genus, p, is given by $p = (3-1)(2-1), = 2$ (see, for example, *Proc. Lond. Math. Soc.*, XI, 1912, p. 286). Conversely, for a surface, of order n, to contain the quintic curve of genus p, the number of linear conditions, for the coefficients in the equation of this surface, when n is large enough, is $5n - p + 1$. For $n = 3$ and $p = 2$, this is 14. This gives $20 - 14$, or six, as the number of linearly independent cubic surfaces containing the curve. The ratios of the polynomials expressing the equations of these surfaces can be used as coordinates in a space of five dimensions.

Interpreted in the original space of five dimensions, the process is the simple one of projecting the original ∞^3 locus, by means of planes passing through a line of the locus, given above by

$$x = y = z = t = 0,$$

upon an arbitrary threefold space, say Π. Such a plane is determined by one point of the locus, and meets Π in a point. Conversely, such a plane, determined by any point of Π, meets each of the given quadrics in another line, beside the common line of these through which the plane is drawn; the intersection of these residual lines is a point of the locus, corresponding to the point taken in Π. An arbitrary prime of the fivefold space, say

$$Ax + By + Cz + Dt + Pu + Qv = 0,$$

meets the locus in a quartic surface; this is projected from the line $x = y = z = t = 0$ by the line-cone given by

$$Ax(xt - yz) + \ldots + Dt(xt - yz) + P(\psi y - \phi t) + Q(\phi z - \psi x) = 0,$$

which is of the third order.

This rationalisation of the locus is given by Klein, *Ges. Math. Abhandl.*, I, p. 89, as due to Noether (1869). By combining the two equations of the locus, we may evidently regard one of them as having the form

$$z't' + ux' + vy' = 0,$$

usual for the relation connecting the coordinates of a line in threefold space. The intersection of two (∞^{n-1}) quadrics, in n-fold space, may be represented by the points of an $(n-2)$-fold, in a similar way, by planes through a line common to the two quadrics (the quadrics not being general when $n=3$). The aggregate of (planar) p-folds existing on the (∞^{n-2}) locus of intersection of two (∞^{n-1}) quadrics, in n-fold space, can be shewn to be of dimension $(p+1)(n-2p-2)$. In particular, when $n = 2p+2$, the number of such p-folds is 2^n, in the most general case; for example, the number of lines on the quartic surface so obtained in fourfold space is sixteen.

We pass now to the case of a Quadratic *Congruence* of lines, defined as the aggregate of the lines common to a quadratic complex and a linear complex. We suppose this to be represented, in the space of five dimensions, by the surface which is the intersection of the preceding ∞^3 locus with a prime of this space. This surface, being the intersection of two quadrics of a fourfold (planar) space, is the same as that studied in Chap. VI. It was shewn to be representable rationally by two parameters (p. 165, above). Any one of the sixteen lines, which were shewn to lie upon it, may be taken to be the line UV of the preceding discussion. Then the prime, whose intersection, with the ∞^3 locus above, determines the congruence, as it contains all the points representing the congruence, and therefore contains this line UV, will have an equation of the form $Ax+By+Cz+Dt=0$. Now, the intersection of a plane, in a threefold space, with a cubic surface passing through a quintic curve, is a cubic curve passing through five fixed points. Thus we see that the line coordinates of a line of a quadratic congruence, in threefold space, are proportional to homogeneous cubic polynomials, in three coordinates, which, equated to zero, represent plane cubic curves passing through five given points. This has been verified in several cases previously examined (pp. 137, 141, 165). Conversely, take five arbitrary points in a plane, and consider what are the relations connecting the polynomials which, equated to zero, represent cubic curves through these five points. If we take six such polynomials, as there are only five linearly independent cubic curves through the five points, these six polynomials are connected by a linear relation. Further, the five ratios of these polynomials, being dependent upon the two ratios of the coordinates in the plane, must be connected by two other algebraic relations. That these may be regarded as arising from two *quadratic* equations connecting the six polynomials, can be made clear by considering six particular polynomials, which are connected by two quadratic relations; for any other six polynomials are expressible linearly in terms of these.

Two ways of obtaining the necessary relations have presented themselves in foregoing work. If, referred to three of the five points, the conic containing these five points be $S=0$, and the line joining the other two points be $P=0$, where $S=\eta\zeta+\zeta\xi+\xi\eta$, and $P=a\xi+b\eta+c\zeta$, then six cubic polynomials which vanish at the five points are x, y, z, x', y', z', given by

$$x=\xi S,\ y=\eta S,\ z=\zeta S,\ x'=(q-r)\eta\zeta P,\ y'=(r-p)\zeta\xi P,\ z'=(p-q)\xi\eta P,$$

in which p, q, r are quite arbitrary constants. These are connected by the relations

$$xx'+yy'+zz'=0,\ \ pxx'+qyy'+rzz'=0,$$
$$-ax-by-cz+(q-r)^{-1}x'+(r-p)^{-1}y'+(p-q)^{-1}z'=0.$$

Or again, if, referred to three of the points, the other two points be (1, 1, 1) and (a, b, c), then, also with three arbitrary constants, p, q, r, six cubic polynomials vanishing at the five points are given by

$$x=(\eta c-\zeta b)(\zeta-\xi)(\xi-\eta),\quad y=(\zeta a-\xi c)(\xi-\eta)(\eta-\zeta),\quad z=(\xi b-\eta a)(\eta-\zeta)(\zeta-\xi),$$
$$x'=(q-r)(\zeta a-\xi c)(\xi b-\eta a)(\eta-\zeta),\quad y'=(r-p)(\xi b-\eta a)(\eta c-\zeta b)(\zeta-\xi),$$
$$z'=(p-q)(\eta c-\zeta b)(\zeta a-\xi c)(\xi-\eta).$$

These are connected by the three equations

$$(b-c)[ax+(q-r)^{-1}x']+(c-a)[by+(r-p)^{-1}y']+(a-b)[cz+(p-q)^{-1}z']=0,$$
$$xx'+yy'+zz'=0,\quad pxx'+qyy'+rzz'=0.$$

Caporali's theorem, given below (p. 240), also deals with the representation of a quadratic congruence.

The generation of a quadric in fivefold space. The tetrahedral complex in threefold space. Let $\alpha, \beta, \gamma, \delta$ be four planes, in fivefold space, meeting in pairs in six points, (β, γ), (γ, α), (α, β), (α, δ), (β, δ), (γ, δ). These points we denote, respectively, by X, Y, Z, X', Y', Z'; and we denote coordinates relative to these points by x, y, z, x', y', z'. It is easy to see that the general quadric containing these four planes has an equation of the form $Axx'+Byy'+Czz'=0$; and that the quadric is made definite by the condition of containing a given fifth plane, which meets each of $\alpha, \beta, \gamma, \delta$ in a point. Let the points of the plane $X'Y'Z'$ be related to the lines of the plane XYZ, so that the point

$$lX'+mY'+nZ',$$

namely the point of coordinates $(0, 0, 0, l, m, n)$, corresponds to the line $Alx+Bmy+Cnz=0$, $x'=0$, $y'=0$, $z'=0$, for all values of l, m, n. Then the quadric is the locus of the plane joining any point of the plane $X'Y'Z'$ to the corresponding line of the plane XYZ.

For, the equation of the quadric, containing the points X, Y, Z, X', Y', Z', will not contain the squares of the coordinates. As the quadric contains the plane α, or YZX', given by $y'=0$, $z'=0$, $x=0$, the equation will not contain terms in yz, $x'y$, $x'z$; for such terms are not reduced to zero by $y'=z'=x=0$. Similarly, the equation will not contain terms in zx, $y'z$, $y'x$, or terms in xy, $z'x$, $z'y$. Lastly, as the quadric contains the plane δ, or $X'Y'Z'$, given by $x=0$, $y=0$, $z=0$, the equation will not contain terms in $y'z'$, $z'x'$, $x'y'$. The equation is, therefore, of the form

$$Axx'+Byy'+Czz'=0,$$

any quadric of this form, conversely, obviously containing the planes $\alpha, \beta, \gamma, \delta$.

Again, a plane meeting each of $\alpha, \beta, \gamma, \delta$ in a point, is, it is easily seen, a plane given by three points with symbols of the forms $a'X'+bZ-cY$, $b'Y'+cX-aZ$, $c'Z'+aY-bX$. The general point of this plane has coordinates of the forms

$$qc-rb,\ ra-pc,\ pb-qa,\ pa',\ qb',\ rc';$$

and this point lies on the quadric $Axx'+Byy'+Czz'=0$, for all values of p, q, r, if, and only if, $Aa'=Bb'=Cc'$. Thus, a, b, c, d being arbitrary, the quadric contains the ∞^3 planes of which one is defined by three points, in α, β, γ, respectively, with symbols

$$A^{-1}dX'+bZ-cY,\quad B^{-1}dY'+cX-aZ,\quad C^{-1}dZ'+aY-bX.$$

When one of these planes is given, the ratios of A, B, C are thereby determined.

Finally, the general point of the plane joining the point $(0,0,0,l,m,n)$ to the line in $x'=y'=z'=0$ given by $Alx+Bmy+Cnz=0$, has coordinates, for variable values of p, q, r, given by

$$p(0,0,0,l,m,n)+q(-Cn,0,Al,0,0,0)+r(-Bm,Al,0,0,0,0);$$

for instance, $x=-qCn-rBm$, $x'=pl$. Thus

$$Axx'+Byy'+Czz'=0,$$

and this plane lies on the quadric whatever l, m, n may be.

Now consider the four planes $Y'Z'X$, $Z'X'Y$, $X'Y'Z$, XYZ, denoting them, respectively, by α', β', γ', δ'. Each of these meets three of the four planes α, β, γ, δ in a line; for instance, δ' contains the points (β,γ), (γ,α), (α,β). Thus these planes lie on the quadric, being planes of the other system from α, β, γ, δ. With these they may be said to constitute a *double four* of planes of the quadric. When two planes meet in a line, a general prime meets the solid, which is defined by these two planes, in a plane; and the prime meets the planes in two intersecting lines lying in this plane. Thus an arbitrary prime meets the planes, of a double four of planes, in eight lines of this fourfold space, forming a system constituted by four lines and the four transversals of threes of these four; or say, in a *double four* of lines in this prime. Let O be any point of the quadric. The solid $O\alpha$, for example, obtained by joining O to the points of the plane α, meets the quadric in another plane beside α, of the opposite system from this, and meeting this in a line; this other plane passes through O, and the line is the intersection of the plane α with the tangent prime of the quadric at O. By considering the solids $O\alpha$, $O\beta$, $O\gamma$, $O\delta$, we thus have four planes through O, of the opposite system of planes of the quadric from α, β, γ, δ, meeting these, respectively, in lines. These are planes of the point-cone in which the quadric is met by the tangent prime at O, of one system; and they meet any plane of this cone, of the other system, in four lines through O. Similarly, four planes of the quadric, meeting the planes α', β', γ', δ' each in a line, pass through O, and meet any plane of the quadric through O, of the same system of planes as α', β', γ', δ', in four lines. These two pencils of four lines, in two planes, of different systems, through O, are *related* pencils.

For, we have remarked that the tangent prime of the quadric, at O, meets the double four of planes in a double four of lines in this prime. Also, it was proved (p. 144, above) that, if a double four of lines, in a fourfold space, be joined to a point, O, of that space, so taken that the eight joining planes belong to a quadric point-cone of vertex O, then the two sets of four joining planes, regarded as belonging to the two systems of planes of the cone, are related. Thus the result stated is obvious. It is the representative, in the fivefold space, of the fact that, in a threefold space, the planes, joining an arbitrary line to four points, are related to the range, on the line, of the four points in which the line meets the planes containing the threes of the four points (Vol. I, p. 30).

Denote the quadric by Ω; suppose that its points represent the lines of a threefold space. Consider a tetrahedral complex of lines of this threefold space. The lines of this complex will be represented, in the fivefold space, by the points common to the quadric Ω and to another quadric, say Ω'. When the equation connecting the coordinates of a line, in the threefold space, is written

$$ll' + mm' + nn' = 0,$$

point coordinates being relative to the four points by which the tetrahedral complex is defined, then the equation for the lines of the tetrahedral complex is of the form $all' + bmm' + cnn' = 0$. Thus it appears, from comparison of the equations, that a tetrahedral complex is represented by two quadrics, Ω and Ω', which have in common four planes of Ω, of the same system; this is evident, otherwise, from the fact that, in the threefold space, every line through any one of the four fundamental points is a line of the tetrahedral complex. In the fivefold space, the four planes of one system common to Ω and Ω', imply four other common planes, of the other system, making, with the first four, a double four of planes; correspondingly, in the threefold space, every line in one of the four planes which are fundamental for the tetrahedral complex, belongs to this complex.

We can, however, interpret, in the fivefold space, the property of a line of the tetrahedral complex, that the planes joining this line to the four fundamental points form a pencil which is related to a definite range. Taking the two quadrics, Ω, Ω', in fivefold space, with the four common planes α, β, γ, δ, meeting in pairs in six points, consider any other plane of Ω, of the same system as α, β, γ, δ. This plane, say ϵ, meets Ω' in a conic. The four points, say A, B, C, D, in which the planes α, β, γ, δ meet the plane ϵ, lie on this conic. Let O be any point of this conic; it lies on both Ω and Ω'. Planes of Ω can be drawn through O, of the system other than that of α, β, γ, δ, to meet these planes each in a line; these planes meet ϵ in lines. Four lines so obtained are OA, OB, OC, OD; and the pencil formed by these is related to the range, *on the conic*, of the four points A, B, C, D. If another plane of Ω, say ϵ', also of the same system as α, β, γ, δ, be drawn through O, meeting α, β, γ, δ, respectively in A', B', C', D', we obtain another pencil OA', OB', OC', OD', likewise lying in the four planes of Ω drawn through O to meet α, β, γ, δ in lines. As the planes of Ω, of the two systems, drawn from O, are planes of the quadric point-cone in which Ω is met by the tangent prime at O, the two pencils $O(A, \ldots, D)$ and $O(A', \ldots, D')$ are related. Thus on the two conics, on Ω', in which this quadric is met by the two planes, ϵ and ϵ', drawn through O, the two ranges $(A, \ldots, D)$ and $(A', \ldots, D')$, are related. A plane of Ω such as ϵ' might, however, have been drawn equally through any point, other than O, of the conic in the plane ϵ. Thus we have the result: Let two quadrics, Ω and Ω', in fivefold space, have four planes of the same system in common, and any plane of Ω, of this system, be drawn, meeting Ω' in a conic; consider the four points of this conic, lying on the four planes common to Ω and Ω', regarded as belonging to the range of the points of this conic. Let a second plane of Ω of the same system be drawn, and consider the four points of the conic, in which this plane meets Ω', which are similarly obtained from the four common planes. Then, the range of this last four points is related to the range of the first four. Moreover, by what is said above, if we take the other four planes, of the opposite system, common to Ω and Ω', say α', β', γ', δ', and a variable plane of Ω, of this system, meeting Ω' in a conic, then the planes α', β', γ', δ' meet this variable plane in four points, of the range of points of this conic on Ω', likewise related to the range of points A, B, C, D, above.

The singular points and planes of a Quadratic Congruence. Caporali's theorem. It has been remarked that a quadratic congruence, consisting of the lines common to a quadratic complex and a linear complex of lines, in threefold space, is represented in fivefold space by the surface which is the intersection of two quadrics and a fourfold space. Any one of the sixteen lines, known to lie on this surface (p. 166 above), represents a plane pencil of lines of the threefold space of the quadratic congruence. In general, only two lines of the congruence pass through an arbitrary point of this threefold space; but, it appears from the present point of view, there are sixteen points through each of which pass an infinity of lines, forming a flat pencil in a plane through this point. Each of the sixteen lines of the quartic surface, in the fourfold space, meets five others; thus, in the threefold space, each of the singular pencils of lines of the quadratic congruence has a ray in common with five others of these pencils; namely, the singular plane of any one of the pencils contains five other singular points, of which the corresponding planes, passing respectively through these points, all pass through the centre of the first pencil. Thus, there are six singular points in any singular plane, and six singular planes through any singular point. It was seen, moreover (p. 171, above), that the sixteen lines of the quartic surface, in the fourfold space, give rise to twenty double-fours of lines, each consisting of four non-intersecting lines, with four others, transversal to the threes of these. The first four lines of such a double-four represent four pencils of lines, in the threefold space; each pencil has a centre, and lies in a plane through this centre. The second four lines of the double-four give rise, similarly, to four pencils, in the threefold space, of which each has rays through three of the four preceding centres, these rays lying in the respective planes through these centres. The double-four of lines, on the quartic surface of the fourfold space, thus gives rise, in the threefold space, to a pair of mutually inscribed Moebius tetrads (Vol. I, p. 61), the rays through one of the points of one tetrad being in a plane containing three points of the other tetrad. From the singular points and planes of the congruence, corresponding to the double-fours of lines for the quartic surface in fourfold space, twenty such pairs of Moebius tetrads can be formed. No two of these pairs have a common tetrad, as we may see from the notation previously given for the double-fours (p. 171, above).

Let us represent the two quadrics, and the prime, in the fivefold space, which define the quadratic congruence, by $\Omega = 0$, $\Omega' = 0$, $\Pi = 0$, respectively, $\Omega = 0$ being the representative of the lines of the threefold space. We may replace Ω', for the definition of the congruence, by a quadric, Ω'', passing through the common points of Ω' and Π; and this may, in fact, be so chosen that the poles of Π in regard to Ω and Ω'' are the same point. For, let $(x_0, \ldots, w_0)$ be the pole of Π in regard to Ω, let Π_0 be the result of substituting $x_0, \ldots, w_0$ for $x, \ldots, w$, in Π, and let D_0 denote the operator $x_0\partial/\partial x + \ldots + w_0\partial/\partial w$. Then we may take $\Omega'' = \Omega'\Pi_0 - \Pi D_0 \Omega'$. Now consider the common self-polar hexad of Ω and Ω''. The six principal primes of the hexad represent six linear congruences of lines in the threefold space, of which every two are conjugate or apolar; and of these Π is one. Thus the twenty pairs of Moebius tetrads obtained above are clearly seen to be part of the figure previously discussed (p. 133, above), containing eighty tetrads, with sets of four tetrads of which every two are mutually inscribed.

The quartic surface, given by $\Omega = 0$, $\Omega' = 0$, $\Pi = 0$, has also (p. 171, above) ten sets of conics. Thus we may infer the existence, in the threefold space, of ten sets of quadric surfaces, of which the generators are lines of the quadratic congruence. And this may be generalised by considering curves of higher order on the quartic surface.

From the present point of view we may also prove easily that the lines of a quadratic congruence, in threefold space, belong to forty *tetrahedral* complexes (Caporali, *Memorie...Lincei* (3°), II, 1878, p. 756). Let $\Omega = 0$, $\Omega' = 0$ be two quadrics, and $\Pi = 0$ be a prime, in fivefold space; we shew that, in forty ways, a linear function of the six coordinates, say A, can be chosen, so that the quadric $\Omega' + A\Pi = 0$ contains four planes of $\Omega = 0$, of the same system; this is sufficient, after what has been said above. To prove this, take one of the double-fours of lines of the surface $\Omega = 0$, $\Omega' = 0$, $\Pi = 0$; through each line, of one of the fours of this, draw the definite plane of Ω, belonging to the first system of planes of Ω. These four planes meet, in pairs, in six points. Let the coefficients, in the linear function A, be chosen so that the quadric $\Omega'' = 0$, where $\Omega'' = \Omega' + A\Pi$, contains these six points of intersection. Then this quadric, $\Omega'' = 0$, wholly contains these four planes. For, any one of these planes contains, beside the three points of Ω'' in which this plane is met by the other three planes, also the line, belonging to the double-four, through which the plane is drawn; and this line, lying on $\Omega' = 0$, $\Pi = 0$, lies on $\Omega'' = 0$. As the intersection of a plane with a quadric is a conic, proper or degenerate, it follows, if we assume that the three points in which this plane is met by the other three planes do not lie in line, that this plane lies on $\Omega'' = 0$; and the same argument applies to the other three planes. This quadric also contains the planes of Ω of the *second* kind, drawn through the four complementary lines of the double-four. Next, beginning again with the four lines of the double-four through which the four planes of the first system of Ω were drawn, we may draw through these lines the four planes of the second system of Ω. Thence we similarly derive a quadric, $\Omega_2 = 0$, containing four, and, therefore, a double-four of planes of Ω. As there are twenty double-fours of lines on the quadric surface $\Omega = 0$, $\Omega' = 0$, $\Pi = 0$, we can derive, in this way, 2.20, or forty, quadrics, each having common with Ω a double-four of planes. Conversely, let Σ be any quadric containing four planes of the same system of Ω. There are then four planes of the other system of Ω, each meeting three of the first four in lines, which also lie on Σ. Any prime, Π, as has been remarked, meets two planes of Ω which meet in a line, in two lines which intersect. The prime Π thus meets the common double-four of planes of Ω and Σ in a double-four of lines. Thus it appears that the only quadrics containing a double-four of planes of Ω which can be found are those constructed as were Ω'' and Ω_2; so that forty is the total number of such quadrics.

A particular case of the theorem would be that in which the quadratic congruence of lines consists of the lines of a quadratic complex which meet a given line. Then the prime Π touches the quadric Ω. It may be seen that the quartic surface given by Ω, Ω', and Π, is not essentially modified.

Ex. 1. The points of a line, in space of three dimensions, are represented, in the fivefold space, by the planes, of the first kind, of the quadric Ω, which pass through a point of this. These planes meet any fixed plane, of the first kind, of Ω, in points lying on a line; these points form a range related to that of the points of the original line in the threefold space.

Consider, in particular, the range, in the threefold space, of points given by $\eta = \zeta = 0$, $\xi = a\tau$, as a varies. These points are represented by planes through the point (1, 0, 0; 1, 0, 0) of the quadric $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$, in the fivefold space, the planes being given by

$$u = x,\ \ v = y\cos\theta - z\sin\theta,\ \ w = y\sin\theta + z\cos\theta,$$

where $\tan\frac{1}{2}\theta = a$. Putting

$$p = m_3 + n_2,\ \ q' = n_1 - l_3,\ \ r = l_2 + m_1,\ \ k = l_1 - m_2 + n_3 - 1,$$

these planes meet the plane of Ω which is given by

$$u = l_1x + m_1y + n_1z,\ \ v = l_2x + m_2y + n_2z,\ \ w = l_3x + m_3y + n_3z,$$

in the range of points whose coordinates, x, y, z, u, v, w, are the six expressed by

$$a(q', -p, -k, q', p, k) - (r, -k, p, r, -k, p),$$

so that, for example, $x = aq' - r$, $w = ak - p$.

Ex. 2. Given two lines in threefold space, which do not intersect, any point, P, determines another point, P', on the transversal from P to the lines, harmonically conjugate to P with respect to these. Thus any line, not meeting the first two, on which P moves, determines a fourth line as the locus of P'. Shew that the representation of four lines so related, in the fivefold space, is by a harmonic range of four points of a conic, lying on the fundamental quadric Ω.

Ex. 3. Let the lines of a quadratic congruence, in threefold space, be given by $ll' + mm' + nn' = 0$, $all' + bmm' + cnn' = 0$, $Al' + A'l + Bm' + B'm + Cn' + C'n = 0$. Let θ, θ' be the two roots of the quadratic equation

$$(a+\phi)^{-1}AA' + (b+\phi)^{-1}BB' + (c+\phi)^{-1}CC' = 0;$$

put $\alpha = (a+\theta)^{-1} AA'$, $\beta = (b+\theta)^{-1} BB'$, etc., and $\alpha' = (a+\theta')^{-1} AA'$, etc., so that $\alpha+\beta+\gamma=0$, $\alpha'+\beta'+\gamma'=0$. Then, with $\lambda = A'l$, $\lambda' = Al'$, etc., we have

$$\lambda\lambda'/\alpha\alpha' = \mu\mu'/\beta\beta' = \nu\nu'/\gamma\gamma', \quad \lambda+\lambda'+\mu+\mu'+\nu+\nu'=0.$$

Further, with $L = A^{-1}l\,(a+\theta)$, $M = B^{-1}m\,(b+\theta)$, etc., we have

$$L\lambda' + M\mu' + N\nu' = 0, \quad L\lambda'\alpha/\alpha' + M\mu'\beta/\beta' + N\nu'\gamma/\gamma' = 0,$$
$$\alpha L + \lambda' + \beta M + \mu' + \gamma N + \nu' = 0,$$

whereby the lines of a quadratic congruence are reduced to the lines of a tetrahedral complex which meet a line (cf. also, Ex. 10, p. 142, above).

Ex. 4. Find the geometrical transformations in fivefold space corresponding, respectively, to the following transformations in threefold space: (1) Harmonic inversion with a fixed point and a fixed plane; (2) Harmonic inversion with two fixed lines; (3) Polarity in regard to a quadric surface; (4) Polarity in regard to a focal system (or linear complex).

CORRECTIONS TO VOLUME III

p. 9, l. 8. *After* ρ *add* and θ.
p. 22, l. 8. *For* (x, y, z, t *read* (x, y, z, t).
p. 46, l. 1. *For* AXX′ *read* OXX′.
p. 47, l. 30. *For* pole, D, of *read* pole, D′, of.
p. 48, l. 8. *For* DFF′ *read* DFE.
p. 55, l. 7. *For* Ulp *read* Ulp + ...
p. 69, l. 11 f. b. *For* quadrics, the *read* quadrics, four of the. Of either system of generators of either quadric there are four touching the curve.
p. 82, l. 8. *For* point *read* points.
p. 87, l. 14. *For* $a+b+c+\Sigma\lambda_r$ *read* $t^2[a+b+c+\Sigma\lambda_r]$.
p. 94, l. 16. *For* $P=(A+\sigma)/(1+\sigma)$ *read* $P=(A+\sigma)/(1+\sigma)$, etc.
p. 102, l. 4 f. b. *For* 193 *read* 163.
p. 112, l. 24. *For* normals of *read* normals at.
p. 120, l. 9 f. b. *For* point *read* points.
p. 124, l. 3. *For* (0, 1, 0, 0) *read* (1, 0, 0, 0).
p. 147, ll. 12 and 11 f. b. *For* A′ and B′ *read* the quadric.
p. 147, l. 8 f. b. *For* these *read* the quadric.
p. 148, ll. 19, 20. *For* the two points where the chord meets the curve *read* any quadric constructed as above for the seven points.
p. 148, l. 26. *For* point-pair constituted by *read* two points of the quadric on the join of. The matter is correctly stated in Ex. 5, p. 155.
p. 177, l. 25. *For* c_{12} *read* c_{34}.
p. 210, l. 23. *For* ten *read* five.
p. 221, l. 15. *For* Bull. Sc. Acad. *read* Bull. Acad.
p. 224, under Vol. II, *for* p. 96 *read* p. 97; also *add* p. 199, l. 4. *For* $(-\xi, -\eta')$ *read* $(-\xi', -\eta')$.
p. 225, under Cayley and Cremona, *add* 142.
p. 226, under Moebius tetrads, *add* 143.

INDEX

Anchor Ring, or Tore, and Dupin's Cyclide, 193, 199
Angle between two circles, in projection from a quadric, 7; unaltered by inversion, 14
Apolar, or conjugate, linear complexes, 42; apolar relation for point of contact of Hart section, 76
Apollonius's problem for circles, 66
Associated lines, five, in fourfold space, 114; all met by planes, 120; planes meeting these through any point, 122; associated system of five planes, 118

Bauer, on a quartic surface, 20
Bennett, theorem on circles through threes of four points, 17; Hart section, equations for, 86
Bertini, on Veronese's surface, 54
Bitangents of Kummer's surface, from lines of Segre's cubic locus, 156
Bobillier, theorem for pedal circles, 22
Burkhardt, the group of the lines of a cubic surface, 105
Burnside: a figure in fourfold space, 105; the group of the lines of a cubic surface, 105; invariants of a linear group, 210

Caporali, tetrahedral complexes containing a quadratic congruence, 242
Cardioid, 99
Cartesian, 99
Castelnuovo and Kronecker, surface with double infinity of reducible plane sections, 55
Caustic, or focal, surface of a quadratic congruence, 225
Cayley, representation of a line by a point of fivefold space, 40; on Veronese's surface, 54; algebraic solution of Malfatti's problem, 68
Centre of a circle, in projection from a quadric, 7
Chords common to two curves, 51
Circle, obtained by projection of plane section of a quadric, 1; circle and two inverse points changed into same by inversion, 14; circle of similitude, obtained by projection, 15; circles of similitude of the pairs of four circles, 15; circles in a plane, chain of theorems for, 29; circle represented by a line of fourfold space, 39; Hart and Feuerbach circles, Chapter II, p. 65; circles meeting given circles at equal, or at given angles, 67; three circles with three concurrent common tangents, 67
Clifford, a theorem for circles, 1, 31, 64, 105
Complex, tetrahedral, determined by planes in fourfold space, 32; linear, determined by planes in fourfold space, 34; linear, represented in fivefold space, 42; six linear complexes mutually conjugate, 42; lines common to three, or to four linear complexes, 43, 44; linear, point and polar plane represented in fivefold space, 48; linear, six conjugate, constructed from six points, 139; linear, six conjugate, from lines of Segre's cubic locus, 155; linear, six conjugate, poles of a plane lie on a conic, 204; quadratic, represented in fivefold space, 213; quadratic, condition for singular point, 214; quadratic, rationality of, 234; tetrahedral, in fivefold space, 239. See also *Focal System*
Confocal Cyclides, cutting at right angles, 181; equation for, 183
Congruence of lines, its order and class, 49; for chords of a curve, 50; number of lines common to two, 51; quadratic, in tangent solid of Segre's quartic locus, 137; six congruences associated with a Kummer surface, 219, 223; quadratic, caustic or focal system of, 225; six quadratic, equations for common singular points, 226; quadratic, singular points and planes, 240; quadratic, lies in

forty tetrahedral complexes, 242; quadratic, and Moebius tetrads, 241

Conic: conics touching fives of six arbitrary lines, 1; triangularly circumscribed to six conics, 3, 5; in an arbitrary plane, determined by the joins of five points, 11; two conics in fivefold space which are mutually conjugate, 43; conics through a point of Segre's quartic locus, 127; conics lying on quartic surface in fourfold space, 171

Conjugate, or apolar, linear complexes, 42

Cremona, triangularly circumscribed conic and line coordinates, 6

Cubic curve: conic touching seven lines associated with, 4; five arbitrary points on, 11; three tetrads of points on, 28; tetrads of points determined by two points, 29; chords of, represented in fivefold space, 52; in plane, through two points, inverse of quartic with two double points, 97

Cubic surface, with four double points, theorem for, 28; group of lines of, 105; cubic envelope, by projection of Segre's figure, 149; by projection of quartic surface in fourfold space, 173

Curvature, lines of; upon a Cyclide, 186; Dupin's theorem, 188; general theorems for, 189; differential equation for, 190

Curve, in n-fold space, intersection of $(n-1)$ quadrics, 185

Cyclide: arising by projection from fourfold space, 174; enveloped by spheres, 175; as locus of limiting points of fixed sphere and variable tangent plane of a quadric, 176; tangent planes of, at a point of double conic, 177; four focal conics of principal quadric surfaces, 178; angle between enveloping spheres of different systems, 179; is inverted into itself from five centres, 179; four points of double curve where tangent planes coincide, 180; three confocal through given point, 181; equations for confocal cyclides, 183; normal of, by projection from fourfold space, 184; lines of curvature upon, 186; spheres having stationary contact with, 187; Dupin's Cyclide, 190, 193, 199; with cuspidal conic, 192; by inversion of a quadric, 192; with planes of symmetry, 193; general, irrational form of equation, 194; with two double points, inversion of, 194; with two double points, equations for, 195; with four double points, equations for, 196; a particular surface, Kummer's equation for, 197; Dupin's, treated tangentially, 201; Dupin's, points expressed by two parameters, 202; lines of curvature correspond to inflexional curves on Kummer's surface, 231

Darboux: surface containing a doubly infinite system of conics, 54; on plane quartic curves, and Cyclides, 172; generalisation of lines of curvature and of geodesic lines, 190; a generalisation of Dupin's Cyclide, 198; inversion of Dupin's Cyclide, 200; on the existence of space of four dimensions, 200; congruences of lines, 225

Del Pezzo, on Veronese's surface, 54, 55

Dickson, the group of the lines of a cubic surface, 105

Director conics, for plane quartic curve with two double points, are confocals, 91

Double four: of lines on quartic surface in fourfold space, twenty such, 170; of planes, 238

Double-six theorem, proved in fivefold space, 63

Double tangents of Kummer's surface, 225

Dupin: theorem on lines of curvature, 188; Dupin's Cyclide, 190, 193, 199, 201, 202. See *Cyclide*

Elte, on a six-dimensional polytope, 105

Feuerbach's theorem, 65; particular case of Hart's theorem, 78

Figure: of twentyseven points in fourfold space, 104; of fifteen lines and points in fourfold space, 113; arising from four arbitrary planes, 117

Focal conics of a confocal system of quadrics, 92; descriptive form of theorem of intervals, 103
Focal system: point and polar plane represented in fivefold space, 48; six conjugate, scheme for, 133; arising in tangent solid of Segre's quartic locus, 135. See *Complex, linear*
Foci of plane quartic curve with two double points, 92
Focus, and focal line, of a singular plane in fivefold space, 219
Fontené, theorem for pedal circles, 22
Fresnel's wave surface, 217

Gergonne, solution of Apollonius's problem, 66
Göpel, equation of Kummer's surface, 216
Grace: generalisation of a theorem for circles, 8; theorem for pedal circles, 23; theorem for spheres touching planes, 58; Wallace's theorem for space of even dimensions, 59; theorem for six lines with a common transversal, 60; on systems of spheres, 105, 112
Group, linear, in four homogeneous variables, 208

Harmonic inversion, with fundamental point and solid, 117; with two fundamental planes, 118; with fundamental line and plane, 119
Harmonically conjugate quadric, 72
Hart: theorem for circles, 65; proof of Steiner's solution of Malfatti's problem, 68; Hart circle of three circles, orthogonal to their orthogonal circle, 71; general case, 72, 75; apolar relation for point of contact, 76; Hart circle for spherical triangle, 77; Feuerbach circle as particular case, 78; radius of Hart section of a quadric, 81; Hart section of a quadric, sufficient condition is necessary, 82; Hart circle of three circles, defined by angles of intersection, 88
Haskell, generalisation of a theorem for circles, 8
Hudson, on Kummer's quartic surface, 208, 227, 233
Humbert, a curve on the Weddle surface, 234

Inflexional lines of a surface, 186; differential equation for, 190; of Kummer's surface, 222, 230; correspond to lines of curvature on Cyclide, 231
Invariants, representative, for an involution in threefold space, 209
Inversion in a plane, deduced from projection in space, 12; of plane quartic curve with two double points, 96; of Cyclide, 179; of Cyclide with two double points, 194; of Dupin's Cyclide, to quadric cone of revolution, 201. See also *Harmonic inversion*
Involution of sets of sixteen points in threefold space, 208

Jessop, theorem for angles of intersection of tangent circles of a quartic curve, 99
Jordan, the group of the lines of a cubic surface, 105

Klein: representation of a line by a point of fivefold space, 40; Lie's correspondence between sphere and lines, 55; six conjugate linear complexes, 202, 208; inflexional lines of Kummer's surface, 233; rationality of quadratic complex, 235
Kronecker and Castelnuovo, surface with double infinity of reducible plane sections, 55
Kühne, generalisation of Wallace's theorem, 59
Kummer, equation for a particular Cyclide, 197
Kummer's quartic surface: arising by section of Segre's quartic locus with tangent solid, 139; bitangents of, from lines of Segre's cubic locus, 156; from profile of Segre's cubic locus, 156; and Weddle's surface, from four quadrics in threefold space, 160; obtained by transformation from fourfold space, 211; as locus of singular points of a quadratic complex, 212; equation expressed as sum of five squares, 215; range of six points associated with, 215; equation shewing double

points, 216; dual of, 217; represented in fivefold space, 218; six congruences associated with, 219, 223; tangent lines at a point, 221; identical with envelope of singular planes, 222; inflexional lines of, 222, 230; bitangents of, 224; from confocal complexes, 227; related range of points and pencil of planes, 228; intersections with a line of fundamental complex, 229; inflexional curves correspond to lines of curvature on Cyclide, 231

Lachlan, Hart circle defined by angles of intersection, 88
Larmor, A., circles arising from the intersection of three circles, 82
Leathem, theorem for four concyclic points of a sphere, 101
Lexell, theorem for four concyclic points of a sphere, 101
Lie, correspondence of a sphere with two lines, 55; applied to Kummer's surface and Cyclide, 233
Limaçon, 99
Line, in threefold space, represented by a point of fivefold space, 41
Line-cone, quadric, and point-cone, in fourfold space, 121
Linear complex. See *Complex, linear*

Malfatti's problem, 68
Mannheim, inversion of Dupin's Cyclide to Anchor Ring, 200
Mantel, symmetry of five points, for pedal property, 25
Maxwell, Dupin's Cyclide, 193, 198
Meridian surface, Plücker's, 142
Miquel, a theorem for circles, 1; generalisation of, 8
Moebius's figure of two tetrads and Wallace's theorem, 18; Moebius tetrads and quadratic congruence, 241
Morley, theorem for Feuerbach circle, 79

Noether, rationality of quadratic complex, 235
Normal of Cyclide, by projection from fourfold space, 184

Orr, theorem for four concyclic points of a sphere, 101
Orthocentric, Absolute conic for which five points are, 11

Pedal circles orthogonal to a fixed circle, 23; pedal property of circumcircle generalised to rational curve, 26; properties in threefold space, 24, 25, 27
Picard, surface whose plane sections are rational, 55
Plücker's Meridian surface, 142
Point-cone, quadric, and line-cone, in fourfold space, 121
Polar system, by plane section of joins of five points, 10; polar line of a line, in regard to a focal system, or linear complex, 42; polar of a line in regard to a quadric, represented in fivefold space, 49; polar lines, in a focal system, corresponding to the same sphere, 55; polar quadric of point of Segre's cubic locus, 156
Poncelet, on circles touching three circles, 66

Quadratic complex. See *Complex, quadratic*
Quadric: in fourfold space, lines of, through a point, 37; in fivefold space, representing lines, 41; quadric surface, generators of, represented by a conic in fivefold space, 43; in fivefold space, planes lying on, 45; each its own polar plane, 47; quadric surface, number of common points of two curves on, 51; quadrics in n-fold space, through given points, 61; quadric surface, sections of, touching three sections, 65; equations of, 69; condition four sections of quadric surface be touched by four planes, 72; quadric surface, harmonically conjugate, 72; equation of, 73; quadric surface, radius of section of, with respect to a point, 80; four sections and four tangent sections, mutual relations, 86; angles between four sections, 100; quadric surfaces with ring contact, theorem of intervals, 103; quadric, in fourfold space, in regard to which Segre's figure is self-polar, 148, 149; quadric surfaces through five points give rise

to Segre's cubic locus, 152; system derived from four general quadrics, 160; quadric surfaces, ten, associated with six conjugate linear complexes, 206; sets of four with two common generators, 207; quadric surface, generators touching curve of intersection with another quadric surface, 94, 170, 220, 223, 224; quadric in fivefold space, generation of, 237

Quartic curve, in a plane, with two double points, 89; as envelope of circles, 90; foci of, 92; number of tangents from a double point, 93; relation between distances to three concyclic foci, 94; inversion into itself, or another such curve, 96; with four foci in a line, 98; angles of two tangent circles with variable tangent circle, 99; general form of theorem for angles between tangent circles, 102

Quartic curve, in fourfold space, 120, 147

Quartic surface, with five double points, from quartic surface with fifteen double points, 160

Quartic surface, in fourfold space: intersection of two quadrics, 161; algebraic treatment, 162; sixteen lines of, 164; representation on a plane, 165; descriptive theory of lines on, 167, 169; self-polar pentad for, 168; double-fours of lines on, 170; notation for lines, 170; conics upon, five pairs through each point of surface, 171; other equations for lines, 185; lines of, by intersection with a quartic locus, 189

Rao, co-residuation on quartic surface in fourfold space, 172

Related planes, three, variable plane containing corresponding points, 145

Reye, triply infinite system of quadric surfaces, 160

Richmond, pedal property for general rational curve, 25; figure arising from six points in fourfold space, 116

Roberts, generalisation of a theorem for circles, 8

Rosenhain, equation of Kummer's surface, 216

Salmon, radius of Hart circle on a sphere, 80

Schoute, on a six-dimensional polytope, 105

Segre, tetrahedral and linear complex, from planes in fourfold space, 36; representation of conics in fivefold space; on Veronese's surface, 54; loci of third order in fourfold space, 116; theorem of five planes meeting six lines, 143; theorem for a Cyclide, 180; general investigation of Cyclides, 172; degenerate forms, 190

Segre's figure, of fifteen points and lines in fourfold space, 113, 203; by transformation from fivefold space, 208; number notation for lines and solids, 114; singular solids, 115, 124, 129; incidences proved algebraically, 115; six systems of planes, two of any system through a point, 123; locus for which two planes coincide, 123; planes found algebraically, 124; planes of a system meeting an arbitrary line, 136; reciprocating quadric for, 148

Segre's cubic locus in fourfold space, regarded as a dual, 151; obtained from quadric surfaces through five points, 152; theorem for lines upon, 155; from fivefold space, 160

Segre's quartic locus in fourfold space, irrational equation of, 125; is rational, 126; six conics through a point met by lines in related ranges, 127; tangent solid, 128; contains a symmetrical figure of sixteen points and planes, 130, 134; thirty-two related ranges in, 131; intersection with tangent solid, 138; particular form of equation, 142; tangential equation, 148; and Weddle surface, 159; representing an involution of threefold space, 210

Severi, surface with no chords through arbitrary point, 55

Similitude, circle of, obtained by projection, 15; centres of similitude for two circles, 66

Sphere, determined by section of quadric in fourfold space, 36; centre of, by projection, 39; represented by a point of fourfold space, 39; angle of intersection of two spheres, 39; touching four given spheres, or planes, 57, 58; enveloping a Cyclide, 175
Steiner: a theorem for two triads of points, 21; Steiner's quartic surface, 54; by projection from fourfold space, or from fivefold space, 191; circles cutting three given circles at given angles, 67; solution of Malfatti's problem, 68
Stéphanos, figure of fifteen circles, in anticipation of Segre, 116, 203
Study, mutual relations of four circles and four tangent circles, 86

Taylor's theorem, for poles of a line in regard to six conics, 5
Tetrads, Moebius, mutually inscribed, theorem for four, 141; fifteen associated with six conjugate linear complexes, 206; property of tetrad in regard to a line, considered in fivefold space, 239
Tetrahedral complex, determined from planes in fourfold space, 32; in fivefold space, 239
Tetrahedroid, or Wave surface, 217
Tore, or Anchor Ring, and Dupin's Cyclide, 193, 199
Transversals of two lines, and points of a quadric surface, 31
Triangularly circumscribed conic, 3, 5

Vaidyanathaswamy, property of a cubic curve, 147
Veronese's surface in fivefold space, conics thereon, 52, 54; projecting to Steiner's quartic surface, 191; theorem of reciprocity of a general figure, 149

Wakeford, on the theorem of a double-six of lines, 6; six conics triangularly circumscribed to another, 32; theorem for six lines with a common transversal, 60
Wallace's theorem, and Moebius's figure of two tetrads, 18; additions to, 21; generalised, 10, 58
Wave surface, or tetrahedroid, 217
Weber's theorem, construction of six conjugate linear complexes from six points, 139
Weddle's surface, origin of, 156; equation of, 158; from Segre's quartic locus, 159; and Kummer's surface, from four quadrics in threefold space, 160; correspondence of conjugate directions, 233
White, equation of a conic triangularly circumscribed to six conics, 5; proof of Clifford's chain of theorems for circles, 64

CAMBRIDGE: PRINTED BY W. LEWIS AT THE UNIVERSITY PRESS

PRINCIPLES OF GEOMETRY

LONDON
Cambridge University Press
FETTER LANE

NEW YORK · TORONTO
BOMBAY · CALCUTTA · MADRAS
Macmillan

TOKYO
Maruzen Company Ltd

All rights reserved

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., LL.D., F.R.S.,
LOWNDEAN PROFESSOR, AND FELLOW OF ST JOHN'S COLLEGE,
IN THE UNIVERSITY

VOLUME V

ANALYTICAL PRINCIPLES OF THE THEORY OF CURVES

CAMBRIDGE
AT THE UNIVERSITY PRESS
1933

PRINTED IN GREAT BRITAIN

PREFACE

THE present volume is an account of the analytic principles of the theory of curves, of the rational functions belonging thereto and of the integrals of these functions, with a brief account of the methods, by loops and by Riemann surfaces, for dealing with the periods of these integrals. But the theory of correspondence, and some necessary references to involutions in a plane, find themselves in the succeeding volume, which is mainly devoted to the theory of surfaces.

It is perhaps desirable to explain the origin of these volumes. In the last fifty years a remarkable advance has been made in the theory of surfaces, and of algebraic loci in general; the English reader may find a description of the nature of this in a Presidential Address to the London Mathematical Society given in November 1912 (*Proceedings*, Vol. XII). But attempts, since the War, to expound these new results have continually shewn the necessity for a precise appreciation of the ideas out of which this advance has developed; in mathematics it is not sufficient to know the enunciation of a result; it is necessary to understand the proof. These two volumes have grown up in the attempt to meet this need. The further need of a volume explaining the applications of topological theory, especially to the periods of the integrals belonging to the higher loci, may, I hope, appeal to another. The volumes are necessarily very incomplete in their inclusion of detail, as the specialist in any branch will easily find; their object is to lay the foundations for a more detailed study.

The pursuit of the analytical principles has a fascination in itself; but since, for reasons of space, these volumes are so largely devoted to this, I may be allowed to add another remark. The study of the fundamental notions of geometry is not itself geometry; this is more an Art than a Science, and requires the constant play of an agile imagination, and a delight in exploring the relations of geometrical figures; only so do the exact ideas find their value. As when, upon a landscape of rugged hill and ruffled water, there breaks the morning sun, scattering the clouds, and anon bathing the whole in a glory of contrasting colour. If these volumes should help to increase the number of those to whom the comparison does not seem an exaggeration, they will have been worth the making.

To the University Press very special acknowledgments are due, for the care, and speed, with which the volumes have been printed.

H. F. B.

29 *August*, 1933.

TABLE OF CONTENTS

CHAPTER I. INTRODUCTORY ACCOUNT OF RATIONAL AND ELLIPTIC CURVES

PAGES

Introductory remarks . . . . 1—2

Linear series on a line . . . . 2—4

Rational curves. Number of double points. Transformation to standard form . . . . 4—10

Greatest possible number of double points of a plane curve 10—11

Elliptic curves. Transformation to standard form. Application of elliptic functions . . . . 11—17

Examples of elliptic curves. Coresiduation. Salmon's identity. Halphen's pencil . . . . 18—23

CHAPTER II. THE ELIMINATION OF THE MULTIPLE POINTS OF A PLANE CURVE

Linear series on a curve. Neutral sets; simple series; series contained in a given one . . . . 24—27

Theory of reduction by contained systems, to non-singular curves . . . . 27—31

Examples of transformation of curves . . . . . 31—34

CHAPTER III. THE BRANCHES OF AN ALGEBRAIC CURVE; THE ORDER OF A RATIONAL FUNCTION; ABEL'S THEOREM

Introductory explanation; meaning of a place; the local parameter . . . . 35—36

Weierstrass's preparatory theorem . . . . . . 36—39

The parametric expression of a branch of a curve . . 39—43

Supplementary remarks, and example . . . . . 43—46

General theorem for infinities of a rational function . . 46—49

Order of a rational function on the curve, and examples . 49—51

Abel's theorem for a sum of algebraic integrals . . . 51—55

Examples of Abel's theorem . . . . 55—57

Note as to definition of a rational function on the curve . 57—58

CHAPTER IV. THE GENUS OF A CURVE. FUNDAMENTALS OF THE THEORY OF LINEAR SERIES

PAGES
Main objects of the chapter. Definition of adjoint curves . 59—63
Expression of rational function by adjoint polynomials . 63—64
Expression of most general everywhere finite integral . . 64—65
Equivalent or coresidual sets of points on the curve . . 65
Complete linear series of sets of points on the curve; examples 65—67
Theorem of coresiduation 67
Sum and difference of complete linear series . . . 67—68
Riemann-Roch formula for linear series, and for rational functions . . . . 68—78
Applications of the Riemann-Roch formula. Clifford's theorem. The canonical curve. Remarks . . . 78—83
A formula for the genus in terms of the branch points. The Jacobian series . . . . 83—86
The existence of a rational function with assigned poles. Weierstrass's gap theorem . . . . . . 86—90
The theory of special sets; an extension of Clifford's theorem for a general curve . . . . 90—94
The number of moduli for a general curve . . . . 94—95
The number of conditions which can be satisfied by a curve 95—96
The number of quadrics containing the canonical curve; the spaces containing the sets of a special series . . 96
Examples of existing special series . . . . . . 97—98
The points common to two plane curves. The Cayley-Bacharach theorem . . . . 99—105
Related theorem as to intersections of plane curves . . 105—107
Canonical forms for the equation of a manifold, or several manifolds . . . . 107—110
Expression of a plane curve by a symmetrical determinant. Rational double planes 110

CHAPTER V. THE PERIODS OF ALGEBRAIC INTEGRALS. LOOPS IN A PLANE. RIEMANN SURFACES

Meaning and number of periods . . . . 111—113
The method of loops in a plane . . . . 113—121
Riemann surfaces . . . . 121—124
Topology of a general Riemann surface. Connectivity . . 124—132
Curve systems on a surface. The Euler-Poincaré invariant . 132—133
Examples of the connectivity, and of curve systems . . 133—135

CHAPTER VI. THE VARIOUS KINDS OF ALGEBRAIC INTEGRALS. RELATIONS AMONG PERIODS

PAGES
The elementary integrals. The expression of any integral thereby . . . . 136—140
The simplified periods on a dissected Riemann surface . 140—144
Interchange of argument and parameter. Converse of Abel's theorem . . . . 144—145
Indication of some general functional theorems . . . 145—146

CHAPTER VII. THE MODULAR EXPRESSION OF RATIONAL FUNCTIONS AND INTEGRALS

Modular expression of a rational function . . . . 147—149
Fundamental integral functions . . . . 149—151
Important properties of the fundamental integral functions 151—155
A significant matrix identity . . . . 155—157
The functions reciprocal to the fundamental integral functions . . . . 157—160
General modular form of the everywhere finite integrals . 160—161
Use of the reciprocal functions for algebraic integrals in general. A particular combination . . . . . 162—165
The structure of a certain fundamental rational function . 165—169
Application to the general rational function. The Riemann-Roch theorem . . . . 169—171
Explicit form of relation for interchange of argument and parameter . . . . 171—174
Weierstrass's prime function for algebraic functions. Converse of Abel's theorem . . . . 174—177
Some particular applications. Return to theory of special series . . . . 177—181

CHAPTER VIII. ENUMERATIVE PROPERTIES OF CURVES

General formulae connecting characteristics of a curve in ordinary space . . . . 182—192
Interpretation of the formulae as illustrating general principles . . . . 192—197
Examples. Formulae for a curve in higher space. Weierstrassian points . . . . 197—201
Curves which are the complete intersection of two surfaces . 201—208
Curves which are the partial intersection of two surfaces . 208—209

PAGES

Examples. Curves not determinable by three surfaces. The simpler cases of existing curves. Curves on a quadric surface. Curves on a cubic surface . . . . . 209—214

Examples. Composite curve intersection of two surfaces. Curves on a ruled surface . . . . 215—217

Linear series upon a curve in space . . . . . . 217—218

Noether's general theorem 218

Preliminary theorem for surface through intersection of two surfaces 219

Lemma in regard to curves in a plane . . . . . . 219—223

Proof of preliminary theorem . . . . 223—224

Proof of Noether's general theorem . . . . . . 224—226

Another proof of the determination of the canonical series on a curve in space . . . . 226—230

Examples for cases of composite intersections . . . 230—231

Connexion with theory of algebraic functions and integrals . 231—233

The greatest genus possible for a curve of given order, in higher space . . . . 234—237

Examples. The form of the everywhere finite integrals in general cases. Lower limit for the order of a manifold of given dimension . . . . 237—241

INDEX . . . . 243—247

CHAPTER I

INTRODUCTORY ACCOUNT OF RATIONAL AND ELLIPTIC CURVES

THE present volumes V, VI are an introduction to the more important of the algebraical and functional relations which are necessary for a clear and precise understanding of the principles of algebraic geometry. These relations are as the bones of the structure, to be clothed finally with a body of purely geometrical doctrine.

For the expression of these relations we make free use of coordinates, of which the justification has been examined in Vols. I and II. With their use we can define an algebraic construct (curve, surface, manifold, etc.) as the aggregate of points whose coordinates satisfy a set of algebraic equations, taken with points (limiting points, and other) which it may be necessary conventionally to add thereto. And, it is to be understood that all coordinates, and parameters, that enter in the equations employed, are capable of complex values; in particular, an aggregate will be said to be of dimension, or freedom, r, or simply to be ∞^r, when it depends on the values of r parameters not restricted to real values. In the elementary geometry two figures, or two manifolds, are regarded as essentially identical when they are projectively related to one another, that is (as we have seen) transformable into one another by equations which are linear in the (homogeneous) coordinates; in general algebraic geometry, two manifolds are regarded as essentially identical when the coordinates of the points of either are expressible as rational algebraic functions of the coordinates of the points of the other, whether linear functions or not.

The simplest algebraic construct from this point of view is then a line, which may be regarded as the locus of a point identified by one parameter fixing the position of the point upon the line. But the points of a conic, of which one point is known, are equally expressible in terms of a parameter, by taking the intersection of the conic with a variable line drawn through the known point. Or again, a plane cubic curve with a double point of known coordinates, is likewise the locus of a point whose coordinates are rational in one parameter, the intersection of the curve with a variable line through the double point. More generally, there is an unlimited family of curves with the property that the coordinates of any point of the curve are expressible as rational functions of a parameter; and this parameter can be chosen so that, conversely, it is a rational function of the coordinates of the point of the curve with which it is associated.

It will be seen, moreover, though this is a subsidiary property at the present stage, that, in the rational functions of the parameter which express the coordinates of a point of the curve, the coefficients which enter may be taken to be rational in the coefficients of the equations by which the curve is given and in the coordinates of one point of the curve; this is illustrated by the case of a conic. Regarding the variable parameter as representing a point of a line, such a curve as we have spoken of is therefore in (1, 1) birational correspondence with the line. Such a curve is called a *rational* curve. In the present chapter we give some fundamental properties of such curves, partly to illustrate general ideas, and partly because these properties should be known. And we treat also, for the same reasons, of curves which, in order of formal difficulty, next follow, those called *Elliptic Curves*.

Linear series on a line. One fundamental notion, which can be stated for a line, and has application not only to rational curves but to algebraic curves in general, is that of a *series of sets of points*, and, in particular, of a *linear* series. If a general point of a line be given by a parameter θ, a *set* of n points will be given by an equation $\theta^n+a_1\theta^{n-1}+\ldots+a_n=0$, with given coefficients $a_1, \ldots, a_n$. These coefficients in turn may depend on other parameters $\xi, \eta, \ldots$, so that when these vary the coefficients vary, and consequently the original set of n points of the line also varies, and gives rise to a *series* of sets, of each n points. The coefficients $a_1, \ldots, a_n$ may be *rational* functions of $\xi, \eta, \ldots$, all of these being capable of independent variation; then we have a *rational* series of sets of n points on the line. As a particular case of this, the coefficients $a_1, \ldots, a_n$ may be linear (fractional) functions of $\xi, \eta, \ldots$, all with the same denominator, of the forms $a_i=u_i/u$, where $u=p\xi+q\eta+\ldots$, $u_i=p_i\xi+q_i\eta+\ldots$, $(i=1, \ldots, n)$, where $p, q, \ldots, p_i, q_i, \ldots$ are constants. Then the sets on the line are given by $u\theta^n+u_1\theta^{n-1}+\ldots+u_n=0$, and hence by $\xi U+\eta V+\ldots=0$, where $U=p\theta^n+p_1\theta^{n-1}+\ldots$, $V=q\theta^n+q_1\theta^{n-1}+\ldots$; these $U, V, \ldots$ are then definite polynomials of order n in θ. Such a series is called a *linear* series of sets of points on the line. But in regard to the polynomials $U, V, \ldots$ two facts must be clear. It may happen that they all vanish for a certain number, say k, of definite values of θ; then, as $\xi, \eta, \ldots$ vary, the sets of the series consist of $n-k$ variable points, and of k points common to all the sets, these being fixed. Further, it may happen that $U, V, \ldots$ are not linearly independent, but connected by one or more linear equations, with constant coefficients, satisfied identically for all values of θ; and this will always happen if the number, say $r+1$, of the polynomials $U, V, \ldots$ is greater than $n+1$. When the $r+1$ polynomials $U, V, \ldots$ are linearly independent, we can obviously determine a set of the linear series, of which r points have arbitrarily assigned positions, the other points of the set being thereby determined; in this case we speak of the series as being of *freedom* (or dimension) r, or as being ∞^r. And we may speak of $n-k$ as the *grade* of the series, this being the number of points which vary when $\xi, \eta, \ldots$ vary. It is obviously necessary that the freedom be not greater than the grade. In particular cases it is convenient to consider linear series of which all the sets have a certain number of fixed common points, and it may be convenient to modify the definition of grade accordingly.

A series of sets of points on the line which is not linear, nor rational, may be *algebraical* in a more general sense. This will be so if the coefficients $a_1, \ldots, a_n$, in the equation which determines the points of a set, be algebraic functions, of one or more parameters $\xi, \eta, \ldots$, of such character as not to be capable of being expressed as rational functions of other parameters. The simplest case of this is when they are algebraic functions of one parameter, ξ. This means, in the first instance, that each coefficient, a_i, satisfies an (irreducible) algebraic equation with coefficients which are rational in ξ; in this case, however, it can be shewn that a single algebraic function of ξ can be chosen, say σ, such that all the coefficients $a_1, \ldots, a_n$ are expressible rationally in terms of ξ and σ, say $a_i = \psi_i(\xi, \sigma)$; this can be done so that the aggregate of values of all these coefficients, each determined by its own equation in terms of ξ, is obtained by taking $a_i = \psi_i(\xi, \sigma)$, and allowing ξ, σ to take all possible simultaneous values consistent with the algebraic equation by which σ is determined from ξ. We may thus *define* an algebraic series of sets of points, upon the line, as that given by an equation $\theta^n + \Sigma\psi_i(\xi, \sigma)\theta^{n-i} = 0$, wherein σ, ξ have all values which satisfy a definite (irreducible) polynomial equation $f(\sigma, \xi) = 0$. And we may similarly have an algebraic series of sets depending on r parameters $\xi_1, \ldots, \xi_r$, wherein each coefficient a_i, of the determining equation, is rationally expressible in terms of $\xi_1, \ldots, \xi_r$ and a further variable σ, satisfying a rational polynomial equation, say $f(\sigma, \xi_1, \ldots, \xi_r) = 0$, all values of σ and $\xi_1, \ldots, \xi_r$ which satisfy this equation being taken; it can be shewn that this covers all cases of an algebraic series.

A general linear series of sets of n points, in which the freedom is also n, is that expressed by the original equation $\theta^n + \Sigma a_i \theta^{n-i} = 0$, in which all the coefficients vary independently of one another. Every algebraic series of sets of n points evidently consists of sets selected from this general series; it may belong to a linear series, of freedom less than n, arising by imposing linear restriction of the values allowed to $a_1, \ldots, a_n$ in the general linear series spoken of.

In an algebraic series of freedom 1, there will generally be more than one set which contains a particular point of the line; for an equation $\theta_0^n + \Sigma\psi_i(\xi, \sigma)\theta_0^{n-i} = 0$ will generally be satisfied, with the same θ_0, by several pairs (ξ_1, σ_1), (ξ_2, σ_2), ... satisfying the fundamental equation $f(\sigma, \xi)=0$. The number of such sets is called the *index* of the series. This is evidently equal to the number of zeros of the rational function, for the curve $f(\sigma, \xi)=0$, which is expressed by $\theta_0{}^n+\Sigma\psi_i(\xi, \sigma)\theta_0{}^{n-i}$. From this it can be proved that the index cannot be 1 unless the curve $f(\sigma, \xi)=0$ is rational. And then, for the index to be 1, each of the rational functions $\psi_i(\xi, \sigma)$ must be expressible as a fractional linear function of the parameter by which the curve $f(\sigma, \xi)=0$ is expressed, with a denominator the same for all. In other words, an algebraic series on the line, of freedom 1 and of index 1, must be given by an equation of the form $\lambda\phi+\mu\psi=0$, where ϕ, ψ are definite polynomials in the parameter θ of the line, and λ, μ are independently variable. More generally, in an algebraic series of freedom r, on a line, there is generally more than one set of the series of which r points are assigned; when the series is linear there is only one such set.

A familiar application of these ideas arises in the definition of a rational curve. Suppose that the coordinates of any point of a given curve are expressible rationally in terms of a parameter, θ, whose variation gives all the points of the curve. For example, for the curve whose equation is $x^2+y^2=z^2$, we may take $x=1-\theta^4$, $y=2\theta^2$, $z=1+\theta^4$; then $\theta^2=y/(x+z)$, and to each point of the curve correspond two points of the line on which θ is represented. We thus have, on the line, an algebraic series of sets of two points, of index 1; but these sets belong to a linear series, expressed by $\lambda+\mu\theta^2=0$, where λ, μ are variable. The coordinates of a point of the curve are then expressible rationally in terms of the single parameter λ/μ; and this is a *representative* parameter, as having only one value for a point of the curve, $-y/(x+z)$. Such a parameter can always be chosen to express a rational curve, as we have indicated. (Cf. Vol. II, p. 136.)

Rational curves. Consider now a plane curve, of which the ratios of the coordinates x, y, z of a point, are expressible rationally in terms of a parameter, θ, so chosen that only one value of this belongs to any general point of the curve. Thus, if ρ denote a factor of proportionality, we may say that each of ρx, ρy, ρz is equal to a polynomial in θ; let one at least of these polynomials be of order n (in general, all of this order); then n is the number of values of θ for which an arbitrary linear form $ax+by+cz$ vanishes, namely the number of points of the curve on an arbitrary line, or the *order* of the curve. For illustration, suppose $n=3$; and imagine the plane (x, y, z) to be in space, wherein the coordinates are X, Y, Z, T. To compare now with the equations $\rho x=f_1$, $\rho y=f_2$, $\rho z=f_3$ for the plane cubic curve, take in the space (X, Y, Z, T) the curve expressed by $\sigma X=f_1$, $\sigma Y=f_2$, $\sigma Z=f_3$, $\sigma T=f_4$, where f_4 is an arbitrary cubic polynomial, such that no identity $Af_1+Bf_2+Cf_3+Df_4=0$ holds, for values of A, B, C, D independent of θ; this is possible since evidently no identity $af_1+bf_2+cf_3=0$ holds, or the points of the plane cubic would be in a line. The curve in space is likewise a cubic curve, having three points in a plane. Since now any point θ, or (x, y, z), of the plane cubic curve is associated with the point θ, or (X, Y, Z, T), of the cubic curve in space by the equations $X/x=Y/y=Z/z$, we have the conclusion that any rational cubic curve in a plane may be regarded as the projection, from a point, of a rational cubic curve in space. Similarly, suppose we have a rational quartic curve in a plane, given by equations $\rho x=\phi_1$, $\rho y=\phi_2$, $\rho z=\phi_3$, where ϕ_1, ϕ_2, ϕ_3 are linearly independent quartic polynomials in a parameter θ; and suppose the plane to lie in a fourfold space of (homogeneous) coordinates X, Y, Z, T, U. Take two other, arbitrary, quartic polynomials in θ, namely ϕ_4, ϕ_5, and consider the rational curve, in this fourfold space, which is given by $\sigma X=\phi_1$, $\sigma Y=\phi_2$, $\sigma Z=\phi_3$, $\sigma T=\phi_4$, $\sigma U=\phi_5$, it being understood that no identity

$$A\phi_1+B\phi_2+C\phi_3+D\phi_4+E\phi_5=0$$

holds, for constant values of A, B, C, D, E. This new curve is related to the given quartic by the equations $X/x=Y/y=Z/z$; and, in space of four dimensions, the three equations $X=0$, $Y=0$, $Z=0$ represent a line. Hence we say, a rational plane quartic curve may be looked upon as arising by projection *from a line*, from a rational curve in space of four dimensions; this curve is also of order 4, since, with a, b, c, d, e arbitrary, the equation

$$aX+bY+cZ+dT+eU=0$$

is satisfied by 4 values of θ. A rational quartic curve in space of three dimensions is similarly derivable from a rational quartic curve in four dimensions, by taking only one additional coordinate, the derivation in this case being by projection from a point.

In general, a rational curve in space of r dimensions, where the homogeneous coordinates are $x_0, x_1, \ldots, x_r$, is given by equations $\rho x_i=f_i$, in which the functions f_i are polynomials in a parameter, θ. One at least of these polynomials must be of order as great as r, since otherwise constants $a_0, \ldots, a_r$ can be found to render the equation $\Sigma a_i f_i=0$ identically true; in which case there exist one or more linear equations $\Sigma a_i x_i=0$ connecting the coordinates of a point of the curve, which then lies in space of less than r dimensions. On the other hand, if one (or all) of the polynomials f_i is of order greater than r, the curve may be regarded as derived by projection from a rational curve lying in space of more than r dimensions, in the manner we have illustrated. We may then suppose that these polynomials are of order r, and are linearly independent. The order of the curve represented by the equations, being, by definition, the number of points lying on a general prime of the space, whose equation is of the form $\Sigma c_i x_i = 0$, is then r. Conversely, any algebraic curve of order r, lying in space of r dimensions, is rational; for a variable prime, drawn through $(r-1)$ fixed points of the curve, meets the curve in one further point; and this point is identified by the single parameter which fixes the particular prime; moreover, by supposing the $(r-1)$ points to coincide at one point of the curve, we see that the only irrationality entering into the rational expression of the points of the curve, beyond those which determine the curve, is that of the coordinates of some particular point of the curve.

We see then that all rational curves are reducible to curves of order r, in space of r dimensions. Such a curve is called a rational normal curve; it is given, so far, by equations of the form $\rho x_i = f_i$, where the $(r+1)$ polynomials f_i are linearly independent; by taking suitable linear functions of $x_0, \ldots, x_r$, say $\xi_0, \ldots, \xi_r$, it is thus capable of being expressed by $\xi_0/\theta^r = \xi_1/\theta^{r-1} = \ldots = \xi_r/1$. Moreover, we see that we can pass from any rational curve of order r, in space of r dimensions, to any other such curve, by linear transformation of the coordinates. Upon such a curve there can be no point such that the value of the parameter θ appropriate to this point occurs as a multiple root in the equation, of order r, which gives the intersections of the curve with a general prime of the space; for such a prime can be put through $r-1$ arbitrary other points of the curve, beside this one. In other words, the curve has no multiple point.

But a rational plane curve, of order greater than 2, must needs have multiple points. We prove in fact that a rational plane curve of order n has $\frac{1}{2}(n-1)(n-2)$ double points, or multiple points equivalent to as many double points; and, conversely, that an algebraic plane curve of order n with $\frac{1}{2}(n-1)(n-2)$ double points, is necessarily rational. To prove the former, assume the curve given by an equation $f(x, y, z) = 0$, and to have δ nodes, and κ cusps. We prove $\delta + \kappa = \frac{1}{2}(n-1)(n-2)$. The equation of the tangent of the curve, at an ordinary point (x, y, z), being known, the t tangent lines of the curve, from an arbitrary point (ξ, η, ζ), touch the curve at the ordinary intersections of the curve with the curve of order $(n-1)$ given by $\xi \partial f/\partial x + \eta \partial f/\partial y + \zeta \partial f/\partial z = 0$; but it is easy to prove that this curve has 2 intersections with the original curve at a node, and has 3 intersections at a cusp. Thus we have $t + 2\delta + 3\kappa = n(n-1)$. On the other hand, assuming the expression of the coordinates of a point of the curve in terms of a parameter, θ, if we join (ξ, η, ζ) to an arbitrary point (x_1, y_1, z_1) of the curve, for which the value of the parameter is θ_1, the parameters of the remaining $(n-1)$ intersections of the joining line, with the curve, are given by the equation $\xi(yz_1 - y_1 z) + \eta(zx_1 - z_1 x) + \zeta(xy_1 - x_1 y) = 0$; if herein, x, y, z and x_1, y_1, z_1 are replaced by their values in terms of θ and θ_1, respectively, and the result divided by $\theta-\theta_1$, there remains an equation $(\theta, \theta_1)=0$, of order $n-1$ in regard both to θ and θ_1, and symmetrical in regard to these. There are therefore $2(n-1)$ values of θ_1 for which the points (θ), (θ_1) coincide, given by the equation $(\theta_1, \theta_1)=0$, which, for a general position of (ξ, η, ζ), will be of aggregate order $2(n-1)$ in θ_1. Such coincidences arise, however, only in two ways, if we assume that the curve has no multiple points beside nodes and cusps: (i) when (θ_1) is a point of the curve at which the tangent line passes through (ξ, η, ζ); (ii) when (θ_1) is a cusp. Thus we infer that $t+\kappa=2(n-1)$. From the former equation obtained we therefore have $\delta+\kappa=\frac{1}{2}(n-1)(n-2)$. The corresponding equation when the curve has higher singularities requires an appropriate definition of δ and κ, into which we do not enter now.

To prove the converse result, that a plane curve of order n, whose only multiple points are nodes and cusps, whose aggregate number is $\frac{1}{2}(n-1)(n-2)$, can have the coordinates of its points expressed rationally by a reversible parameter, we shew that such a curve can be changed, by transformations which are rational in the coordinates, and also rational in the coefficients in the given equation of the curve, either to a straight line, or to a conic. For this, consider, in conjunction with the given curve f, of order n, the most general plane curve ψ, of order $n-2$, which passes through each of the $\frac{1}{2}(n-1)(n-2)$ double points of f. The number of terms in the equation of the general plane curve of order $n-2$ is $\frac{1}{2}n(n-1)$; for this to contain a double point of f one linear condition must be imposed upon the coefficients in ψ. The form of ψ under consideration will thus contain $\frac{1}{2}n(n-1)-\frac{1}{2}(n-1)(n-2)$, or $n-1$, homogeneously entering arbitrary coefficients, or *more* if the $\frac{1}{2}(n-1)(n-2)$ conditions for the double points are not independent; thus the curve ψ will have an equation of the form

$$\lambda_0\psi_0+\ldots+\lambda_{n-2}\psi_{n-2}+\mu_1V_1+\mu_2V_2+\ldots=0,$$

where $\psi_0, \ldots, \psi_{n-2}, V_1, V_2, \ldots$ are definite polynomials of order $n-2$ in the coordinates, linearly independent of one another upon the curve f, and $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}, \mu_1, \mu_2, \ldots$ are arbitrary. As the double points of f are the common solutions of three rational equations $\partial f/\partial x=0$, $\partial f/\partial y=0$, $\partial f/\partial z=0$, symmetrical functions of the coordinates of all these points are expressible rationally by the coefficients in the equation of f, and therefore the coefficients in the polynomials $\psi_0, \ldots, \psi_{n-2}, V_1, \ldots$ are rational in the coefficients in f. The curve ψ will have intersections with f not at the double points, of number $n(n-2)-(n-1)(n-2)$, or $n-2$; the number of coefficients left arbitrary in ψ cannot therefore be enough to enable us to prescribe a particular ψ having, other than at the double points, more than $n-2$ intersections with the given curve f; thus the terms

$\mu_1V_1+\mu_2V_2+\ldots$ are unnecessary; and the double points of f do furnish independent conditions for ψ. Moreover, the curves $\psi_0=0, \ldots, \psi_{n-2}=0$ cannot have common zeros on the curve f, other than at the double points of f, because the number of intersections remaining would then be less than the number which can be prescribed arbitrarily by proper choice of $\lambda_0, \ldots, \lambda_{n-2}$. In particular $\psi_0, \ldots, \psi_{n-2}$ have no common factor.

We remark in passing that we can now at once see that the coordinates of a point of the curve f are expressible rationally by a parameter, if the coordinates of some arbitrarily taken point of this curve be assumed known. For, if, first, $n-3$ arbitrary points be taken on f, the curves ψ through these points and through the double points will, by what we have seen, have an equation of the form $\theta u-v=0$, where u, v are definite polynomials in x, y, z, involving the coordinates of the $n-3$ points taken, and θ is variable. The combination of this equation with the equation of f will lead to the coordinates of the only remaining intersection of this curve with f, expressed rationally in terms of θ. If we now suppose the $n-3$ arbitrarily taken fixed points of f to be made to coincide at one point of f, we thus have an expression rational in θ, and in the coordinates of one point of f; and θ is conversely rational in the coordinates of the point which it represents, being equal to v/u.

But we may proceed by a succession of steps, from the general equation of ψ involving $n-1$ homogeneous parameters $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}$. Take $(n-4)$ arbitrary points of the plane, whose coordinates may then be reckoned rational. The curves ψ passing through these will then have an equation of the form $c_0\Psi_0+c_1\Psi_1+c_2\Psi_2=0$, where Ψ_0, Ψ_1, Ψ_2 are definite linear functions of $\psi_0, \ldots, \psi_{n-2}$, likewise, therefore, rational in the coefficients in f; while c_0, c_1, c_2 are arbitrary. Take now ξ, η, ζ, so that $\xi/\Psi_0=\eta/\Psi_1=\zeta/\Psi_2$; then, as (x, y, z) describes the curve f, the point of which ξ, η, ζ are the coordinates will describe another curve, ϕ. The order of this curve ϕ, equal to the number of zeros of a general form $u\xi+v\eta+w\zeta=0$, on ϕ, is equal to the number of variable zeros of $u\Psi_0+v\Psi_1+w\Psi_2$ on f, other than the common zeros of Ψ_0, Ψ_1, Ψ_2 at the double points of f, namely is $n-2$ in general, that is, when the double points are distinct. To any ordinary point of f will correspond a single point of ϕ; to a node of f will in general correspond two points of ϕ, each obtained by one of the modes of approach on f to this double point; but these will be fixed points on ϕ. To a general point of ϕ, say $(\xi. \eta, \zeta)$, there will, by the construction, correspond a point (x, y, z) of the curve f; but there will not correspond *two* points. For equations of the form

$$\Psi_0(x', y', z')/\Psi_0(x, y, z)=\Psi_1(x', y', z')/\Psi_1(x, y, z)$$
$$=\Psi_2(x', y', z')/\Psi_2(x, y, z)$$

would involve that every curve $c_0\Psi'_0+c_1\Psi'_1+c_2\Psi'_2=0$ which passes through the point (x, y, z) of f passes likewise through (x', y', z'); it is not true that every curve ψ of order $n-2$, through the double points of f, which is drawn through a general point of f likewise passes through another point of f determined thereby, since, as we have seen, such a curve ψ involves a number $n-2$ of arbitrary parameters equal to the number of its unassigned intersections with f; we may therefore assume, if the $n-4$ fixed points of the plane are taken with sufficient generality, that the same is true of the system $c_0\Psi'_0+c_1\Psi'_1+c_2\Psi'_2=0$. Wherefore, the two equations

$$\xi/\Psi'_0=\eta/\Psi'_1=\zeta/\Psi'_2,$$

taken with $f=0$, lead, for a general point (ξ, η, ζ) of the curve ϕ, to a single point (x, y, z) of f. This point may be determined by rational processes of elimination; so that the ratios of x, y, z are not only rational in (ξ, η, ζ), but equally rational in the coefficients in $\Psi'_0, \Psi'_1, \Psi'_2$, and hence also in the coefficients in f.

We have thus found a new curve ϕ, of points (ξ, η, ζ), of order m say, where m is in general $n-2$, which is wholly in birational correspondence with f. We have seen above, in passing, that the coordinates of the points of f are expressible rationally in terms of a reversible parameter (the expressions involving the coordinates of a point of f); the points of ϕ are therefore expressible rationally by a parameter. From this, by what was proved above, it follows that ϕ has the equivalent of $\frac{1}{2}(m-1)(m-2)$, in general $\frac{1}{2}(n-3)(n-4)$ double points. If we assume that these are distinct, the same process can be applied to ϕ as was applied to f; it can be placed in $(1, 1)$ birational correspondence with a curve of order $n-4$; and the argument can be repeated, until finally we reach either a line (when n is odd), or a conic. The coordinates of the points of a conic can be expressed rationally in terms of a parameter and the coordinates of one particular point arbitrarily taken on the conic; the coordinates of the points of a line are wholly rational in terms of a parameter. The conclusion which has been stated thus follows in general. But the reasoning assumes that at each stage the curve obtained has distinct nodes or cusps. This condition is evidently unnecessary when, for instance, there arises, instead, a k-ple point with separated tangents; for by prescribing the reducing curve to have there a $(k-1)$-ple point, equivalent to $\frac{1}{2}k(k-1)$ conditions, we thereby prescribe $k(k-1)$ intersections, or twice the number of conditions; the multiple point thus has just the effect of $\frac{1}{2}k(k-1)$ separated double points. The examination of the corresponding necessary modification of the reasoning in more complicated cases must be omitted at this stage. Another possibility may be illustrated by considering the simple case where the process is applied to reduce a curve of order k which has a general $(k-1)$-ple point O. The reducing curves would then be taken to be curves of order $k-2$ with a $(k-2)$-ple point at the given multiple point O; these reducing curves would then consist of $k-2$ lines through this point, and the system $u\Psi_0+v\Psi_1+w\Psi_2=0$ would consist of two variable lines through O, together with $(k-4)$ fixed lines through O. The process thus reduces the given curve to the conic $\xi\zeta-\eta^2=0$, which is rational without the assignment of any point thereon. The original curve is in fact obviously rational, being met in one variable point by a variable line through O, even when k is even. The proof that essentially no other case needs remark must be omitted here.

Greatest possible number of double points of a plane curve. It is natural to suppose, since the condition for a curve of order n to be rational has been shewn to be the possession of $\frac{1}{2}(n-1)(n-2)$ double points, in general, that this is the maximum number possible. We prove now definitely that this is so. More generally, for an irreducible curve of order n with multiple points, of which the general one is of multiplicity denoted by k (varying from point to point), we prove that $\frac{1}{2}\Sigma k(k-1)\leqslant\frac{1}{2}(n-1)(n-2)$. This inequality we prove in the equivalent form

$$n(n-1)\geqslant\Sigma k(k-1)+\tfrac{1}{2}(n-1)(n+2)-\tfrac{1}{2}\Sigma k(k-1).$$

For the curve $f=0$, of order n, it can easily be proved that the so-called first polar curve, whose equation is

$$\xi\,\partial f/\partial x+\eta\,\partial f/\partial y+\zeta\,\partial f/\partial z=0,$$

has $k(k-1)$ intersections with $f=0$ at an ordinary k-ple point of f; it may have more when the tangents at the multiple point are not distinct (for instance, at a cusp there are 3 intersections). As the total number of intersections is $n(n-1)$, we infer that

$$\Sigma k(k-1)\leqslant n(n-1),$$

so that we have $\frac{1}{2}(n-1)(n+2)-\frac{1}{2}\Sigma k(k-1)\geqslant n-1$, and, for $n>1$, the number, say ξ, occurring on the left is positive. Now consider the most general curve of order $n-1$, prescribed to have a $(k-1)$-ple point at every k-ple point of the curve f. If these conditions at the multiple points are independent, the curve will have $\xi+1$ homogeneously entering arbitrary coefficients; for a curve of order m has $\frac{1}{2}m(m+3)+1$ homogeneously entering coefficients, and, for a curve to have an h-ple point at a given point requires $\frac{1}{2}h(h+1)$ conditions.

Thus such curve of order $n-1$ has always at least $\xi+1$ homogeneous coefficients, and, by choice of these, can be made to pass through at least ξ other points of the curve f. By being made to have a $(k-1)$-ple point at a k-ple point of f, a number of intersections at least $k(k-1)$ is secured; but it may happen, for special kinds of the multiple point, that there are more than this. Thus the curve of order $n-1$ has in all at least $\xi+\Sigma k(k-1)$ intersections; its total number is however $n(n-1)$, which is thus equal to or greater than $\xi+\Sigma k(k-1)$. And this is the fact we set out to prove.

The application of the result is most generally to the case when every one of the k-ple points has distinct tangents; but it may be noticed that the argument does not require this. And further, when $\frac{1}{2}\Sigma k(k-1)=\frac{1}{2}(n-1)(n-2)$, the argument shews (*a*) that, for curves of order $n-1$, the conditions of having a $(k-1)$-ple point at every k-ple point are independent, (*b*) that the number of other intersections of such curve with f is precisely equal to the number of disposable coefficients in such curve, so that all these other intersections can be taken arbitrarily, (*c*) that all the k-ple points of f are such that the curve of order $n-1$ prescribed to have thereat a $(k-1)$-ple point has just $k(k-1)$ intersections with f at this point, and no more. The whole argument, however, as in previous cases, assumes that f is an irreducible curve; otherwise the curve of order $n-1$, by containing a part of this, may have infinity, and not $n(n-1)$, as the greatest possible number of its intersections with f.

Elliptic curves. Having proved that $\frac{1}{2}(n-1)(n-2)$ is the greatest possible number of double points for an irreducible curve of order n, and considered the case when this is the actual number, we consider now the case of a curve with $\frac{1}{2}(n-1)(n-2)-1$ double points $(n>2)$. We prove that such a curve is birationally equivalent with a plane cubic curve.

Let a general curve, ψ, of order $n-2$ be described through the $\frac{1}{2}(n-1)(n-2)-1$ double points of the given curve f; this will have $n(n-2)-(n-1)(n-2)+2$, or n further intersections, and its equation, containing at least $\frac{1}{2}n(n-1)-\frac{1}{2}(n-1)(n-2)+1$, or n arbitrary coefficients, will be of the form

$$\lambda_0\psi_0+\dots+\lambda_{n-1}\psi_{n-1}+\mu_1 U_1+\mu_2 U_2+\dots=0,$$

where $\psi_0, \dots, \psi_{n-1}, U_1, U_2, \dots$ are definite polynomials of order $n-2$, linearly independent on the given curve, and the terms in $U_1, U_2, \dots$ may be omitted if the double points of f furnish independent conditions for the curve of order $n-2$ described through them. In fact there are not more than n terms in ψ; for, if so, take a form of ψ involving $n+1$ terms (say $\mu_2=\mu_3=\dots=0$); and then, that particular ψ which further passes through $n-1$ prescribed general points of f; this ψ will have an equation of the form $u+v\theta=0$, where u, v are definite, and θ is variable, and this ψ will have one further intersection with f. There is then a correspondence between this point, and the value of θ, either determining the other; by elimination it then follows that the coordinates of this point are rational functions of θ. But this is not the case, or the curve f would have $\frac{1}{2}(n-1)(n-2)$ double points. We may thus suppose, in the equation of ψ, that $\mu_1 = \mu_2 = \ldots = 0$. Further, it is not the case that if ψ be made to pass through an arbitrary general point of f, it will necessarily pass through a further point of f determined by the first point; for, by suitable choice of the ratios of $\lambda_0, \ldots, \lambda_{n-1}$, the curve ψ can be made to pass through $n-1$ arbitrary points of f, and if each of these determined another point of f lying on ψ, we should thus have $2(n-1)$ intersections, which is greater than the number n, of intersections of ψ with f which are not at the double points of f, if $n > 2$.

Now, of such curves ψ, consider those which have $n-3$ prescribed general points of intersection with the curve f; such curves will have 3 further intersections with f, and will have an equation of the form $c_0\Psi_0 + c_1\Psi_1 + c_2\Psi_2 = 0$, where c_0, c_1, c_2 are variable. Such a curve can, by proper choice of the ratios of c_0, c_1, c_2, be made to pass through two arbitrary points of f; thus, the assignment of one arbitrary point of such a curve, upon f, does not involve another intersection determined by the former. Hence, if we put

$$\xi/\Psi_0 = \eta/\Psi_1 = \zeta/\Psi_2,$$

the point (ξ, η, ζ), as (x, y, z) describes f, describes another curve, to any point of which corresponds conversely only a single point of f; this curve, in other words, is birationally equivalent with f. The order of this curve is 3, unless Ψ_0, Ψ_1, Ψ_2 have a common intersection with f in addition to the $n-3$ prescribed intersections of ψ; as this curve cannot be a conic or a line, not being rational, the order is therefore really 3. This shews that the curve f is birationally equivalent with a cubic curve, as stated. The equations of transformation, however, involve the coordinates of the $n-3$ points through which ψ was made to go; by taking these points to coincide at one point, the equations of transformation become rational in the original coefficients of f, and the coordinates of one arbitrary point thereon.

The simplest example of this argument is that of a quartic curve with two double points. Conics through these, and one further arbitrary point of the quartic curve, suffice to transform this curve into a cubic curve. If the two double points, and the third given point, be the vertices of the triangle of reference for (x, y, z), the transformation is effectively $\xi x = \eta y = \zeta z$, as is easy to see. But we may take another point of view, which is suggestive of generalisation. With a, b as constants and u_1, v_1 as homogeneous of order 1 in y, z only, and u_2 homogeneously quadratic in y, z, the equation of the quartic curve may be taken to be

$$ax^4 + x^3u_1 + x^2u_2 + xyzv_1 + by^2z^2 = 0,$$

(the double points being at the vertices (y), (z)). The curve is therefore the projection, on to the original plane, of the curve of intersection of two quadric surfaces; namely the projection, from $(0, 0, 0, 1)$, of the intersection of the two quadric surfaces whose equations are $xt-yz=0$ and $ax^2+xu_1+u_2+tv_1+bt^2=0$. The cubic curve, to which the plane quartic curve is transformable, may be taken to be the projection of the quartic curve in the space (x, y, z, t), from any point of itself. The plane quartic curve is not transformable into a cubic curve with equations rational only in the coefficients in the equation of the quartic curve.

Returning to the plane curve of order n, with $\frac{1}{2}(n-1)(n-2)-1$ double points, if the equation of the general curve of order $n-2$ through these double points be written as before in the form $\lambda_0\psi_0+\ldots+\lambda_{n-1}\psi_{n-1}=0$, and if we take

$$\frac{\xi_0}{\psi_0}=\frac{\xi_1}{\psi_1}=\ldots=\frac{\xi_{n-1}}{\psi_{n-1}},$$

and regard $(\xi_0, \ldots, \xi_{n-1})$ as coordinates of a point in space $[n-1]$, of $n-1$ dimensions, we may shew that as (x, y, z) describes the curve f, the point $(\xi_0, \ldots, \xi_{n-1})$ describes a curve in the space $[n-1]$ which is in birational $(1, 1)$ correspondence with f. The order of this curve, defined as the number of its intersections with a general *prime* locus of this space, given by an equation of the form

$$\lambda_0\xi_0+\ldots+\lambda_{n-1}\xi_{n-1}=0,$$

is n. Conversely, a curve of order n, in space $[n-1]$, is necessarily either rational, or in $(1, 1)$ birational correspondence with a plane cubic curve (without double point). This we see by projecting the curve, in the space $[n-1]$, on to a plane, by means of variable spaces $[n-3]$ drawn through $n-3$ fixed arbitrary points of the curve. The projection is a cubic curve.

Curves which are in $(1, 1)$ birational correspondence with a non-singular plane cubic curve are generally called *Elliptic*. They are evidently distinct from rational curves. But whereas any two rational curves, being in $(1, 1)$ correspondence with a line, are in $(1, 1)$ correspondence with one another, and, in that sense, essentially identical, it is not the case that any two elliptic curves, though they have the same character, are in such $(1, 1)$ correspondence. There belongs in fact to an elliptic curve a single numerical constant, called its modulus; and only when the moduli of two elliptic curves are the same are they essentially identical in the sense explained. To make these statements quite clear, it is desirable to have a detailed theory of a plane cubic curve; and to this end it is proper to reduce its equation to as simple form as possible. The equation of a general plane cubic curve contains 10 homogeneous coefficients, and so depends on 9 constants. The general linear homogeneous transformation in a plane contains 8 constants; it is to be expected therefore that the equation of a plane cubic curve may be reduced, by linear transformation only, to contain only one constant. It may in fact be reduced to the form $\xi^3+\eta^3+\zeta^3+6\mu\xi\eta\zeta=0$, involving the constant μ. But the nature of the irrationalities, in the original coefficients, involved in this reduction, is not thereby made clear. In fact this form is easily deduced when the points of inflexion of the curve are known. Such an inflexion would be obtainable by substituting, in the original general equation, for z, an expression $mx+ny$, and expressing the conditions, for m and n, that the resulting cubic equation for x/y should have three equal roots. The examination shews that we are thus led to require the solution of an equation of order 9. The theory of the solution of this equation is indeed interesting. But we shall not enter into it here. Instead, we shew how, by a birational transformation which is not linear in the original coordinates, we can reduce the equation of the cubic curve to the form $Y^2Z=4X^3-g_2XZ^2-g_3Z^3$, by equations rational in the original coefficients and the coordinates of one point assumed to be given (arbitrarily) on the curve. It is easy to see that this equation essentially depends only on the one modulus $g_2{}^3/g_3{}^2$. A knowledge of the curve given by this equation then enables us to determine all the properties of the original curve.

If the equation of the cubic curve be given in general form, and the coordinates of a point O thereon, we can determine, rationally, the coordinates of the point Q where the tangent line at O meets the curve again. By a linear transformation we can then take O for the point (y) of a triangle of reference, and Q for the point (x). Take now an arbitrary line through O for $x=0$, and an arbitrary line through Q for $y=0$. The equation of the cubic curve thus takes a form $pz^3+ax^2y+bx^2z+cy^2z+dz^2x+ez^2y+fxyz=0$, $z=0$ being the tangent at (y). It is assumed here that O is not a point of inflexion (the case when this is so is dealt with below); also it is assumed that the curve is irreducible and has no double point; in particular there is no double point at O or Q and the coefficients c and a do not vanish. The general conic through the point Q, or (x), and through the two points, other than O, in which $x=0$ meets the curve, has then an equation $l(pz^2+cy^2+eyz)+hxy+kxz=0$, where l, h, k are arbitrary; and such conic has three other intersections with the curve, of which two may be arbitrarily assigned by choice of l, h, k. As in the preceding cases, we define new (non-homogeneous) coordinates by the ratios of the coefficients of l, h, k, putting $\xi=y/z$, $\eta=(pz^2+cy^2+eyz)/xz$. Considering these as rational functions on the curve, ξ becomes infinite to the second order at O, and has for zeros the two points, other than Q, where $y=0$ meets the curve, while η becomes infinite to the third order at O, and vanishes at the three points constituted by Q, and the two points, not on

$x=0$, where the lines joining Q to the intersections of the curve with $x=0$ meet the curve again. For the reverse transformation we then have $x/z=(p+e\xi+c\xi^2)/\eta$, $y/z=\xi$. If now we make the linear transformation to coordinates ξ_1, η_1 which is expressed by $-\xi_1=ac\xi+\frac{1}{3}(ae+bc-\frac{1}{4}f^2)$, $\eta_1/ac=f\xi+2\eta+d$, it is easy to verify that the equation of the curve is the same as $\eta_1{}^2=4\xi_1{}^3-g_2\xi_1-g_3$, where, in terms of the three constants given by $u=ae+bc-\frac{1}{4}f^2$, $v=be+ap-\frac{1}{2}df$, $w=bp-\frac{1}{4}d^2$, the values of g_2, g_3 are given by $\frac{1}{4}g_2=\frac{1}{3}u^2-acv$, $\frac{1}{4}g_3=\frac{2}{27}u^3-\frac{1}{3}acuv+a^2c^2w$. As the coefficient ac is not zero, ξ and η are reversely expressible in terms of ξ_1 and η_1.

The construction for the line $y=0$ fails if the point O be a point of inflexion. In this case, it is easy to see that if a variable line through O meet the curve again in H and K, the locus of the harmonic conjugate of O in regard to H and K is a line. If this line be taken for $y=0$, and a suitable line through O be taken for $x=0$, it is at once verified that the equation is reducible, by linear transformation only, to the form obtained in general.

Combining the result now obtained, with what was proved above, we can assert that a plane curve of order n, with $\frac{1}{2}(n-1)(n-2)-1$ double points, can be birationally reduced to have its equation in the form $\eta^2=4\xi^3-g_2\xi-g_3$, the necessary equations being rational in the coefficients of the given curve and the coordinates of an arbitrary point thereon. This enables us to specify in detail the difference between a rational curve and the curves now under consideration. A rational curve has its coordinates reversibly expressible by a parameter θ; this statement may be modified by saying that its coordinates are reversibly expressible by singly periodic functions (trigonometrical functions) of an argument u; for we have only to take u so that $\tan\frac{1}{2}u=\theta$, or $\cos u=(1-\theta^2)/(1+\theta^2)$, $\sin u=2\theta/(1+\theta^2)$, leading to $\theta=\sin u/(1+\cos u)$; then to any point of the curve corresponds a definite value of $\sin u$ and $\cos u$, and hence a definite value of u, save for multiples of 2π; while conversely any value of u gives a definite point of the curve. For the curve of order n with $\frac{1}{2}(n-1)(n-2)-1$ double points, there is an analogous reversible expression of the coordinates of a point of the curve by single-valued functions of an argument u, the functions namely which are called elliptic functions. These differ however from the trigonometrical functions in having *two* periods (whose ratio is not real), say Ω and Ω'; to any value of u belongs a definite point of the curve; to any point of the curve belongs a definite value of u save for an additive term $k\Omega+k'\Omega'$, in which k and k' are integers. We refer to books dealing with these elliptic functions for the establishment of their properties; but the theory, in its simplest form, is directly associated with the equation $\eta^2=4\xi^3-g_2\xi-g_3$ which we have obtained, it being assumed that this curve has no double point. (The condition for this curve to have a double point is easily verified to be $g_3^2=(\frac{1}{3}g_2)^3$.) So much of the elements of the theory as will indicate the connexion with the equation to which we have reduced the cubic curve may be curtly stated. Let two values of the argument u whose difference is of the form $k\Omega+k'\Omega'$, in which k and k' are integers, be called equivalent. Any of the elliptic functions considered is expressible, in the neighbourhood of any finite value, u_0, of the argument, in the form

$$A_r(u-u_0)^{-m_r}+\ldots+A_1(u-u_0)^{-m_1}+B+B_1(u-u_0)+B_2(u-u_0)^2+\ldots,$$

in which m_r, m_{r-1}, ..., m_1 are positive integers with

$$m_r>m_{r-1}>\ldots>m_1>0, \text{ and } A_r,\ldots,A_1,B,B_1,\ldots$$

are constants, in which there is a converging infinite series of positive integer powers of $u-u_0$, and a finite number of negative integer powers. It is only for a finite number of non-equivalent values of the particular argument u_0 that these negative powers of $u-u_0$ are actually present in the expansion; when they are present, the argument u_0 is called a *pole* of the function, and the integer m_r is called the order of the pole. The sum of the orders of the poles actually existing, at non-equivalent values of u_0, is called the *order of the function*. For such a function there is also a number of non-equivalent values of the argument for which the function vanishes, the *zeros* of the function, and the sum of the orders of vanishing at all the non-equivalent zeros is equal to the order of the function. More generally, if A be any arbitrary value, and ϕ denote the function, the sum of the orders of vanishing of $\phi-A$ at all non-equivalent values of the argument is equal to the order of the function ϕ. We may agree to say that if the difference $\phi-A$ vanishes to order m for a particular value of the argument, then ϕ takes the value A for this argument m times. With this phraseology there is the further property, that the sum of the non-equivalent values of the argument u at which the function takes any specified value, A, is independent of A, being the same as when A is infinite. There exists no such function whose order is less than 2; there exists, however, a function with simple poles for two arbitrary non-equivalent values of the argument, and no other non-equivalent poles; and there exists a function with a pole of the second order for an arbitrary value of the argument, whose only other poles are for equivalent values of the argument. This latter function, suitably specialised, is the simplest doubly periodic, or elliptic, function. The pole being taken to be at $u=0$, the function can in fact be chosen to have an expression, in the neighbourhood of this value, of the form $u^{-2}+A_1u^2+A_2u^4+\ldots$, where only even powers of u arise, and there is no term independent of u. This function being denoted by $\wp(u)$, it can be shewn that its differential coefficient, $\wp'(u)$, is connected with it by an equation of the form

$$[\wp'(u)]^2 = 4\wp^3(u) - g_2\wp(u) - g_3,$$

where g_2, g_3 are definite functions of the periods Ω, Ω'. It can further be shewn that every elliptic function, having the character we have described, which has the periods Ω, Ω', can be expressed as a rational function of $\wp(u)$ and $\wp'(u)$. Thus the points of the cubic curve above obtained can be expressed, in the form $\xi = \wp(u)$, $\eta = -\wp'(u)$, in terms of u, the value of u which belongs to any point of the curve being given by the equation

$$u = \int_{(\xi,\eta)}^{\infty} \frac{d\xi}{\eta},$$

wherein the general value of the integral, for all possible paths of integration, is of the form $u + k\Omega + k'\Omega'$, in which k, k' are integers.

Many of the geometrical properties of a cubic curve, and, more generally, of any curve which is birationally represented by this cubic curve, become very simple with the help of the ideas which we have sketched. For instance, consider the intersections of the cubic curve with a general plane curve of order m, whose equation we may represent by $f(\xi, \eta) = 0$. These intersections correspond to the non-equivalent values of the argument u for which the doubly periodic function $f(\wp u, -\wp' u)$ vanishes. If the term in η^m is present in $f(\xi, \eta)$, this function has a pole of order $3m$ at $u = 0$, and no other non-equivalent pole, since $\wp'(u)$ has a pole of order 3 at $u = 0$, and no other non-equivalent pole, while $\wp(u)$ has a pole of order 2 at $u = 0$, and no other non-equivalent pole. Hence, by what we have said, the function $f(\wp u, -\wp' u)$ vanishes for $3m$ non-equivalent values of u, say $u_1, \ldots, u_{3m}$, and the sum $u_1 + \ldots + u_{3m}$, save for an additive term $k\Omega + k'\Omega'$, in which k, k' are integers, is equal to the sum of the values of u at the non-equivalent poles of the function, that is, equal to zero. Conversely, it can be shewn that, if $3m - 1$ arbitrary points be taken on the cubic curve, then a curve of order m can be put through these, and is thereby determined, so far as its intersections with the cubic curve are concerned, the argument of its remaining intersection being given by the equation remarked.

Of the intersections with a given *rational* curve, however, of another curve, *all* can be arbitrarily assigned. For a given curve which is neither rational, nor elliptic, it will be seen, in what follows, that *more than one* of its intersections with another curve are determined by the others; the number of such intersections determined by the others will be seen to be a distinguishing mark of the character of the given curve.

Examples of the properties of elliptic curves. 1. All plane cubic curves which pass through 8 of the nine intersections of two given cubic curves $U=0$, $V=0$, pass through their remaining intersection. In fact these cubics all have an equation of the form $U+\lambda V=0$. We have remarked above on the general case of the intersection of a given cubic curve with curves of order m. A particular consequence is that, if three lines meet a given cubic curve respectively in P, Q, R, in P', Q', R', and in P'', Q'', R'', and $PP'P''$ be in line and $QQ'Q''$ be in line, then also $RR'R''$ are in line.

2. If the tangent line of the cubic curve $f(x, y, z)=0$, at a point (x, y, z), meet the curve again in (ξ, η, ζ), we have $\xi\partial f/\partial x+\eta\partial f/\partial y+\zeta\partial f/\partial z=0$. Regarding (ξ, η, ζ) as given, and (x, y, z) as current coordinates, this equation represents a conic, called the *polar conic* of (ξ, η, ζ); it can easily be shewn to touch the cubic curve at (ξ, η, ζ); thus it has 4 other intersections with the curve; so that 4 tangents can be drawn to the curve from any point of itself. We have seen that the coordinates of the points of the cubic curve can be represented by elliptic functions of an argument u, in such a way that the three values of u at the intersections of the curve with a line, have a sum which vanishes, or has a form $k\Omega+k'\Omega'$, where k, k' are integers, Ω, Ω' being the periods of the functions. In particular, if the tangent of the curve, at a point u, meet the curve again in v, we have $2u+v=k\Omega+k'\Omega'$. When v is given, this leads only to 4 non-equivalent values of u, namely $-\tfrac{1}{2}v$, $-\tfrac{1}{2}(v+\Omega)$, $-\tfrac{1}{2}(v+\Omega')$, $-\tfrac{1}{2}(v+\Omega+\Omega')$; these give the 4 points of contact of tangents to the curve from the point v. Denote these points in order by A, B, C, D; then AB meets the curve again in the point $v+\tfrac{1}{2}\Omega$, and CD meets the curve again in the point $v+\tfrac{1}{2}\Omega+\Omega'$, which is equivalent to $v+\tfrac{1}{2}\Omega$, and is the same as the former point. Similarly AC, BD meet on the curve, in the point $v+\tfrac{1}{2}\Omega'$, and AD, BC meet on the curve, in the point $v+\tfrac{1}{2}(\Omega+\Omega')$. And the tangents of the curve at the four points v, $v+\tfrac{1}{2}\Omega$, $v+\tfrac{1}{2}\Omega'$, $v+\tfrac{1}{2}(\Omega+\Omega')$ all meet the curve again in the same point, whose argument is $-2v$.

3. Let X, Y, T be three collinear points of the cubic curve, and Z be the point of contact of any one of the four tangents which can be drawn to the curve from T. From what is remarked (and proved algebraically) in Ex. 1, it follows that if an arbitrary line from X meet the curve again in P and P', and ZP, ZP' meet the curve again respectively in Q and Q', then Q, Q' are in line with Y. The line YQQ' is thus determined algebraically and unambiguously from the line XPP', and conversely. The pencil of all lines XPP' through X is thus projectively related to the pencil of corresponding lines YQQ'. The locus of the point of intersection of XPP' and YQQ' can in fact be shewn to be a conic which touches XZ, YZ respectively at X and Y, and passes through the points of contact of the four tangents of the curve which can be drawn from Z. A particular consequence is that the pencil of four tangents of the curve from any point X lying thereon is related to the pencil of four tangents from any other point Y. We may in particular suppose Y to be one of the inflexions of the curve, and then that the equation of the curve is reduced (by linear transformation), as is explained above, to the form $y^2z=4x^3-g_2xz^2-g_3z^3$, (y) being the inflexion Y, and $z=0$ the inflexional tangent. Then the pencil of the 4 tangents from Y is that given by the values of x/z which satisfy the quartic equation $4x^3z-g_2xz^3-g_3z^4=0$. Assuming algebraic properties, the quadratic and cubic invariants of this equation are g_2 and g_3 and the cross ratio of its roots satisfies a sextic equation whose coefficients are rational in $g_2{}^3/g_3{}^2$. This ratio is the absolute

invariant of the elliptic functions in terms of which the curve is expressible; and the necessary and sufficient condition for two cubic curves to be birationally transformable into one another is that this invariant have the same value for both curves.

4. The following fundamental so-called theorem of coresiduation for a cubic curve follows at once from the result stated above in regard to the sum of the elliptic arguments of the intersections of any curve with a given cubic curve: let (A), (P) be two sets of points of the cubic curve, forming together the complete intersection of this curve with another curve; let (A), (Q) be likewise, together, the complete intersection of the cubic curve with another curve of the same order as before. The sets (P), (Q) are each said to be *residual* to (A), and *coresidual* with one another. Now draw any other curve, of sufficiently high order, through the points (P), its residual set of intersections with the cubic curve being (B). Then the theorem is that the sets (Q), (B) form together the complete intersection of the cubic curve with another curve. Stated briefly, of two coresidual sets, on the cubic curve, either is the residual of an arbitrary residual of the other. Algebraically, there exists a function, rational in the coordinates by which the cubic curve is expressed, which, considered on the curve, has the set (P) for its zeros, and the set (Q) for its poles (or infinities).

5. By eliminating y between the equation of a general cubic curve, and the equation of a line $ly = \mu x + z$, and expressing that the resulting cubic equation in x/z has three equal roots, we may prove that the determination of the inflexions of the cubic curve depends upon an equation of order 9. There are thus 9 inflexions, and, when the coefficients in the equation of the cubic curve are real, it appears at once that at least one of these inflexions is real. Assuming one inflexion known, the equation of the curve is at once reducible, as we have seen, to the form

$$y^2z = x^3 + axz^2 + bz^3.$$

Taking this form, prove that the conditions for $ly = \mu x + z$ to be an inflexional tangent are $a^2\lambda^2 + \lambda(3b\mu^2 - 4a\mu) + \mu^2 = 0$, $3a\lambda^2 - 6\lambda\mu - \mu^4 = 0$, where $\lambda = l^2$. Putting c for $a^3/27b^2$, using instead of μ the value $\theta = \frac{1}{3}a/b\mu$, and putting $f(\theta) = 3\theta^4 - 2\theta^3 + 6c\theta^2 - 6c\theta + c - c^2$, prove that the conditions are equivalent to $f(\theta) = 0$, $\lambda = 2(\theta^2 + c)^2/b\theta^2 f'(\theta)$, where $f'(\theta) = \partial f/\partial\theta$. The form $f(\theta)$ is one for which the quadratic invariant vanishes. When c is real the equation $f(\theta) = 0$ has two real, and two imaginary roots, as we may verify, using $(\theta - \frac{1}{6})f'(\theta) = f(\theta) + (3\theta^2 - \theta + c)^2$. A corresponding inflexion is then given by $x/z = a^2 f'(\theta)/54b(\theta^2 + c)^2$, $y\lambda^{\frac{1}{2}} = \mu x + z$. Thus, through any inflexion of the curve there can be drawn four straight lines each of which contains two further inflexions; and if the original inflexion be real, and a, b be real, two of these lines are real, and two are conjugate imaginaries; of the real lines, one contains two real inflexions beside the original, and the other contains two conjugate imaginary inflexions (the signs of $f'(\theta)$ being opposite for the two real roots of $f(\theta)$, as is easy to see). Let A, B, C be the real inflexions on the first line, and APP' the second real line containing the two conjugate imaginary inflexions P and P'; through B there will similarly be a real line BQQ', containing the two conjugate imaginary inflexions Q, Q', which will therefore be, one on each of the two conjugate imaginary lines through A; through C there will similarly pass a real line CRR', with the imaginary inflexion R, say, on AQ, and the imaginary inflexion R' on AQ'. The three real lines APP', BQQ', CRR' thus contain all the nine inflexions, each line containing one real, and two conjugate imaginary inflexions. But the line joining any two inflexions whatever contains a third inflexion. This follows from the remark made in Ex. 1. Here, to use the equations, we may prove this anew. Let $z=0$ be the line joining two inflexions, $P=0$, $Q=0$ being the inflexional tangents at these; the equation of the cubic curve is expressible by the vanishing of a homogeneous cubic polynomial in z, P, Q, and here this reduces to the single term z^3 both when $P=0$ and when $Q=0$. The equation of the curve is thus of the form $z^3+PQ(aP+bQ+cz)=0$, where a, b, c are constants. There is thus a third inflexion on $z=0$, given by $aP+bQ=0$. Now write this equation more symmetrically in the form $z^3+PQR=0$, and take the constants l, m, n, k, and the linear functions of P, Q, z denoted by X and Y, in such a way that $lP+mQ+nR=3kz$, and also $lP=X+Y+kz$, $mQ=\omega X+\omega^2Y+kz$, $nR=\omega^2X+\omega Y+kz$, where $\omega=\exp(2\pi i/3)$; and then take $Z=(lmn+k^3)^{\frac{1}{3}}z$, $\mu=k/(lmn+k^3)^{\frac{1}{3}}$. Then the equation of the cubic curve takes the form

$$X^3+Y^3+Z^3-3\mu XYZ=0.$$

Each of the lines $X=0$, $Y=0$, $Z=0$ contains three inflexions, and, in particular, the inflexional tangents at the three inflexions on $Z=0$ are $X+Y+\mu Z=0$, $\omega X+\omega^2Y+\mu Z=0$, $\omega^2X+\omega Y+\mu Z=0$. We thus see that any two inflexions determine a triangle whose sides contain all the nine inflexions, and that there are four lines through any inflexion each containing two others. The inflexions thus lie, by threes, on twelve lines; and there are four triangles for each of which the sides contain all the inflexions. Beside $XYZ=0$, found above, it is easy to see that the other three triangles are $(X+Y+Z)(\omega X+\omega^2Y+Z)(\omega^2X+\omega Y+Z)=0$, and the two whose equations are obtainable from this last by replacing X by ωX and by ω^2X respectively, where $\omega=\exp(2\pi i/3)$.

The number and configuration of the inflexions may also be obtained by using the representation in terms of elliptic functions. The argument, u, of such a point, is such that $3u=k\Omega+k'\Omega'$, where k and k' are integers. The inflexions have thus the nine non-equivalent arguments $\frac{1}{3}k\Omega+\frac{1}{3}k'\Omega'$, where $k=0, 1, 2$ and $k'=0, 1, 2$.

The inflexions are the intersections of the cubic curve $f=0$ with another cubic curve, commonly called the Hessian of the original, whose equation is obtained by equating to zero the three-rowed determinant whose general element is f_{ij}, in which $f_{11}=\partial^2f/\partial x^2$, $f_{12}=\partial^2f/\partial x\partial y$, etc. The Hessian is a covariant of the cubic form, giving the same curve if formed, by the same rule, from any linear transformation of the original equation of the cubic.

Of the three methods of considering the inflexions here suggested, the first, by direct algebra, requires modification when the cubic curve has a double point; the second, by elliptic functions, is inapplicable; but the third may be used. The Hessian of a cubic curve which has a double point, is a cubic curve also with a double point whereat the tangents are the same as for the original curve—the other three intersections give the only existing inflexions, three in number, which are in line. The inflexions of $xyz+u(x,y)=0$ lie on $2z+\partial^2u/\partial x\partial y=0$.

In the direct algebraic consideration, we have assumed one of the inflexions to be known. It is, however, the case that the equation of the ninth order on which the determination of the inflexions depends is one which is capable of algebraic solution. This can be seen from what has been proved if we add thereto the facts (a) that the Hessian of a given general cubic curve has its inflexions at the inflexions of the given cubic and (b) that every cubic curve through the inflexions of a given general cubic curve has likewise its inflexions at these points; these follow by remarking that the Hessian of a cubic curve given by $y^2z+(x,z)_3=0$ is given by $y^2L+[x,z]_3=0$, where L is of the form $px+qz$. From this it follows, if $f=0$, $H=0$ be a general cubic curve and its Hessian respectively, that the Hessian of a curve $\lambda f+\mu H=0$, passing through the intersections of $f=0$, $H=0$ has an equation $\lambda' f+\mu' H=0$, where then λ', μ' are polynomials in λ, μ, of the third order, as we see by considering the determinantal formation of the Hessian. A triangle containing the nine inflexions of $f=0$, as passing through the intersections with $H=0$, has an equation $\lambda f+\mu H=0$; and the Hessian of the cubic curve constituted by three lines consists of these lines, as we see at once; thus, when $\lambda f+\mu H=0$ is a triangle, $\lambda\mu'-\lambda'\mu=0$. This leads to four values of λ/μ. We have seen, however, that there are just four triangles containing the inflexions. Whence the condition that the Hessian of $\lambda f+\mu H=0$ coincides with this is a sufficient as well as necessary condition that $\lambda f+\mu H=0$ should be a triangle. The cubic polynomials λ', μ' in λ, μ can be computed by using a particular form of f, for example the form $y^2z+(x, z)_3$. Whence the quartic equation $\lambda\mu'-\lambda'\mu=0$ is known in terms of g_2 and g_3, which are the invariants of the form f, as may be proved. Whence, given f in its general form, and solving a quartic equation whose coefficients are functions of the invariants of f, we can determine the triangles each containing the nine inflexions. Moreover, the formation of the Hessian of a general cubic, the formation of the Hessian of $\lambda f+\mu H$, and the expression of the conditions that this Hessian agrees with $\lambda f+\mu H$, are direct algebraic problems, from which, by elimination, the quartic equation $\lambda\mu'-\lambda'\mu=0$ can be formed without knowledge of the invariants of f. When finally we have a cubic form which is known to be a triangle, the vertices of this triangle are obtainable by solving the algebraic problem of finding the three common points of three conics, known to have three points in common (these conics being the first polars of the cubic). Thence the sides of two such triangles are known, and the inflexions of the original cubic are the intersections of these sides. Full consideration of the cubic form may be found in Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie*, II, Paris, 1880. That the equation of the ninth degree on which the inflexions depend can be solved algebraically is a consequence, however, of the relation of three roots of this equation which follows from the fact that the line joining two of the inflexions contains a third. For this see Weber, *Algebra*, II, 1896, p. 322. Also, Dickson, *Linear Groups*, 1901, p. 77; Steiner, *Werke*, II, p. 435; Netto, *Combinatorik*, 1901, p. 202.

6. A problem for the cubic curve, suggested by the use of the elliptic functions, is that of finding a point on the curve such that the conic described to have there 5 intersections may have its sixth intersection also there. Of these points, called *sextactic* points, there are 27, lying in threes upon 9 lines, a set of 3 being the points of contact of the three tangents to the curve from an inflexion (other than the inflexion itself). The elliptic argument of a sextactic point is in fact clearly given by

$$u=\tfrac{1}{6}k\Omega+\tfrac{1}{6}k'\Omega', \qquad k=0, 1, \ldots, 5;\ k'=0, 1, \ldots, 5,$$

where simultaneously even values of k, k' are to be excluded. The remaining $36-9$ or 27 possibilities are in 9 such sets of three as

$$\tfrac{1}{6}(2h+3)\Omega+\tfrac{1}{3}h'\Omega',\ \ \tfrac{1}{3}h\Omega+\tfrac{1}{6}(2h'+3)\Omega',\ \ \tfrac{1}{6}(2h+3)\Omega+\tfrac{1}{6}(2h'+3)\Omega',$$
$$h,\ h'=0, 1, 2.$$

More generally, it may easily be proved that if the tangent at a point A of the cubic curve meet the curve again in B, and the tangent at B meet the curve again in C, and the chord AC meet the curve again in P, then the conic with five intersections at A contains P.

Cf. Cayley, *Papers*, IV, p. 227; V, p. 221. For the application of general methods to determine the sextactic points of a general curve, see Segre, *Ann. d. Mat.* XXII, 1894, p. 90 and Enriques-Chisini, *Teoria geometrica*, I, p. 277; II, p. 289.

7. A fundamental theorem is that the triads of points in which an arbitrary line meets a cubic curve, and its Hessian, are apolar to one another. It can be proved at once by verifying that the two curves $x^3+y^3+z^3=0$, $xyz=0$ are met in apolar triads by an arbitrary line, and then noticing that the Hessian of the general cubic curve expressed by $x^3+y^3+z^3-3\mu xyz=0$ has an equation of the form $x^3+y^3+z^3-3\mu_1 xyz=0$.

If two binary cubics be symbolically expressed by $a_x{}^3$, $b_x{}^3$, the condition that they should be apolar is symbolically expressed by $(ab)^3=0$; when this is so, if p, q, r be the roots of $a_x{}^3=0$, we not only have $a_p a_q a_r=0$, but also $b_p b_q b_r=0$. From this it follows that, if $f_x{}^3$, $h_x{}^3$ be the symbolical forms of a ternary cubic and its Hessian, and (u_1, u_2, u_3) be arbitrary, we have $(fhu)^3=0$; and if (x), (y), (z) be the three points of intersection of the line $u_x=0$ with the cubic curve $f_x{}^3=0$, we have $f_x f_y f_z=0$ and $h_x h_y h_z=0$. Another form of the theorem is that, if (x), (y) be any two points of the cubic curve, we have $f_x{}^2 f_y/h_x{}^2 h_y=f_x f_y{}^2/h_x h_y{}^2$.

8. It may be proved, if (c) be an arbitrary point, and $(cx\,dx)$ denote the three-rowed determinant whose rows are (c_1, c_2, c_3), (x_1, x_2, x_3), (dx_1, dx_2, dx_3), that the integral

$$w=\int_{(y)}^{(x)}(cx\,dx)/f_x{}^2 f_c,$$

which is independent of (c), is finite for all positions of (x), (y) upon the cubic curve $f_x{}^3=0$, along which it is taken. And then that the elliptic function $\wp(w)$, defined to have the periods obtained from this integral, is given by

$$3(cxy)^2\wp(w)=(f_x f_y f_c)^2+2f_x{}^2 f_c\cdot f_y{}^2 f_c-f_x f_c{}^2\cdot f_x f_y{}^2-f_y f_c{}^2\cdot f_y f_x{}^2.$$

In particular, taking

$$f_x{}^3=\tfrac{3}{2}\,[x_2{}^2 x_3-4x_1{}^3+g_2 x_1 x_3{}^2+g_3 x_3{}^3],$$

and also

$$(c)=(0, 1, 0),\quad \xi=x_1/x_3,\quad \eta=-x_2/x_3,\quad \xi'=y_1/y_3,\quad \eta'=-y_2/y_3,$$

with also

$$u=\int_{(\xi)}^{\infty}\frac{d\xi}{\eta},\quad u'=\int_{(\xi')}^{\infty}\frac{d\xi}{\eta},$$

$(\xi, \eta, 1)$ and $(\xi', \eta', 1)$ being points of $f_x{}^3=0$, the integral w is $u-u'$, and the formula for $\wp(w)$ becomes $\wp(u-u')+\xi+\xi'=\tfrac{1}{4}(\eta+\eta')^2(\xi-\xi')^{-2}$, leading when ξ', η' are infinite to $\wp(u)=\xi$.

The general formula for $\wp(w)$ was given by G. Pick, *Math. Annal.* XXVIII, 1887, p. 309. It can, however, also be shewn that

$$\wp(u-u')=h_y{}^2 h_x/f_y{}^2 f_x,\quad =h_y h_x{}^2/f_y f_x{}^2.$$

In the quotient $h_y{}^2 h_x/f_y{}^2 f_x$, if we regard (y) as a fixed point of the cubic curve, the denominator, $f_y{}^2 f_x$, regarded as a function of (x), vanishes to the second order at (y); and the numerator $h_y{}^2 h_x$ vanishes at the tangential point of (y), where the tangent of the cubic at (y) meets the curve again. Thus this quotient, regarded as an algebraic function of (x) on the cubic curve, has only the pole (y), of the second order. Considering the value of the function when (x) is near to (y), it may be verified that its infinite part is $(u-u')^{-2}$, and that $(u-u')^2 h_y{}^2 h_x/f_y{}^2 f_x$ is of the form $1+H$, where H vanishes to the fourth order when (x) approaches to (y). Hence the identity $\wp(u-u')=h_y{}^2 h_x/f_y{}^2 f_x$ follows from the theory of the function $\wp$. An algebraic proof of this, and other formulae of this character, will be found in Halphen, *Fonctions elliptiques*, 2$^{\text{me}}$ partie, Chap. XI, 1888, p. 433, etc.

9. If $\phi=0$, $\psi=0$ be two independent cubic curves through 8 arbitrary points, the sextic curves, with these 8 points as double points, whose number should be $\frac{1}{2}7.8-24$, or 4, are given by an equation

$$a\phi^2+2h\phi\psi+b\psi^2+cU=0,$$

where a, h, b, c are variable parameters, and $U=0$ is a particular sextic curve with these double points. These sextic curves meet the cubic $\phi=0$ in $18-16$ or 2 further points, determining thereon ∞^1 sets of pairs of points given by $b\psi^2+cU=0$. These pairs are then determinable by a variable line passing through a fixed point of the cubic curve $\phi=0$, and there are 4 points of this curve at which the points of a pair coincide; one of these is however the ninth intersection of the curves ψ and ϕ, the pair given by $b\psi^2+cU=0$ coinciding when $c=0$. Let $\psi^2-eU=0$ be one of the three other curves $b\psi^2+cU=0$ whose two remaining intersections with $\phi=0$ coincide, say in A. Then there are ∞^1 sextic curves, given by the equation $\lambda\phi^2+\psi^2-eU=0$, in which λ is variable, which have 9 double points, consisting of A and the original 8 points. By what is proved in this chapter, for curves of order n with $\frac{1}{2}(n-1)(n-2)-1$ double points, all these curves are elliptic. More generally, it can be proved in a similar way, that it is possible to have ∞^1 curves, of order $3n$, all having 9 multiple points, not double points but multiple points of order n. And these will be elliptic curves, since $\frac{1}{2}(3n-1)(3n-2)-\frac{9}{2}n(n-1)=1$. This circumstance was remarked by Halphen, *Bull. d. l. Soc. Math. d. France*, x, 1882, p. 162.

CHAPTER II

THE ELIMINATION OF THE MULTIPLE POINTS OF A PLANE CURVE

As a large part of the difficulty of the theory of curves depends upon the possible intricacy of their multiple points, it seems desirable to shew at once, with the most elementary considerations, that any plane curve is in (1, 1) birational correspondence with a curve, in space of three or more dimensions, which is without multiple points. Some of the ideas involved have already been met with in Chap. I; but the present chapter is designed to be complete in itself.

Consider a given plane curve $f(x, y, z)=0$, of order μ, with any multiple points; the curve is supposed to be irreducible, that is, the polynomial $f(x, y, z)$ cannot be written as the product of other polynomials also rational in x, y, z. A curve whose equation is of the form $\lambda_0\phi_0+\lambda_1\phi_1+\ldots+\lambda_r\phi_r=0$, wherein $\phi_0=0, \ldots, \phi_r=0$ are definite curves of the same order, which may have common points, upon $f=0$, or elsewhere in the plane, determines, by its intersections with $f=0$, a *set* of points thereon. Of these points, some may be the same for all values of the parametric coefficients $\lambda_0, \ldots, \lambda_r$, being common intersections with $f=0$ of all the curves $\phi_0=0, \ldots, \phi_r=0$; the others will, in general, vary when these coefficients vary. It is naturally assumed that there exists no identical linear homogeneous relation, with constant coefficients, connecting the polynomials $\phi_0, \ldots, \phi_r$; it is also assumed that there is no such relation which is true in virtue of $f=0$, as would be the case if there were an identity of the form $c_0\phi_0+\ldots+c_r\phi_r=\psi f$, in which $c_0, \ldots, c_r$ are constants, and ψ is a polynomial in x, y, z. Thus we say that $\phi_0, \ldots, \phi_r$ are assumed to be linearly independent upon $f=0$. Then we say that, as $\lambda_0, \ldots, \lambda_r$ vary, the curve $\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$ cuts upon $f=0$ a *linear series of sets of points of freedom* r; the number of points of a set which vary when $\lambda_0, \ldots, \lambda_r$ vary is called the *grade* of the series, and will be denoted by n; as, by proper choice of $\lambda_0, \ldots, \lambda_r$, a set of the series can be found including r arbitrary points of $f=0$, it is clear that $n \geqslant r$.

There are two properties of such a linear system in regard to which it is desirable to be clear. First, there may exist, on $f=0$, a batch of, say, s points, of definite positions, such that the curves of the linear system $\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$ which are made, by suitable limitation of $\lambda_0, \ldots, \lambda_r$, to pass through some number less than s of the points of this batch, necessarily pass through the other points;

a simple illustration is that of a double point of $f=0$, with distinct tangents, so that we can distinguish the two points of the curve which coincide thereat; the curves of the linear system passing through one of these points necessarily passes through the other. In general, to make a curve of the linear system pass through a point of $f=0$ not common to all of $\phi_0=0, \dots, \phi_r=0$, we must prescribe one linear relation connecting $\lambda_0, \dots, \lambda_r$; a batch of s points of the kind under consideration is one which imposes less than s conditions for the curves of the linear system which are to contain the points of this batch. Such a batch is sometimes spoken of as *neutral*. The other possible property spoken of, refers to a batch of points on $f=0$ of which one point is arbitrary thereon; it may be that the curves of the linear system which are made to pass through *any* arbitrary point of $f=0$ necessarily then pass through other points, which vary when the first point is varied. A simple illustration is that of the linear system of lines (of freedom 1) which pass through a fixed point; any such line, made to pass through an arbitrary point of the curve $f=0$, necessarily passes through the other intersections of this line with the curve. We say, of a linear system $\lambda_0\phi_0+\dots+\lambda_r\phi_r=0$, that it is *stiff*, when any curve of it which passes through a general point of $f=0$, then necessarily passes through other, say $(k-1)$, points of $f=0$, determined by the first point, and variable with it. As a curve of the system can be put through r points of $f=0$, each eligible arbitrarily, independently of the others, the value of k must be such that $kr \leqslant n$. A linear system which is not stiff may be called *pliable*; the series determined on $f=0$ by this system is most often said to be *simple*.

We now consider linear systems *contained* in the given system; such a system is given by $\mu_0\psi_0+\dots+\mu_\rho\psi_\rho=0$, with $\rho+1$, $<r+1$, homogeneously entering parameters $\mu_0, \dots, \mu_\rho$, wherein $\psi_0, \dots, \psi_\rho$ are definite linear functions of $\phi_0, \phi_1, \dots, \phi_r$ with constant coefficients, which are linearly independent of one another. For instance all the curves of the original system which pass through an assigned point form such a contained system, with $\rho=r-1$. What we are interested in here is the possibility of such contained systems for which there is a lessening of the grade, or number of variable intersections with $f=0$, which is *at least twice* the lessening of the freedom. This arises, for instance, if there be a double point of $f=0$ which is not a common zero of $\phi_0, \dots, \phi_r$, when we form, from the original system, the system of freedom $r-1$, of which all curves pass through this double point; the new system has a grade which is less by two, at least, than that of the original. If we denote $n-r$ by s, and call it momentarily, the *sequence*, and also introduce $s-r$, or $n-2r$, and call it the *efficiency* of the system, what we are interested in here is whether the given system contains another system whose

efficiency is less than, or at most equal to its own. We may momentarily say that a system containing such another system is *capable of reduction*.

We can shew that a linear system of freedom $r \geqslant 2$ which is pliable, or cuts on $f=0$ a linear series which is simple, and is not capable of reduction, enables us to put the curve $f=0$ into (1, 1) birational correspondence with another curve, in space of r dimensions, which has no multiple point. For perfect clearness we shall make the assumption, fully justified independently in a following chapter, that in every possible method of approach along the algebraic curve $f=0$, to a simple or multiple point of this curve, the coordinates of the approaching point may be given by converging series of a parameter vanishing at the multiple point. Let the linear system be given as before by $\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$, and consider the point $(\xi_0, \ldots, \xi_r)$, in space $[r]$, of dimension r, which is given by $\xi_0/\phi_0=\xi_1/\phi_1=\ldots=\xi_r/\phi_r$; as (x, y, z) describes $f=0$, this point describes a curve in the space $[r]$; for any general (simple) point of $f=0$, this point (ξ) is definite; for any point of $f=0$ at which all of $\phi_0, \ldots, \phi_r$ vanish, the values of $\xi_0 : \xi_1 : \ldots : \xi_r$ are to be found by a limiting process, from the expansions of x, y, z referred to; and when the point of $f=0$ is a multiple point there will correspond in general several points (ξ), each found from an appropriate expansion. Conversely, to any point (ξ) of the new curve, corresponds the point (x, y, z) of $f=0$ from which it arose, but there corresponds no other point (x', y', z') of $f=0$, unless, when ϕ_i' denotes what ϕ_i becomes when x', y', z' are put for x, y, z, we have

$$\phi_0'/\phi_0=\phi_1'/\phi_1=\ldots=\phi_r'/\phi_r;$$

these equations would involve, however, that any curve

$$\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$$

which passes through the point (x, y, z) necessarily passes also through the point (x', y', z'). This possibility is expressly excluded, for an arbitrary point (x, y, z), by the hypothesis that the linear system is pliable. It is not excluded thereby for definite fixed positions of (x, y, z), which might then give rise to points (ξ) through which several branches of the curve in the space $[r]$ would pass. We deal with this below. The algebraic equations $\xi_0/\phi_0=\ldots=\xi_r/\phi_r$, $f(x, y, z)=0$, thus lead only to one set of values for (x, y, z), in terms of $(\xi_0, \ldots, \xi_r)$, for general values of these latter; and these, being obtainable by rational processes, must be capable of rational (necessarily homogeneous) expression in terms of $\xi_0, \ldots, \xi_r$. The new curve is thus in (1, 1) birational correspondence with $f=0$. The new curve, however, can have no multiple point, namely cannot have a point such that the general prime of the space $[r]$, given by an equation, $c_0\xi_0+\ldots+c_r\xi_r=0$, which is made to pass through this point, has a number of other intersections, with the curve, less by 2, or more, than the number of intersections of a general prime (the order of the curve). This order is clearly the number of intersections, with $f=0$, of the curve $\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$, which are not common to all of $\phi_0=0, \ldots, \phi_r=0$, that is, to the grade n of the series on $f=0$ determined by this system. If the new curve, in $[r]$, had a multiple point, this system would contain another, of one freedom less, and of grade at least two less—and this we have excluded by the hypothesis that the system is not capable of reduction.

Passing now to the consideration of the reduction of the given linear system, of grade n and freedom r, suppose it contains another system, of grade n_1 and freedom r_1, (with $r_1<r$), with not greater efficiency than itself, namely $n_1-2r_1\leqslant n-2r$, the conditions $1\leqslant r_1\leqslant n_1$ being satisfied. Suppose this new system likewise contains a further system of grade n_2 and freedom r_2, with

$$r_2<r_1, \quad n_2-2r_2\leqslant n_1-2r_1, \quad 1\leqslant r_2\leqslant n_2;$$

and so on. Each system is contained in the preceding, and, therefore, in the original system. There must be a final system beyond which reduction cannot go, either because a further step would lead to increased efficiency (as would be the case, for example, if a lessening of one in the freedom led only to a lessening of one in the grade, $\nu-1-2(\rho-1)$ being equal to $\nu-2\rho+1$), or because the freedom, having sunk to 1, cannot be further lessened, and still leave a variable system. Let this final system be of freedom r' and grade n', with $1\leqslant r'\leqslant n'$ and $n'-2r'\leqslant\ldots\leqslant n_1-2r_1\leqslant n-2r$. When $r'\geqslant 2$, this final system is pliable, since else, taking the system contained therein of which all the curves pass through a definite point of $f=0$, we should have a new system, of freedom at least 1, with grade at least two less than n'. Thus, when $r'\geqslant 2$, the final system can be used to put $f=0$ into (1, 1) birational correspondence with a curve, in space $[r']$, having no multiple points. But $r'\geqslant 2r-n+s'$, where $s'=n'-r'$ is the sequence of the final system; thus $r'\geqslant 2$ when $s'\geqslant 2$, provided $2r-n\geqslant 0$. We prove below that if $s'=0$ or $s'=1$, the curve $f=0$ has a special character, being either rational (for $s'=0$), or of a kind called *hyperelliptic*. But whatever be s' ($\geqslant 0$), we have $r'\geqslant 2$ if $2r-n\geqslant 2$.

And we can always take the original system so that $2r-n\geqslant 2$. For let the system be given by the most general polynomial of order N, where $N>\mu-3$, in which μ is the order of the curve $f=0$. Such a polynomial U has $\frac{1}{2}(N+1)(N+2)$ variable coefficients, entering homogeneously; when $N\geqslant\mu$, this polynomial vanishes on $f=0$ at the same points as a polynomial $U-uf$, where u is any homogeneous polynomial of order $N-\mu$, containing therefore

$$\tfrac{1}{2}(N-\mu+1)(N-\mu+2)$$

reduced is stiff, determining on $f=0$ a series which is not simple. The answer is immediate in case, for the original system, $N > \mu-3$, and therefore $2r-n \geqslant 2$. For this involves

$$2r'-n' \geqslant \ldots \geqslant 2r_1-n_1 \geqslant 2r-n;$$

and we have remarked that, for a system of grade ν and freedom ρ to be stiff, we must have $2\rho-\nu \leqslant 0$. In this case, therefore, neither the original system, nor any contained system obtained in the method of reduction, is stiff. If for the original system $N=\mu-3$ and $2r-n=0$, the answer to the question appears implicitly in subsequent more detailed theory; but the case of a rational or hyperelliptic curve evidently stands apart from the general case.

Suppose then that we have placed the original curve $f=0$ in (1, 1) birational correspondence with a curve Ψ, having no multiple points, in space $[r']$, starting with polynomials of order $\geqslant \mu-3$ in general, but with polynomials, of order $> \mu-3$ when the curve is rational or hyperelliptic. We proceed to deduce that when $r'>2$ we can obtain a (1, 1) birational representation by means of a curve without multiple points in ordinary space of three dimensions, and also such a representation by a plane curve having no other multiple points than double points with distinct tangents. The first statement is proven when $r'=3$; suppose then $r'>3$. The points of the chords and tangents of a curve Ψ in space $[r']$ form a locus, M_3, of 3 dimensions, there being ∞^2 chords of the curve (including tangents), and ∞^1 points on each chord, at most. In a space of r' dimensions, a space of $r'-4$ dimensions, say a $[r'-4]$, the intersection of four primes of the space (cf. the general explanation in Vol. I of the writer's *Principles of Geometry*), can be chosen to have no point in common with M_3; in general, two spaces $[p]$, $[q]$, in $[r]$, meet in a point only if $p+q \geqslant r$. Take now an arbitrary threefold space [3], say S, in the space $[r']$, and join the chosen $[r'-4]$ to any point P of the curve Ψ, by means of a space $[r'-3]$. This will meet the space S in a point, P', which we speak of as obtained from P by projection from the chosen $[r'-4]$. As P describes the curve Ψ we thus obtain a curve Ψ', in the space S, described by the point P'. Conversely, there are not two points P, Q, of the curve Ψ, thus giving rise to the same point P' of Ψ'; for, if so, the chord PQ, which lies in a definite $[r'-3]$ through the chosen $[r'-4]$, would meet this $[r'-4]$, while we have supposed this space to be so chosen as not to contain any point of any chord or tangent of Ψ. There is thus a definite (1, 1) correspondence between the two curves Ψ, Ψ'; and as Ψ is without multiple points, so also is Ψ'. From the nature of the process, this correspondence may be expressed by rational equations, the coordinates of a point of either curve being rational in the coordinates of the point of the other curve to which it corresponds. Proceeding now further to place the curve Ψ' in correspondence with a plane curve, we assume that this curve, in space [3], has ∞^1 trisecants, that is chords meeting the curve in three points, which then form a ruled surface; further we assume that there are ∞^1 chords of Ψ', likewise forming a ruled surface, each chord having the property that the tangents of the curve, at the two ends of the chord, intersect one another; finally there are ∞^1 tangents of the curve, also forming a surface. We can then find points of the space in which Ψ' lies which do not lie on any one of these three surfaces. Let O be such a point, and ϖ be a general plane. Then the curve Ψ' projects from O into a curve Ψ'' of the plane ϖ; this curve Ψ'' will be without cusps (which would arise if O were on a tangent of Ψ'), and will be without multiple points of the third or higher order (which would arise if O were on a trisecant or a line meeting Ψ' in more than three points). There may, and there will in general be double points of the curve Ψ'', corresponding to the finite number of (bisecant) chords of Ψ' which pass through O; but as O is not on a chord of Ψ' whereat the tangents at the two ends intersect one another, the tangents at such a double point of the curve Ψ'' will be distinct. Thus, to any general point of Ψ' will correspond a single point of Ψ'', and conversely; but a double point of Ψ'', which may be reached along either branch of the curve passing through it, will correspond to two points of Ψ'. Thus again the coordinates of a point of either curve are rational functions of the coordinates of the point of the other to which it corresponds; but the coordinates of a point of Ψ' are (ratios of) four polynomials, in the coordinates for Ψ'', having the property that two at least of these polynomials vanish at any double point of Ψ'', so that their ratio may be capable of taking two values as we approach the double point of Ψ''. By use of the preceding transformation to Ψ', we can thus obtain a direct (1, 1) birational transformation between Ψ'' and the original curve $f=0$; and in particular, the passage which we have made from the curve Ψ to Ψ'' in two steps, a projection from a $[r'-4]$, and a projection from a point O, may be made in one step, by projection from a suitably chosen space $[r'-3]$.

Remark. It will appear, from the greater detail of a subsequent chapter, that the linear system by which, after reduction, we pass to the curve Ψ, can be so taken that in the final system with which we make the transformation from $f=0$ to Ψ, say $\mu_0\psi_0+\ldots+\mu_{r'}\psi_{r'}=0$, $(r'>2)$ three of the polynomials ψ, say ψ_0, ψ_1, ψ_2, are of the forms $x\psi$, $y\psi$, $z\psi$, where x, y, z are the coordinates in $f=0$. The corresponding coordinates ξ_0, ξ_1, ξ_2 of the space $[r']$ have then the ratios of x, y, z. In this case the transformation from $f=0$ to Ψ is a projection on to the original plane from the space $[r'-3]$ for which

$\xi_0=0=\xi_1=\xi_2$. Thus any plane curve $f=0$, however intricate its multiple points may be, may be regarded as a *projection*, from a curve having no multiple points, lying in higher space (or is in (1, 1) correspondence with a plane curve having no multiple points, if $r'=2$).

For the preceding theory cf. Albanese, *Rend. Lincei*, XXXIII, 1924, pp. 13, 14; also Severi, *Trattato di geom. alg.* I, 1, 1927, p. 75. A valuable history is given on pp. 332–5 of the same volume.

Ex. 1. As a very simple application of the theory of this chapter we may take a plane quintic curve with 3 double points, and consider the system of all conics in the plane. By the process of reduction explained these lead to the contained system of conics through the double points, for which $n'=4$, $r'=2$. This is not further reducible, and leads to the transformation of the quintic curve to a plane quartic curve having no multiple points.

If however we take a plane quintic curve with 4 double points, the system of conics through these has $n'=2$, $r'=1$. The curve is therefore hyperelliptic, and cannot be transformed to a curve without multiple points with this system of conics. In this case we may take the cubic curves through the 4 double points, for which $n'=7$, $r'=5$; this system appears not to be further reducible in the sense of the text, and leads to a birational representation of the curve by a curve of order 7 in space of 5 dimensions, not having multiple points. But this curve can be projected from a line into a curve in ordinary space, and if the line be taken to be a general chord of this curve, this curve in [3] will be of order 5. It can be obtained directly from the original plane curve by considering cubic curves through the 4 double points and two arbitrary general points of the curve; the system of these cubic curves is of freedom 3 and grade 5; we can indeed consider cubic curves through the double points of the original curve and 3 arbitrary general points of the curve, forming a system of freedom 2 and grade 4, and so transform the original curve to a plane quartic curve; but this will have one double point, being obtained by projection of the quintic curve in [3] from a point of itself, and it can be shewn that there is one trisecant of the quintic curve through any point of itself. In fact it will appear that the quintic curve in [3] is the residual intersection of a quadric surface with a cubic surface containing one generator of the quadric, and the generator of the quadric surface of the same system through any point of the quintic curve is a trisecant. It will also appear that the septimic curve in [5] is the residual intersection of a rational ruled quartic surface with a quadric containing one generator of this.

Ex. 2. A more particular example, having however an intrinsic interest, is the curve $z(x^4-y^3z)^2-9x^5y^4=0$. The system

$$\lambda_0 xy^3+\lambda_1 zy^3+\lambda_2(x^4-y^3z)=0$$

effects a birational transformation to the curve

$$3^6X^7(Y+Z)^2-Y^3Z^6=0,$$

the equations $$X/xy^3=Y/zy^3=Z/(x^4-y^3z)$$

leading to $$x/X^2(Y+Z)=9y/YZ^2=z/XY(Y+Z),$$

but this system is reducible in the sense of the text, containing the system $\mu_0xy^3+\mu_2(x^4-y^3z)=0$ of grade $n_1=3$, for which $2r_1-n_1=-1$ is greater than $2.2-9$ or -5, of the original system. The new curve, in X, Y, Z, has accordingly multiple points.

If however we take the system

$$\lambda_0 x^2 y^3 + \lambda_1 x(x^4 - y^3 z) + 9\lambda_2 y(x^4 - y^3 z) = 0,$$

putting $\quad \xi/x^2y^3 = \eta/x(x^4 - y^3z) = \zeta/9y(x^4 - y^3z),$

which lead to $\quad x/\eta(\eta^3 - \xi^2\zeta) = 9y/\zeta(\eta^3 - \xi^2\zeta) = z/\xi\eta^2\zeta,$

we obtain a birational correspondence with the curve, without multiple points, $3^{-6}\eta\zeta^3 + \zeta\xi^3 - \xi\eta^3 = 0$. If we put $\xi = \xi_1$, $\eta = \omega\eta_1$, $\zeta = -\omega^3\zeta_1$, with $\omega^7 = 3^6$, the last assumes a symmetric form $\eta_1\zeta_1{}^3 + \zeta_1\xi_1{}^3 + \xi_1\eta_1{}^3 = 0$. For properties of this curve, the reader may consult Klein-Fricke, *Modulfunctionen*, I, 1890, p. 702, and also the writer's *Multiply-periodic Functions*, Cambridge, 1907, p. 269.

Ex. 3. We may employ the case of Ex. 2 to illustrate the expansion of the branches of a curve near a multiple point which has been referred to in the text. For the curve $3^{-6}\eta\zeta^3 + \zeta - \eta^3 = 0$, when t, $= \eta/3$, is sufficiently small, it can be proved that $\zeta = 3^3t^3 - 3^4t^{10} + 3^6t^{17} - \ldots$, and hence, for the curve $(x^4 - y^3)^2 - 9x^5y^4 = 0$, when x, y are sufficiently small, $x = t^6 - 6t^{13}\ldots$, $y = t^8 - 3^2t^{15}\ldots$. Denoting the left side of the equation of this curve by $f(x, y)$, we may shew that, if ϕ denote a general sextic polynomial in x, y, 1, the number of linearly independent conditions for the coefficients of ϕ in order that the quotient $\phi x/\partial f/\partial y$ may vanish as we approach $x = 0 = y$ along the curve $f = 0$, is 21.

Cf. Cayley, *Papers*, v, p. 524 and C. A. Scott, *Amer. J. of Maths.* xv, 1892, p. 318.

Ex. 4. It will appear subsequently that, for a curve of order μ which is not hyperelliptic or rational, and has only double points, the final reduced system, when we start with polynomials of order $\mu - 3$, is that determined by the curves of this order, passing through the double points; and that, for this system, the efficiency, $n' - 2r'$ vanishes. Any system contained in this final system has efficiency greater than zero (Clifford's theorem).

Ex. 5. If the fundamental curve be $y^2 = (x, y)_4$, a quartic with a self contact at $x = 0$, $y = 0$, it can be shewn that the system of conics is reducible, in the sense of the text, to conics touching the quartic curve at $x = 0$, $y = 0$, given by $y + (x, y)_2 = 0$. We therefore make a transformation to a space [3], by means of

$$\frac{x_0}{y} = \frac{y_0}{x^2} = \frac{y_1}{xy} = \frac{y_2}{y^2},$$

whereby the plane is transformed to the quadric cone in this space (x_0, y_0, y_1, y_2) which is expressed by

$$\frac{x_0}{\phi} = \frac{y_0}{\theta^2} = \frac{y_1}{\theta} = \frac{y_2}{1}$$

(where $\theta = x/y$, $\phi = 1/y$). The original curve, given by $\phi^2 = (\theta, 1)_4$, is represented by the intersection of this cone with the quadric surface $x_0{}^2 = (y_0, y_1, y_2)_2$, the multiple point being replaced by the two simple points where this quadric meets the line (x_0, y_0), or $y_1 = y_2 = 0$. The original curve is obtainable by projection, of the curve on the cone, from the point $x_0 = 0$, $y_1 = 0$, $y_2 = 0$. From an arbitrary point of the space (x_0, y_0, y_1, y_2) the curve on the cone is projected into a plane quartic curve with two double points.

Ex. 6. More generally, it will appear subsequently that the most general plane hyperelliptic curve can be given by an equation $y^{2p} = (x, y)_{2p+2}$; and that, if we begin with a general polynomial of order $n - 2$, that is $2p$, and effect the possible reduction explained in this chapter (due in fact to the character of the multiple point at $x=0$, $y=0$), then the final system obtained is expressed by an equation of the form $y^p(x, y)_{p-1}+(x, y)_{2p}=0$, where $(x, y)_{p-1}$, $(x, y)_{2p}$ are homogeneous polynomials of the orders indicated, with arbitrary coefficients. In accordance with the text we therefore make a transformation to a space $[3p]$, with coordinates $x_0, \ldots, x_{p-1}, y_0, \ldots, y_{2p}$, by means of the equations

$$\frac{x_0}{x^{p-1}y^p}=\frac{x_1}{x^{p-2}y^{p+1}}=\ldots=\frac{x_{p-1}}{y^{2p-1}}=\frac{y_0}{x^{2p}}=\frac{y_1}{x^{2p-1}y}=\ldots=\frac{y_{2p}}{y^{2p}}.$$

Putting $x/y=\theta$ and $1/y=\phi$, these are the same as

$$\frac{x_0}{\phi\theta^{p-1}}=\frac{x_1}{\phi\theta^{p-2}}=\ldots=\frac{x_{p-1}}{\phi}=\frac{y_0}{\theta^{2p}}=\frac{y_1}{\theta^{2p-1}}=\ldots=\frac{y_{2p}}{1},$$

and represent in fact a rational ruled surface of order $3p-1$ obtained by joining every point in turn of the normal rational curve of order $p-1$ in space $(x_0, \ldots, x_{p-1})$ to the corresponding point (given by the same value of θ) of the normal rational curve of order $2p$ in space $(y_0, \ldots, y_{2p})$. This surface is the representative of the original plane. The curve of this plane corresponds to an equation $\phi^2=(\theta, 1)_{2p+2}$, and its transformation on the ruled surface is the intersection of this with the quadric cone expressed by an equation $x^2_{p-1}=(y_{p-1}, y_p, \ldots, y_{2p})_2$. This quadric has, as base or vertex, the space $[2p-3]$ on which all the coordinates except $x_0, \ldots, x_{p-2}, y_0, \ldots, y_{p-2}$ are zero. The multiple point of the original plane curve is represented by the two simple points in which the curve on the ruled surface meets the generator $\theta=\infty$. This is the generator joining the vertices (x_0), (y_0) of the coordinate system, which lie respectively on the two rational curves spoken of; this generator lies on the base—or vertex—of the quadric cone specified. It can be shewn that this generator counts $2p-2$ times in the intersection of the quadric cone with the ruled surface, the remaining intersection, of order $4p$, being the transformation of the original curve. The original curve can be recovered by projection of the curve on the ruled surface, from any one of p spaces $[3p-3]$ such as

$$x_i=0,\quad y_{p+i}=0,\quad y_{p+1+i}=0, \qquad (i=0, \ldots, p-1).$$

In particular, we may take $x_0=0$, $y_p=0$, $y_{p+1}=0$, which contains, as its intersection with the ruled surface, the generator $\theta=0$, taken $p-1$ times over, accounting therefore for $2p-2$ intersections with the curve of order $4p$, which thus projects into the original curve, of order $2p+2$. And the generator $\theta=\infty$ meets the space $x_0=0$, $y_p=0$, $y_{p+1}=0$, in the point (y_0), so that the two intersections of the curve with the generator $\theta=\infty$ project into the one multiple point in the plane, at $x=0$, $y=0$.

More generally, in space $[n-p]$, with $n \geqslant 2p+2$, a quadric containing $n-2p-2$ generators of a rational ruled surface of order $(n-p-1)$ meets this ruled surface further in a hyperelliptic curve of order n (Segre, *Math. Ann.* XXX, 1887, p. 203). This is easily proved by combining equations $x_0/\phi\theta^\lambda=\ldots=x_\lambda/\phi=y_0/\theta^\mu=\ldots=y_\mu/1$ with the equation of a general quadric in the coordinates $(x_0, \ldots, x_\lambda, y_0, \ldots, y_\mu)$.

It may be proved however that a hyperelliptic curve is equivalent with a curve of order $p+3$ on a quadric surface in ordinary space, obtained as the residual intersection with a surface of order $p+1$ passing through $p-1$ generators of the quadric, of the same system; this is without singularities. Or may be represented by a plane curve of order $p+2$ having a single p-fold multiple point of distinct tangents.

Ex. 7. For the curve expressed by $y^3+y^2(x, y)_2+y(x, y)_4+(x, y)_6=0$, if we begin with polynomials of order 3, we can shew that the final reduced system is given by $y[y+(x, y)_2]=0$. Putting then

$$x_0/y=y_0/x^2=y_1/xy=y_2/y^2,$$

it is easily proved that the curve is transformed into the intersection of a quadric cone, in the space (x_0, y_0, y_1, y_2), expressed by $y_0y_2-y_1^2=0$ with a general cubic surface of that space. The multiple point at the origin is transformed into the three intersections of the surface with the generator $y_1=0$, $y_2=0$ of the quadric cone. At these points, as at the three intersections of the curve with any generator of the cone, the tangent lines of the curve are in one plane. The original curve can be obtained again by projection of the space curve from a point of this generator. By projection from a point of the space curve, there is obtained a plane quintic curve having one point of self contact. By projection from a general point of the threefold space there is obtained a plane sextic curve with 6 double points.

CHAPTER III

THE BRANCHES OF AN ALGEBRAIC CURVE; THE ORDER OF A RATIONAL FUNCTION; ABEL'S THEOREM

Introductory explanation. The present chapter deals with important fundamental conceptions in the theory of algebraic curves and functions, and, for greater precision, some familiarity with the elements of the theory of functions is assumed.

The preceding chapter incidentally brings out the want of definiteness in regarding a curve as defined by its points, each given by one set of values of the coordinates; for we have seen that a multiple point may be replaced, on another curve which is in (1, 1) birational correspondence with it, by several distinct points. In the present chapter we are led to consider a point of a curve as belonging to a definite range of points, lying on, and forming all the points of a so-called *branch* of the curve. For clearness, a point so considered, in association with a branch of the curve to which it belongs, will be called a *place*. A point may belong to several branches, but when this is so it is accidental, the distinction of the branches being the essential fact. The point of view which is reached will ultimately be found to be of great importance.

The fundamental curve considered, with its equation expressed by non-homogeneous coordinates, $f(x, y)=0$, may be spoken of as containing; (i), points for which both x and y are finite; (ii), points for which x is finite, but y is infinite; (iii), points for which x is infinite, but y is finite; (iv), points for which both x and y are infinite. If $f(x, y)=0$ be regarded as arising from an equation $F(x_0, x_1. x_2)=0$, homogeneous in x_0, x_1, x_2, by putting $x=x_0/x_2$, $y=x_1/x_2$, then points (i) are those for which x_2 is not zero; points (ii) are those for which $x_2=0$ and $x_0=0$; points (iii) are those for which $x_2=0$ and $x_1=0$; while points (iv) are those for which $x_2=0$ but neither x_0 nor x_1 vanishes. We may therefore deal with points (ii) by putting $\eta=1/y$, and considering the equation $f(x, \eta^{-1})=0$, in x and η; with points (iii) by putting $\xi=1/x$, and considering the equation $f(\xi^{-1}, y)=0$, in ξ and y; and with points (iv) by putting $\xi=1/x$, $\eta=1/y$, and considering the equation $f(\xi^{-1}, \eta^{-1})=0$, in ξ and η.

If $x=a$, $y=b$ be a point of the curve, so that $f(a, b)=0$, where a, b are finite, it will be proved that there are points of the curve given by $x=a+t^r$, $y=b+b_1t+b_2t^2+\ldots$, for all sufficiently small, real, and complex, values of the parameter t, where the second expression is a power series, converging for such values of t, and r is a positive integer, and this in such a way that the value of t thus leading to any point (x, y) of the curve, sufficiently near to (a, b), is unique. Save for a finite number of values of a and b, this number r is unity, and the expression for y is a power series in $x-a$; otherwise, as we see by replacing t by ωt, where $\omega=\exp(2\pi i/r)$, the expression gives r values of y, corresponding to a single value of x; these values however can be continuously changed into one another by continuous variation of t, and form a *cycle*. The points (x, y) so obtained, for all sufficiently small values of t, are the points of a *branch* of the curve which are in the neighbourhood of (a, b). The branch does not cease to exist when t is so large that the power series for y fails to converge, but may be *continued*, by starting afresh from a new origin lying on the branch, instead of (a, b). It will further be proved that *all* values of x, y satisfying the equation $f(x, y)=0$, in which x is sufficiently near to a, are given by a finite number of pairs of expressions, for x and y, of the same form as the pair above taken, the sum of the values of r which arise in all these pairs being n, the order in which y enters into the equation $f(x, y)=0$. For points of the categories (ii), (iii), (iv), named above, there are similar pairs of expressions, respectively for (x, η), (ξ, y), (ξ, η).

Two simple examples may be given, to explain these statements:

(a) For the curve represented by $xy^2=(x-1)^3$, when x is near to 1, we find at once, with $x=1+t^2$, that $y=t^3-\frac{1}{2}t^5+\frac{3}{8}t^7+\ldots$, the two existing values of y being both included in this by change of t into $-t$; we say then that at $x=1$, $y=0$ there is only one place. When x is near to 0, by $x=t_1{}^2$ and $\eta=1/y$, we find $\eta=it_1+\frac{3}{2}it_1{}^3+\frac{15}{8}it_1{}^5+\ldots$, there being again only one place. When x is infinite, and therefore also y is infinite, putting $x=1/\xi$, $y=1/\eta$, and $\xi=t_2$, there are two expansions of η when ξ is small,

$$\eta=t_2+\tfrac{3}{2}t_2{}^2+\tfrac{15}{8}t_2{}^3+\ldots, \quad \eta=-(t_2+\tfrac{3}{2}t_2{}^2+\tfrac{15}{8}t_2{}^3+\ldots),$$

which do not pass continuously into one another by continuous variation of t_2; they do not belong to a single cycle, and there are *two* places, at this single point of the curve.

(b) For the curve represented by $x^3y^2=(x-1)^3$, when x is infinite y is finite, and we put only $\xi=1/x$, and thence $y^2=(1-\xi)^3$; near $\xi=0$ there are two distinct expansions for y, and there are two places as this point of the curve.

Weierstrass's preparatory theorem. To prove the general result we enter into the proof of a theorem stated by Weierstrass (*Werke*, II, p. 135). Let $A_0+A_1y+A_2y^2+\ldots$ be a power series in y, and $A_0, A_1, A_2, \ldots$ be power series in x, so that the whole is a power series in x and y, which is supposed to converge for all sufficiently small values of x and y. It is further supposed that the function represented by this series vanishes for $x=0$, $y=0$. It is necessary then that A_0 vanish for $x=0$; for the sake of generality it is supposed that also $A_1, A_2, \dots, A_{n-1}$ vanish for $x=0$, but that A_n does not. It is then proved that the given power series may be regarded as a product ϖU, where U is a power series in x and y, converging for sufficiently small values of these, but *not vanishing for* $x=0$, $y=0$, while ϖ is of the form $\varpi=y^n+p_1y^{n-1}+\dots+p_n$, wherein $p_1, p_2, \dots, p_n$ are power series in x, converging for sufficiently small values of x, and all vanishing for $x=0$. In the application of the theorem which we make the power series is only a polynomial in x and y, namely $f(x, y)$, where $f(x, y)=0$ is the curve considered; and the consequence of the theorem is that, in considering the small values of x and y which satisfy the equation $f(x, y)=0$, where $f(0, 0)=0$, it is sufficient to consider only the factor ϖ; the values of y satisfying the equation $f(x, y)=0$, which reduce to zero when $x=0$, are then n in number, and given by $\varpi=0$. The particular case when $n=1$, the equation $f(x, y)=0$ being then of the form

$$ax+y+(x, y)_2+(x, y)_3+\dots=0,$$

where $(x, y)_r$ is a homogeneous polynomial of order r in x and y, is familiar; the only value of y satisfying this equation which reduces to zero when x vanishes is given by a power series $y=-ax+bx^2+\dots$.

In general, under the hypothesis made, the power series considered, which we may denote by $f(x, y)$, is of the form

$$f(x, y)=x\,(B_0+B_1y+\dots+B_{n-1}y^{n-1})+(C+xB_n)y^n+A_{n+1}y^{n+1}+\dots,$$

where $B_0, B_1, \dots, B_n, A_{n+1}, \dots$ are converging power series in x, *but C is a non-vanishing constant.* We denote $f(0, y)$ by ϕ, and $\phi-f(x, y)$ by ψ, so that ψ vanishes when $x=0$, for all values of y. The series ϕ will be of the form $\phi=Cy^n+Dy^{n+1}+\dots$, in which $C, D, \dots$ are constants, and C is not zero. It may be assumed, as familiar, that the power series $f(x, y)$, converging for all sufficiently small values of x and y, by hypothesis, converges absolutely for sufficiently small values of these, and, if necessary, may be rearranged; and, further, that, for a power series in y, converging for all values near $y=0$, which vanishes at $y=0$, there is a finite neighbourhood of $y=0$ within which no other zero of the series exists. Thus, denoting the absolute value of y by $|\,y\,|$, as usual, we may suppose a (real positive) σ chosen to satisfy the two conditions, (a) that ϕ does not vanish for $0<|\,y\,|\leqslant\sigma$; (b) that $f(x, y)$ converges, if x be sufficiently small, for $|\,y\,|<\sigma$. Here ϕ is a function of y only, and ψ vanishes when $x=0$ for every (small) y; if, therefore, we take x small enough, and take care that y is not too small, we can suppose ψ less in absolute value than ϕ. Precisely, a real positive number σ_1 exists, with $\sigma_1 < \sigma$, and a real positive ρ, such that, for $|x| < \rho$, and $\sigma_1 < |y| < \sigma$, we have $|\psi| < |\phi|$. Whence, for a definite value of x, since $f(x, y) = \phi(1 - \psi/\phi)$, we have (putting f for $f(x, y)$)

$$\frac{1}{f}\frac{\partial f}{\partial y} = \frac{1}{\phi}\frac{d\phi}{dy} - \frac{\partial}{\partial y}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\left(\frac{\psi}{\phi}\right)^{\lambda};$$

here, since $1/\phi = 1/(Cy^n + Dy^{n+1} + \ldots)$, and $(\psi/\phi)^\lambda$ is expressible as a series of positive and negative powers of y, converging, with the definite x, for $\sigma_1 < |y| < \sigma$, while the series $\Sigma\lambda^{-1}(\psi/\phi)^\lambda$ is uniformly convergent, it follows, by a well-known theorem (Weierstrass, *Werke*, II, p. 205), that we can write the preceding equation in the form

$$\frac{1}{f}\frac{\partial f}{\partial y} = \frac{1}{\phi}\frac{d\phi}{dy} - \frac{\partial}{\partial y}\sum_{\mu=-\infty}^{\infty} G_\mu(x) y^\mu,$$

and, hence, in the form

$$\frac{1}{f}\frac{\partial f}{\partial y} = \frac{n}{y} - \frac{\partial}{\partial y}\sum_{\mu=-\infty}^{-1} G_\mu(x) y^\mu + G(y) - \frac{\partial}{\partial y}\sum_{\mu=0}^{\infty} G_\mu(x) y^\mu,$$

where $G_\mu(x)$ arises as a power series in x which vanishes for $x = 0$, and $G(y)$ is a power series in y.

Now let the number of values of y, less than σ in absolute value, which, for the definite value of x chosen, are zeros of $f(x, y)$, be m; as $f(x, y)$ is a power series in y, this number is finite. Denote these values of y, not necessarily all different, by $y_1, y_2, \ldots, y_m$; then, also for $|y| < \sigma$, the difference

$$\frac{1}{f}\frac{\partial f}{\partial y} - \left(\frac{1}{y - y_1} + \ldots + \frac{1}{y - y_m}\right)$$

is expressible as a power series in y; denote this by $\Phi(y)$. Thus, taking a value of y greater than all of $y_1, \ldots, y_m$ in absolute value, subject also to $\sigma_1 < |y| < \sigma$, we have

$$\frac{1}{f}\frac{\partial f}{\partial y} = \frac{m}{y} + \sum_{\nu=1}^{\infty} y^{-(\nu+1)}[y_1{}^\nu + \ldots + y_m{}^\nu] + \Phi(y).$$

This must agree with the previous expression of $f^{-1}\partial f/\partial y$ in powers of y; thus we have $m = n$, and $y_1{}^\nu + \ldots + y_m{}^\nu = \nu G_{-\nu}(x)$; defining then ϖ by $\varpi = (y - y_1)\ldots(y - y_n)$, $= y^n + p_1 y^{n-1} + \ldots + p_n$, and recalling an ordinary theorem of the theory of equations, we infer that $p_1 + G_{-1}(x) = 0$, $2p_2 + p_1 G_{-1}(x) + 2G_{-2}(x) = 0$, etc., whereby $p_1, p_2, \ldots$ are expressed as power series in x, vanishing for $x = 0$. Also we have

$$\frac{1}{f}\frac{\partial f}{\partial y} = \frac{1}{\varpi}\frac{\partial \varpi}{\partial y} + \Phi(y) = \frac{1}{\varpi}\frac{\partial \varpi}{\partial y} + G(y) - \frac{\partial}{\partial y}\sum_{\mu=0}^{\infty} G_\mu(x) y^\mu;$$

taking then U so that the right side is $\varpi^{-1}\partial\varpi/\partial y + U^{-1}\partial U/\partial y$, namely

$$U = A \exp\left[\int_0^y G(y)\,dy - \sum_{\mu=0}^{\infty} G_\mu(x) y^\mu\right],$$

where A is independent of y (for the definite value of x), we can infer the equation $f(x, y) = \varpi U$. The form of U is

$$U = A(\lambda_0 + \lambda_1 y + \lambda_2 y^2 + \ldots),$$

in which $\lambda_0, \lambda_1, \lambda_2, \ldots$ arise as power series in x, of which λ_0 reduces to unity for $x=0$; comparing coefficients of y^n in the identity $A_0 + A_1 y + A_2 y^2 + \ldots = A(p_n + p_{n-1} y + \ldots + p_1 y^{n-1} + y^n)(\lambda_0 + \lambda_1 y + \ldots)$, we see that A is given by $A = A_n(\lambda_0 + \lambda_1 p_1 + \ldots + \lambda_n p_n)^{-1}$; thus A is a power series in x, which reduces, when $x=0$, to the constant term C in the power series A_n. For the values of x and y for which the two series $G(y)$ and $\sum\limits_{\mu=0}^{\infty} G_\mu(x) y^\mu$ converge, the factor U does not vanish, as its exponential form shews.

Ex. For the curve $y^2 = x^4 + 2Ax^3y + Bx^2y^2 + 2Cxy^3 + y^4$, prove that
$$\varpi = [y - Ax^3 - (AB+C)x^5 - \ldots]^2 - [x^2 + \tfrac{1}{2}(A^2+B)x^4 + \ldots]^2.$$

The parametric expression of a branch of a curve. We consider now, for small values of x and y, the form of the solution of the equation $\varpi(x, y) = 0$, where $\varpi(x, y) = y^n + p_1 y^{n-1} + \ldots + p_n$, wherein $p_1, \ldots, p_n$ are power series in x, converging when x is sufficiently small, all vanishing for $x=0$. We suppose that $\varpi(x, y)$ is incapable of being written as a product

$$(y^m + q_1 y^{m-1} + \ldots + q_m)(y^h + r_1 y^{h-1} + \ldots + r_h),$$

in which $q_1, \ldots, q_m, r_1, \ldots, r_h$ are converging power series in x (which, as all the roots of $\varpi(x, y) = 0$ vanish when $x=0$, would all vanish for $x=0$); for instance $y^2 + x$ is so incapable. Otherwise we should deal with the factors separately.

Under this hypothesis, we shew that the n roots of $\varpi(x, y) = 0$ form a single cycle, being all given by one converging series, $y = a_1 t + a_2 t^2 + \ldots$, in the parameter t, defined by $x^n = t$, by allowing t to vary continuously in the neighbourhood of $t=0$. Naturally, complex values are allowed for t; so that we can pass continuously from t to ϵt, where $\epsilon = \exp(2\pi i/n)$, and thence to $\epsilon^2 t$, and so on up to $\epsilon^{n-1} t$, and thence back to t; thereby we obtain n values of y corresponding to any value of x which is near enough to $x=0$ to secure the convergence of the series.

For definiteness, let the Resultant of two integral polynomials in y, of respective degrees m and n, mean the determinant of $m+n$ rows and columns, whose rows consist first, for the first n rows, of n repetitions of the coefficients in the first polynomial, beginning in turn in the first, second, ..., n-th columns, the remaining elements being zeros, and then, for the remaining m rows, of m repetitions of the coefficients in the second polynomial, likewise beginning in turn in the first, second, ..., m-th columns, the other elements being zeros; for instance, for the polynomials $a_0 y^2 + a_1 y + a_2$, $b_0 y + b_1$, the rows of the determinant would be in turn $a_0, a_1, a_2; b_0, b_1, 0; 0, b_0, b_1$. Further, let the Discriminant of such a polynomial, of degree n in y, made homogeneous in y and z by inserting proper powers of z, so becoming $u(y,z)$, mean the Resultant of $\partial u/\partial y$ and $\partial u/\partial z$ (z being put equal to 1 after differentiation). The discriminant is a polynomial homogeneously of degree $2(n-1)$ in the coefficients of the original polynomial. Thus the discriminant of the polynomial in y which we have denoted by $\varpi(x, y)$ will be a homogeneous polynomial of degree $2(n-1)$ in the coefficients $1, p_1, \ldots, p_n$; it will thus be capable of being written as a convergent power series in x, when x is sufficiently small. This discriminant of $\varpi(x, y)$ we may denote by Δ. In general, as is known, the vanishing of the resultant of two polynomials in y is the necessary and sufficient condition that they should have a common factor, linear in y, whose coefficients would then be rational in the coefficients of the two polynomials; and the vanishing of the discriminant of a polynomial in y is the necessary and sufficient condition that the polynomial contain as factor the square of a linear polynomial in y, likewise having therefore coefficients rational in the coefficients of the polynomial. In our case, as $\varpi(x, y)$ is incapable of being written as a product of factors of its own form, it cannot have such a square factor. Thus the discriminant Δ does not vanish for all small values of x. But Δ vanishes for $x=0$, for which $\varpi(x, y)$ reduces to y^n. Thus Δ is a power series in x vanishing for $x=0$; and, by a property of power series already remarked, there is a neighbourhood of $x=0$ within which no other zeros of Δ are found. We suppose the (real and complex) values of x represented on a Euclidean plane, in the familiar way; the neighbourhood in question will then be defined by a circle whose centre is the point $x=0$. For clearness we may speak of the interior of this circle as the *domain* of the origin, $x=0$; it is supposed that this circle is so taken that the series Δ, and all the series $p_1, p_2, \ldots$ in ϖ, converge therein.

Now, let x_0 be a value of x within this domain, not at $x=0$; and let y_0 be one of the n roots of $\varpi(x_0, y)=0$. The polynomial of order $(n-1)$ in y, $\partial\varpi(x, y)/\partial y$, or, say, $\varpi_y'(x, y)$, will not vanish for $x=x_0$, $y=y_0$, since the two equations $\varpi(x_0, y_0)=0$, $\varpi_y'(x_0, y_0)=0$ would involve that Δ was zero for $x=x_0$, which is contrary to the definition of the domain. Next consider values $x=x_0+\xi$, $y=y_0+\eta$, which are near, respectively, to x_0 and y_0, $x_0+\xi$ being within the domain, and denote the values of $\partial\varpi(x, y)/\partial x$ and $\partial\varpi(x, y)/\partial y$, for $x=x_0$, $y=y_0$, respectively by A and B. The substitution of these values of x and y in $\varpi(x, y)$ then gives a result which, if ξ, η be sufficiently small, can be arranged in the form, a power series in ξ and η, $A\xi+B\eta+(\xi, \eta)_2+(\xi, \eta)_3+\ldots$. It is then a simple application of the general result proved earlier that, since B is not zero, there is, for small ξ, one and only one value of η, satisfying the equation $\varpi(x_0+\xi, y_0+\eta)=0$, which reduces to zero when $\xi=0$, this being expressible as a convergent power series in ξ. We may denote this solution of $\varpi(x, y)=0$, reducing to y_0 when x approaches x_0, by $y=y_0+C_1(x-x_0)+C_2(x-x_0)^2+\ldots$. If y_1 be another root of $\varpi(x_0, y)=0$, there will similarly be a unique solution of $\varpi(x, y)=0$ which reduces to y_1 when $x=x_0$, likewise expressible by a convergent power series in $(x-x_0)$. Of the n series in $x-x_0$, so obtainable, representing the n roots of $\varpi(x, y)=0$ when x is near to x_0, let r be the least radius of convergence. This number r is not zero; it varies (presumably) as x_0 varies in the domain, and may be regarded as a real function of the real and imaginary parts of x_0. We desire to prove that as x_0 varies, this function r does not approach indefinitely near to zero, but has a lower bound greater than zero. The definition of r has assumed that x_0 is not at $x=0$; we suppose then a small circle put about $x=0$, and suppose x_0 to be without this. We likewise suppose another circle taken near to but within the original circle bounding the domain, also with centre at $x=0$. Then we restrict x_0 to the annulus, lying within the domain, which is bounded by these concentric circles.

We can then see that r varies continuously as x_0 varies within the domain. For, if x_0' be a point, within the annulus, in the immediate neighbourhood of x_0, while we could obtain the power series in $x-x_0'$, which give the roots of $\varpi(x, y)=0$ when x is near to x_0', directly from $\varpi(x, y)$, as we obtained the power series in $x-x_0$, we can also obtain them by the known process of analytical continuation from the power series in $x-x_0$, previously obtained, provided x_0' be sufficiently near to x_0. Hence, from the method of this continuation, if r' be the least radius of convergence for the n series in $x-x_0'$, we have $r' \geqslant r-|x_0'-x_0|$. If, however, $|x_0'-x_0|<r'$, we could equally begin with x_0' instead of x_0, and obtain the power series in $x-x_0$ by continuation from those in $x-x_0'$; thus we equally have $r \geqslant r'-|x_0-x_0'|$, and the inequalities

$$r-|x_0'-x_0| \leqslant r' \leqslant r+|x_0-x_0'|$$

shew that r varies continuously with x_0. As a continuous function of two real variables, in a limited domain, actually reaches its lower bound, and r is not zero at any point within the annulus, it follows that the lower bound of r is greater than zero. There exists therefore a real positive number ρ, greater than zero, such that for every point x_0, within the annulus, the radius of convergence of the series in $x-x_0$, which represents any root of $\varpi(x_0, y)=0$, is greater than ρ.

We start then with a particular x_0, within the annulus, and form the series in $x-x_0$ which represents the root of $\varpi(x_0, y)=0$ which reduces to y_0 when $x=x_0$, where y_0 is any of the n values satisfying $\varpi(x_0, y_0)=0$. Then, within the circle of convergence of this series, and within the annulus, we take a point x_1, and form the analytic continuation of this particular series in powers of $x-x_1$. Since, for this derived series, the radius of convergence is $>\rho$, we can, by suitable choice of x_1, near to the circumference of the circle of convergence of the original series, secure that the circle of convergence of the derived series in $x-x_1$ contains a region lying outside the original circle of convergence. In this region, and within the annulus, we can take a point x_2, and transform the series in $x-x_1$ into a series in $x-x_2$, in such a way, by proper choice of x_2, that the derived series in $x-x_2$ contains a region outside the two former circles of convergence. It is thus clear that the original series in $x-x_0$ can be continued completely round the origin $x=0$, the various centres $x_0, x_1, x_2, \ldots$, of the successive series, lying on a curve enclosing $x=0$, which, after a finite number of steps, again reaches x_0. We thus obtain another series in $x-x_0$, giving a root of $\varpi(x_0, y)=0$, obtained, as explained, by continuation of the original root, which was constructed to reduce to y_0 for $x=x_0$. This second root of $\varpi(x_0, y)=0$ may not reduce to y_0, but to another root of $\varpi(x_0, y)=0$, when $x=x_0$. In such case we again make the circuit of $x=0$, and find a third root of $\varpi(x_0, y)=0$, as a power series in $x-x_0$. As the number of roots of $\varpi(x_0, y)=0$ is finite, we must, after, say μ circuits of $x=0$, finally reach again the original root which reduces to y_0 for $x=x_0$.

We then take a variable t, similarly representable on a new plane of complex variables, such that $x=t^\mu$. As t makes a circuit about $t=0$ in its own plane, the point x will make a circuit about $x=0$ in the plane of x; but an unclosed path for t, by which it changes to $te^{2\pi i/\mu}$, will correspond to a complete circuit for x; and the complete circuit of t, about $t=0$, will correspond to μ complete circuits by x, described in succession. Thus, if, instead of $\varpi(x, y)=0$, we consider the equation $\varpi(t^\mu, y)=0$, the root of the equation $\varpi(x_0, y)=0$ which we have discussed, which reproduces its value after μ circuits about $x=0$, will correspond to a root of $\varpi(t_0{}^\mu, y)=0$, where $x_0=t_0{}^\mu$, which reproduces itself by continuation, after one circuit of $t=0$ in the plane of t. This root, being developable about every point of the annulus in the t-plane which corresponds to the annulus taken in the x-plane, as is clear from what has been said, is representable*, in a form valid for the whole of the annulus in the t-plane, by a series $\sum_{h=-\infty}^{\infty} c_h t^h$. As, however, all the roots of $\varpi(x, y)=0$ tend to zero when x tends to zero, the series can only contain terms for which $h>0$, and the root is representable over the whole of the t-annulus by a power series $y=c_1t+c_2t^2+\dots$. Considering this root again in the annulus in the x-plane, we obtain $\mu-1$ roots, into which it can successively be continued by circuit of $x=0$, by replacing t in this series successively by $\epsilon t, \epsilon^2 t, \dots, \epsilon^{\mu-1}t$, where $\epsilon=\exp(2\pi i/\mu)$.

---

* Weierstrass, *Werke*, I, p. 51. For the history of Laurent's theorem, cf. Mittag-Leffler, *Acta Math.* XXXIX, 1923, p. 34.

If then these roots, regarded as depending on x, be denoted by $\eta_1, \dots, \eta_\mu$, the product $(y-\eta_1)\dots(y-\eta_\mu)$ will be of the form $y^\mu+q_1y^{\mu-1}+\dots+q_\mu$, where every coefficient q, symmetrical in $x^{1/\mu}, \epsilon x^{1/\mu}, \dots, \epsilon^{\mu-1}x^{1/\mu}$, is expressible by integral powers only of x, and arises as a power series; thus every coefficient q is a power series in x; and, as all the roots of $\varpi(x, y)=0$ vanish when $x=0$, this power series vanishes for $x=0$. The function $y^\mu+q_1y^{\mu-1}+\dots+q_\mu$ is thus a factor of $\varpi(x, y)$, which, however, by hypothesis, has no such factor, unless $\mu=n$. Thus, this is the case, and all the roots of $\varpi(x, y)=0$ form a single cycle, and are all derivable from a single series, $y=c_1t+c_2t^2+\dots$, where $x=t^n$, by continuous variation of t. The proof of the statements made at starting is thus completed.

Supplementary remarks. The expression of the points of a branch of a curve in terms of a parameter is ascribed to Puiseux (*Journ. de Math.* XV, 1850). Some further remarks, without complete proof, and suggestions for alternative methods of reaching the result, may be made:

(*a*) In the expressions $x=a+t^r$, $y=b+c_1t+c_2t^2+\dots$, it is clear from the proof we have given that, to any values of x, y, in sufficient nearness to a, b, respectively, there corresponds only a single value of t. And it is not necessary for this that the exponent of the lowest power of t which is actually present in the series for $y-b$, should be unity, or even be prime to the exponent r. For instance, for the curve given by the equation $x^2-2xy^2+y^4-4m^2x^3y-m^4x^5=0$, there is only one place at $x=0$, $y=0$, with a cycle of four values, expressed by $x=t^4$, $y=t^2+mt^5$; and t is actually given by

$$t=(-x+y^2+m^2x^2y)/mx(2y+m^2x^2).$$

It is clear too that, instead of t, we may use any other parameter τ which is expressible by a convergent power series in t, of the form $\tau=At+Bt^2+\dots$, provided the coefficient A be not zero; then both x and y appear as expressed by a power series in τ.

(*b*) The number of pairs of values of a, b satisfying the equation $f(x, y)=0$ for which, in the expression

$$x=a+t^r, \quad y=b+c_1t+c_2t^2+\dots,$$

the value of r is greater than unity, is necessarily finite. For all such pairs equally satisfy the equation $\partial f/\partial y=0$, and, by elimination of y between this equation and the equation of the curve, only a finite number of values of x arises. If we take account also, as is proper, of the associated equations $f(x,\eta^{-1})=0, f(\xi^{-1},y)=0, f(\xi^{-1},\eta^{-1})=0$, as explained above, when this is necessary, the number of places for which $r > 1$ is still finite. It is convenient to have a name for the number $r-1$, at any place, when it is not zero; it may perhaps, for reasons which appear below, be called the *winding index* at the place. The sum of the values of this index, for $f(x, y)=0$ and the associated equations such as $f(x, \eta^{-1})=0$, etc., at the finite number of places where this index is not zero, will be found to be of great importance.

(*c*) The number of pairs of expressions

$$x=a+t^r, \quad y=b+c_1t+c_2t^2+\dots,$$

with $r=1$ and $r>1$, taken with the analogous expressions arising for the associated equations such as $f(x, \eta^{-1})=0$, which are *necessary*, in order to give every pair of values of x and y (or of (x, η), (ξ, y), (ξ, η)) which satisfy the equation of the curve (or its associated equations), *is only finite*. This may be seen by regarding the complex variable x as represented upon a Riemann sphere (replacing the plane on which x is represented); every one of the expressions in question has then a region of existence on this sphere of which the radius has a lower bound not infinitesimally small.

(*d*) Moreover, all the pairs of expressions such as those in (*c*), which are necessary to represent all pairs of associated values of x and y satisfying the equation of the curve, are derivable by analytic continuation from any one such pair of expressions (provided we may assume all the coefficients in this one expression to be known). Thus any one such pair of expressions theoretically contains the equation of the curve. For this however the condition that the equation $f(x, y)=0$ is incapable of being obtained by multiplication of other such equations, equally rational in x and y, or, that the curve $f(x, y)=0$ is irreducible, is fundamental.

(*e*) It may be proved directly that, for the neighbourhood of a simple point $(a, b, c, \dots)$ on an algebraic curve in space in which the (non-homogeneous) coordinates are $x, y, z, \dots$, the values of $y-b, z-c, \dots$ are expressible as converging power series in $x-a$ (if x be chosen appropriately among the coordinates). If therefore we assume, from Chapter II above, that any given curve can be put in (1, 1) birational correspondence with a non-singular curve, it follows that any place of the given curve can be represented by power series in a parameter; and further that, for this parameter we may choose one which is rationally expressible by the coordinates of the point of the given curve. This is evidently the simplest way in which the general theorem may be reached. But also, conversely, it appears at once on consideration that, in a birational transformation between two curves, a branch of one necessarily changes into a single branch of the other, and consequently a place of one into a single place of the other. Thus, in the birational transformation of a curve into one without multiple points, the number of points of the latter curve which correspond to a multiple point of the original, is the number of places which exist at this multiple point.

(*f*) But the object of representing the neighbourhood of a particular multiple point of a plane curve by an aggregate of power series would also be achieved if any transformation, birationally reversible, were found which replaced this point by a number of simple points on the transformed curve, even if thereby new multiple points were introduced not corresponding to the neighbourhood of the original multiple point. There exists an extensive theory of this possibility, in which the transformations employed are birational in the whole plane (so-called Cremona transformations); and this suffices to establish the general theorem referred to. The theory was first elaborated in detail by Noether, using a succession of quadratic transformations to build up the necessary Cremona transformation. In regard to the theory in question, which will not be fully treated in this volume, the reader may consult: The Brill-Noether *Bericht*; *Deut. Math. Ver.* 1894, p. 377, etc. (cf. Noether, *Math. Ann.* IX, 1876; and Segre, *Ann. d. Mat.* XXV, 1897, p. 1); also Segre and Castelnuovo, *Atti...Torino*, XXXVI, 1901, pp. 645 and 861; and, thereto Alexander, *Trans. Amer. Math. Soc.* XVII, 1916, p. 295; and further, the very full account in Enriques-Chisini, *Teoria geometrica*, II, with the bibliography, pp. 535 ff. As a simple example of the application of the method we may quote the case of the curve represented by $y^{2p}=(x, y)_{2p+2}$, for which the neighbourhood of $x=0$, $y=0$ may be discussed by making the transformation $x=\xi\eta^{p-1}$, $y=\xi\eta^{p}$, reversible by $\xi=x^p/y^{p-1}$ and $\eta=y/x$, whereby the curve becomes $\eta^2=\xi^2(1, \eta)_{2p+2}$, with the two branches of a double point at $\xi=0$, $\eta=0$.

(*g*) It follows from the theory given here that the number of *places* existing at a multiple point of a plane curve is the same as the number of irreducible factors of the function denoted above, in the text, by $\varpi(x, y)$. Indications of real value, giving also the effective terms in the relative factor, to the first approximation, are very easily found by use of Newton's polygon. As a simple example, consider the curve represented by the equation

$$y^3+y^2u_2+yu_4+u_6=0,$$

where u_r is homogeneous of order r in x and y. The Newton polygon indicates at once that there are three (branches) places for this curve at $x=0$, $y=0$, each given to the first approximation by an equation of the form $y=mx^2$; namely, there are three ordinary branches with a common tangent. The multiple point is described in the Noether theory, referred to under (f), as consisting of a triple point, and a further triple point in the immediate neighbourhood of this. In regard to the use of Newton's polygon reference may be made to a paper, *Trans. Camb. Phil. Soc.* XV, Part IV, Oct. 1893, p. 403.

Ex. The transformation $\xi/xy^{n-1}=\eta/y^n=\zeta/[zy^{n-1}+(x, y)_n]$ is uniquely reversible, leading to $x/\xi\eta^{n-1}=y/\eta^n=z/[\zeta\eta^{n-1}-(\xi, \eta)_n]$. The curve $y^{n-1}+(x, y)_n=0$ has one place at $x=0$, $y=0$, representable by a pair $x=t^{n-1}$, $y=mt^n+\dots$; the curve is in (1, 1) correspondence with a curve in space $[n+1]$ obtainable by a prime section of the cone which joins an arbitrary point, O, to the rational normal curve given by

$$y_0/\theta^n=y_1/\theta^{n-1}=\dots=y_n,$$

the single place at $x=0$, $y=0$ corresponding to a simple point on the line joining O to the point $\theta=\infty$ of the rational curve. Any two curves $axy^{n-1}+by^n+c[zy^{n-1}+(x, y)_n]=0$, for different values of a, b, c, have, beside a single variable intersection, common points at $x=0$, $y=0$ which are described in Noether's phraseology as consisting of a $(n-1)$-ple point, followed by $(2n-2)$ simple points, on a branch of order $(n-1)$, in successive neighbourhoods (Segre, *Atti...Torino*, XXXVI, 1901). In fact $n^2=(n-1)^2+2n-2+1$.

General theorem for infinities of a rational function. We pass now to make application of the parametric expression we have established, to the theory of the rational functions of x and y, these variables being supposed to be connected by the equation of the fundamental curve, $f(x, y)=0$. These functions will be briefly referred to as rational functions on the curve. For clearness, we repeat that, in speaking of the places belonging to the equation $f(x, y)=0$, we mean not only those for whose neighbourhood there is an expression of the form $x-a=t^r$, $y-b=c_1t+c_2t^2+\dots$, but also those, derived from the associated equations, for which, in this expression, $x-a$ is to be replaced by $1/x$, or $y-b$ replaced by $1/y$, or both.

A function, explicitly rational in x and y (cf. the Note at the end of the chapter), is expressible in the neighbourhood of any place belonging to the curve $f(x, y)=0$, after substitution for x and y of the appropriate series in the parameter t, in the form $t^\lambda P(t)$, where $P(t)$ denotes a power series in t, not vanishing when t is zero, and λ is an integer, which may be positive, zero, or negative. If λ is positive, the rational function is said to vanish to order λ at this place; if λ is negative, equal to $-\mu$, the function is said to have a *pole* of order μ at this place. In general λ will be zero; it will indeed appear that there is only a finite number of places for which λ is >0, and only a finite number at which $\mu>0$; and that the sum of the values of λ at the former places (often called the number of zeros of the function) is equal to the sum of the values of μ at the latter places (often called the number of poles of the function). Either of these sums is then called the *order of the rational function*. Further, if the function be denoted by Φ, and C be an arbitrary constant, the difference $\Phi - C$, at a place which is not a pole of Φ, may vanish to a certain order; for brevity, this is often spoken of as the number of times for which Φ is equal to C at this place. It will appear that the sum of the numbers of times for which Φ is equal to C, at all the places where this happens (including, if necessary, those for which x, or y, is infinite), is equal to the order of the function Φ.

In order to prove these results, and others of still greater importance, we proceed to prove a theorem which we express in the form

$$\left[\Phi \frac{dx}{dt}\right]_{t^{-1}} = 0.$$

For the neighbourhood of any place satisfying $f(x, y) = 0$ we suppose Φ expressed, in the manner explained, in terms of the parameter t, and the expression multiplied by dx/dt; where $x = a + t^r$, $y = b + c_1 t + c_2 t^2 + \dots$, this dx/dt will be rt^{r-1}; but, where $1/x = t^{r_1}$, and $y - b$, or $1/y$, is $c_1 t + c_2 t^2 + \dots$, this dx/dt will be $-r_1 t^{-r_1 - 1}$. In the product $\Phi\, dx/dt$, so formed, arranged as an ascending series in t, there will be only a finite number of negative powers of t; if the power t^{-1} occur, we take the coefficient of this. It will appear that there is only a finite number of places where negative powers of t arise, in this expansion of $\Phi\, dx/dt$. The theorem expresses that the sum of the coefficients of t^{-1}, at all such places, is zero.

We first remark a corresponding, but much simpler, theorem, for a rational function, ϕ, of a *single independent* variable x. For convenience we express this in the same form $[\phi\, dx/dt]_{t^{-1}} = 0$. Any such function, ϕ, is necessarily of the form

$$Ax^m + Bx^{m-1} + \dots + Mx + N + \sum_{i=1}^{k} [P_i (x - a_i)^{-n_i} \\ + Q_i (x - a_i)^{-n_i + 1} + \dots + R_i (x - a_i)^{-1}],$$

where k is the number of places at which the function becomes infinite. In the neighbourhood of a finite value of x, say $x = c$, we can put $x = c + t$, and expand the function in ascending powers of t. For general values of c, no negative powers of t will arise; for $c = a_i$, there will be the negative powers $P_i t^{-n_i} + Q_i t^{-n_i + 1} + \dots + R_i t^{-1}$, in which the coefficient of t^{-1} is R_i. In the neighbourhood of $x = \infty$, we can put $x = 1/t$, and similarly develop the function in ascending powers of t; the result will again be a power series with the addition of terms involving negative powers of t, the number of such negative powers being only finite; in this case, if we multiply the function by dx/dt, or $-1/t^2$, the coefficient of t^{-1} in the product will be the negative of the coefficient of t in the function ϕ itself, namely will be $-\Sigma R_i$. Thus, it appears that the sum of the coefficients of t^{-1} in the expansions of the product $\phi\, dx/dt$, for all the places where this contains negative powers, is zero. This is what is expressed by the equation put down, for this case of a rational function of a single variable.

Now take the theorem for the rational function $\Phi(x, y)$, where $f(x, y)=0$. Consider a place at which $x=a+t^r$, and $y-b$, or $1/y$, is a power series in t. Denoting the values of y, belonging to the cycle, when x is near to a, by $y_1, \ldots, y_r$, form the sum

$$\Phi(x, y_1)+\ldots+\Phi(x, y_r),$$

regarded as a function of t. Each constituent of this sum is a series of integral powers of t, with only a finite number of negative powers; if ϵ denote $\exp(2\pi i/r)$, the series are the same in the respective quantities $t, \epsilon t, \epsilon^2 t, \ldots, \epsilon^{r-1}t$. Thus the sum is a series of integral powers of t^r or $x-a$. In this series, the coefficient of $(x-a)^{-1}$, or t^{-r}, if not zero, arises by equal contributions from each constituent of the sum, and is thus equal to the coefficient of t^{-1} in the series of powers of t given by $\Phi(x, y)rt^{r-1}$, or $\Phi(x, y)dx/dt$, where y denotes any one of $y_1, \ldots, y_r$. Next consider a place arising for an infinite value of x, for the neighbourhood of which $x=t^{-k}$, the appropriate corresponding values of y, as series in t, being $y_1, \ldots, y_k$ (which may become infinite for $t=0$). The sum $\Phi(x, y_1)+\ldots+\Phi(x, y_k)$ is then, similarly, a series of integral ascending powers of x^{-1}, in which the term in t^k, or x^{-1}, if present, arises by equal contributions from each constituent of the sum. This is then equal to the coefficient of t^{-1} in $\Phi(x, y)kt^{-k-1}$, where y denotes any one of $y_1, \ldots, y_k$. The *negative* coefficient of x^{-1} in the sum is thus equal to the coefficient of t^{-1} in $\Phi(x, y)dx/dt$.

Lastly consider, for any value of x, finite or infinite, the sum $\phi(x)$ defined by $\phi(x)=\Phi(x, y_1)+\ldots+\Phi(x, y_n)$, where $y_1, \ldots, y_n$ are all the values of y satisfying the equation $f(x, y)=0$, for the value of x taken. As a symmetrical function of these, $\phi(x)$ is expressible as a rational function of x only. For a finite value of x, say $x=a$, if the sum be regarded as an aggregate of partial sums, each extended to all the values of y belonging to one place, and forming a cycle, the coefficient of $(x-a)^{-1}$ in the complete sum is, by what we have shewn, equal to the sum of the coefficients of t^{-1} in $\Phi(x, y)dx/dt$, evaluated in turn at the various places arising for this value of x, the symbol t denoting in turn the parameters at these places. Likewise for infinite x; the negative coefficient of x^{-1} in $\phi(x)$ is equal to the sum of the coefficients of t^{-1} in $\Phi(x, y)dx/dt$, at the various places arising for $x=\infty$. As the number of values of x, giving contributions, when we are considering the function $\phi(x)$, is finite, it is clear that the number of places providing a term to be taken into account is also finite for $\Phi(x, y)dx/dt$. And, from the theorem proved for $\phi(x)$, the sum of the coefficients of t^{-1}, at this finite number of places, in the appropriate expressions for

$\Phi(x, y)\,dx/dt$, evidently vanishes. Which is the theorem to be proved.

Order of a rational function on the curve. An immediate corollary is the theorem justifying the definition of the order of an algebraic function which we have given. Let $\Psi(x, y)$ be any rational function, and K any constant. Then the function

$$\Phi(x, y) = [\Psi(x, y) - K]^{-1}\frac{d}{dx}[\Psi(x, y) - K],$$

where d/dx means total differentiation, equal to

$$\partial/\partial x + (-\partial f/\partial x \div \partial f/\partial y)\,\partial/\partial y,$$

is a rational function. Also, at a place where $\Psi(x, y)$ has a pole of order μ, being expressible in the neighbourhood of this place in the form $\Psi(x, y) = t^{-\mu}P(t)$, where $P(t)$ is a power series not vanishing for $t = 0$, the expression $\Phi(x, y)\,dx/dt$, say $(\Psi - K)^{-1}d(\Psi - K)/dt$, has the form

$$\frac{d}{dt}\log(\Psi - K), \text{ or } \frac{d}{dt}\log[(P(t) - Kt^{\mu})/t^{\mu}];$$

and in this, as the power series $P(t)$ does not vanish for $t = 0$, the coefficient of t^{-1} is $-\mu$. Again, at a place where $\Psi(x, y) - K$ vanishes to order ρ, having an expression for the neighbourhood of this place, if the parameter here be also denoted by t, which we may write $t^{\rho}P(t)$, the coefficient of t^{-1} in $d\log(\Psi - K)/dt$ is ρ. Hence, by the theorem $[\Phi(x, y)\,dx/dt]_{t^{-1}} = 0$, we see that the sum of the values of μ which arise, for all finite and infinite values of x, is equal to the sum of the values of ρ; and this, therefore, is the same for all values of the constant K. This is the theorem in question, justifying the definition of the order of the rational function $\Psi(x, y)$. That the sum of these values of ρ is finite has been remarked; it is obvious also because the places where $\Psi(x, y) = 0$, subject to $f(x, y) = 0$, are obtainable by algebraic elimination, say of y, from these two equations, leading to an equation for x of finite order.

The theorem that a rational function assumes, on $f(x, y) = 0$, any assigned value at a number of places which is independent of the value, is, in its simpler cases at least, a well-known theorem for the intersection of curves. If the explicit expression of the function be $u(x, y)/v(x, y)$, where u, v are polynomials, the theorem is that the curve $u(x, y) - Kv(x, y) = 0$ intersects $f(x, y) = 0$ in a number of points which is independent of K. But it is clear that this familiar result may be difficult of interpretation when some of the intersections are at multiple points of $f(x, y) = 0$, or if, at an intersection which is ordinary on $f(x, y) = 0$, there is multiple contact. The intersections at infinity present no special difficulty if homogeneous coordinates be employed. The computation of the exact number of intersections, in any case of difficulty, can only be made by some method equivalent to that we have employed, with the precedent expression of the places of the curve in terms of a representative parameter.

Ex. 1. As a very simple illustration, suppose $f(x, y)$ to be given by $f(x, y)=y^2-4(x-a)(x-b)(x-c)$. On the curve $f(x, y)=0$ there is then a single place at each of $x=a$, $x=b$, $x=c$, $x=\infty$, there being a cycle of two values; for all other values of x there are two places. Thus the rational function $x-m$, for general values of the constant m, is infinite to the second order at $x=\infty$, and vanishes to the first order at each of the two places for which $x=m$. But these zeros coincide in one zero of the second order if $m=a$, or b, or c. Or again, the rational function y is infinite to the third order at the place given by $x=\infty$, and vanishes to the first order at each of the places given by $x=a$, or $x=b$, or $x=c$. Or, further, if z be not a, nor b, nor c, and s be either of the two values for which $f(z, s)=0$, the rational function, of x and y, $(y+s)/(x-z)$, is infinite to the first order at one (and only one) of the two places where $x=z$, and is infinite to the first order at $x=\infty$; its zeros are for the two values of x satisfying the quadratic equation

$$[(x-a)(x-b)(x-c)-(z-a)(z-b)(z-c)]/(x-z),$$

each of these values of x being associated with *one* of the two corresponding values of y which satisfy the equation $f(x, y)=0$. The function $(y+s)/(x-z)$ is thus of the second order.

Ex. 2. In further illustration of the order of a rational function, we may prove that a curve for which there exists a rational function of order 2 is of special character, being in fact hyperelliptic, see Chap. II; or is rational if there be a rational function of order 1. It is clear that if there exist a rational function of order 1, the explicit expression of this function being $u(x, y)$, then the elimination of y from the two equations $u(x, y)=\xi$, $f(x, y)=0$, must lead to an equation for x, giving only one value, and hence, as the process is rational, x may be expressed rationally in terms of ξ; and, similarly, so may y; and ξ by definition is rational in x and y. This is the definition of a rational curve.

If there exists a rational function $u(x, y)$, of order 2, the two equations $u(x, y)=\xi$, $f(x, y)=0$, in which ξ is a general value, must, on elimination of y, lead to a quadratic equation $x^2-2xr+s=0$, where r and s are rational in ξ, unless the two places, where $u(x, y)=\xi$, have the same value of x for all values of ξ. If this exception arises, these two places cannot also have the same value of y for all values of ξ, because then, taking x_0, y_0 arbitrarily, subject to $f(x_0, y_0)=0$, the function $u(x, y)-u(x_0, y_0)$ vanishes to the second order at the arbitrary place (x_0, y_0), and hence the differential coefficient of $u(x, y)$ vanishes at every place (x_0, y_0), or $u(x, y)$ is a constant. Thus we can use, instead of x, a coordinate $x+\lambda y$, where λ is constant; and, that done, we can make the statement that the elimination of y between $u(x, y)=\xi$, and $f(x, y)=0$, leads to a quadratic equation $x^2-2xr+s=0$, in which r^2-s vanishes only for particular values of ξ. Take now the rational function of x and y given by $\eta=x-r$, the value of ξ in r being expressed by x and y. Then, when ξ and η are given, we have an expression for x in terms of these, namely $x=\eta+r$, with also $\eta^2=r^2-s$, in which r^2-s is rational in ξ. Further, if (x_1, y_1) be one of the two places where $u(x, y)=\xi$, the equations $u(x_1, y)=\xi$, $f(x_1, y)=0$, must on elimination of x_1, lead to $y=y_1$. If $x_1=\eta+r$, the value of x at the other place at which $u(x, y)=\xi$ must be $x_2=-\eta+r$;

thus the equations $u(\eta+r, y)=\xi$, $f(\eta+r, y)=0$ lead to a rational expression of y in terms of ξ and η. There exists therefore an equation, $\eta^2=$ rational function of ξ, to which the original equation $f(x, y)=0$ is birationally equivalent. If this equation be $\eta^2=p(\xi)/q(\xi)$, in which $p(\xi)$, $q(\xi)$ are polynomials, by taking $\eta_1=\eta q(x)$, we obtain $\eta_1{}^2=p(\xi)q(\xi)$. If the polynomial $p(\xi)q(\xi)$ be of the first or second order, it is familiar that ξ and η_1 can be rationally expressed by a proper rational function of themselves, and hence the original curve $f(x, y)=0$ is rational; if $p(\xi)q(\xi)$ be a cubic or quartic polynomial, the original curve is elliptic (see Chap. I).

Ex. 3. More generally the following may also be stated, but without complete proof at this stage. Let ξ and η be two rational functions of x and y for the curve $f(x, y)=0$; let σ be the order of ξ. To any general value a there are σ places of $f(x, y)=0$ at which $\xi=a$, and, save for a finite number of values of a, these places will be different. Let $b_1, \ldots, b_\sigma$ be the values of η at these places. Then there are two possibilities; either these values $b_1, \ldots, b_\sigma$ are different from one another except for a finite number of values of a; or, these values consist, *for every value of* a, of q different values, each repeated p times, where $\sigma=pq$. In the former case there is a relation connecting ξ and η, rational in both, and irreducible, of order σ in η; in the latter case there is also such an irreducible rational relation, but of order q in η. In the former case, this relation may be obtained by eliminating x, y between $f(x, y)=0$, $u(x, y)=\xi$, $v(x, y)=\eta$, where u, v represent the explicit forms of ξ and η; in the latter case, the corresponding result of the elimination appears as the p-th power of the single irreducible relation connecting ξ and η. In the former case, the rational relation connecting ξ and η may be regarded as that of a curve in (1, 1) birational correspondence with $f(x, y)=0$; in the latter case, the new curve is such that to every point of it there correspond p points of $f(x, y)=0$; or, the new curve regarded as repeated p times may be considered as in (1, 1) correspondence with the original. The former case arises certainly if there be a single value of a for which the values $b_1, b_2, \ldots, b_\sigma$ are all different.

Abel's theorem for a sum of algebraic integrals. We now deduce, from our general theorem, another consequence of great importance.

Let $\Phi(x, y)$, $\Psi(x, y)$ be two rational functions, on $f(x, y)=0$. If C be a constant, the function $\Psi(x, y)-C$ has a definite number of zeros, say m zeros, for every value of C. As C is taken differently, the positions of these zeros on $f(x, y)=0$ change, and these zeros may thus be regarded as depending on C. We shall suppose that these zeros are all of the first order, for all the values of C which we consider, and that none of them coincides with any of the places where the expansion of $\Phi(x, y)\,dx/dt$ contains negative powers of t. We shall also put

$$I=\int_{(a)}^{(x)}\Phi(x, y)\,dx,$$

the integral being extended along a continuous range, on $f(x, y)=0$, given by a variation of x, and the consequent variation of an associated value of y satisfying $f(x, y)=0$, from a place (a), which is fixed, to the place (x). Further, denoting the m places where $\Psi(x, y)=C$ by $(x_1), \ldots, (x_m)$, and the m places where $\Psi(x, y)$ has a particular value C_0 by $(a_1), \ldots, (a_m)$, it will be convenient to denote the sum

$$\int_{(a_1)}^{(x_1)} \Phi(x,y)\,dx+\ldots+\int_{(a_m)}^{(x_m)} \Phi(x, y)\,dx$$

by a symbol, say Σ. Denoting $\Phi(x, y)$, $\Psi(x, y)$ by Φ, Ψ, the function $[\Psi(x, y)-C]^{-1}\Phi(x, y)$ is a rational function, and, by what we have proved, the equation expressed by

$$\left[\frac{1}{\Psi-C}\Phi\frac{dx}{dt}\right]_{t^{-1}}=0$$

holds. This we can write in the form

$$\left[\left(\frac{1}{\Psi-C}\right)\frac{dI}{dt}\right]_{t^{-1}}=-\left[\left(\frac{dI}{dt}\right)\frac{1}{\Psi-C}\right]_{t^{-1}},$$

in which the sum on the left refers to the m places where $\Psi(x, y)=C$, and only to these, and the sum on the right is for places where dI/dt contains negative powers of t. We have assumed that, for all the values of C considered, these two sets of places are different. For a point (x_i+dx_i, y_i+dy_i), near to a zero (x_i, y_i) of $\Psi(x, y)-C$, and on a particular branch of $f(x, y)=0$ issuing from (x_i, y_i), for which the points are given in terms of the parameter t, the value of $\Psi(x, y)$, expressed in powers of t, will be $C+tA+\ldots$, where A is not zero because we have assumed that the zeros of $\Psi(x, y)-C$ are simple for the values of C which we consider. We may denote this value by $C+dC$, regarding x_i and y_i as functions of C. The contribution to the sum on the left side, corresponding to this zero, will then be the value at (x_i, y_i) of

$$\frac{1}{A}\frac{dI}{dx_i}\cdot\frac{dx_i}{dt}, \text{ or } \frac{dI}{dx_i}\cdot\frac{dx_i}{dC},$$

and the whole value of the left side of the equation will be $d\Sigma/dC$, where Σ is defined above. On the right side, the places to be considered are those for which dI/dt contains negative powers of t; at all of these, by hypothesis, $\Psi(x, y)-C$ differs from zero, while dI/dt does not depend on C. For the neighbourhood of such a place, at which $\Psi(x, y)$ is Ψ_0, suppose, we have an expression of the form

$$-\frac{1}{\Psi-C}=\frac{d}{dC}\log(\Psi-C)=\frac{d}{dC}[\log(\Psi_0-C)+Pt+Qt^2+\ldots],$$

where P, Q, ... depend on C, and the series converges, uniformly for C, when t is small enough. Here

$$-\left[\left(\frac{dI}{dt}\right)\frac{1}{\Psi-C}\right]_{t^{-1}}=\frac{d}{dC}\left[\left(\frac{dI}{dt}\right)\log(\Psi-C)\right]_{t^{-1}}.$$

Thus, varying C from C_0 to C, we have the equation

$$\int_{(a_1)}^{(x_1)}\Phi(x, y)\,dx+\dots+\int_{(a_m)}^{(x_m)}\Phi(x, y)\,dx=\left[\left(\frac{dI}{dt}\right)\log\frac{\Psi-C}{\Psi-C_0}\right]_{t^{-1}}.$$

Here $(\Psi-C)/(\Psi-C_0)$ is a rational function, say $\Theta(x, y)$, whose zeros are the upper limits $(x_1), \dots, (x_m)$ of the integrals on the left, and whose poles are the lower limits, $(a_1), \dots, (a_m)$ of these integrals; the right side can then be written

$$\left[\left(\frac{dI}{dt}\right)\log\Theta(x, y)\right]_{t^{-1}}.$$

This is the result, equivalent to the famous theorem due to Abel, at which we desired to arrive. A simple example will serve to make the matter clear: Suppose the curve $f(x, y)=0$ is given by $y^2-4x^3+8ax+b=0$; that the function $\Phi(x, y)$ is x/y; and the function $\Theta(x, y)$ is $(y-mx-c)/(y-m_0x-c_0)$, where m, c, m_0, c_0 are constants of general values. This function $\Theta(x, y)$ evidently vanishes at three places, say (x_1, y_1), (x_2, y_2), (x_3, y_3), and has poles at three places, say (x_1^0, y_1^0), (x_2^0, y_2^0), (x_3^0, y_3^0), so that the sum on the left may be denoted by

$$\int_{(x_1^0)}^{(x_1)}\frac{x\,dx}{y}+\int_{(x_2^0)}^{(x_2)}\frac{x\,dx}{y}+\int_{(x_3^0)}^{(x_3)}\frac{x\,dx}{y}.$$

To evaluate the right side, notice that dI/dt, or $y^{-1}x\,dx/dt$, can only have negative powers of t in its expansion when either y vanishes or x becomes infinite. At a place, $x=e$, where y vanishes, the equation $f(x, y)=0$, of the form $y^2=4(x-e)(x^2+a'x+b')$, shews that the two values of y form a cycle, and we must put $x=e+t^2$, giving for y a form $y=2t(A+Bt+\dots)$, so that no negative power of t arises in $y^{-1}x\,dx/dt$. At $x=\infty$, the two values of y equally form a cycle, and the appropriate expression is to be found by putting $x=t^{-2}$, giving $y=-2t^{-3}(1-at^4-\dots)$, and hence $dI/dt=t^{-2}(1-at^4-\dots)$. Also, if, in Θ, $=(y-mx-c)/(y-m_0x-c_0)$, we substitute $x=t^{-2}$, and

$$y=-2t^{-3}(1-at^4-\dots),$$

we obtain

$$\begin{aligned}\log\Theta&=\log(1+\tfrac{1}{2}mt+\tfrac{1}{2}ct^3+\dots)/(1+\tfrac{1}{2}m_0t+\tfrac{1}{2}c_0t^3+\dots),\\&=\tfrac{1}{2}(m-m_0)t+\dots.\end{aligned}$$

Wherefore $[(dI/dt)\log\Theta]_{t^{-1}}=\frac{1}{2}(m-m_0)$. The three equations $y_i=mx_i+c$ give $m=(y_1-y_2)/(x_1-x_2)$; thus finally the sum of the three integrals is equal to $\frac{1}{2}(y_1-y_2)/(x_1-x_2)-\frac{1}{2}(y_1^0-y_2^0)/(x_1^0-x_2^0)$.

It can be shewn, as will appear, that, except when the curve $f(x, y)=0$ is rational, there are rational functions $\Phi(x, y)$, such that, for the integral $I=\int\Phi(x, y)\,dx$, there are no places at which dI/dt contains negative powers of t. Then the right side, in our general theorem, vanishes. The integral I is then spoken of as an everywhere finite integral; and for every one such integral we have an equation

$$\int_{(a_1)}^{(x_1)} \Phi(x, y)\,dx + \ldots + \int_{(a_m)}^{(x_m)} \Phi(x, y)\,dx = 0,$$

where $(a_1), \ldots, (a_m)$ are the places at which an arbitrary rational function $\Psi(x, y)$ takes any assigned value, for instance ∞, and $(x_1), \ldots, (x_m)$ are the places at which the same function takes any other assigned value, for instance 0; the paths of integration are defined by continuous variation of $\Psi(x, y)$ from one of these values to the other. It has been assumed for simplicity in the proof given here that, for the range considered, no two of the places, at which $\Psi(x, y)$ takes any of the values passed over, coincide with one another; but this appears evidently unnecessary for the final result obtained. It should be remarked however that, if the path of integration be not assigned, the integral $\int_{(a)}^{(x)} \Phi(x, y)\,dx$ is not definitely determined by the extreme points (a) and (x); it is ambiguous by additive multiples of certain constants, the so-called periods of the integral. The theory of such periods is considered in subsequent chapters (v, vi).

It can also be shewn that there exists a rational function $\Phi(x, y)$ such that, for the integral $I = \int\Phi(x, y)\,dx$, there are two places only at which dI/dt contains negative powers of t, each of the first order, so that the expansions at these places are of the forms $dI/dt = \alpha t^{-1} + P(t)$, $dI/dt = \beta t^{-1} + P_1(t)$, where $P(t)$, $P_1(t)$ denote power series in the (unconnected) parameters t belonging to these places. The theorem $[\Phi(x, y)\,dx/dt]_{t^{-1}} = 0$, or $[dI/dt]_{t^{-1}} = 0$ shews then that $\beta = -\alpha$; so that, multiplying $\Phi(x, y)$ by a constant, we may suppose $\alpha = 1$, $\beta = -1$. It will be shewn later how to construct such an integral, with its infinities at any two assigned places of $f(x, y) = 0$. If $\Phi(x, y)$ be such that these infinities are at (z) and (c), whereat we have respectively $dI/dt = t^{-1} + P(t)$, $dI/dt = -t^{-1} + P_1(t)$, and Θ denote an arbitrary rational function with simple zeros at $(x_1), \ldots, (x_m)$, and simple poles at $(a_1), \ldots, (a_m)$, the general formula leads to

$$\int_{(a_1)}^{(x_1)} \Phi(x, y)\,dx + \ldots + \int_{(a_m)}^{(x_m)} \Phi(x, y)\,dx = \log[\Theta(z)/\Theta(c)].$$

In this formula it is assumed that the paths of the m places under the integrals on the left do not pass through either of the places (z), (c); but, further, an appropriate integral multiple of $2\pi i$ must be added on the right. For the integral $\int_{(a)}^{(x)} \Phi(x, y)\,dx$, regarded as a function of (x), when (x) is near to (z), is of the form $\log t + P(t)$.

It can also be shewn that there exists a rational function $\Phi(x, y)$, so that for the resulting integral $I = \int\Phi(x, y)\,dx$, there is only one place, which may be arbitrarily taken, at which dI/dt has negative powers of t in its expansion, if this power be the second or a higher power; in particular, for which $dI/dt = -t^{-2} + P(t)$.

The possibilities and relations of the algebraic integrals for an elliptic curve (represented by a quartic plane curve with a point of self-contact) were investigated, after Abel had given the general theorem obtained above, by Legendre, whose work, extending over forty years, was entirely algebraical. The classification of the algebraic integrals, for any curve, by reference to the nature of their possible infinities, is essentially due to Riemann; the theory of the conduction of heat in a metal plate depends on a potential function which may either have two logarithmic infinities (for the case of a single source and a single sink) or one algebraic infinity (for the case of a single doublet); the two possible kinds of algebraic integrals with infinities which we have referred to above, have analogous behaviour; there is ground for believing that the analogy was in Riemann's mind. The exact, and simple, treatment of the theory which we have given (entirely different from Riemann's) is essentially due to Weierstrass.

A very important character of a curve, indeed the most important, is the number, p, of algebraic integrals $\int \Phi_i(x, y)\, dx$ existing for it which are everywhere finite in the sense we have explained, under the condition that there exists no linear equation, with constant coefficients, c_i, of the form $c_1\Phi_1(x, y) + \ldots + c_p\Phi_p(x, y) = 0$, satisfied in virtue of $f(x, y) = 0$. To emphasise this character we remark as follows: If an arbitrary rational function $\Psi(x, y)$, of order m, take the same value at the places $(x_1), \ldots, (x_m)$ of $f(x, y) = 0$, and the increments $dx_1, \ldots, dx_m$ refer to the passage to a consecutive set of such places, for the same function $\Psi(x, y)$, there exist, by what we have proved, p equations $\Phi_i(x_1, y_1)\, dx_1 + \ldots + \Phi_i(x_m, y_m)\, dx_m = 0$; it will be proved later, however, that, conversely, the existence of these p equations, with the p definite functions Φ_i, is sufficient to shew that there exists a rational function assuming the same value at all of them. Thus the problem of integrating this set of p total differential equations is that of finding the most general rational function of order m belonging to the given curve $f(x, y) = 0$. It will appear that when $m > p$ the number of arbitrary constants entering in the general integral of these equations is p.

Ex. 1. Suppose it be required to integrate the single differential equation

$$[u(x_1)]^{-\frac{2}{3}}\, dx_1 + [u(x_2)]^{-\frac{2}{3}}\, dx_2 + [u(x_3)]^{-\frac{2}{3}}\, dx_3 = 0,$$

where $u(x)$ denotes a given cubic polynomial in x. It can be proved that, for the cubic curve represented by the equation $y^3 - u(x) = 0$, there is one everywhere finite integral, namely $\int y^{-2}\, dx$, which is $\int [u(x)]^{-\frac{2}{3}}\, dx$. The problem of integrating this differential equation is therefore that of finding the general rational function of order 3 for this curve.

A particular function of order 3 is given by $(y - mx - n)/(y - m_0x - n_0)$, where m, n, m_0, n_0 are constants; and the intersections (x_1, y_1), (x_2, y_2), (x_3, y_3) of the cubic curve with any line are such that

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0.$$

This equation then is a particular integral of the given differential equation. If $u(x)=a+3bx+3cx^2+dx^3$, this equation, expressed in terms of x_1, x_2, x_3 only, which are the roots of an equation of the form $u(x)-(mx+n)^3=0$, is $(a+bp_1+cp_2+dp_3)^3=u(x_1)\,u(x_2)\,u(x_3)$, where $p_1=x_1+x_2+x_3$, $p_2=x_2x_3+x_3x_1+x_1x_2$, $p_3=x_1x_2x_3$. It is not necessary, however, that the points (x_1, y_1), (x_2, y_2), (x_3, y_3), of the curve $y^3=u(x)$, which satisfy the differential equation, should lie in line. For if $U=0$, $V=0$ represent two conics having three common intersections with the curve, the latter meeting the curve also in (a_1), (a_2), (a_3), while $U=0$ meets the curve also in (x_1), (x_2), (x_3), then U/V is a rational function of third order with (x_1), (x_2), (x_3) as its zeros. Conversely, it will be proved in the next chapter that any rational function with given poles, (a_1), (a_2), (a_3), is capable of expression, with help of the equation of the given cubic curve, in this form U/V. Thus, the most general set (x_1, y_1), (x_2, y_2), (x_3, y_3) satisfying the differential equation are the intersections with the given cubic curve of a conic having three fixed intersections therewith; of these three fixed points, however, two may be taken to be the same for all sets (x_1, y_1), (x_2, y_2), (x_3, y_3), and the coordinates of these may be regarded as absolutely fixed, the single arbitrary constant of the general integral of the differential equation being the position of the third base point of the conics. This will be clearer after the discussion of coresidual sets of points in the next chapter. The particular integral already found arises when the third base point of the conics is in line with the two absolutely fixed points. To express the general integral, suppose that the cubic curve is met by $x=0$ in $(0, p)$, $(0, \omega p)$, $(0, \omega^2 p)$; then the general conic through the two latter of these points has an equation

$$Ax^2+2Hxy+2Gx+y^2+yp+p^2=0.$$

The equation which expresses that this conic contains the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_0, y_0), of the cubic curve, is the general integral in question, with (x_0, y_0) as the arbitrary constant of integration. In particular when $(x_0, y_0)=(0, p)$, this integral, after removal of the factor $x_1x_2x_3$, reduces to the particular integral found above. For this example, the reader may compare Cayley, *Papers*, XII, p. 30.

Ex. 2. The curve given by $y^3=az^3+3bz^2x+3czx^2+dx^3$ has inflexions at its three intersections with $y=0$. If $ez+x$ be any one of the linear factors of the cubic polynomial on the right, prove that $y(ez+x)^{-1}$ and $z(ez+x)^{-1}$ are rational functions, on the cubic curve, of orders respectively 2 and 3. Hence reduce the equation of the cubic curve, birationally, to the form $\eta^2=\xi^3+m$.

Ex. 3. If $y^2=4x^3+ax+b$, shew that the general integral of the differential equation $y_1^{-1}dx_1+y_2^{-1}dx_2+y_3^{-1}dx_3=0$, is obtained by expressing that the conic $Ax^2+Bx+Cy+D=0$ contains the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_0, y_0), the last point furnishing the arbitrary constant of integration. Obtain the particular integral when $(x_0, y_0)=(\infty, \infty)$.

Ex. 4. For the curve represented by $y^2=u(x)$, where $u(x)$ is a polynomial in x of order $2p+1$, having only simple roots, shew that the complete integral of the p simultaneous differential equations

$$y_1^{-1}x_1^{i-1}dx_1+\dots+y_{p+1}^{-1}x_{p+1}^{i-1}dx_{p+1}=0, \qquad (i=1, 2, \dots, p),$$

is expressed by the p equations

$$F(\alpha_i)\left[\sum_{r=1}^{p+1} y_r/(\alpha_i-x_r)\,F'(x_r)\right]^2=C_i, \qquad (i=1, 2, \dots, p),$$

where $F(\theta)$ denotes $(\theta - x_1)\dots(\theta - x_{p+1})$, and $\alpha_1, \dots, \alpha_p$ are any p roots of $u(x)$, the $C_1, \dots, C_p$ being the arbitrary constants of the integration. The general everywhere finite integral for the given curve is in fact $\int \phi(x)\,dx/y$, where ϕ is an integral polynomial in x of order $p-1$.

Ex. 5. The expression of the branches of a plane curve in terms of a parameter, and the formula for Abel's theorem, may be extended to a curve in space.

As a very simple example, prove that, for the quartic curve given by the two equations $y^2 = x(1-x)$, $z^2 = x$, wherein x is regarded as the independent variable, there are: (i), four places on the curve for any value of x other than $x=0$, $x=1$, $x=\infty$; (ii), two places for $x=0$ (where the curve has a double point), expressed by the expansions

$$x=t^2,\quad y=t(1-\tfrac{1}{2}t^2-\dots),\quad z=t,\quad \text{and}\quad x=t^2,\quad y=-t(1-\tfrac{1}{2}t^2-\dots),\quad z=t;$$

(iii), two places for $x=1$, expressed by the expansions

$$x=1+t^2,\quad y=it+\dots,\quad z=1+\tfrac{1}{2}t^2+\dots,$$

and

$$x=1+t^2,\quad y=it+\dots,\quad z=-(1+\tfrac{1}{2}t^2+\dots);$$

(iv), two places for $x=\infty$, expressed by

$$x=t^{-2},\quad y=it^{-2}+\dots,\quad z=t^{-1}\quad \text{and}\quad x=t^{-2},\quad y=-it^{-2}+\dots,\quad z=t^{-1}.$$

Prove also that the integral $I = \frac{1}{2}\int y^{-1}z^{-1}dx$ is such that dI/dt contains a negative power of t on each of the two branches at $x=0$, but not elsewhere; further that the sum of the two integrals I, taken from the two places, other than $x=0$, $y=0$, $z=0$, where the plane $z+ax+by=0$ meets the curve, to the two places where $z+Ax+By=0$ meets it, is

$$\log[(1+B)(1-b)/(1-B)(1+b)].$$

Verify the identity

$$\int^{x_1}\frac{dx}{2x(1-x)^{\frac{1}{2}}} + \int^{x_2}\frac{dx}{2x(1-x)^{\frac{1}{2}}} = \log[(\phi + x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}})/(\phi - x_2^{\frac{1}{2}} + x_1^{\frac{1}{2}})] + \text{const.},$$

where $\phi = x_1^{\frac{1}{2}}(1-x_2)^{\frac{1}{2}} - x_2^{\frac{1}{2}}(1-x_1)^{\frac{1}{2}}$.

Note as to definition of a rational function on the curve. We have defined as a rational function on the curve $f(x, y)=0$ one which is given explicitly rationally in x and y. But, with the help of a fundamental theorem for functions of one independent variable, we can prove that if there be given a function R which has a perfectly definite value for every point of the curve (including those for which x, or y, is infinite), and on every branch at such point, being expressible in terms of the parameter appropriate to the branch, for the immediate neighbourhood of the origin of the branch, in the form $t^\lambda P(t)$, where $P(t)$ is a power series in t not vanishing for $t=0$, and λ is a finite integer, positive, zero, or negative, then this function is expressible explicitly as a rational function of x and y. In brief, the conditions are that the function be single valued on the curve, and analytic with no discontinuities other than poles. There is then a finite neighbourhood about every pole within which the function is finite.

For first, for the values of x in the neighbourhood of a finite value $x=a$, within which all the values of R are finite, save possibly at $x=a$ itself, let $y_1, y_2, \ldots, y_n$ be the values of y satisfying $f(x, y)=0$, not necessarily all different, and $R_1, \ldots, R_n$ be the values of the function R at the places (x, y_1), (x, y_2), $\ldots$, (x, y_n). Consider the sum $y_1{}^k R_1+\ldots+y_n{}^k R_n$, where k is a positive integer. This is the sum of portions, such as $y_1{}^k R_1+\ldots+y_r{}^k R_r$, where $y_1, \ldots, y_r$ belong to a cycle, there being a single place, on the curve $f(x, y)=0$, for which $x=a$, in the neighbourhood of which the values of x, y are given by $x=a+t^r$, $y_i=b+c_1t+c_2t^2+\ldots$, for $i=1, 2, \ldots, r$. The portion $y_1{}^k R_1+\ldots+y_r{}^k R_r$ is thence expressible by a series in t, involving only integer powers, and only a finite number of negative powers. As, however, $y_1, \ldots, y_r$ and the values $t, \epsilon t, \ldots, \epsilon^{r-1}t$, (where $\epsilon=\exp(2\pi i/r)$) enter symmetrically in this partial sum, the series will only involve the powers of t^r, namely will be a series in integer powers of $x-a$, with a finite number of negative powers. The whole sum $y_1{}^k R_1+\ldots+y_n{}^k R_n$ is thence a function of x, expressible by series of integer powers about every finite value of x, single valued, and having no singularities other than poles. For the neighbourhood of $x=\infty$, by a similar argument, this same sum is single valued, and has no singularity other than perhaps a pole. Hence by the theory of functions of a single variable, this sum has only a finite number of poles, and is a rational function of x. Thus we may put $y_1{}^k R_1+\ldots+y_n{}^k R_n=U_k(x)$. From the equations of this form for $k=0, 1, \ldots, n-1$, we can express R_1 as M_1/Δ, where Δ is the determinant of n rows whose general row is $y_1{}^k, y_2{}^k, \ldots, y_n{}^k$, and M_1 is the determinant obtainable from Δ by replacing the first column by $U_0(x), U_1(x), \ldots, U_{n-1}(x)$. Put in the form $M_1\Delta/\Delta^2$, the denominator may be expressed as a rational function of x only, and $M_1\Delta$ as a rational function of y_1 and x. This rational expression of R_1 by x and y_1, involves the like expression of R by x and y, which we desired to establish.

CHAPTER IV

THE GENUS OF A CURVE. FUNDAMENTALS OF THE THEORY OF LINEAR SERIES

Main objects of the chapter. The present chapter has the main purposes of establishing the definition of that most important number called the *genus* of a curve, which is invariant in all birational transformations of the curve; and of obtaining the fundamental results of the theory of linear series of sets of points upon the curve. Both these are intimately related to the theory of the rational functions existing on the curve, and to the theory of the algebraic integrals belonging thereto, especially the so-called everywhere finite integrals. But it is desired that the account given shall be logically sound, shall be brief, and shall be simple; these conditions seem best satisfied by employing, together, ideas from several different modes of approach, due mainly to Abel, Riemann, Weierstrass, and Brill and Noether. A further method, developed by Kronecker and by Dedekind, in extension of the arithmetical theory of integer numbers, is explained at length in a subsequent chapter (VII). Unless the contrary is stated, the curve considered is supposed to be a plane curve.

We distinguish provisionally between what we may call the *integral genus*, and the *arithmetical genus*; the former is easy to explain in general, the latter can be computed for a curve of sufficient simplicity; it is part of our task to shew that these are the same.

As regards the former, it has already been remarked that, if the fundamental curve $f(x, y)=0$ be not rational, there exists at least one algebraic integral attached thereto, $\int R(x, y)\,dx$, where $R(x, y)$ is rational in x and y, which is everywhere finite on the curve. It is clear that if the given curve be transformed birationally into another curve, such an integral gives rise to an algebraic integral for the other curve, likewise finite everywhere on this. The number of such integrals, $\int R_i(x, y)\,dx$, for which there exists no linear equation $\Sigma\lambda_i R_i(x, y)=0$, with constant coefficients λ_i, is thus the same for two curves which are in (1, 1) birational correspondence. It is this number which is the integral genus. For a rational curve this number is zero, every integral $\int r(t)\,dt$, in which $r(t)$ is a rational function of t, becoming infinite for a finite or infinite value of t; for an elliptic curve this number is unity.

This definition suggests at once the enquiry, in fact begun by Abel in his great paper *Mémoire sur une propriété générale d'une* *classe très étendue de fonctions transcendantes* (1826, cf. *Oeuvres complètes*), as to the behaviour, at infinity and at the multiple points of the curve $f(x, y)=0$, necessary for the function $R(x, y)$ in order that the integral should be everywhere finite; and this enquiry is intimately related with the further one of the behaviour of the polynomials which form the numerator and denominator in the expression of a rational function, at the multiple points of a curve, in order that the function should be quite general. It has been seen that it may be possible to transform the given curve birationally so that a multiple point is replaced by several points of the new curve; it appears probable then that a rational function belonging to the curve should assume different values at these; this can only happen when, in the expression of the function by the coordinates on the original curve, both the denominator and numerator vanish at the multiple point of the original curve. The different values of the function thereat may then be obtainable by a limiting process, approaching the multiple point on the different branches of which it is the origin. For the sake of brevity, we answer this enquiry by defining what is meant by saying that a polynomial, in the coordinates, is *adjoint* at a multiple point of the given curve $f(x, y)=0$; then we shew that any rational function is expressible as a quotient of two such adjoint polynomials. The definition we give covers all cases; later we shall see that for simple multiple points it can be replaced by an easy geometrical definition.

In order that it may be clear that the definition is not limited to points of the curve of which the coordinates are finite, we suppose the curve first given in homogeneous coordinates x_0, x_1, x_2, say $F(x_0, x_1, x_2)=0$, of order n. If, beside (x_0, x_1, x_2), the consecutive point $(x_0+dx_0, x_1+dx_1, x_2+dx_2)$ also belong to the curve, we have the two equations $F_0dx_0+F_1dx_1+F_2dx_2=0$, $x_0F_0+x_1F_1+x_2F_2=0$, where

$$F_i=\frac{1}{n}\partial F/\partial x_i;$$

and hence

$$(x_1dx_2-x_2dx_1)/F_0=(x_2dx_0-x_0dx_2)/F_1=(x_0dx_1-x_1dx_0)/F_2,$$

so that, if c_0, c_1, c_2 be quite arbitrary, each of these fractions is equal to $(c, x, dx)/F_x^{n-1}F_c$, where (c, x, dx) denotes the determinant whose rows are (c_0, c_1, c_2), (x_0, x_1, x_2), (dx_0, dx_1, dx_2), and the ordinary symbolical notation is used, $F(x_0, x_1, x_2)$ being written F_x^n. We say now that any integral form $\Psi(x_0, x_1, x_2)$, homogeneously of order m in x_0, x_1, x_2, is *adjoint* to F when the curve $\Psi=0$ intersects $F=0$ in such a way that the differential

$$\Psi.(cxdx)/F_x^{n-1}F_c$$

is everywhere finite. In other words, Ψ must vanish appropriately at the zeros of $F_x^{n-1}F_c$ upon $F=0$, in so far as these are not compensated by the zeros of $(cxdx)$.

To see the meaning of this, consider first a point of the curve $F=0$ at which the coordinate x_2 does not vanish. The quotient $x_2^{-(m-n+3)}\Psi.(cxdx)/F_x^{n-1}F_c$ is a functional differential, unaltered by replacing x_0, x_1, x_2 by any constant multiples of themselves. Herein put $x=x_0/x_2$, $y=x_1/x_2$, and take $c_0=0$, $c_2=0$; denote $x_2^{-m}\Psi$ by $\psi(x, y)$, or ψ, and $x_2^{-n}F$ by $f(x, y)$, so that $nF_x^{n-1}F_1$ is $x_2^{n-1}\partial f/\partial y$, or, say, $x_2^{n-1}f'(y)$. Save for the factor n, the differential is then $\psi dx/f'(y)$, and the meaning is that, in the neighbourhood of any point for which x is finite, given by such a pair of formulae as $x=a+t^r$, $y-b=c_1t+c_2t^2+\ldots$ (as in the preceding chapter), the expression $[\psi/f'(y)]\,dx/dt$, written as an ascending series of powers of t, must contain no negative powers of t, this being so for every branch of the curve having origin at the point, (a, b). As $dx/dt=rt^{r-1}$, this is exactly the same as saying that $(x-a)\psi/f'(y)$ must vanish at every finite place for which $x=a$.

Consider next a point at which $x_2=0$. If also $x_0=0$, we can choose a constant λ so that $x_0-\lambda x_1$ does not vanish, and then use $x_0'=x_0-\lambda x_1$, $c_0'=c_0-\lambda c_1$ instead of x_0 and c_0; we suppose then x_0 is not zero. Then the quotient $x_0^{-(m-n+3)}\Psi.(cxdx)/F_x^{n-1}F_c$ is a functional differential. Herein put $\xi=x_2/x_0$, $\eta=x_1/x_0$, and take $c_0=0$, $c_2=0$; denote $x_0^{-n}\Psi$ by $\psi_1(\xi, \eta)$, or ψ_1, and $x_0^{-n}F$ by $f_1(\xi, \eta)$, so that $nF_x^{n-1}F_1$ is $x_0^{n-1}\partial f_1/\partial\eta$, or, say, $x_0^{n-1}f_1'(\eta)$. Save for the factor n, the differential is then $-\psi_1 d\xi/f_1'(\eta)$, and the meaning is that, for every place in the neighbourhood of $\xi=0$, given by such a pair of expansions as $\xi=t^r$, $\eta-b=c_1t+c_2t^2+\ldots$ (as in the preceding chapter), the expression of $[\psi_1/f_1'(\eta)]\,d\xi/dt$ as an ascending series in t must contain no negative powers; or, what is the same, that $\xi\psi_1/f_1'(\eta)$ must vanish. Still supposing x_0 not zero, the equation $f_1(\xi, \eta)=0$ is derived directly from $f(x, y)=0$ by $\xi=x^{-1}$, $\eta=yx^{-1}$, which lead to $x^{-(m-n+3)}\psi dx/f'(y)=-\psi_1 d\xi/f_1'(\eta)$; thus the condition of adjointness for $x=\infty$ is that the differential on the left here must be finite. The case of a point for which $x_2=0$ and $x_0=0$, of which we have evaded the detailed consideration, arises, for the non-homogeneous equation $f(x, y)=0$, when $x=0$, $y=\infty$.

The quotient $(x-a)\psi(x, y)/f'(y)$ certainly vanishes, and the condition of adjointness is nugatory, at every finite value of x, unless $f'(y)=0$. The condition is thus effective only, (i), at a point of the curve $f(x, y)=0$ at which x is finite but y is infinite, namely for a value of x which reduces to zero the coefficient of the highest power of y, when this power is not y^n. We can easily evade this by writing for x, in the equation, $x+hy$, where h is a constant; and may thus suppose the term y^n to be present; or (ii), when $f'(y)=0$ for a finite value of x. This happens, as we see by eliminating y between $f'(y)=0$ and $f(x, y)=0$, only for a finite number of values of x. The equation $f'(y)=0$ expresses that two, or more, of the n values of y corresponding to the same value of x coincide.

It is easy to give a geometrical interpretation in the simplest cases. For instance, if at $x=0$, $y=0$ there be a multiple point of order k, with distinct tangents; so that the terms of lowest order in $f(x, y)=0$ are of order k, having k distinct linear factors, which, first, we suppose to be all of the form $y-mx$: then in $f'(y)$, the terms of lowest order are of order $k-1$, and it is sufficient that the terms of lowest order in ψ should equally be of order $k-1$, namely, that $\psi=0$ should have a multiple point of order $k-1$ at $x=0$, $y=0$. In this case the quotient $\psi/f'(y)$ is finite on all branches of $f(x, y)=0$ as we approach the origin. If, however, next, one of the tangents of $f(x, y)=0$ be $x=0$, and along the corresponding branch, to the first approximation, $x \propto y^2$, it is still true that the condition of $x\psi/f'(y)$ vanishing is satisfied if $\psi=0$ have a multiple point of order $k-1$ at the origin. Another simple case is that of an ordinary cusp; when the equation of the curve near the origin, $(x, y)_r$ denoting a homogeneous polynomial in x, y of order r, is $y^2+(x, y)_3+\ldots=0$, the term in x^3 being present; in this case $f'(y)$, to the first approximation, $\propto x^{\frac{3}{2}}$, and it is sufficient that $\psi/x^{\frac{1}{2}}$ should vanish, or that the curve $\psi=0$ should pass through the cusp.

Ex. 1. When in $f(x, y)=0$, the terms of lowest order are of the form $y^2+y(x, y)_2+y(x, y)_3+\ldots+y(x, y)_{r-2}+(x, y)_r+\ldots$ with $r>3$, the term in yx^2 being present, prove that there are two branches of the curve at the origin, but it is sufficient for adjointness that the curve $\psi=0$ should pass through the origin with $y=0$ for tangent.

Ex. 2. When in $f(x, y)=0$, the terms of lowest order are of the form $x+x(x, y)_1+x(x, y)_2+\ldots+x(x, y)_{r-2}+(x, y)_r+\ldots$, with $r>2$, the term in yx being present, it is not necessary for adjointness that $\psi=0$ should pass through the origin, though $f'(y)$ there vanishes.

Ex. 3. For the sextic curve $y^3+y^2(x, y)_2+y(x, y)_4+(x, y)_6=0$, the condition of adjointness of $\psi=0$, at the origin, is that $\psi=0$ should have a self-contact, with $y=0$ as the tangent.

Further, it is capable of proof, though we do not enter into this in detail at this stage, that the conditions of adjointness of a polynomial ψ, at a multiple point of the curve $f(x, y)=0$, are those which we should employ in carrying out the process of reduction described in Chap. II, so far as this reduction is made possible by the existence of the multiple point. Consider the case of a multiple point of $f(x, y)=0$, of multiplicity k, with k distinct tangents. In order that $\psi=0$ should have there a multiple point of order λ, it is necessary to impose $\frac{1}{2}\lambda(\lambda+1)$ linear conditions for ψ; by this we secure λk intersections of $\psi=0$ with $f=0$; the condition that this should be at least twice the number of conditions, is that

$\lambda k \geqslant \lambda(\lambda+1)$, or $\lambda \leqslant k-1$; and we have shewn that for $\psi=0$ to be adjoint at this point, we have $\lambda=k-1$. It is an incidental consequence of the theory that a polynomial ψ satisfying the conditions of adjointness wherever, on $f(x, y)=0$, they are effective, cannot be further reduced in the sense of Chap. II.

Expression of rational function by adjoint polynomials. Now let $R(x, y)$ be any rational function for the curve, of order n, $f(x, y)=0$; and suppose, (a), that the term in y^n is present in $f(x, y)$, a substitution of the form $x+hy$ for x being made, if necessary, to secure this; (b), that the terms of order n in $f(x, y)$ are the product of n distinct linear factors, as may be secured by a suitable (fractional) linear substitution for x and y. Further, suppose that the poles of the function R are all at places for which x is finite, as can also be secured in a similar way. Next choose a polynomial $\psi(x, y)$ quite arbitrarily subject to the condition that the product $R\psi$ is finite at all the poles of R; or that ψ vanishes on $f(x, y)=0$, at any pole of R, to an order at least equal to the order of this pole, and subject further to the condition that ψ is adjoint at all the places of $f(x, y)=0$, arising for finite values of x, at which $f'(y)$ vanishes, where $f'(y)=\partial f/\partial y$. For a place at which $x=a$, the second condition is that $(x-a)\psi/f'(y)$ shall vanish at this place. If it happen that $R(x, y)$ becomes infinite at a place where $f'(y)=0$, the two conditions for ψ are understood to require that $(x-a)R\psi/f'(y)$ shall vanish at this place. By taking ψ of sufficiently high order, the conditions can be satisfied. Consider then the value, on $f(x, y)=0$, of the sum

$$\sum_{i=1}^{n} \frac{(\theta-y_1)\ldots(\theta-y_n)}{\theta-y_i} M \frac{R(x, y_i)\,\psi(x, y_i)}{f'(y_i)},$$

where M is the product of all the different factors $x-a$ at all the finite values of x for which $f'(y)=0$, θ is an undetermined quantity, and, for a general value of x, the $y_1, \ldots, y_n$ are the n values of y which satisfy $f(x, y)=0$, all of which, by the hypothesis, are finite. This sum, from its symmetry in regard to $y_1, \ldots, y_n$, is expressible as a rational function of x only; and it can only become infinite, for a finite value of x, when $f'(y_i)$ vanishes, or $R(x, y_i)$ becomes infinite. By the conditions for ψ, $MR\psi/f'(y)$ vanishes for every such value of x. The sum is therefore a rational function of x with no poles for finite values of x, and is therefore a polynomial in x; moreover it vanishes for all the values of x which make $M=0$, so that it divides by M. Thus, on $f(x, y)=0$, it is capable of the form

$$M(\theta^{n-1}u_1+\theta^{n-2}u_2+\ldots+u_n),$$

where $u_1, u_2, \ldots, u_n$ are integral polynomials in x. If we now put y_1 for θ, and divide by M, since $f'(y_1)=(y_1-y_2)\ldots(y_1-y_n)$, we obtain $R(x, y_1)=(y_1{}^{n-1}u_1+y_1{}^{n-2}u_2+\ldots+u_n)/\psi(x, y_1)$, and in this we may put y for y_1. Hence it is proved that any rational function R, for the curve $f(x, y)=0$, is expressible as the quotient of two polynomials of which the denominator is an arbitrary adjoint polynomial ψ chosen to vanish at the poles of R, so that $R\psi$ is finite at such a pole, or, if such a pole be also a point, with $x=a$, for which $f'(y)$ vanishes, chosen to compensate this zero, so that

$$(x-a)R\psi/f'(y)$$

vanishes. The numerator polynomial must needs vanish at all the zeros of the denominator which are not poles of the function, and thus in particular must also be adjoint; and since the function is taken here not to have a pole for infinite values of x, the order of the numerator polynomial will not be higher than that of the denominator.

We have defined adjointness with use of homogeneous variables; thus the references to infinite values of x in this enunciation can easily be eliminated.

Expression of most general everywhere finite integral. Before applying the preceding result to discuss further the properties of rational functions, we apply an almost identical reasoning to prove that all the everywhere finite integrals belonging to the curve $f(x, y)=0$ can be expressed in the form $\int \phi\, dx/f'(y)$, where ϕ is an integral polynomial of suitably limited order. Suppose, first, as before, that the equation $f(x, y)=0$ is taken so that y becomes infinite only when x is infinite, and that all the places arising for $x=\infty$ are distinct; let u denote any everywhere finite integral; then du/dx is a rational function only becoming infinite, for finite x, at a place where the parametric expression of the form $x=a+t^r$, $y=b+c_1t+c_2t^2+\dots$, has $r>0$; at such a place

$$du/dx=\frac{1}{r}t^{-(r-1)}\,du/dt,$$

and du/dt is finite; thus $(x-a)du/dx$ vanishes. Hence if M denote the product of all the different factors $x-a$ so arising, and $(du/dx)_i$ denote the value of du/dx for $y=y_i$, we have, as above, the equation

$$\sum_{i=1}^{n}\frac{(\theta-y_1)\dots(\theta-y_n)}{\theta-y_i}M\left(\frac{du}{dx}\right)_i=M(\theta^{n-1}u_1+\dots+u_n),$$

and can infer an equation of the form $du/dx=(y^{n-1}u_1+\dots+u_n)/f'(y)$. At any one of the places arising for $x=\infty$, however, we have $x=t^{-1}$, $y=mt^{-1}+\dots$; thus du/dx vanishes to the second order. From this it appears that the aggregate order of the polynomial $y^{n-1}u_1+\dots+u_n$ in x and y does not exceed $n-3$. The deduction shews that this polynomial is adjoint at all places where $f'(y)=0$. Conversely, any such adjoint polynomial, of aggregate order $n-3$, is easily seen to give rise to an everywhere finite integral.

And as before, the references to infinite values of x are easily eliminated, the everywhere finite integrals belonging to the curve $F(x_0, x_1, x_2)=0$ being given by $\int \phi(x_0, x_1, x_2)(cxdx)/F_x^{n-1}F_c$, where $\phi(x_0, x_1, x_2)$ is an everywhere adjoint polynomial homogeneously of order $n-3$. The number we have defined as the integral genus is thus the number of undetermined coefficients in such an everywhere adjoint polynomial.

Equivalent or coresidual sets of points on the curve. We now introduce another technical term, and say that two sets, each of m places, on the curve $f(x, y)=0$, are *equivalent*, or *coresidual*, when there exists a rational function for this curve, say $K(x, y)$, which has one value, say A, at the places of one of these sets (and not elsewhere), and has another value, say B, at the places of the other set (and not elsewhere). As we may consider the rational function $[K(x, y)-A]/[K(x, y)-B]$, we see that this is the same as saying that two sets of places are coresidual when they constitute respectively the zeros and poles of the same rational function. Hence, the most general set, N, of places coresidual with a given set D, is obtained by finding the most general rational function of which the places of the latter set, D, are the poles, and taking the zeros, N, of this function. By what we have proved above, this is done by first taking a perfectly arbitrary adjoint polynomial (D) which vanishes in the places of the set D; this polynomial (D) will in general vanish, on $f(x, y)=0$, at places other than those of the set D, say in places of a set O, which is then said to constitute a set residual to D. The formation of the rational function is then to be completed by finding the most general adjoint polynomial, (N), vanishing in the set O, in each case to the same order as does (D), which is of the same order as (D). The rational function is then given by $(N)/(D)$. In general the polynomial (N) is not completely determined by the conditions imposed upon it, but the most general rational function with given poles contains a number of linearly entering arbitrary coefficients.

The statements made seem easiest of application if we think of the curve $f(x, y)=0$ as given by the homogeneous equation

$$F(x_0, x_1, x_2)=0;$$

then the conditions of adjointness are effective only at the zeros of the polar $F_x^{n-1}F_c$ which are independent of the arbitrary point (c_0, c_1, c_2), that is, at the multiple points of the curve; they require the adjoint curve to have a certain multiplicity at such a point; in general the set O spoken of will consist of the residual intersections of the curve $(D)=0$, other than those which this has at the multiple points of $F=0$.

Complete linear series of sets of points on the curve. Now suppose there are $(r+1)$ homogeneously entering arbitrary coefficients in the numerator polynomial (N). Then the sets of places determined on the curve $F=0$ by $(N)=0$, other than those required by the conditions of adjointness (at the multiple points of the curve), and other than the set O which are also zeros of $(D)=0$, are said to form, on $F=0$, a linear series of sets of places, of freedom r; one such set exists for every definite set of ratios of the undetermined coefficients in (N); thus one such set, of the linear series of sets coresidual with the given zeros of (D), exists with arbitrary positions for r places of the set. This number r is necessarily less than the whole number, m, of zeros of (D). Further, we may form a partial series, of sets each of m places, of freedom less than r, defined by the zeros (other than at the multiple points of $F=0$) of a polynomial found from (N) by assigning definite values (say zero) to some of the undetermined coefficients therein. But the series of sets of m places which we have found is said to be *complete*, because, by the manner of its construction, it is not contained in any other series, of sets of the same number, m, of places, having freedom greater than r. It may be remarked that among the sets of this complete series, the set of zeros of (D) occurs, one of the curves $(N)=0$ through the points O being evidently $(D)=0$. We may say then that a complete series of sets of coresidual places is uniquely determined by any one of its sets, which may be taken arbitrarily.

Ex. 1. Consider a plane quartic curve with two double points, and the set D determined by the intersections of the curve with a line u; coresidual sets of four points, forming a series *of freedom* 2, are evidently obtained by the intersections of the curve with a variable line v. But the complete series is to be obtained by taking an adjoint curve through the points of the set u, meeting the quartic again in a set O, and then taking the most general adjoint curve of the same order through the points of the set O. If, for instance, we take the adjoint conic through the set u, which breaks up into the line u and the line joining the double points, in which case there exists no residual set O, then the most general adjoint conic is conditioned only by passing through the double points, and determines on the quartic curve a coresidual series of sets of four points *of freedom* 3. Likewise if we take an adjoint cubic curve through the original set u, the residual set O consists of four points lying on a particular conic through the double points; the series of coresidual sets is then determined by cubic curves through the double points and the four points O, and is *of freedom* 3 (since the equation of a cubic curve contains ten terms).

If, however, upon the original quartic curve with two double points we start with a set of 4 points which do not lie on a conic containing the double points of the curve, we determine a residual set O of four points by a cubic curve through the double points and the original set of 4 points; and the coresidual series, of sets of 4 points, is obtained by cubic curves through the double points and the set O. This series is likewise of freedom 3, but it is distinct from the series found before; it cannot contain a set of 4 points which lie on a line unless there is a conic through the set O and the double points, which is not generally the case.

Ex. 2. Prove that on a plane quartic curve with one double point, the complete series of sets coresidual with a set of four collinear points of the curve is given by the intersections of the curve with a variable line. Examine whether this series is distinct from that of points coresidual with four general points.

Theorem of coresiduation. The theorem we have proved, that in the expression of a rational function the denominator polynomial is arbitrary, so long as it is adjoint and vanishes at the poles of the function, is a remarkable result, and leads to an important geometrical property, namely: Suppose there be, upon the curve $f(x, y)=0$, two sets N, D, of the same number of places, which have the same residual set of places O, so that there is an adjoint curve $\psi=0$ whose aggregate set of intersections with $f=0$, other than at the multiple points, consists of the sets O, D; and, also, another adjoint curve $\phi=0$, of the same order as $\psi=0$, whose aggregate set of intersections, other than at the multiple points, consists of the sets O, N. Let then any other adjoint curve $\psi'=0$ be taken, which has the same intersections D with $f=0$ as has $\psi=0$; let its residual set of intersections with $f=0$, other than at the multiple points, be O'; *then* there exists an adjoint curve $\phi'=0$, of the same order as $\psi'=0$, whose aggregate intersections, other than the multiple points, consist of the sets O', N together. This is called the *theorem of coresiduation*. It is sometimes important to bear in mind that in this theorem there may be points which are common to the sets N and D.

The equality of the ratios ϕ/ψ and ϕ'/ψ', which holds in virtue of $f(x, y)=0$, must involve a polynomial identity of the form $\phi\psi'=\phi'\psi+\theta f$, where θ is equally a polynomial in the coordinates. In a later chapter a theorem of Noether's, in regard to the expression of the equation of a curve which passes through the common points of two given curves, will be considered (Chap. VIII), which can be used to furnish an alternative proof of the theorem of coresiduation. The method we have followed has a wide bearing; it is used in the exposition of Galois' theory of the invariants of a group of permutations of the roots of an algebraic equation (cf. Kronecker, *Werke*, II, p. 202).

Sum and difference of complete linear series. The sets of a linear series are given by the intersections with $f=0$ of the curves of a system with equation $\lambda_0\phi_0+\lambda_1\phi_1+\ldots+\lambda_r\phi_r=0$, wherein $\lambda_0, \ldots, \lambda_r$ are variable parameters, and $\phi_0, \ldots, \phi_r$ are definite polynomials not connected, in virtue of $f=0$, by any linear equation with constant coefficients. The places of a set of this series are the intersections, with $f=0$, other than those which are independent of $\lambda_0, \ldots, \lambda_r$, namely, other than those common to all of $\phi_0=0, \ldots, \phi_r=0$ (among which are the intersections at the multiple points). It is sometimes convenient however to adjoin, to the variable sets thus defined, certain of the intersections which are not variable. If a particular set of the series is denoted by A, it is usual to denote by

$|A|$ the complete series determined thereby. If $|B|$ be another complete series, given by a system $\lambda_0\psi_0+\ldots+\lambda_s\psi_s=0$, there exists a further complete series $|A+B|$, defined by the aggregate of a set A taken with a set B. This contains all sets given by the system $\Sigma\lambda_{ij}\phi_i\psi_j=0$, but may contain also other sets not so given. Conversely, given two complete series $|C|$, $|A|$, it may be that there are sets of $|C|$ which contain all the places of a particular set A_0 of $|A|$; if $|C|$, of freedom k, be given by $\mu_0\vartheta_0+\ldots+\mu_k\vartheta_k=0$, the curves of $|C|$ which contain the places A_0 will be obtained by imposing proper linear conditions upon $\mu_0, \ldots, \mu_k$. These curves define a complete system $|C-A_0|$, which can be shewn, utilising the theorem of coresiduation proved above, to be equally definable from any other set of $|A|$ instead of A_0. The resulting series is therefore denoted by $|C-A|$. If this be denoted by $|B|$, we have $|C|=|A+B|$. It is clear that these notions, of the sum or difference of two linear series, can be explained in another way, speaking of the product or quotient of two rational functions belonging to the curve $f=0$.

Riemann-Roch formula for linear series, and for rational functions. There exists an important relation connecting the freedom of a linear series, upon a curve of given genus, with the number of places in a set of the series, which we develop in connexion with a numerical definition of the genus; such a definition is possible when the fundamental curve $F=0$ has no multiple points other than those with distinct tangents. The genus defined by the number of everywhere finite integrals belonging to the curve, has been proved (p. 65) to be the same as the number of linearly independent curves of order $n-3$ (if n be the order of $F=0$) which are everywhere adjoint to $F=0$; but, even when $F=0$ has the simple form referred to, this is not final without an examination of whether the conditions of adjointness at all the multiple points of $F=0$ are linearly independent. We proceed then with care.

Let $F=0$ be a curve of order n, *which at every multiple point, say of multiplicity k, has k distinct tangents.* We define the number P by the equation $P=\frac{1}{2}(n-1)(n-2)-\frac{1}{2}\Sigma k(k-1)$, where the summation extends to all the multiple points. It has already been shewn (Chap. I, p. 10) that this number is not negative.

Now consider a curve, $\psi=0$, of order m, subject to no conditions other than of having a $(k-1)$-ple point at every k-ple point of $F=0$; it is clearly possible to construct such a curve if m be great enough. Let N be the whole number of intersections of $\psi=0$ with $F=0$ other than these prescribed ones, and $R+1$ the number of homogeneously entering coefficients which remain arbitrary in ψ after satisfying the conditions. We cannot obtain R by adding the number of conditions imposed upon ψ at the separate multiple points, because we

when $\alpha = 1$ or 2, for curves $\psi = 0$ of order m, greater than $n-3$, including curves for which $m \geqslant n$, as we have seen above, our result is $R \geqslant N-P$.

The sign of inequality arises from the possibility that the linear conditions to be satisfied by the coefficients in ψ, at the multiple points of $F=0$, may not be linearly independent. The number of such conditions depends however only on $F=0$, and not on $\psi=0$; and if the number of coefficients in ψ be great enough, that is, if ψ be of sufficiently high order, these linear conditions must certainly be linearly independent. Thus we may say:

For curves $\psi=0$ of sufficiently high order, we have $R=N-P$; for curves $\psi=0$ of less order, but greater than $n-3$, we have $R \geqslant N-P$; finally, for curves $\psi=0$ of order $n-3$, we have $R > N-P$.

Consider in particular the last case, $m=n-3$. Then $N=2P-2$, and we have $R > P-2$. We can prove in an instructive way that $R=P-1$, as follows: Denote $R+1$ by p, so that $p > P-1$; say $p = P+\epsilon$, with $\epsilon \geqslant 0$. Denote the general adjoint polynomial of order $n-3$ by

$$\lambda_1\phi_1 + \ldots + \lambda_p\phi_p,$$

where $\lambda_1, \ldots, \lambda_p$ are arbitrary coefficients; the zeros on $F=0$, other than those prescribed at the multiple points, of any one of the polynomials $\phi_1, \ldots, \phi_p$ are $2P-2$ in number. Further,

$$N-R = 2P-2-(p-1) = p-1-2\epsilon,$$

so that $N-R<R$ if $\epsilon>0$. Now, first, suppose that, save for the prescribed points, there is on $F=0$ no zero common to all the polynomials $\phi_1, \ldots, \phi_p$. Take R, or $p-1$, arbitrary general points, $M_1, \ldots, M_{p-1}$, on $F=0$, and determine the particular curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$ which passes through these, say Φ; let its remaining intersections with $F=0$, other than at the multiple points, be $A_1, A_2, \ldots, A_{N-R}$. When $\epsilon>0$ we have $N-R<p-1$, so that, beside Φ, at least one other curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$, linearly independent of Φ, can be drawn through the derived points $A_1, \ldots, A_{N-R}$. We prove that any such curve, say $\Psi=0$, necessarily contains also the points $M_1, \ldots, M_{p-1}$; this we prove by shewing that $\Psi=0$ contains M_{p-1}, any one of $M_1, \ldots, M_{p-1}$. To this end remark, that there exists, for $F=0$, a rational function having M_{p-1} as a pole of the first order, and having also $A_1, \ldots, A_{N-R}$, or some of these, as poles of the first order. For, since $M_1, \ldots, M_{p-1}$ are arbitrary and general, we can suppose that the most general curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$ through $M_1, \ldots, M_{p-2}$, only, has an equation of the form $\mu\Phi+\mu_1\Phi_1=0$, where Φ is as above, passing through $M_1, \ldots, M_{p-1}$, and μ, μ_1 are arbitrary parameters, and can suppose that this curve does not pass through M_{p-1} for all values of μ, μ_1, that is, that $\Phi_1=0$ does not pass through M_{p-1}. Then the

quotient $(\mu\Phi+\mu_1\Phi_1)/\Phi$ is a rational function, for $F=0$, with M_{p-1} as pole, which has also for poles such of $A_1, \ldots, A_{N-R}$ as do not happen to be zeros of Φ_1. On the other hand, to form such a rational function with pole at M_{p-1}, and other poles occurring only among $A_1, \ldots, A_{N-R}$, all of the first order, we may, by the theorem of coresiduation, proceed thus: draw through M_{p-1} an arbitrary line $L=0$, meeting $F=0$ in $n-1$ other points (B); draw through $A_1, \ldots, A_{N-R}$ a general adjoint curve of order $n-3$, say $\Psi=0$, meeting $F=0$ in remaining points (C); thus $L\Psi=0$ is an adjoint curve of order $n-2$ containing the points $M_{p-1}, A_1, \ldots, A_{N-R}$, with residual intersection consisting of (B), (C) together; then draw an adjoint curve of order $n-2$ through these points (B), (C); we have proved that the function exists, so that such a curve is possible, other than $L\Psi=0$. This curve, taken as numerator, with $L\Psi$ as denominator, will express the most general rational function of the description. But a curve of order $n-2$ containing the $n-1$ points (B) will contain the line L as factor; the function will therefore have for numerator a curve of order $n-3$, and have Ψ for denominator. The function has thus for poles only the zeros of Ψ. But by hypothesis it has M_{p-1} for pole. Whence $\Psi=0$, which is any curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$ drawn through $A_1, \ldots, A_{N-R}$, contains M_{p-1}; and, therefore, contains all of $M_1, M_2, \ldots, M_{p-1}$. This, however, is inconsistent with $R>P-1$, or $\epsilon>0$, and shews that $p=P$. For the $2P-2$ points $M_1, \ldots, M_{p-1}, A_1, \ldots, A_{N-R}$ are determined by the $p-1$ perfectly arbitrary points $M_1, \ldots, M_{p-1}$; while the points $A_1, \ldots, A_{N-R}$, in number $p-1-2\epsilon$, are not only fewer than $p-1$ if $\epsilon>0$, but are, we may say, less general in character than $M_1, \ldots, M_{p-1}$, as being derived from these, and not assigned arbitrarily. It is therefore impossible that the whole $2P-2$ points, which depend upon $p-1$ quite arbitrary general points, should be determined by these fewer and less general points $A_1, \ldots, A_{N-R}$, as they would be if every curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$, through these, also contained $M_1, \ldots, M_{p-1}$. We can only conclude therefore that $\epsilon=0$, and $p=P$; in which case $N-R=R$, and the points $A_1, \ldots, A_{N-R}$ are as many as $M_1, \ldots, M_{p-1}$. This conclusion is reached on the hypothesis that the curves $\phi_1=0, \ldots, \phi_p=0$ have no common intersection with $F=0$, save at the multiple points. If there are such points, they will be among the points $A_1, \ldots, A_{N-R}$, the derived zeros of the curve $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$ drawn through $M_1, \ldots, M_{p-1}$; say they are $A_1, \ldots, A_k$. These points can then be excluded from consideration, the same argument proceeding with $A_{k+1}, \ldots, A_{N-R}$ instead of $A_1, \ldots, A_{N-R}$. The conclusion now will be that $N-R-k=p-1$, leading to $2P=2p+k$. This can only be consistent with $p\geqslant P$, if $p=P$, $\epsilon=0$, and $k=0$.

The sign of inequality, for the freedom of adjoint curves of order

$n-3$, in the formula $R \geqslant P-1$, is thus unnecessary; and the conditions of adjointness, at the multiple points of $F=0$, for curves of this order, are linearly independent. These conditions are therefore independent for all adjoint curves of order $m \geqslant n-3$. *This is a capital result.* The formulae we found are thus replaced by $R=N-p$ for all adjoint curves of order $>n-3$, while for adjoint curves of order $n-3$ we have $R=p-1$. This is $R=N-P+1$ and replaces $R>N-P$, because $N=2p-2$, $P=p$*.

The preceding argument is for curves of assigned order, intersecting the given curve $F=0$, which are subject to no conditions

---

* Some remarks of importance may be placed in a footnote, so as not to interrupt the general argument. (a) The general adjoint curve of order $n-3$, $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$, though not having zeros on $F=0$, save at the multiple points, which are common to $\phi_1=0, \ldots, \phi_p=0$, may have such zeros elsewhere in the plane. For example, for a plane sextic curve with 8 double points, the adjoint cubic curves $\lambda_1\phi_1+\lambda_2\phi_2=0$ all have a point in common; unless the sextic curve is further specialised this ninth point does not lie thereon. (b) The curves $\lambda_1\phi_1+\ldots+\lambda_p\phi_p=0$ may all have a common part; by the theorem of the text, this part is then a curve intersecting $F=0$ only at its multiple points. For instance, for the sextic curve given by an equation of the form

$$y^3z^3+y^2z^2u_2+yzu_4+u_6=0,$$

where u_i is homogeneous of order i in y, x, the adjoint curves of order $n-3$ have an equation of the form $y(yz+v_2)=0$, where v_2 is homogeneous of order 2 in y and x; the partial curve $y=0$ meets the sextic curve only at the multiple point $x=0$, $y=0$. The linear series, of sets of $2p-2$ points, with freedom $p-1$, is then given by variable curves of order less than $n-3$, constituting the *pure*, or *reduced*, adjoint system. It may be shewn that, in any birational transformation *of the plane*, the pure adjoint system is transformed into the pure adjoint system of the curve into which the given curve is transformed. This is clear here from the association with the everywhere finite algebraic integrals; a simple independent proof is given by Bertini, *Ann. d. Mat.* XXII, 1894, p. 14. Conversely, it may be shewn that, when the multiple points of the given curve $F=0$ are such that it is possible to put through them a curve γ, having no other intersections with the given curve, then this curve γ is part of the general adjoint curve of order $n-3$, provided the multiplicities of γ at the multiple points of $F=0$ are such as would constitute independent conditions in defining γ (Castelnuovo, *Memorie Torino*, XLII, 1892). For instance, for a sextic curve with two triple points, the variable part of the adjoint system of cubics is given by conics passing simply through each triple point; but, for a sextic curve having six double points which lie on a conic, the system of adjoint cubics has no common part. (c) Reference may be made here to a general theorem for a linear system of curves in a plane, $\lambda_0\phi_0+\ldots+\lambda_r\phi_r=0$, not necessarily adjoint to any given curve, that if the variable part of every curve of the system is composite, the general curve of the system is given by an equation of the form

$$\psi\left[\lambda_0u^r+\lambda_1u^{r-1}v+\ldots+\lambda_rv^r\right]=0,$$

where $u=0$, $v=0$ are curves of the same order (Bertini, *Rend. Lombardo*, XV, 1882; Lüroth, *Math. Ann.* XLII, XLIV; 1893, 1894). For the system consisting of the adjoint curves of order $n-3$ of a curve $F=0$ of order n, this happens if $F=0$ be hyperelliptic (see Chaps. II, III, above), but only then. For instance, for the curve expressed by $y^{2p}z^2=(y, x)_{2p+2}$, the adjoint system in question is given by $y^p(y, x)_{p-1}=0$.

save the conditions of adjointness at the multiple points of $F=0$. Now consider adjoint curves which, beside, are required to pass through certain specified simple points of $F=0$. Let the number of these points be k, and let ν, $=N-k$, be the number of remaining intersections which are not assigned (other than at the multiple points of $F=0$). If the k prescribed points give linearly independent conditions for the coefficients of the curve to be obtained, this curve will define a linear series of freedom $\rho=R-k$; of this, however, we cannot be sure unless the curve be of sufficiently high order; when the conditions are not independent, we shall have $\rho>R-k$. Thus we conclude that, for the linear series defined by adjoint curves with prescribed fixed base points on $F=0$, we have, for the freedom, $\rho=\nu-p$, for all curves of sufficiently high order; but for curves of less order ($>n-3$ however), $\rho\geqslant\nu-p$; and, for curves of order $n-3$, $\rho\geqslant\nu-p+1$, that is $\rho>\nu-p$.

It can however be shewn, for adjoint curves of order $>n-3$, with k fixed base points, that, if these k points do not furnish independent conditions, so that, for the linear series of sets of ν points, of freedom ρ, formed by the sets of residual intersections with $F=0$, we have $\rho>\nu-p$, then the points of any one of the sets of the linear series lie upon an adjoint curve of order $n-3$; and that the series can be given by variable adjoint curves of order $n-3$, with possibly fixed prescribed base points; of these curves there will then be ν intersections not prescribed, and the general curve of the system will contain $\rho+1$ homogeneously entering arbitrary coefficients. This we proceed to prove. In saying that the k prescribed base points for the adjoint curves in question are not independent, or that some are determined by the others, we have not said that there may not be other points of $F=0$ also necessarily lying on all the adjoint curves through the k prescribed points. The linear system of adjoint curves of the assigned order through the k prescribed points, say (Ψ), given by an equation $\lambda\Psi+\lambda_1\Psi_1+\ldots+\lambda_\rho\Psi_\rho=0$, may be such that there are, say, l points of $F=0$, common to all the curves $\Psi=0, \ldots, \Psi_\rho=0$, in addition to the k prescribed base points. Choose now ρ arbitrary general points of $F=0$, say $M_1, \ldots, M_\rho$, and let the definite curve (Ψ) passing through these, and the k prescribed base points, meet $F=0$ further in points $A_1, \ldots, A_{\nu-\rho}$ (beside the multiple points of $F=0$); the l common zeros of $\Psi=0, \ldots, \Psi_\rho=0$, if existent, will be among $A_1, \ldots, A_{\nu-\rho}$. Since $\nu-\rho<p$, an adjoint curve of order $n-3$, or several linearly independent such curves, can be put through $A_1, \ldots, A_{\nu-\rho}$. We proceed to prove that every such adjoint curve, of order $n-3$, contains M_ρ, any one of $M_1, \ldots, M_\rho$, and hence contains all of these. The argument is the same as one used above. As $M_1, \ldots, M_\rho$ are independent points for curves (Ψ), there are such curves, given, say, by

$\mu\Psi' + \mu_1\Psi'_1 = 0$, containing only $M_1, \ldots, M_{\rho-1}$, where μ, μ_1 are arbitrary, and $\Psi' = 0$, suppose, is the (Ψ') curve containing $M_1, \ldots, M_\rho$, $A_1, \ldots, A_{\nu-\rho}$. Hence there is a rational function, $(\mu\Psi' + \mu_1\Psi'_1)/\Psi'$, having M_ρ as pole, having beside, for poles, such of the points from $A_1, \ldots, A_{\nu-\rho}$ as are not common zeros of Ψ' and Ψ'_1. By the theorem of coresiduation, however, such function must be capable of expression by taking an arbitrary adjoint curve through $M_\rho, A_1, \ldots, A_{\nu-\rho}$, and then an adjoint curve of the same order through the residual intersections (other than the multiple points) of this curve with $F=0$. Thus, draw a line $L=0$ through M_ρ, meeting $F=0$ in $(n-1)$ points (B); and take an adjoint curve of order $n-3$ through $A_1, \ldots, A_{\nu-\rho}$, say $\phi=0$, whose residual intersections with $F=0$ are denoted by (C). The denominator in the expression of the function may then be taken to be $L\phi$, the numerator being found from an adjoint curve of order $n-2$ containing the sets (B), (C). But such a curve, containing the $(n-1)$ points (B) upon the line $L=0$, is of the form $L\psi=0$, where $\psi=0$ is an adjoint curve of order $n-3$. The function is then ψ/ϕ; and as it has M_ρ as pole, the curve $\phi=0$ must pass through this point (and the curve $\psi=0$ will not). Thus we see that an adjoint curve of order $n-3$ contains all the points $M_1, \ldots, M_\rho, A_1, \ldots, A_{\nu-\rho}$; let $\phi=0$ be such a curve, and the general adjoint curve $\psi=0$ be taken passing through the residual intersections (other than the multiple points) of $\phi=0$ with $F=0$; the function ψ/ϕ must be the same, by the theorem of coresiduation, save for a constant multiplier, as the function $(\Psi')/\Psi'$, where $\Psi'=0$ is a particular curve of the original system, containing $M_1, \ldots, M_\rho, A_1, \ldots, A_{\nu-\rho}$, and $(\Psi')=0$ is the general adjoint curve, of the same order as $\Psi'=0$, formed, similarly to $\psi=0$, from the residual set on $\Psi'=0$. And as the common points, other than the k prescribed points, of all the curves $\Psi'=0, \Psi'_1=0, \ldots, \Psi'_\rho=0$ are eliminated in considering the rational function (being common to numerator and denominator), and are zeros of $\phi=0$, they will also be zeros of $\psi=0$. Thus, the system $\psi=0$ will have the same fixed points as the system $(\Psi')=0$. In the argument given, we have allowed, as possible, that there may be l points among $A_1, \ldots, A_{\nu-\rho}$ which are zeros of all the adjoint curves $(\Psi')=0$ put through the k points originally prescribed; if we denote $\nu-l$ by m, the function $(\Psi')/\Psi'$ will be of the m-th order, and ρ will, *a fortiori*, be $> m-p$. But, conversely, if a function with m given poles have a number $\rho+1$ of homogeneously entering arbitrary coefficients in its expression, where $\rho > m-p$, the function can be expressed as a quotient of adjoint polynomials of order $n-3$; for we can suppose the l points spoken of to be included among the k prescribed points, and then apply the preceding argument. From this investigation we conclude:

(*a*) *That a linear series of sets of ν points, obtained by general variable adjoint curves of order $> n-3$, passing through certain prescribed points of $F=0$, has a freedom ρ given by $\rho=\nu-p$, unless every one of the sets of ν points lies entirely upon an adjoint curve of order $n-3$; and*

(*b*) *For a series of sets of ν points, determined by general adjoint curves of order $n-3$, which pass through k prescribed points, not necessarily all independent, the freedom ρ is such that $\rho>\nu-p$. There may be points common to all the sets of ν points of the series, and, hence, common to all the adjoint curves of order $n-3$ through the k prescribed points.*

A linear series of sets of ν points in which all the ν points of any set lie upon an adjoint curve of order $n-3$ is said to be *special*. Such a series can, by what we have proved, be determined by the free intersections, with $F=0$, of a system of adjoint curves of order $n-3$, in general with prescribed fixed points, say k in number. We proceed to prove that, if the number of linearly independent adjoint curves of order $n-3$, conditioned only by their adjointness, which pass through the points of any set of the series be $\rho'+1$, then the freedom of the series is given by $\rho=\nu-p+\rho'+1$. This formula then gives definiteness to the result (*b*) above, and may be understood also to contain the result (*a*), with $\rho'+1=0$.

Take the particular set of the series which contains ρ independent general points of $F=0$, say $M_1, \ldots, M_\rho$; let $A_1, \ldots, A_{\nu-\rho}$ be the other points of this set. In the first place define $\rho'+1$ as the number of linearly independent adjoint curves of order $n-3$, conditioned only by their adjointness, which contain $A_1, \ldots, A_{\nu-\rho}$; as $\nu-\rho<p$ such curves exist. We have proved that all these curves pass through $M_1, \ldots, M_\rho$. We clearly have $\rho'+1\geqslant p-(\nu-\rho)$, the sign of inequality being unnecessary if the points $A_1, \ldots, A_{\nu-\rho}$ furnish independent conditions for such curves. These $\rho'+1$ adjoint curves of order $n-3$, which contain all of $M_1, \ldots, M_\rho, A_1, \ldots, A_{\nu-\rho}$, define a new linear series of sets of ν' points, where $\nu'=2p-2-\nu$, which is of freedom ρ'. It follows from the theorem of coresiduation that this new series is complete, namely that there is no series of sets of ν' points, of freedom greater than ρ', which contains all the sets of this series. Likewise, the whole original complete series of freedom ρ, of sets of ν points, is determined by all the adjoint curves of order $n-3$ which can be drawn through any one of the sets of ν' residual points. These two complete series are thus reciprocal; and the number of linearly independent adjoint curves of order $n-3$, conditioned only by their adjointness, which pass through the particular set of ν' residual points which we select, is $\rho+1$. Thus, beside $\rho'+1\geqslant p-(\nu-\rho)$, we also have $\rho+1\geqslant p-(\nu'-\rho')$. Since $\nu'+\nu=2p-2$, these lead to the two equations

$$\rho=\nu-p+\rho'+1, \quad \rho'=\nu'-p+\rho+1.$$

A particular consequence, also, is that the $\nu-\rho$ points $A_1, \ldots, A_{\nu-\rho}$ used in the argument furnish independent conditions for adjoint curves of order $n-3$, conditioned only by their adjointness, which pass through them. In the nature of the case, and it must be borne in mind, the equation $\rho=\nu-p+\rho'+1$ is proved on the hypothesis that the linear series is complete.

We have introduced the theory of linear series by considering the numerator polynomial of a rational function with given poles, the freedom of the series depending on the number of arbitrary constants in the function; and we have obtained a relation connecting the freedom of the series with the number of points in a set of the series. But the order of a rational function may prove less than was intended owing to undesigned coincidences in the zeros of the numerator and denominator. It is necessary then to examine whether the relation found for a complete linear series is the same as that between the number of constants in the expression of a rational function of given poles, and the order of this function.

Consider a linear series, of sets of ν points, of freedom ρ, given by a system of adjoint curves of order $n-3$, $\lambda\phi+\lambda_1\phi_1+\ldots+\lambda_\rho\phi_\rho=0$, having k prescribed common zeros (other than at the multiple points of the given curve $F=0$); and suppose that all of $\phi=0$, $\phi_1=0, \ldots, \phi_\rho=0$, have, beside, l common zeros on $F=0$. Then the quotient $(\lambda\phi+\ldots+\lambda_\rho\phi_\rho)/\phi$ is a rational function of order $\nu_0=\nu-l$, having poles only at the unprescribed zeros of $\phi=0$ other than these l common zeros. By what we have proved, the number, $\rho+1$, of the homogeneously entering arbitrary coefficients in the function is given by $\rho=\nu_0-p+\rho'+1+l$, where $\rho'+1$ is the number of linearly independent adjoint curves, of order $n-3$, otherwise unconditioned, which pass through the points of a set of the linear series. We have shewn that if the particular curve $\lambda\phi+\ldots+\lambda_\rho\phi_\rho=0$ which passes through ρ independent general points of $F=0$, $M_1, \ldots, M_\rho$, have for its remaining zeros (other than the multiple points, and the k prescribed points) the set $A_1, \ldots, A_{\nu-\rho}$, then the adjoint curves of order $n-3$ which pass through $A_1, \ldots, A_{\nu-\rho}$, and are otherwise unconditioned, all pass through $M_1, \ldots, M_\rho$, and that $A_1, \ldots, A_{\nu-\rho}$ furnish independent conditions for these curves. Among $A_1, \ldots,$ $A_{\nu-\rho}$ will be found the l undesigned common zeros spoken of, say these are $A_1, \ldots, A_l$; and, from the independence of $A_1, \ldots, A_{\nu-\rho}$, there will be $\rho'+1+l$ linearly independent adjoint curves of order $n-3$, otherwise unconditioned, which pass through $A_{l+1}, \ldots, A_{\nu-\rho}$. These $\rho'+1+l$ curves will all contain $M_1, \ldots, M_\rho$. The proof of this is the same as the proof of the former statement, and depends on the fact that there exists a rational function having M_ρ for pole and having its other poles among the points $A_{l+1}, \ldots, A_{\nu-\rho}$. Thus there are $\rho'+1+l$ adjoint curves of order $n-3$, otherwise unrestricted, passing through $M_1, \ldots, M_\rho, A_{l+1}, \ldots, A_{\nu-\rho}$. Put now

$$\rho_0'+1=\rho'+1+l;$$

then we have the conclusion that a rational function of order ν_0 contains $\rho+1$ homogeneously arbitrary constants, where $\rho=\nu_0-p+\rho_0'+1$, and $\rho_0'+1$ is the number of adjoint curves of order $n-3$, otherwise unconditioned, which pass through the points which are the poles of the function. The same result can be proved to hold for a rational function expressible as the quotient of adjoint polynomials of order $>n-3$, the number $\rho_0'+1$ being replaced by zero.

Ex. 1. A linear series of sets of points being said to be *simple* when it is not the case that every set which contains an arbitrary general point of the given curve necessarily contains one or more points of the curve, prove that a simple series which is *special*, that is, determined by adjoint curves of order $n-3$, which has freedom ρ, is such that *any* ρ points of a set of the series suffices to individualise the set (cf. Bertini, *Ann. Mat.* XXII, 1894, p. 25). Thus, if the number of points in a set be ν, *every* $\nu-\rho$ points of a set are independent conditions for adjoint curves of order $n-3$.

Ex. 2. The series, of freedom $p-1$, with $2p-2$ points in a set, which is determined by unconditioned adjoint curves of order $n-3$, is always simple, unless the given curve $F=0$ be hyperelliptic. This series is called the *canonical series* on $F=0$. The theorem is proved below (p. 80).

In the preceding theory the number p was effectively defined as the number of linearly independent everywhere finite algebraic integrals belonging to the given curve $F=0$; for it was shewn previously that these integrals are those of the form $\int \psi dx/f'(y)$ (or the equivalent of this in homogeneous variables), in which $\psi=0$ is an adjoint curve of order $n-3$. It has now been shewn, when the multiple points of the curve are of the simplest kind, k-ple points with k distinct tangents, in which case the condition of adjointness for another curve is the possession of a $(k-1)$-ple point at such a multiple point, that the conditions of adjointness at all the multiple points, for curves of order $n-3$, or more, are independent; and accordingly that the value of p in this case is given by

$$\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma k(k-1).$$

Further, arguing with this simple case in view, we have found a relation connecting the number of points in a set of a linear series and the freedom of the series, and, correspondingly, a relation connecting the order of a rational function belonging to the given curve and the number of constants left arbitrary in such a function when the poles of the function are given. It is clear, however, when two curves are in (1, 1) birational correspondence, that not only do everywhere finite integrals associated therewith transform into such integrals, but also rational functions and linear series; in the latter cases the number of arbitrary constants, or the freedom of the linear series, is unaltered by such transformation; so also is the order of a rational function, and likewise, with due consideration of fixed points of a linear series which may undesignedly become merged with a multiple point of a transformed curve, is the number of points in a set of a linear series. Thus we may say that the equation $\rho = \nu - p + \rho' + 1$, as applied to a linear series or a rational function, with inclusion of the case when $\rho' + 1 = 0$, is universally true, p being *defined* by the number of everywhere finite integrals, and *computed*, for instance, from a simple transformed equation. This relation is known as the Riemann-Roch theorem, having been given, in its application to rational functions, by Riemann, with neglect of the possibility expressed by the term $\rho' + 1$, which was supplied by Roch. The interpretation in terms of linear series was given by Brill and Noether. While, in Riemann's theory, rational functions are built up from algebraic integrals having algebraic infinities (as is explained in Chap. VI below), in Brill and Noether the theorem of coresiduation is based on a theorem for the general curve of assigned order which passes through the common points of two given curves. (Riemann, *Ges. Werke*, 1876, p. 100, etc.; Roch, *Crelle*, LXIV, 1864, p. 372; Brill u. Noether, *Math. Ann.* VII, 1873, p. 269. For an elementary exposition of Riemann's theory, see C. Neumann, *Riemann's Theorie*, 1884; for the transition to the geometrical theory, see Clebsch u. Gordan, *Abelsche Functionen*, 1866; for a comprehensive history, Brill u. Noether, *Entwicklung der Theorie der algebraischen Functionen, Deutsch. Math. Ver. Bericht*, III, 1894, pp. 109–566.)

Applications of the Riemann-Roch formula. We now develop in turn various consequences of the theory which has been established. Most often we denote by r the freedom of a linear series of sets of points upon the given fundamental curve, and by n the number of points in a set of the series; when it is necessary to refer to the order of the fundamental curve, a symbol different from n will be employed, say N; the adjoint curves of order $N-3$ will often be called simply ϕ-curves. The series itself will often be denoted* by g_r^n. There may be points common to every set of the linear series (beside the multiple points of the fundamental curve); to allow for this we may speak of n as the *complete grade* of the series, denoting the number of points of a set which actually vary from set to set as the *grade*. When every set of the series consists of points which lie upon a ϕ-curve, so that $r > n - p$, the series is called a *special* series; the series of sets of $2p-2$ points, of freedom $p-1$, which is determined by the complete series of ϕ-curves, is called

---

[*] The established notation is g_n^r, which however conflicts with the equally established notation whereby a manifold of order n and dimension r is denoted by M_r^n.

the canonical series; the Riemann-Roch theorem, which we now write in the form $r=n-p+r'+1$, enables us to prove easily that the canonical series is the only existing series g_{p-1}^{2p-2}. This series is very fundamental, the canonical series of any curve changing into the canonical series of any other curve obtained from the former by (1, 1) birational transformation. We have proved in the foregoing that, if $\phi_1=0, \dots, \phi_p=0$ be linearly independent ϕ-curves, there is no point of the fundamental curve common to all these; thus, a rational function expressed by $(\lambda_1\phi_1+\dots+\lambda_p\phi_p)/\phi_i$ is of order $2p-2$ for general values of the constants $\lambda_1, \dots, \lambda_p$; or, we may say, the canonical series has no 'fixed' points. The theory has been developed on the hypothesis that the fundamental curve is not rational; when this is so, however, we have $p=0$, and there exists no canonical series, while for every linear series $r=n$ (as we have seen in earlier chapters). Conversely, it is easy to see that the existence of a series g_r^r involves that the curve is rational. Upon such a curve there exist series g_1^1; there also exist series of sets of 2 points of freedom 1 (quadratic involutions), but such a series is not complete, being contained in a series of sets of 2 points of which both are arbitrary. When $p=1$, there exists one everywhere finite integral, but the canonical series has no freedom, the existing ϕ-curve having no intersections with the fundamental curve other than at the multiple points (for instance, for a plane quartic curve with two double points, the ϕ-curve is the line joining these). Upon a curve for which $p=1$ there exist complete series g_1^2 (for instance, upon a plane cubic curve, the lines drawn through a fixed point of the curve determine such a series); likewise upon a curve for which $p=2$, there exists one complete series g_1^2, the canonical series (for instance, upon a plane curve of order m with a single $(m-2)$-ple point, this series is given by the lines drawn through this point). In general, however, for $p>2$, the existence of a linear series of sets of two points with freedom 1, involves that the fundamental curve is not the most general curve of its genus; it involves the existence of a rational function of order 2; and we have shewn (Chap. III, p. 50) that then the fundamental curve is hyperelliptic, and can be birationally transformed to a simple form. The Riemann-Roch theorem, applied to a complete series g_1^2 on a curve of genus p, giving $1=2-p+r'+1$, shews that $p-1$ of the ϕ-curves pass through every set of the series g_1^2, namely that every ϕ-curve through one point of this necessarily contains the other. For a curve which is not hyperelliptic, the canonical series is simple, that is, it is not the case that every ϕ-curve, otherwise unconditioned, which is made to pass through an arbitrary general point of the fundamental curve, necessarily passes then through one or more other points of the curve, determined by the arbitrary point. For suppose it possible that every

ϕ-curve through an arbitrary point contained in consequence $(\mu-1)$ others; such a curve can be put through $(p-1)$ perfectly general independent points, and would therefore contain $(\mu-1)(p-1)$ others, or, in all, $\mu(p-1)$ points of the fundamental curve. But a ϕ-curve has only $2p-2$ zeros; thus μ, if >1, is 2. Suppose this is so. Then the ϕ-curves through $(p-2)$ arbitrary general points (which are expressible in the form $\mu\phi+\mu_1\phi_1=0$, where ϕ, ϕ_1 are definite and μ, μ_1 are arbitrary constants) will contain $p-2$ other definite points; the curves $\mu\phi+\mu_1\phi_1=0$ have therefore two intersections with the fundamental curve variable with μ and μ_1. There exists therefore a rational function, expressible in the form ϕ_1/ϕ, which is of the second order. The fundamental curve is therefore hyperelliptic. Suppose now that the fundamental curve is not hyperelliptic. Then the canonical series is not only simple, as we have just proved, but is further incapable of being *reduced*, in the sense of Chap. II; namely, it is not possible, by considering the ϕ-curves with k prescribed zeros, to obtain a linear series of sets of ν points, with freedom ρ, for which $\nu-2\rho$ is less than the value $[2p-2-2(p-1)$, or$]$ zero, arising for the original canonical series defined by the ϕ-curves without prescribed zeros. The ϕ-curves through k general prescribed points may, let us suppose, all pass through l other points determined by the k points (beside the multiple points of the curve $F=0$); these curves will then determine a linear series of sets of $2p-2-k-l$, say ν, variable points, with freedom $\rho=p-1-k$. We can prove that, when the curve is not hyperelliptic, and $k>0$, we always have, for such a special series, $\nu>2\rho$; that is $k>l$. This result, in the form $\nu\geqslant 2\rho$, is usually called Clifford's theorem (see Clifford, *Math. Papers*, 1882, p. 329, where general results are assumed which are obtained here much later). The argument is one we have employed before: when the series $g_\rho{}^\nu$ is established, a particular set is identified by assigning ρ points thereof, namely, of the ϕ-curves conditioned by the fixed points belonging to all of them, there is one, determined by ρ points, which contains all the points of a set. On the other hand, we have shewn above that ϕ-curves conditioned only by their adjointness which pass through $\nu-\rho$ arbitrary points of a set of the series likewise contain all the points of the set. These unconditioned ϕ-curves must evidently require at least as many conditions, in order that they may contain all the points of a set of the series, as are required by the conditioned ϕ-curves. Thus $\nu-\rho\geqslant\rho$. This argument is under the hypothesis that the series $g_\rho{}^\nu$ is complete; the result holds *a fortiori* for a series $g_\sigma{}^\nu$ contained therein with $\sigma<\rho$. That the series $g_\rho{}^\nu$, obtained as described, is in fact complete is easily proved. From the fact that the canonical series g_{p-1}^{2p-2} is simple, it follows, as in Chap. II, above, that, with coordinates $(\xi_1, \ldots, \xi_p)$, of a space of $p-1$ dimensions $[p-1]$, the equations $\xi_1/\phi_1 = \ldots = \xi_p/\phi_p$ transform the original *non-hyperelliptic* curve $F=0$, with a $(1, 1)$ birational correspondence, into a curve of order $2p-2$ in this space. And from the inequality $\nu \geqslant 2\rho$, for any series g_ρ^ν contained in the canonical series, it follows that the curve so obtained is without multiple points. Moreover this curve is not the projection of another curve of the same order, in a space of dimension $p+q-1$, $>p-1$; for such a curve, being in $(1, 1)$ correspondence with this, and therefore with the original curve, would have the coordinates of a point expressible by equations

$$\xi_1/u\phi_1 = \ldots = \xi_p/u\phi_p = \xi_{p+1}/\psi_1 = \ldots = \xi_{p+q}/\psi_q,$$

where $u, \psi_1, \ldots, \psi_q$ are polynomials in the coordinates of the original plane; the canonical series on the original curve would then be incomplete, being contained in a series of sets of $2p-2$ points, of freedom $p+q-1$, given by an equation

$$u(\lambda_1\phi_1 + \ldots + \lambda_p\phi_p) + \mu_1\psi_1 + \ldots + \mu_q\psi_q = 0,$$

the zeros of u, on the original curve $F=0$, being also zeros of all of $\psi_1, \ldots, \psi_q$. This is contrary to what we have proved. The curve, of order $2p-2$, in space of dimension $p-1$, thus obtained as representative of any non-hyperelliptic curve, is called the *canonical curve*; and (like any other curve with the same property) is said to be *normal* in the space $[p-1]$ because it cannot be obtained by projection from another curve of the same order in space of higher dimension. This curve has the great simplicity that the canonical series thereon, that is the series of sets corresponding to the canonical sets of the original curve, are determined by its intersections with the *prime spaces* of the space $[p-1]$, given by linear equations $\lambda_1\xi_1 + \ldots + \lambda_p\xi_p = 0$; and thence, any special series contained in the canonical series, is given by its intersections with primes passing through fixed points of the canonical curve. In particular, consider a special series g_ρ^ν; we have shewn that all the points of a set of this series lie on unconditioned ϕ-curves passing through any $\nu-\rho$ points of the set. In the space $[p-1]$, this number of points of the canonical curve determine a linear space $[\nu-\rho-1]$. The sets of the series g_ρ^ν on the canonical curve thus lie each in such a space. For instance, for $p=4$, there is a canonical curve of order 6 lying in ordinary space; and thereon there is a special series g_1^3 of which the sets lie upon trisecant lines of the curve; it will appear in fact that the curve is the complete intersection of a cubic surface and a quadric surface; the trisecants in question are one system of generators of the quadric surface (and there is another g_1^3). If we now assume that, for a curve c^m, of order m, in space of k dimensions $[k]$, a space of dimension $s-1$, put through s *arbitrary* points of the curve, has no further intersections with the curve, unless $s=k$, and the space is a prime (for instance, a curve, not lying in a plane, cannot be such that every chord is a trisecant), then we can infer that Clifford's theorem $\nu \geqslant 2\rho$, for a special series $g_\rho{}^\nu$, can be replaced by $\nu > 2\rho$, except when $\rho = p-1$, $\nu = 2p-2$, the special series being then the whole canonical series. For when $\nu-\rho=\rho$, a set of the series $g_\rho{}^\nu$ being determined by ρ quite arbitrary points of the canonical curve, a space $[\nu-\rho-1]$ put through $\nu-\rho$ quite arbitrary points will wholly contain a set of $g_\rho{}^\nu$, and will thus meet the curve again. By the assumption made, this is so only when $\nu-\rho=\rho=p-1$.

Remark I. A prime through a set of a special series $g_\rho{}^\nu$, on the canonical curve, meets the curve again in a set of ν' points $(\nu'=2p-2-\nu)$, determining a complementary special series $g_{\rho'}{}^{\nu'}$ where $\rho=\nu-p+\rho'+1$, so that $\nu'-2\rho'=\nu-2\rho$; and any set of this complementary series lies in a space $[\nu'-\rho'-1]$. The two spaces $[\nu-\rho-1]$, $[\nu'-\rho'-1]$, containing complementary sets, as they both lie in a prime $[p-2]$, meet one another in a space of dimension $\nu-\rho-1+\nu'-\rho'-1-(p-2)$, which is a $[\nu-2\rho-1]$ or $[\nu'-2\rho'-1]$. When $\rho>0$ and $\rho'>0$, this space is not coextensive either with the $[\nu-\rho-1]$ or the $[\nu'-\rho'-1]$. For instance when $p=4$, two complementary sets of the two series $g_1{}^3$ lie in generators of opposite systems of a quadric surface. These have a point in common (not lying on the canonical curve).

Remark II. If $p-3$ arbitrary points be taken on the canonical curve, the space $[p-4]$ determined thereby does not meet the curve further. By means of the ∞^2 spaces $[p-3]$ through this $[p-4]$, the canonical curve can be projected into a plane curve of order $p+1$. Any curve of genus p, which is not hyperelliptic, can thus be transformed birationally into a plane curve of order $p+1$ (with appropriate multiple points). It has already been remarked that a hyperelliptic curve of genus p can be represented by a plane curve of order $p+2$ (with one p-ple multiple point).

Remark III. It was proved above that unless a curve be hyperelliptic there exists no rational function of order 2. There exists, however, always a rational function of order $p+1$, with its poles at quite arbitrary places, as is easy to see. And this is the lowest order for which a rational function can always be constructed with quite arbitrary poles. For particular positions of the poles, a rational function may be constructed of order $\frac{1}{2}p+1$, or $\frac{1}{2}(p+1)+1$ (according as p is even or odd); and this is the absolutely lowest order for a rational function unless the curve be in some way less general than the most general curve of genus p. This will be proved.

Remark IV. We have proved that the canonical series g_{p-1}^{2p-2} is simple, and not further reducible in the sense of Chap. II. We can prove that the same is true of any series g_r^n in which $n > 2p$ (so that this series is not special); by such a series then the curve can be put in (1, 1) birational correspondence with a curve of order n, in space $[r]$, which is normal in this space if the series g_r^n is complete, as we suppose. As the series is not special we have $r=n-p$; thus $n>2p$ is the same as $n<2r$. That the series is simple, follows because, if every set containing an arbitrary point of the fundamental curve contained $\mu-1$ other points of this curve, we could, by taking r arbitrary points, find a set of the series containing μr points, so that $\mu r \leqslant n$, and this, by $n<2r$, involves $\mu=1$. Again there are no points of the curve $F=0$ common to all the sets of n points of the series; for if there were only $n-l$ variable points in these sets, then $n-l<n<2r$, shewing (by Clifford's theorem) that the series g_r^{n-l}, of these sets of variable points, is not special; so that $r=n-l-p$, or $l=0$. Lastly, if we fix a single point, and consider the residual series g_{r-1}^{n-k}, then $k=1$; for, if $k \geqslant 2$, the condition $n<2r$ involves $n-k<2(r-1)$, and the series g_{r-1}^{n-k} (by Clifford's theorem) is not special, so that $r-1=n-k-p$. Hence, as in Chap. II, if the system of curves determining the series be

$$\lambda_0\psi_0+\ldots+\lambda_r\psi_r=0,$$

the equations $\xi_0/\psi_0=\ldots=\xi_r/\psi_r$ define a (1, 1) birational transformation to a non-singular curve. This argument holds if the original curve be hyperelliptic. By projection on to a plane, from $r-2$ general points of this curve in $[r]$, we thus obtain a plane curve of order $p+2$ in (1, 1) correspondence with the original.

Proof of a formula for the genus in terms of the branch points. Jacobian series of a linear series. There is a very important formula, illustrating the relation between the theory of linear series, and the functional point of view, which seems to find its appropriate place here. The formula is moreover often of practical use for determining the genus of a given curve.

Let ξ be any rational function upon the fundamental curve, say of order m. At a place of the curve where ξ has a finite value, say α, the expression of ξ, in terms of the usual local parameter t, may be of the form $\xi=\alpha+\alpha_1t^r+\alpha_2t^{r+1}+\ldots$, where α_1 is not zero. At a place where ξ is infinite, with local parameter t_0, the expression may be of the form $\xi^{-1}=k_1t_0^{r_0}+k_2t_0^{r_0+1}+\ldots$, where k_1 is not zero. We have then the formula $\Sigma(r-1)+\Sigma(r_0-1)=2m+2p-2$, where p is the genus of the fundamental curve, and the summation extends to all places of this at which $r>1$, or $r_0>1$.

To illustrate, first, the geometrical meaning of this result, suppose that every r which is >1 is 2; and let the expression of ξ as a quotient of adjoint polynomials be $\xi = \vartheta/\psi$. Consider the linear series of sets of m variable points on the fundamental curve given by $\vartheta - \lambda\psi = 0$, as λ varies. For any set of this series, corresponding to a definite value of λ, there may be one or more coincidences of 2 of the m points of the set, the curve $\vartheta - \lambda\psi = 0$ having (in general) contact with the fundamental curve at such a coincidence. The aggregate of the points, on the fundamental curve, at which such coincidences occur, in all the sets of the linear series, is called the *Jacobian set* of the series, and the formula states that the number of points in this set is $2m + 2p - 2$. It will in fact be proved that the Jacobian set, on the fundamental curve, is equivalent with, or coresidual to, a set constituted by the sum of two sets of the linear series, and a set of the canonical series. Conversely, this fact enables us to define the canonical series when we have determined the Jacobian set of any given linear series of freedom 1, on the fundamental curve, and the complete series defined by this set. For if this complete series be given by the system of adjoint curves, $\lambda_0 \Phi_0 + \lambda_1 \Phi_1 + \ldots = 0$, the curves of this system which pass through the points of any two sets of the given series (or pass doubly through the points of one such set) have, for their residual intersections with the fundamental curve, the (complete) canonical series.

For instance, if the fundamental curve have for multiple points only nodal double points, the complete series on this curve determined by the points of contact of tangents to the curve from an arbitrary point, O, of the plane, is given by the intersections, other than at the nodes, of a linear system of curves all passing through the nodes. The curves of this system which contain the intersections of the curve with two arbitrary lines through O, have, as residual intersection, the sets of the canonical series. When the curve has cusps, as well as nodes, the Jacobian set of the series determined by lines through O, will include the cusps. In this case, the system of curves employed must not only be adjoint at each cusp (or pass simply through this), but must have 3 intersections, namely touch the cuspidal tangent.

To prove the theorem in general, let u be any everywhere finite integral of the fundamental curve. Then $du/d\xi$ is a rational function, and, by what was shewn as to the construction of u, this function has a canonical set of the fundamental curve as part of the set of its zeros. Its only other zeros arise when ξ is infinite; at a place where, in the notation above, $\xi = k_1^{-1} t_0^{-r_0} + \ldots$, we have, effectively, $du/d\xi = -k_1 r_0^{-1} t_0^{r_0+1} du/dt_0$, so that there is a zero of order $r_0 + 1$. The poles of $du/d\xi$ arise only from places whereat, in the notation above, $\xi = \alpha + \alpha_1 t^r + \ldots$, with $r > 1$; this form gives in fact, effectively,

$du/d\xi = (r\alpha_1 t^{r-1})^{-1} du/dt$, or a pole of $du/d\xi$ of order $r-1$. As the number of zeros of the rational function $du/d\xi$ is equal to the number of poles, and a canonical set consists of $2p-2$ simple points, we thus have $\Sigma(r-1)=\Sigma(r_0+1)+2p-2$. By hypothesis ξ is of order m, so that $\Sigma r_0 = m$. Hence we obtain the formula originally stated $\Sigma(r-1)+\Sigma(r_0-1)=2m+2p-2$. If K denote a canonical set on the fundamental curve, C denote a set where the function ξ has a given arbitrary value, and J denote a set of places whose orders are given, each with its appropriate multiplicity, by the terms of $\Sigma(r-1)+\Sigma(r_0-1)$, and the sign $\equiv$ denote that two sets of places are coresidual or equivalent, the proof we have given establishes that $K+2C\equiv J$. The set J is the aggregate of all coincidences in the sets given by $\xi=$ constant (including $\xi=\infty$), a coincidence of r points of a set being regarded as $r-1$ coincidences; or J is the generalised form of the Jacobian set.

It may be remarked here that if, from any linear series $g_r{}^n$ with $r>1$, defined by a system of curves $\lambda\psi+\lambda_1\psi_1+\ldots+\lambda_r\psi_r=0$, we choose a series of freedom 1, which is given, with variable μ, and fixed $a, \ldots, a_r, b, \ldots, b_r$, by curves $a\psi+\ldots+a_r\psi_r+\mu(b\psi+\ldots+b_r\psi_r)=0$, and take the Jacobian set of this series $g_1{}^n$, then all the Jacobian sets so obtained by different choice of $a, \ldots, a_r, b, \ldots, b_r$ are equivalent with one another, or belong to the same linear series. In fact the Jacobian set of a series given by curves $\vartheta-\lambda\psi=0$ is at once seen to be found from the intersections of the fundamental curve $F=0$ with the curve expressed by the vanishing of the Jacobian determinant of the three polynomials ϑ, ψ, F (other than at the multiple points of $F=0$); and if in this determinant ϑ, ψ are replaced respectively by $a\psi+\ldots+a_r\psi_r$ and $b\psi+\ldots+b_r\psi_r$, the curve expressed by the vanishing of the determinant is one of a linear system of curves, in which the part of the variable parameters is played by binary determinants of the form $a_r b_s - a_s b_r$.

When the curve $F=0$ has only multiple points with distinct tangents, it is easy to prove directly that the intersections with $F=0$, other than at the base points common to all curves $\vartheta-\lambda\psi=0$, of the curve represented by the vanishing of the Jacobian determinant $(\vartheta_1, \psi_2, F_3)$, are in number $2m+2p-2$, where m is the number of intersections, other than at the base points, of any curve of the system $\vartheta-\lambda\psi=0$ with $F=0$. For let $F=0$ be of order N, and $\vartheta=0$, $\psi=0$ of order M, and have, beside a $(k-1)$-ple point at every k-ple point of $F=0$, a number q of simple intersections with $F=0$. It is easy to prove that, at a k-ple point of $F=0$, the curve represented by the vanishing of the Jacobian determinant has a multiple point of order $3k-4$, with k tangents coinciding with those of $F=0$, and has therefore $3k(k-1)$ intersections with $F=0$; and that this curve has two intersections with $F=0$ at each of the q

simple base points of $\vartheta - \lambda\psi$. The determinant is of order $2M+N-3$. Thus the number of intersections, not at the multiple points, or common points of $\vartheta=0$, $\psi=0$ or $F=0$, is

$$N(2M+N-3)-3\Sigma k(k-1)-2q.$$

This, however, is $2m+2p-2$, because $m=NM-\Sigma k(k-1)-q$, $p=\frac{1}{2}(N-1)(N-2)-\frac{1}{2}\Sigma k(k-1)$.

Ex. 1. Supposing $F=0$ without multiple points, the degree in the coefficients of a curve $\Theta=0$, of order M, of the function whose vanishing expresses that $\Theta=0$ touches $F=0$ (the *tact invariant*), is $N(2M+N-3)$. And, if these curves be written symbolically as $F_x^N=0$, $\Theta_x^M=0$, and $u_x=0$ be an arbitrary line, this tact invariant is the product of the values, at all the NM intersections of these curves, of the expression

$$(F\Theta u)F_x^{N-1}\Theta_x^{M-1}/u_x.$$

For a line to touch $F=0$ is a condition of degree $N(N-1)$ in the coefficients of the line, and degree $2(N-1)$ in the coefficients of $F=0$.

Ex. 2. The relation $\Sigma(r-1)+\Sigma(r_0-1)=2m+2p-2$ is often convenient for computing the genus of a curve whose equation is given. For instance, when the equation of the curve is of the form

$$y^m=(x-a_1)^{m_1}\dots(x-a_k)^{m_k},$$

the rational function x is of order m, and the values of $r-1$, r_0-1 are obtained by considering only the places for which $x=a_1, \dots, x=a_k, x=\infty$. For example, the curve represented by $y^{10}=x(x-a)^4(x-b)^5$ is of genus 2.

Note I. The existence of a rational function with assigned poles. Weierstrass's gap theorem. It seems worth while to give greater definiteness to some of the results of this chapter by reconsidering them in connexion with the question whether, given a set of points on the fundamental curve $F=0$, there exists a rational function having every one of these as a pole of the first order.

It was shewn that if $T_1, \dots, T_k$ be points lying on a ϕ-curve (an adjoint curve of order $N-3$, if N be the order of $F=0$), then it is impossible to construct a rational function having T_k as an actual pole of the first order, with its other poles lying among $T_1, \dots, T_{k-1}$, unless it is the case that every ϕ-curve passing through $T_1, \dots, T_{k-1}$ also passes* through T_k. Thus, where there are ϕ-curves through $T_1, \dots, T_{k-1}$ which do not contain T_k, the linear series defined by $T_1, \dots, T_{k-1}, T_k$ is such that T_k belongs to every set of the series. When $k \leqslant p-1$ a ϕ-curve can always be put through $T_1, \dots, T_k$; but the remark remains equally true for $k=p, \dots, 2p-2$, if

---

* The argument, depending on the theorem of coresiduation, proceeded by the attempt to construct a function of the form $\psi/L\phi$, where $L=0$ is a line through T_k. The deduced theorem of Riemann-Roch shews that the freedom $k-p+r'+1$, of the sets coresidual with $T_1, \dots, T_k$, can only be as great as 1, when, with $k \leqslant p$, we have $r'+1>p-k$, shewing that the k points are not independent conditions for unconditioned ϕ-curves.

$T_1, \ldots, T_k$ lie on a ϕ-curve. If $\phi_1=0, \ldots, \phi_p=0$ denote independent ϕ-curves, the condition that the ϕ-curves containing $T_1, \ldots, T_{k-1}$, which we now denote by $(c_1), \ldots, (c_{k-1})$, should contain (c_k), or T_k, is the existence of the p equations

$$\phi_i(c_k)=\mu_1\phi_i(c_1)+\ldots+\mu_{k-1}\phi_i(c_{k-1}), \qquad i=1, \ldots, p,$$

where $\mu_1, \ldots, \mu_{k-1}$ are certain constants, the same for all values of i. For $k<p+1$, these equations are necessary for the existence of a rational function having (c_k) as a simple pole with its other poles among $(c_1), \ldots, (c_{k-1})$. For $k=p+1$ a rational function can be constructed with (c_k) as pole and its other poles among $(c_1), \ldots, (c_{k-1})$ (a function with $p+1$ arbitrary poles, by the formula $\rho=\nu-p$, always exists, of the form $\lambda+\mu\zeta$, where λ, μ are arbitrary constants); and, for $k=p+1$, values of $\mu_1, \ldots, \mu_{k-1}$ can be found to satisfy the p equations whatever $(c_1), \ldots, (c_{k-1}), (c_k)$ be, it being understood, in both these statements, that $(c_1), \ldots, (c_{k-1})$ do not themselves lie on a ϕ-curve. Similarly for $k>p+1$. Now suppose we construct an array of p rows and k columns, in which the elements of the sth column are $\phi_1(c_s), \phi_2(c_s), \ldots, \phi_p(c_s)$. Then, without further statement as to the value of k, we may express our conclusion by saying that a *necessary* condition for the existence of a rational function having (c_k) as pole, with its other poles among $(c_1), \ldots, (c_{k-1})$, is that the kth *column* of this array should be linearly expressible in terms of the other columns. The converse of this result is true in the form: If the kth column of this array is expressible linearly by certain of the other preceding columns, so that there exist the p equations

$$\phi_i(c_k)=A_m\phi_i(c_m)+A_t\phi_i(c_t)+\ldots+A_s\phi_i(c_s), \qquad i=1, \ldots, p,$$

where $(c_m), (c_t), \ldots, (c_s)$ are certain points from $(c_1), \ldots, (c_{k-1})$, taken for convenience in their natural order, and $A_m, A_t, \ldots, A_s$ are constants, *all different from zero*, having the same values for all values of i, *then* there exists a rational function having (c_k) and (c_m), $(c_t), \ldots, (c_s)$ as actual poles of the first order, and no other poles. It is understood that the columns containing $(c_m), (c_t), \ldots, (c_s)$ are linearly independent; otherwise the equations put down could be expressed without one or more of these.

When the number, say q, of the points $(c_m), (c_t), \ldots, (c_s)$, which precede (c_k), is at most $p-1$, we can determine $p-q$ ϕ-curves passing through these q points—these points furnishing independent conditions for ϕ-curves because the columns $(c_m), (c_t), \ldots, (c_s)$, of the array, are independent by hypothesis. In virtue of the equations, all these $p-q$ ϕ-curves pass through (c_k), and $p-q$ is the total number of linearly independent ϕ-curves passing through the $q+1$ points. Thus the linear series of sets of $q+1$ points determined by the set $(c_m), (c_t), \ldots, (c_s), (c_k)$ has freedom $q+1-p+p-q$, or 1. There is, therefore, just one rational function having these $q+1$

points as poles of the first order; and the function is actually infinite at every one of these $q+1$ points—for, if the function were infinite only at some of these, the ϕ-curve drawn through all but one of this partial set would pass through this one, and the linear equations put down would involve less than the $q+1$ points.

When the number of points $(c_m), (c_t), \ldots, (c_s)$ is p, there is no ϕ-curve passing through these points, since this would involve a linear equation connecting the p corresponding columns of the array. The linear series determined by $(c_m), (c_t), \ldots, (c_s), (c_k)$ thus has freedom $q+1-p$, or 1, and there is one rational function with these as poles; the function is actually infinite at every one of these, for the same reason as before.

When we take more than p columns $(c_m), \ldots, (c_s)$, we cannot maintain the hypothesis that the columns are independent, any $p+1$ columns being necessarily connected linearly. But we have the result: If $p+f$ points be given, there being no rational function having simple poles chosen from the first p points (and no other poles), then there are f rational functions, all having actual poles of the first order at the first p points, having respectively each a single pole at one of the other f points. The sum of these functions, each multiplied by an arbitrary constant, is a function effectively infinite at each of these f points, with also poles among the p first points. This function contains $1+f$ homogeneously entering arbitrary constants.

This result can be stated differently: As the p ϕ-curves $\phi_1(x)=0, \ldots, \phi_p(x)=0$, do not all vanish for any point (x), on the fundamental curve $F=0$, other than at the multiple points, as we have proved, and the ratios of the functions $\phi_1(x), \ldots, \phi_p(x)$ have definite values at any point (x) of $F=0$, these functions can be regarded as the homogeneous coordinates of a point in space $[p-1]$, of $p-1$ dimensions. Any set of points $(c_1), \ldots, (c_h)$, of the curve $F=0$, thus give rise to points, say $C_1, \ldots, C_h$, in this space $[p-1]$. The expression of the column (c_k) in terms of the columns $(c_m), (c_t), \ldots, (c_s)$ involves that the linear space determined by $C_m, C_t, \ldots, C_s$ contains the point C_k (cf. p. 79 preceding). When $F=0$ is hyperelliptic, the points $C_m, C_t, \ldots$ lie on a rational curve; of the two points of $F=0$ corresponding to any one of these, only one is represented by a column in the array.

There are theorems, analogous to those preceding, which relate to rational functions having their poles all at one place of the curve $F=0$, this being a multiple pole. A function, when expressed by the local parameter for the neighbourhood of the place where it is infinite, contains negative powers, $A_r t^{-r} + A_{r-1} t^{-r+1} + \ldots + A_1 t^{-1}$; when we say that the function is infinite to order r, we mean that the coefficient A_r is not zero, but we make no statement in regard to the coefficients $A_{r-1}, \ldots, A_1$. Consider now rational functions having (multiple) poles only at one place (x) of the curve. If $u_1, \ldots, u_p$ denote linearly independent everywhere finite integrals, and t the parameter for the neighbourhood of (x), we can consider the limiting values of $du_1/dt, \ldots, du_p/dt$, at (x); these we denote by $\Omega_1(x), \ldots, \Omega_p(x)$; similarly the limiting values of $d\Omega_1/dt, \ldots, d\Omega_p/dt$ may be denoted by $\Omega_1'(x), \ldots, \Omega_p'(x)$. We then form an array of p rows, in which the elements of the first column consist of $\Omega_1(x), \ldots, \Omega_p(x)$, the elements of the second column consist of $\Omega_1'(x), \ldots, \Omega_p'(x)$, and so on. From what has been said above, we can infer, by a limiting process, that the necessary and sufficient condition that there should exist a rational function, infinite only at the place (x), of order $q+1$, is that the $(q+1)$th column of this array should be linearly dependent upon all, or some, of the preceding columns. If this dependence does not require all the preceding columns, this will be because there exists a function, infinite only at (x), of lower than the $(q+1)$th order; the expression of the function of order $q+1$, in the neighbourhood of (x), will then not involve all the negative powers $t^{-q}, t^{-q+1}, \ldots, t^{-1}$, of order less than t^{-q-1}. If no one of the first p columns of the array depends on those preceding, in whole or in part, then the $(p+1)$th column depends on these p columns (the array having only p rows) and involves them all. In this case there is no function, of the kind considered, of any of the first p orders, but there is a function of order $p+1$; this indeed is the case for a general position of (x). We have proved that all the functions $\Omega_1(x), \ldots, \Omega_p(x)$ do not vanish, at any place (x); it may in fact be proved that all the functions $\Omega_1'(x), \ldots, \Omega_p'(x)$, equally, do not vanish at any place (x); or in general, not all the functions $\Omega_1^{(k)}(x), \ldots, \Omega_p^{(k)}(x)$. Hence we may regard the elements of any column of the array as the homogeneous coordinates of a point in space $[p-1]$. Clearly not all the indefinite number of points representing the columns of the array can lie in the same prime of this space, since this would mean the existence of an expression $C_1\Omega_1(x)+\ldots+C_p\Omega_p(x)$ which is constant. There must therefore be, in this indefinite series of points, just p points which are independent, and suffice to determine the space, while all the others depend on these p. There exists therefore a rational function, infinite only at the place (x), of every order except p orders. This result, called Weierstrass's *gap theorem*, is found in Weierstrass, *Math. Werke*, IV, p. 225. See also Band II, p. 235.

That there should exist a rational function, infinite only at (x), of less order than $p+1$, evidently involves that the determinant formed from the first p columns of the array, is zero; and clearly this is so only for a limited number of places (x). It can be shewn that this number is at most $(p-1)p(p+1)$, but the places need not be all distinct. For $p>1$, however, there are at least $2p+2$ distinct places; there are more unless the fundamental curve be hyperelliptic (Hurwitz, *Math. Annal.* XLI, 1893, p. 409; an exposition will be found in the writer's *Abel's Theorem*, pp. 41, 90). Geometrically, these exceptional places, for which there exists a rational function of order less than $p+1$, are characterised by the existence of a ϕ-curve vanishing thereat to the pth order (so having only $p-2$ other zeros, at most). For instance, for the curve of genus 2 represented by $y^2z^4=(x, z)_6$, it may be proved that the ϕ-curves are $z^2(x, z)_1=0$, and there are 6 cases where $(x, z)_1$ has a zero of the second order, namely when it is one of the (supposed different) factors of $(x, z)_6$. Here $6=2p+2=(p-1)p(p+1)$. For the general plane curve of order 4 (genus 3), the ϕ-curves are the lines of the plane, and the points in question are the $24, =(p-1)p(p+1)$, inflexions. But, for example, for the particular quartic curve represented by $x^4+y^4+z^4=0$, the number of such points which are distinct is $12, =2p+6$. In general, it can be proved, for $p>3$, that the number of distinct points of the kind is

$$\geqslant 2p+6+8(p-3)/(p^2-3p+4)$$

(see Segre, *Rend. Lincei*, VIII, 1899, p. 89; and Berzolari, *Enzykl. Math.* III_2, 3, p. 436). When the fundamental curve is not hyperelliptic, and is representable by a canonical curve of order $2p-2$ in space $[p-1]$ (see p. 81 preceding), the points in question, on this canonical curve, are those at which a prime has p coincident intersections therewith. For instance, on the complete intersection of a quadric and cubic surface, in ordinary space ($p=4$), there are $(p-1)p(p+1), =60$, points, at which the osculating plane of the curve has 4 coincident intersections. More generally, upon any curve of genus p, a set of a linear series g_r^n exists having r points coincident at one point of the curve. It will be proved later that there are $(r+1)(n+pr-r)$ positions of this point for which one of the $n-r$ remaining points of the set coincides with this point. For $n=2p-2$, $r=p-1$, this number is $(p-1)p(p+1)$.

Ex. Prove that for a hyperelliptic curve whose genus p is odd, there exists no rational function of order p.

Note II. On the theory of special sets, and an extension of Clifford's theorem, in the case when the fundamental curve is general for its genus. By a special set of points, as has been explained, is meant a set lying on an adjoint curve of order $N-3$, if N be the order of the fundamental curve, or, as we say, on a ϕ-curve. It is understood that the points of the set are generally not at the multiple points of the fundamental curve. When this curve is not a plane curve, a special set is one whose points form part of a canonical set thereon. We have seen that in any set of a linear series g_r^n, consisting of coresidual special sets, of freedom r, with n points in a set, there are $n-r$ points giving independent conditions for unconditioned ϕ-curves passing through them, and that all such ϕ-curves contain the remaining r points of the set. Thus, denoting the points of such a set by $(a_1), \ldots, (a_n)$, and forming the matrix of p rows and n columns, in which the jth column consists of $\phi_1(a_j), \ldots, \phi_p(a_j)$, the values of a set of independent ϕ-polynomials at (a_j), this matrix must be of rank $n-r$. The series g_r^n being complete, and $r=n-p+r'+1$, this, with appropriate notation, is the same as saying that every one of the $r(r'+1)$ determinants, expressed, for $k=1, 2, \ldots, r'+1$ and $\sigma=1, \ldots, r$, by

$$\begin{vmatrix} \phi_1(a_1), & \ldots, & \phi_1(a_{n-r}), & \phi_1(a_{n-r+\sigma}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n-r}(a_1), & \ldots, & \phi_{n-r}(a_{n-r}), & \phi_{n-r}(a_{n-r+\sigma}) \\ \phi_{n-r+k}(a_1), & \ldots, & \phi_{n-r+k}(a_{n-r}), & \phi_{n-r+k}(a_{n-r+\sigma}) \end{vmatrix},$$

must vanish; this is only equivalent to saying that there are $r'+1$ ϕ-curves containing $(a_1), \ldots, (a_n)$. *When these* $r(r'+1)$ *conditions connecting the points* $(a_1), \ldots, (a_n)$ *are independent*, we can take $n-r(r'+1)$ arbitrary points of the fundamental curve, and thence determine $r(r'+1)$ others, so that just $r'+1$ ϕ-curves contain the n points, where r is determined from n and r' by $r=n-p+r'+1$. When one such set of n points is found, it determines a series g_r^n, of which any particular set is fixed by r points of it. This number, defining a particular set of an established series, cannot, clearly, be greater than the number, $n-r(r'+1)$, of arbitrary points of the curve, from which the initial set of n points is found. Thus we have $n-r(r'+1)\geqslant r$, or $p\geqslant(r+1)(r'+1)$, or $n-r\geqslant rp/(r+1)$, which, unless r is as great as $p-1$, involves Clifford's equation $n-2r>0$. Since the initial set of n points has $n-r(r'+1)$ arbitrary points, the series g_r^n, of which, in every set, r points are arbitrary, depends only on $n-r(r'+1)-r$, that is, $p-(r+1)(r'+1)$ parameters.

The preceding determination of a special series g_r^n from $n-r(r'+1)$ arbitrary points of the curve, can be stated more geometrically in terms of the canonical curve of order $2p-2$ in space $[p-1]$, which is applicable because it is assumed that the curve is general for its genus, and in particular is not hyperelliptic. Two facts are then to be used; (i), that in a space $[s]$, of s dimensions, a space $[k]$ depends on $(k+1)(s-k)$ constants (as, for instance, a line in ordinary space depends on 4 constants); (ii), that the number of conditions for a space $[k]$, in a space $[s]$, to meet a given curve of this space, is $s-k-1$. The problem of a special series g_r^n on the canonical curve, is that of finding a set of n points on this curve lying in $r'+1$, or $p-n+r$, primes of the space $[p-1]$, namely lying in a space $[n-r-1]$. In the space $[p-1]$, spaces $[n-r-1]$ depend on parameters whose number is $(n-r)[p-1-(n-r-1)]$, or

$$nr'+n-r(r'+1);$$

for such a space to meet the canonical curve it must satisfy $p-1-(n-r)$, or r' conditions for each meeting; to meet the curve n times, if all the meetings are independent, it must satisfy nr' conditions. Thus spaces $[n-r-1]$, meeting the canonical curve in n points, if these require independent conditions, depend on $n-r(r'+1)$ constants. As the position of a point on the canonical curve depends on one parameter, we infer that, of the n intersections, $n-r(r'+1)$ may be taken arbitrarily.

Consider for example a g_1^n, for which $r'+1=p+1-n$, and $n-r \geqslant rp/(r+1)$, that is $n \geqslant 1+\frac{1}{2}p$. Taking $n-r(r'+1)$, or $2n-(p+1)$ arbitrary points of the curve, the other $p+1-n$ points of a set can be determined so that $p+1-n$ ϕ-polynomials vanish in all the points of the set. The set of $p+1-n$ supplementary points is not unique; it was found by Brill indeed (*Math. Ann.* XXXVI, 1890, p. 354) that, to put with the set of $2n-(p+1)$ arbitrary points, this supplementary set of $p+1-n$ points can be taken in

$$\sum_{\lambda=0}^{\mu} (-1)^{\lambda} \binom{p}{\lambda} \binom{2p-1-n-2\lambda}{p-2}$$

ways, where μ is the greatest integer in $\frac{1}{2}(p+1-n)$. When n has its least value, which is $1+\frac{1}{2}p$ when p is even, the number of quite arbitrary points of the set being 1, this number is $\frac{2}{p}\binom{p}{n}$; or when n has its least value for p odd which is $1+\frac{1}{2}(p+1)$, the number of arbitrary points of the set being 2, this number is $\frac{4}{p-1}\binom{p}{n}$. Thus on a curve which is general for its genus, there exists a special linear series, of freedom 1, of sets of $1+\frac{1}{2}p$, or $1+\frac{1}{2}(p+1)$ points, but not less. Thus the sets of this series can have no point in common, and these numbers are the least order of existing rational function on the curve, and, of such function, respectively 1 or 2 poles may be taken arbitrarily. We may consider some simple illustrations: (*a*) On a plane quartic curve $(p=3)$, there exists a rational function of 3 poles, of which 2 may be taken arbitrarily, the other being either of the two remaining intersections, with the curve, of the line joining these two; (*b*) On the curve of intersection of a quadric and a cubic surface $(p=4)$, there exist two series g_1^3, a single point of a set being arbitrary, the other two points of the set being on one of the two generators of the quadric which pass through this point; (*c*) On the canonical curve of order 8 and genus 5 in space [4], which we may denote by ${}^5c^8[4]$, there exists a series g_1^4, of which a set is determined by taking two arbitrary points of the curve, and drawing a plane through these to meet the curve in 2 other points, which, by the formula quoted, is possible in $\frac{4}{p-1}\binom{p}{n}$ or 5 ways. And in aggregate there are ∞^τ such series g_1^4, where $\tau = p-(r+1)(r'+1)$, namely ∞^1 such series. That there are 5 planes through two arbitrary points of the curve each meeting the curve again in 2 points, is obvious directly by remarking that, by planes through the 2 arbitrary points, the curve is projected into a plane sextic curve, which, being of genus 5, has 5 double points. We may obtain a set of 4 points on this sextic lying on $(r'+1=)\,2$ adjoint cubic curves, of which two points A, B are arbitrary, by describing cubics through the 5 double points, the points A, B of the curve, and a further point P of the curve, and choosing P so that the further intersection of these cubics also lies on the sextic curve. Particular series g_1^4 on the sextic are obtained by the lines through any one of the double points, or by the conics through any 4 of the double points. As examples of the general theorem for series g_1^n when n has not its least value, we may instance a g_1^{p-2}; if $p-5$ arbitrary points be taken, 3 others can be associated with them in $\frac{1}{6}p(p-1)(p-5)$ ways, thus making a set of $p-2$ points through which 3 ϕ-curves pass. Or a g_1^{p-1}; if $p-3$ arbitrary points be taken, 2 others can be associated with them in $\frac{1}{2}p(p-3)$ ways, thus making a set of $p-1$ points through which there pass 2 ϕ-curves.

More generally, to find a set of n points lying on 3 ϕ-curves, defining then a series g_r^n in which $r=n-p+3$, we may take $n-3(n-p+3)$, or $3p-9-2n$ points arbitrarily; but this number must be $\geqslant n-p+3$, namely $n \leqslant \frac{4}{3}(p-3)$. Considering the canonical curve in space $[p-1]$, through the n points so determined, which lie in a space $[p-4]$, can be put spaces $[p-3]$, by means of which the curve can be projected birationally into a plane curve of genus p and order $2p-2-n$; this order is then $\geqslant \frac{2}{3}(p+3)$. Considering in turn the cases in which p is of the forms 3ϖ, $3\varpi+1$, $3\varpi+2$, we find that the order of the plane curve is in all cases $\geqslant p+2-\varpi$, namely $p+2-[p/3]$, where $[p/3]$ is the integer part of $p/3$. In all cases $\frac{2}{3}(p+3)$ is greater than $1+\frac{1}{2}p$; but $p+2-\varpi$ is less than p when $p \geqslant 9$. The case when $n=p-3$ may be regarded as included in this result; by projection of the canonical curve from $p-3$ arbitrary points thereon, we have a representation of the curve as a plane curve of order $p+1$, with the equivalent of $\frac{1}{2}p(p-3)$ double points (as found above, in discussing series g_1^{p-1}). A corollary may be noted: In order that a plane curve of order n should represent a curve general for its genus, it must have multiple points equivalent to at least $\frac{1}{2}(n-2)(n-4)$ double points (for instance, a plane quintic curve is incompetent to represent a general curve of genus 6, when it is without a multiple point; or a general curve of genus 5, when it has one double point. When it has two double points it may represent a curve of genus 4; it is then obtainable by projection of the canonical sextic curve).

Some further indications of general theorems find their proper place here:

(*a*) As we know the number of coefficients entering into the plane curve of order n', or $2p-2-n$, which we have just obtained by projection of the canonical curve, from the points of a special set g_r^n thereon, we can form an estimate of the number of fixed constants, or *moduli*, on which a general curve of genus p depends. The special set on the canonical curve was found from k, $=3p-9-2n$, arbitrary points, and there are ∞^r coresidual sets, where $r=n-p+3$. Thus the series g_r^n depends on $k-r$, or $4p-3n-12$ constants. The plane curve of order n' has $\frac{1}{2}n'(n'+3)$ coefficients, of which 8 are disposable by suitable linear transformation of the coordinates in the plane; this plane curve has the equivalent of $\frac{1}{2}(n'-1)(n'-2)-p$ double points, of which the existence by itself of any one would impose one condition for the coefficients of the curve. There are then *at least* $\frac{1}{2}n'(n'+3)-8-[\frac{1}{2}(n'-1)(n'-2)-p]$, or $3n'+p-9$ independent constants; of which, however, we can dispose of $4p-3n-12$ by suitable choice of the set of points of the canonical curve from which we project. There remain then, at least,

$$3n'+p-9-(4p-3n-12)$$

absolute constants, namely $3p-3$, for the general curve of genus p. This provisional reasoning, however, (1) is only for curves general for their genus, and, in particular, excludes the case of hyperelliptic curves; thus it supposes $p>2$, though it happens that 3 is the right number of constants, in this case, for a general curve represented by $y^2z^4=(x, z)_6$, being the number of independent cross ratios of the roots of the sextic $(x, z)_6$; (2) supposes that, for a general curve, the existence of d double points requires d conditions for the constants of the curve; (3) supposes that the curve has no birational transformations into itself involving an arbitrary parameter; such a transformation would involve a lessening of the number of constants upon which a series g_r^n depends. It will in fact be proved below that no such transformations exist for $p>1$. For $p=0$ there are such transformations depending on 3 parameters; for $p=1$, there is a transformation depending on one parameter. The number $3p-3$ was found by Riemann (1857, Werke, *Theorie der Abel'schen Fctnen*, § 12), who estimates the total number of constants in a rational function of order m as $2m-p+1$, since if its m poles be assigned there are further $m-p+1$ constants; and, then, regarding the curve as represented by an equation $F(\eta, \xi)=0$, proves that there are $2m+2p-2$ branchings of η as a function of ξ (as above, p. 84). There are then $2m+2p-2-(2m-p+1)$ of these which cannot be arbitrarily assigned by suitable choice of ξ. Recent criticisms of the argument for the number $3p-3$ will be found in Severi-Löffler, *Alg. Geom.* Leipzig, 1921, pp. 321, 394; and in Enriques-Chisini, *Teoria geom.* III, Bologna, 1924, pp. 112, 359.

Ex. Consider the curve $y^3z^3+y^2z^2u_2+yzu_4+u_6=0$, where u_m is homogeneous of order m in x and z. Let (a, b, c) be any point of this curve. Shew that by the transformation obtained by putting ξ, η, ζ proportional respectively to $zc(xc-za)$, $zc(yc-zb)$, $(xc-za)^2$, of which the reverse forms give x, y, z proportional respectively to $\xi(\xi a+\zeta c)$, $\xi^2 b+\eta\zeta c$, $\xi^2 c$, the curve is changed to a quintic curve with a point of self-contact, whose equation, with a further suitable change, of the form, $\eta=\eta_1+\lambda\xi$, may be taken to be $\eta_1{}^3\zeta^2+\eta_1{}^2\zeta v_2+\eta_1 v_4+\zeta w_4=0$, where v_i, w_i are homogeneous in ξ, ζ, of order i. For the sextic and quintic, respectively, the ϕ-polynomials are of the forms $z[yz+(x, z)_2]$, $\eta_1\zeta+(\xi, \zeta)_2$. A ϕ-curve which touches the fundamental curve at each of its intersections is given, in these cases respectively, by $z(x-\mu z)^2=0$, $(\xi-\sigma\zeta)^2=0$, where μ, σ are arbitrary constants; namely there are ∞^1 such ϕ-curves. Hence, by a theory which cannot be given here, the curves each depend, not on $3p-3$ or 9, but on 8 moduli (cf. the writer's *Abel's Theorem*, 1897, pp. 270, 94). There are, as in general, $2^{p-1}(2^p-1)$ or 120, other ϕ-curves having contact with the fundamental curve at all their intersections therewith. (For an elementary geometrical theory, in this case, cf. W. P. Milne, *Proc. Lond. Math. Soc.* XXV, 1926, p. 174.)

It may happen that there are ∞^2 (or more) such particular contact ϕ-curves; for instance for a general plane quintic curve, every line of the plane, repeated, is such a curve, and the aggregate is ∞^2. In this case the number $2^{p-1}(2^p-1)$, of proper contact ϕ-curves, or 2016, is reduced to 2015, in accordance with the general theory (cf. W. P. Milne, *Journ. Lond. Math. Soc.* II, 1927, p. 79, for this particular case). In the notation of the general theory referred to, the two exceptional ϕ-systems are associated respectively with an even and an odd characteristic.

(*b*) Another question is what is the number of constants upon which a general curve of order n and genus p normal in space $[r]$ depends. Thus, though all rational curves are birationally equivalent, a conic *in ordinary space* depends on 8 constants (for example, a single conic exists which meets an arbitrary line and also seven other lines which meet this, and do not meet one another); and, more generally, the rational curve of order r in space $[r]$ depends on $(r-1)(r+3)$ constants, as we shall see. While the canonical curve of order $2p-2$ in space $[p-1]$ depends on $(p-1)(p+4)$ constants (for instance, a plane quartic curve on 14; and the intersection of a quadric surface and a cubic surface on $9+(19-4)$, or 24 constants). It can be proved that the number in general is $(r+1)[n-r-rp/(r+1)]+(r+1)^2-1+3p-3$, the first term being the number τ, $=p-(r+1)(r'+1)$, of constants upon which a *complete* series $g_r{}^n$, where $r=n-p+r'+1$, depends ($\tau \geqslant 0$; p. 91); the second term, the number of constants in a general linear transformation of coordinates in space $[r]$; and the last term, the number of absolute constants of the class of all birationally equivalent curves of genus p; namely is $n(r+1)-(r-3)(p-1)$.

If the series of prime sections of the curve be of deficiency h, and specialness $r'+1$, the number $(n-r)(r+1)-rp$ must be diminished by $h(r'+1)$; e.g. the octavic curve of genus 5, intersection (save for two skew conics) of a cubic and quartic, depends on 31 constants. When the curve is not general, this number may be negative and need increase (p. 91). Examples are, the plane quintic curve $(p=6)$, depending on $3n+p-1$ constants; the space septimic $(p=6)$, partial intersection of a quadric and quartic, depending on $4n$; and the space octavic $(p=9)$, complete intersection of a quadric and quartic, depending on $4n+1$. In general, a complete intersection on a quadric depends on $2n+p+8$ constants $(n>4)$. See Noether, *Berlin. Abh.* 1882, pp. 18, 58.

As to surfaces, a Veronese quartic surface, in space [5], depends on 27 constants, and can be put through 9 points of its space; a Del-Pezzo surface, of order r, in space $[r]$, depends on r^2+10. Cf. Room, *Proc. Camb. Phil. Soc.* XXVII, 1931, p. 518. In general, see Castelnuovo-Enriques, *Enzykl. Math.* III, C. 6*b*, p. 713.

(*c*) It was proved by Noether that the canonical curve of order $2p-2$ in space of $p-1$ dimensions lies on $\frac{1}{2}(p-2)(p-3)$ quadrics (an exposition, of the proof, for quadrics, and manifolds of higher dimension, will be found in the writer's *Abel's Theorem*, Cambridge, 1897, p. 155). For general values of p, these quadrics have no other common intersection than the canonical curve, unless the canonical curve possesses a special series g_1^3 $(p>4)$, in which case the quadrics intersect in a rational ruled surface of order $p-2$. But, in particular for $p=6$, if the canonical curve contain a special series g_2^5, being the representative of a plane quintic curve, the quadrics intersect in a Veronese surface (Enriques, *Rend....Bologna*, XXIII, 4 May 1919).

For a curve of order n and genus p, in space $[r]$, when $r=n-p$, if also $\frac{1}{2}r(r-1)-p>0$, it can be shewn that the curve lies on $\frac{1}{2}r(r-1)-p$ quadrics *at least*.

(*d*) We have seen that a set of a special series g_r^n, on the canonical curve, lies on a space $[n-r-1]$, the sets of the complementary series $g_{r'}^{n'}$, where $n'=2p-2-n$, and $r=n-p+r'+1$, lying similarly each on a $[n'-r'-1]$. Supposing $r'\geqslant r$, it can be shewn that all the spaces $[n-r-1]$ generate a manifold of dimension $n-1$, and order $\binom{r'+1}{r}$, which likewise can be defined from the spaces $[n'-r'-1]$. If τ, given by $p-(r+1)(r'+1)$ be $\geqslant 1$, all the spaces of this manifold contain a space $[\tau-1]$; and the locus of these spaces $[\tau-1]$, for all the ∞^τ series g_r^n, is a manifold of dimension $2\tau-1$. The theorem arises very naturally from the expression of the everywhere finite integrals given in Chap. VII following, to which we refer.

Ex. 1. Suppose $p=6$. In general, by the preceding theory, there exist, on the canonical curve of order 10, in space [5], five special series g_1^4, any such series being established by taking an arbitrary point on the curve, and thence determining three other points, so that the 4 points lie on a plane. The curve can be represented on a plane by a sextic curve, with 4 nodes, obtainable by projecting the canonical curve from a set of 4 coplanar points thereof. The ϕ-curves of the plane sextic, 6 in number, proportional to coordinates in the space [5], not only transform the sextic curve into the canonical curve, but transform the plane into a rational surface in the space [5], containing the canonical curve, which is of order 5, because two of the ϕ-curves in the plane have 5 intersections (other than the 4 nodes). This is called a Del-Pezzo surface. The five series g_1^4 on the plane sextic are obtained, either by lines through a node, or by conics through the 4 nodes; in each case the determining curves meet a ϕ-curve (which is represented on the surface by a prime section) in two variable points. Thus there are 5 systems of conics on the Del-Pezzo surface, one conic of each system through an arbitrary point of the surface, the planes of the conics meeting the canonical curve in sets g_1^4. The complementary series to the g_1^4 is a g_2^6, of which the sets lie in spaces [3], or solids; any set of g_2^6 lies with any set of the complementary g_1^4 in a prime, or space [4], so that the plane of the latter meets the solid of the former in a line. To establish a series g_2^6 we may take $n'-r'(r+1)$, with $n'=6$, $r'=2$, $r=1$, that is two arbitrary points of the canonical curve, and find thence 4 points of the curve lying in a solid with the 2 arbitrary points. In the plane representation, the five series g_2^6 are given, either by a conic through 3 of the nodes of the plane sextic curve, or by a line of the plane, and in every case the determining curve meets an adjoint cubic curve in the plane (which represents a prime section of the Del-Pezzo surface) in 3 points; thus the rational cubic curve through the points of a set of a g_2^6, in the solid containing this set, is one of a system of cubic curves lying on the Del-Pezzo surface, of which one curve can be put through 2 arbitrary points of the surface. Further, in the plane any curve determining a set of a series g_1^4 meets any curve determining a set of the complementary g_2^6 in two points; thus on the Del-Pezzo surface, any conic, of either system, meets any cubic curve, of the complementary system, in two points.

Besides the g_1^4 or g_1^{p-2}, on the canonical curve, there are series g_1^5 or g_1^{p-1}, of which each set lies in a solid, a set being obtainable by taking $p-3$, or 3, arbitrary points of the curve, and determining the other 2, so that the five lie in a solid, which is possible in $\frac{1}{2}p(p-3)$, or 9, ways. In the plane representation, we take three arbitrary points of the plane sextic curve, A, B, C, and a further point P so that the remaining intersection Q of the two adjoint cubic curves through A, B, C, P may also lie on the sextic curve. The complementary series of a g_1^5 is likewise a g_1^5. Beyond these series, and the canonical series g_1^{10}, and the g_1^4, g_2^6 discussed, there appear to be no special series on the canonical curve, when this is perfectly general.

But when the curve is particular there may be other series. Consider plane sextic curves, (a) which have 4 nodes of which 3 are in line; (b) which have a triple point and a single double node; (c) which have 4 nodes all in a line. In case (a), the line has no further intersection with the sextic curve, but the 3 nodes are not independent conditions for the line (cf. footnote, p. 72, under (b)); the six adjoint cubic curves contain 5 consisting of the line and an arbitrary conic through the fourth node; two such conics have 3 further intersections. Thus it may be proved that the Del-Pezzo surface has a node, and lies on 5 quadrics passing through this; the canonical curve lies also on another quadric not containing the node. There are now 4 special series g_1^4, two of the 5 generally existing coinciding with one another. In case (*b*), the adjoint cubics have one fixed double point, and one simple base point, and the Del-Pezzo surface degenerates, containing as part a rational ruled quartic surface, which lies on all quadrics containing the canonical curve. This curve has now a g_1^3 (corresponding to lines in the plane through the triple point), of which the complementary series, g_3^7, is determined by primes through any generator, containing one set of g_1^3; the sets of g_3^7 lie on rational cubic curves. The ruled surface contains ∞^1 conics, each of which contains a set of a g_1^4 lying on the canonical curve; any prime through such a conic meets the surface beside in two generators. In case (*c*), where the 4 nodes of the plane sextic are in line, the sextic curve now contains this line, and the adjoint cubic curves break up into this line and a variable conic. The Del-Pezzo surface is again degenerate, consisting of a Veronese surface, of order 4, and a plane meeting this in a conic. The canonical curve is obtained by the intersection of the Veronese surface with a cubic primal passing through a conic of the surface; and now contains a g_2^5, corresponding to the intersections of the plane quintic curve with a variable line. The six quadrics containing the canonical curve contain also the Veronese surface (cf. (*c*), p. 96 above).

Ex. 2. The reader may similarly consider the particular cases for $p=7$, taking a plane septimic curve with 8 double points as representative curve.

Ex. 3. By remark (*d*) above, p. 96, the ∞^1 spaces $[\frac{1}{2}(p-1)]$ containing the sets of one series g_1^n on the canonical curve of odd genus p, where $n=1+\frac{1}{2}(p+1)$, meet in a point, generating a cone of order $\frac{1}{2}(p-1)$. There are ∞^1 such series, and the locus of this point is a curve; D. W. Babbage (*Proc. Camb. Phil. Soc.* XXVIII, 1932, p. 426) finds for $p=5$ that this curve is of genus 6 and order 10.

Ex. 4. On a non-singular plane curve of order n there is a special series g_1^{n-1} determined by lines through a fixed point O of the curve. The complementary series consists of sets obtained by adjoining O to the set of intersections of the fundamental curve with a general curve of order $n-4$.

Ex. 5. A plane curve of order n with less than $n-2$ double points $(p>\frac{1}{2}(n-2)(n-3))$ is unobtainable by projection from a curve of order n in higher space; namely is a *normal* curve. But a curve of order n with more than $\frac{1}{2}(n-2)(n-3)$ double points $(p<n-2)$ can always be so obtained. When $p=n-2$, a complete series g_2^n is obtainable by variable adjoint curves of order $n-2$ through $2p-2$ (or $2n-6$) arbitrary fixed points; this series agrees with that determined by the lines of the plane when the $2p-2$ fixed points are a canonical set.

The enumerative problem of finding the number of curves satisfying a number of conditions equal to that of the constants on which such a curve depends arises below (Vol. VI, Chap. II). For a geometrical discussion of this problem for conics (and quadrics) see Ursell, ***Proc. Lond. Math. Soc.*** XXX, 1929, p. 322. For rational cubics see Todd, ***Proc. Royal Soc.*** A, CXXXI, 1931. For a rational quartic curve in space [4], see White, ***Journ. Lond. Math. Soc.*** IV, 1929; Welchman, ***Proc. Camb. Phil. Soc.*** XXVIII, 1932; Todd, ***Proc. Camb. Phil. Soc.*** XXVI, 1930, p. 332; Babbage, ***Journ. Lond. Math. Soc.*** VIII, 1933.

Note III. On the points common to two plane curves. The Cayley-Bacharach theorem. It is a familiar fact, which attracted attention very early in the study of plane curves, that, of the mn common points of two curves of orders m and n, which have no multiple points, not all can be arbitrarily assigned. When $m \geqslant n$, the equation of the curve of order m can be so modified, with the help of the equation of the other curve, as to contain only

$$mn - \tfrac{1}{2}(n-1)(n-2)$$

assignable coefficients, so far as its intersections with the curve of less order are concerned. Thus we can assume (what is a particular case of the subsequently proved Riemann-Roch theorem) that, among the mn intersections, there are $\frac{1}{2}(n-1)(n-2)$ such that all curves of order $m(\geqslant n)$ drawn through the others, pass through these.

Assuming this, we can prove that, in order that mn given points should be the intersections of two curves of orders m and n, with $m > n$, the coordinates of the mn points must be connected by $mn - 3n + 1$ conditions; when $m = n$, they must be connected by $n^2 - 3n + 2$ conditions (e.g. 6 points, to be the common points of a conic and a cubic curve, must satisfy one condition; 9 points, to be common to two cubic curves, must have their coordinates subject to two conditions). To formulate the proof we mentally divide the mn points, when $m > n$, into three sets: A, of $\frac{1}{2}n(n+3)$ points; B, of $mn - [\frac{1}{2}n(n+3) + \frac{1}{2}(n-1)(n-2)]$ points; C, of $\frac{1}{2}(n-1)(n-2)$ points. We can suppose the points A taken arbitrarily, and therefore of sufficient generality to determine a curve, $f = 0$, of order n, passing through them. That the points B should lie on this curve $f = 0$ requires $mn - \frac{1}{2}n(n+3) - \frac{1}{2}(n-1)(n-2)$ conditions connecting the coordinates of the points $A + B$, whose aggregate number is $mn - \frac{1}{2}(n-1)(n-2)$. This number is less than the number, $\frac{1}{2}m(m+3)$, of arbitrary points through which a curve of order m can be described, being less than this by $\frac{1}{2}(m-n+1)(m-n+2)$. We put then a curve, $\phi = 0$, of order m, through the points $A + B$; that its remaining $\frac{1}{2}(n-1)(n-2)$ intersections with $f = 0$ should agree with the points C requires $(n-1)(n-2)$ conditions connecting the coordinates of the points $A + B + C$. The total number of conditions required is thus $mn - \frac{1}{2}n(n+3) - \frac{1}{2}(n-1)(n-2) + (n-1)(n-2)$. This is the number $mn - 3n + 1$ enunciated. The number of points in the set B is, we easily see, $(m-n)n - 1$, and the set B does not exist unless $m > n$. Indeed, when $m = n$, there remain, after taking the set A, of $\frac{1}{2}n(n+3)$ points, only $\frac{1}{2}(n-1)(n-2) - 1$ of the given n^2 points. In this case, therefore, we begin by selecting $\frac{1}{2}n(n+3) - 1$ arbitrary points; the $\frac{1}{2}(n-1)(n-2)$ remaining intersections of the two curves, of order n, which can be put through the selected points, must agree with the $\frac{1}{2}(n-1)(n-2)$ remaining given points. This requires n^2-3n+2 conditions connecting the coordinates of the given points, as was said.

Relations among the mn intersections of two curves were investigated by Jacobi (*Werke*, III, pp. 285, 610); these can be used to prove Abel's Theorem (Chap. III, above), as in Clebsch-Gordan, *Abelsche Fctnen*, 1866, p. 44. Proof of Jacobi's relations (for functions of any number of variables) is given in Netto's *Algebra*, II, 1900, p. 165, with references to Kronecker. See also Harnack, *Math. Ann.* IX, 1876, p. 371, who obtains the results in the following form (likewise capable of extension to any number of variables): let $f=0$, $\phi=0$ be two curves, of orders n and m, and $u=0$ an arbitrary line; also, the coordinates being x, y, z, let J be the Jacobian determinant $\partial(f, \phi, u)/\partial(x, y, z)$. Then, if Φ be any polynomial, homogeneously of order $m+n-3$, the sum of the values of the fraction $u\Phi/J$, at all the mn common points of the curves, is zero. For *the two given curves*, whose coefficients enter into these equations, this gives, in general, $\frac{1}{2}(m+n-1)(m+n-2)$ relations.

We now proceed to consider the conditions to be satisfied by a general curve, $\psi=0$, of order r (without multiple points), in order that it may pass through the mn points common to two given curves, $f=0$, of order n, and $\phi=0$, of order m, also without multiple points. We put $r=m+n-3-\alpha$, taking $\alpha=0$ when $r \geqslant m+n-3$, and suppose $r \geqslant m$, $r \geqslant n$. We prove that, when $r > m+n-3$, the mn common points of $f=0$, $\phi=0$ give independent conditions for the curve $\psi=0$, of order r, to pass through these points; but that, when $r \leqslant m+n-3$, these mn points give only $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$ independent conditions; in this latter case there are in fact

$$mn-\tfrac{1}{2}(\alpha+1)(\alpha+2)$$

points, among the mn, such that curves $\psi=0$ through these necessarily contain the others*.

The results we reach are, *in part*, contained in a much more general theorem, which will be considered in some detail in Chap. VIII, below. For the sake of clearness we enunciate this general theorem: In space $[s]$, of s dimensions, consider the manifold which is the intersection of

---

* In particular, curves of order $m+n-3$ through all but one of the intersections of two given irreducible curves of orders n and m necessarily contain the remaining intersection. This is in accordance with the identity

$$\tfrac{1}{2}(m+n-1)(m+n-2)=mn-1+\tfrac{1}{2}(m-1)(m-2)+\tfrac{1}{2}(n-1)(n-2).$$

It may be compared with the known result that, for a non-degenerate curve of order $m+n$, having double points, these double points give independent conditions for curves of order $m+n-3$ required to pass through them. For the general case of the adjoint ϕ-curves of a degenerate curve, see Noether, *Acta Math.* VIII, 1886, p. 161.

h primals, each given by the vanishing of a single polynomial homogeneous in the $s+1$ homogeneous coordinates; suppose that this manifold of intersection has no multiple part of dimension $s-h$. Let the orders of the intersecting primals be $n_1, n_2, \ldots, n_h$. Then the number of independent linear conditions for the coefficients of a primal of order ρ, in order that it may contain the manifold of intersection, is obtainable as the coefficient of t^ρ in the ascending expansion of the function

$$(1-t^{n_1})(1-t^{n_2})\ldots(1-t^{n_h})/(1-t)^{s+1}.$$

This number is called the postulation of the manifold for primals of order ρ. The proof of the statement may be found in Bertini, *Geom. d. Iperspazi*, 1907, p. 263; with which cf. F. S. Macaulay, *Modular Systems*, Cambridge Tracts, No. 19, 1916, p. 65. If we apply this general theorem to the case now under consideration, for which $s=2$, and the manifold of intersection is that of two curves of orders n, m, whose common points are simple, the postulation of these mn common points, for a curve of order r, is the coefficient of t^r in the ascending expansion of $(1-t^n)(1-t^m)(1-t)^{-3}$, namely, is $(r+2)_2-(r-m+2)_2-(r-n+2)_2+(r-m-n+2)_2$, with the omission of any binomial coefficient, $(\lambda+2)_2$, of these for which $\lambda<0$. Whence for $r \geqslant m+n$, all these terms enter, and it is easy to verify that their sum is mn; for $r \geqslant m$, $r \geqslant n$, but $r < m+n$, the postulation is, thus, $mn-\frac{1}{2}(r-m-n+1)(r-m-n+2)$, which remains equal to mn for $r=m+n-1$ or $m+n-2$; for $r=m+n-3-\alpha$, with $\alpha \geqslant 0$, the postulation is $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$. These results are in accordance with the theorem, also part of the general theorem referred to, that when $\psi=0$ passes through all the (simple) intersections of $f=0$, $\phi=0$, there exist two curves $f'=0$, $\phi'=0$, of appropriate orders, such that we have an identity $\psi=f'f+\phi'\phi$.

In the present note we prove a more detailed theorem: *For $r \geqslant n$, $r \geqslant m$, $r=m+n-3-\alpha$, with $\alpha \geqslant 0$, the curves $\psi=0$, of order r, put through $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$ of the intersections of two non-singular curves $f=0$, $\phi=0$, of orders n, m, with simple intersections, have a freedom $i+1+\frac{1}{2}(r-n)(r-n+3)+\frac{1}{2}(r-m)(r-m+3)$, that is are, in tale, $i+(r-n+2)_2+(r-m+2)_2$, where i is the number of linearly independent curves of order α containing the $\frac{1}{2}(\alpha+1)(\alpha+2)$ excepted intersections (in general no curves of order α can be put through this number of points); and further that the curves $\psi=0$, so described, will pass through these excepted points if made to contain i of them, suitably chosen (when $i<\frac{1}{2}(\alpha+1)(\alpha+2)$).* The

$$mn-\tfrac{1}{2}(\alpha+1)(\alpha+2)$$

chosen points are thus equivalent only to $mn-\frac{1}{2}(\alpha+1)(\alpha+2)-i$ independent points as conditions for curves of order r put through them; and the choice of i further independent points, among the $\frac{1}{2}(\alpha+1)(\alpha+2)$ excepted points, makes the number of conditions, in all, equal to $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$; thus, curves $\psi=0$, of order r, put through $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$ points, of the intersections of $f=0$, $\phi=0$, which provide independent conditions, pass through the remaining intersections. Obviously, the $i+(r-n+2)_2+(r-m+2)_2$ curves through the $mn-\frac{1}{2}(\alpha+1)(\alpha+2)$ selected points may contain, necessarily, *some* of the $\frac{1}{2}(\alpha+1)(\alpha+2)$ remaining points, as we shall illustrate by examples. The theory we give was initiated by Cayley (*Papers*, I, 1843, p. 27), and developed by Bacharach (*Math. Ann.* XXVI, 1886, pp. 275–99); it serves as an application of the general theory of this chapter. From this theory we assume two lemmas: (1), Among the intersections of a curve $f=0$, of order n, which may have multiple points, with an adjoint curve of order μ, let there be a set of p points (not at the multiple points) which lie on h adjoint curves ϕ, of order $n-3$; denote by H a set of h points suitably chosen from these p points, and by K the remaining $p-h$ points of these; also denote by M the intersections (other than the multiple points) of the curve of order μ with $f=0$ other than the p points $H+K$. Then every adjoint curve of order μ through the sets M, H contains the set K (the set $H+K$ defines a series $g_h{}^p$). (2), An adjoint curve of order μ, with $\mu>n$, put through $n\mu-p+1$ of the intersections of $f=0$ with a particular adjoint curve of order μ (other than the multiple points), contains $f=0$ entirely, if these $n\mu-p+1$ points are independent conditions for such adjoint curves. Here p as usual denotes the genus of $f=0$.

Now consider the whole set of intersections, with $f=0$, of order n, beside those at the multiple points, of an adjoint curve of order m (which we may denote by c^m), in which $m>n-3$. These points are of number $n(m-n+3)+2p-2$. Divide this set into two sets: one set A, and a remainder set B; this latter consisting of β points, where

$$\beta=p-[kn-\tfrac{1}{2}k(k+3)],\ =\tfrac{1}{2}(s+1)(s+2)-[\tfrac{1}{2}(n-1)(n-2)-p],$$

in which k is any number >0 and $<n-3$, small enough to make $\beta\geqslant 0$, and $s=n-3-k$. It is easy to see that the two forms of β are equal, and, from the second form, one more than the number of independent points through which we could put an adjoint curve, c^s, of order s, if the multiple points of $f=0$ were independent conditions for this c^s. Take, further, a set A_0, of $\frac{1}{2}k(k+3)$ points of $f=0$, so chosen as to be independent conditions for non-adjoint curves of order k put through them; let $\omega=0$ be the curve, c^k, through A_0; denote the residual intersections of $\omega=0$ with $f=0$ by B_0, so that B_0 is of $kn-\frac{1}{2}k(k+3)$, or $p-\beta$ points.

The composite set $B+B_0$ consists then of p points. The set B_0 is of number $kn-\frac{1}{2}k(k+3)$, which is greater than the number, $\frac{1}{2}k(k+3)$, of independent points through which a non-adjoint curve of order k could be drawn, because $k<n-3$; and the number of points in B_0 is $k(n-3)-\frac{1}{2}(k-1)(k-2)+1$, or $k(n-3)-q+1$, where q is the genus of the curve $\omega=0$. Thus, by lemma 2, all curves of order $n-3$ through the set B_0 contain $\omega=0$ as a part, provided these points give independent conditions on $\omega=0$ for curves of order

$n-3$ not necessarily adjoint to $f=0$. The result certainly follows for $k=1$, or $k=2$, the number of points being then $>k(n-3)$.

Now denote by $i\,(\geqslant 0)$ the number of curves, c^s, of order s, adjoint to $f=0$, if any, which can be drawn through the set B $(0<s<n-3)$. If $\psi=0$ be such a curve, the curve $\omega\psi=0$ is a curve, adjoint to $f=0$, of order $n-3$, through the p points $B+B_0$, and there are i such curves at least, since there are i curves $\psi=0$. Let $i+j$ be the total number of curves of order $n-3$, adjoint to $f=0$, through the set $B+B_0$. Certainly $j=0$ when $k=1$, or $k=2$, and more generally when the $p-\beta$ points B_0 are independent for curves of order $n-3$, as we have remarked. This is ensured if, but does not require* that, the $p-\beta$ points B_0 should be independent for curves of order $n-3$ adjoint to $f=0$; and this sufficient condition is satisfied if there are no other sets of $p-\beta$ points of $f=0$ which are coresidual with the set B_0; this again is true if $f=0$ have no multiple points, since the $\frac{1}{2}k(k+3)$ points A_0 are supposed independent for curves of order k.

The sets $A+A_0$, $B+B_0$, together, form the complete intersection, with $f=0$, of a composite adjoint curve of order $m+k$, consisting of the adjoint curve of order m through $A+B$, together with the non-adjoint curve $\omega=0$; and the set $B+B_0$ consists of p points, and lies on $i+j$ adjoint ϕ-curves of $f=0$. Wherefore (see lemma 1), the general adjoint curve of order $m+k$, put through the set $A+A_0$, gives on $f=0$ a series of freedom $i+j$; and this curve will contain all the points $B+B_0$ if made to contain $i+j$ of them which give independent conditions (when $i+j<\beta$). Now, the set A_0 gave independent conditions for non-adjoint curves of order k; thus it likewise gives independent conditions for curves of order $m+k$ adjoint to $f=0$. Wherefore, discarding the set A_0, we infer that a curve of order $m+k$, adjoint to $f=0$, which contains all but β of the intersections with $f=0$ of an adjoint curve of order m (namely, passes through the set A) has, on $f=0$, a freedom $i+j+\frac{1}{2}k(k+3)$. Here i is the number of curves of order s, adjoint to $f=0$, which pass through these β remaining intersections $(s=n-3-k)$, and $i+j$ is the number of adjoint curves of order $n-3$, passing through the p points constituted by these β remaining intersections, and through $p-\beta$ points of $f=0$, which are residual, by a non-adjoint curve $\omega=0$ of order k, to $\frac{1}{2}k\,(k+3)$ arbitrary general points of $f=0$. If we denote $m+k$ by r, so that $s=m+n-3-r$, and recognise that the absolute freedom, in the plane, of curves of order r, $> n-3$, is greater than the freedom on $f=0$ by $\frac{1}{2}(r-n)$ $(r-n+3)+1$, we reach the conclusion that the tale of the curves of order r, adjoint to $f=0$, under consideration, is $i+j+(r-m+2)_2+(r-n+2)_2$. We have stated the proof subject to $n-3<m<r<m+n-3$. When $f=0$ has no multiple points, we have proved that $j=0$; in this case also $\beta=\frac{1}{2}(s+1)(s+2)$. We can apparently allow the possibility $s=0\,(k=n-3,\ \beta=1)$, and hence $r=m+n-3$; and also the possibility $k=0$, with $r=m$; and remove the condition $m>n-3$. Then the conditions are $n\leqslant r$, $m\leqslant r$, $r\leqslant m+n-3$, and we obtain the result as stated above, when $f=0$ has no multiple points.

---

* For a plane sextic curve $f=0$, with $p=4$, having its six double points on a conic, taking $k=1$, and hence $\beta=0$, a set A_0, of 2 points, gives 4 collinear points B_0, which are independent for general cubic curves, but equivalent only to 3 for cubic curves adjoint to $f=0$. Thus, here also, $j=0$, the adjoint cubics through B_0 reducing to the line and the conic containing the six double points $(i=1)$.

But though, when $i>0$, it does not follow that non-singular curves of order r, put through any $mn-\beta$ intersections of the non-singular curves of orders n and m, necessarily pass through all the other β points, unless put also through i independent points of these, it may well be that there are *some* of the β points through which they do all pass. Namely, going back to the argument, the linear series of sets of p points determined on $f=0$ by the set $B+B_0$, which is special since $i>0$, may have points common to all sets of the series. It may be, in particular, that, among the β points B there are β_1 points, such that curves of order $n-3$ through these β_1 points (otherwise unrestricted, $f=0$ being now without multiple points), and through the set B_0, do not necessarily pass through the remaining $\beta-\beta_1$ points of B, these latter furnishing exactly $\beta-\beta_1$ conditions for curves of order $n-3$ through B_0 and the set of β_1 points. We have remarked ($f=0$ being non-singular) that curves of order $n-3$ through the set B_0 contain the curve $\omega=0$. Hence, the possibility is that, among the β points, there are β_1 points such that curves of order s through these do not necessarily pass through the remaining $\beta-\beta_1$ points of B. If this is so, the series determined by curves of order $m+k$ through the set $A+A_0$ has $\beta-\beta_1$ points of B common to all sets of the series (p. 87 preceding); and this would be so if curves of order $m+k$, through the point A only, had this character. From the arbitrariness of the set A_0, we shall assume that the converse is true, namely that, under the assumed property of the set B, with regard to curves of order s through them, all curves of order $m+k$, through the set A only, necessarily contain the $\beta-\beta_1$ points of the set B.

For example (Bacharach, *loc. cit.* p. 284) let $n=6$, $m=7$, $k=1$; consider curves of order 8, through a set A of 36 points, common to curves of orders 6 and 7, when these curves are such that the remaining 6 intersections, forming the set B, have 5 points lying in a line. The conics through these 5 points are 3 in number, and do not necessarily contain the sixth point. Hence: Curves of order 8, through the 36 points A, and through two arbitrary points, A_0, of the sextic curve, will determine, on this curve, the linear series defined by the 10 points which consist of B and the 4 remaining intersections, B_0, with the sextic, of the line through A_0. These octavic curves will all pass through the sixth point of the set B (other than the 5 collinear points); we assume that octavic curves, through the set A only, have the same property. More generally, if, among the $\frac{1}{2}(s+1)(s+2)$ points B, there be $s+i+1$ in line ($i>0$), curves of order s through these collinear points consist of this line, and an arbitrary curve of order $s-1$, not necessarily containing the $\frac{1}{2}s(s+1)-i$ points remaining from the set B. Or, if, of the

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

excepted intersections of the given curves $f=0$, $\phi=0$, there be $i+m+n-r-2$ which are collinear ($i>0$), then curves of order r through the $mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$ specified intersections, all pass through the $\frac{1}{2}(m+n-r-2)(m+n-r-3)-i$ non-collinear remaining points of the excepted intersections. These points lie on, at least, i curves* of order $m+n-r-4$.

We also include the following example of the application of the principles developed in this chapter. *Let* n, h, ρ *be positive integers, such that* $\rho<n-3$, $1<h<\frac{1}{2}n(n+3)$, $\frac{1}{2}\rho(\rho-3)<h-1\leqslant\rho(\rho+3)$. *Thus the number* q, *given by* $q=h+n\rho-\frac{1}{2}(\rho+1)(\rho+2)$, *is* $>\frac{1}{2}\rho(\rho+3)$ *and* $>\rho(n-3)$, *while also* $n^2-q>\frac{1}{2}n(n+3)-h$; *for these involve, respectively, only* $h-1+\rho(n-3-\rho)>0$, $h>\frac{1}{2}(\rho-1)(\rho-2)$, *and*

$$(n-\rho-1)(n-\rho-2)>0.$$

Now suppose that, of the n^2 *intersections of two non-singular curves of order* n, *there are* q *points, forming a set* Q, *which lie on a curve of order* ρ, *itself not passing through any of the* n^2-q *other common points. If, from these other* n^2-q *points, we choose a set* B, *of* $\frac{1}{2}n(n+3)-h$ *points, which give independent conditions for curves of order* n *passing through them, then, all curves of order* n *through these points* B, *pass of themselves through the set,* R, *of*

$$\tfrac{1}{2}(n-\rho-1)(n-\rho-2)$$

points which, with B, *make up the set of* n^2-q *points; and these points* R *do not lie on a curve of order* $n-\rho-3$. *Also, the remaining* q *intersections of any two of the* ∞^h *curves, of order* n, *which pass through the points* $B+R$, *lie upon a curve of order* ρ.

For, by the theorem just proved, the curves of order $n+\rho$ put through all but a set R of $\frac{1}{2}(n-\rho-1)(n-\rho-2)$ points, among the intersections of two curves of order n, have absolute freedom

---

* The general question is: In what way must $\frac{1}{2}(s+1)(s+2)$ points of a curve of order s be situated, in order to contain among them the maximum number which is possible, of points which do not lie necessarily on curves of order s put through the others? Cf. also Bacharach, *loc. cit.* p. 287.

$i+1+\rho(\rho+3)$, where i is the number of curves of order $n-\rho-3$ which pass through the set R. Suppose that a set, Q, of q, or

$$h+n\rho-\tfrac{1}{2}(\rho+1)(\rho+2)$$

points, among the points through which the curves of order $n+\rho$ are put, lie upon a curve of order ρ, which does not contain any of the other n^2-q intersections of the two curves of order n. Consider, on one of the two curves of order n, say $f=0$, the series of sets of points coresidual with the $q+\frac{1}{2}(n-\rho-1)(n-\rho-2)$ points $Q+R$; the number of points in the set $Q+R$ is also $h+\frac{1}{2}n(n-3)$, and the freedom of the series in question is thence

$$h+\tfrac{1}{2}n(n-3)-\tfrac{1}{2}(n-1)(n-2)+j,$$

where j is the number of curves of order $n-3$ which pass through the set $Q+R$; in virtue of $q>(n-3)\rho$, remarked above*, such curves of order $n-3$, containing the q points Q, must contain entirely the curve of order ρ; thus, j is the number, say i, of curves of order $n-3-\rho$ which contain the set R. Wherefore, the freedom, on $f=0$, of the series defined by the set $Q+R$ is $h-1+i$; this series is given, however, by curves of order n through the remaining points, B, of intersection of the two curves of order n, whose number is $\frac{1}{2}n(n+3)-h$. These points are therefore equivalent only to $\frac{1}{2}n(n+3)-h-i$ independent points for curves of order n, such curves through them having an equation of the form

$$\lambda f+\lambda_1 f_1+\ldots+\lambda_h f_h+\mu_1\psi_1+\ldots+\mu_i\psi_i=0.$$

We have, however, supposed these points to be independent for curves of order n; thus $i=0$, there are no curves of order $n-\rho-3$ through the points R, and all curves of order $n+\rho$ through the points B, and through the q points Q, pass through the points R. The freedom of these curves is $1+\rho(\rho+3)$, which we have supposed $\geqslant h$; among such curves will therefore be the composite curves consisting of the curve of order ρ through the points Q, taken with the $h+1$ curves of order n through the $\frac{1}{2}n(n+3)-h$ independent points B. Thus, all these curves of order n pass through the points R.

Now, again on $f=0$, consider the freedom of the linear series defined by the set, Q, of q points. We have proved that a curve of order $n-3$ through the points of this set contains, as part, the curve of order ρ on which the set lies; the number of such curves of order $n-3$ is thus $\frac{1}{2}(n-\rho-1)(n-\rho-2)$, and the freedom of the series on $f=0$ defined by these points is thus

$$h+n\rho-\tfrac{1}{2}(\rho+1)(\rho+2)-\tfrac{1}{2}(n-1)(n-2)+\tfrac{1}{2}(n-\rho-1)(n-\rho-2),$$

which is $h-1$, the same as the freedom on $f=0$ of the series determined by curves of order n through the points B; by what has been proved, this should be so. But, by the theorem of coresiduation, the series can also be obtained by drawing curves of order ρ through the further intersections, with $f=0$, of the curve of order ρ which contains the defining set Q. If these residual points give independent conditions for curves of order ρ, the freedom of this system of curves of order ρ is

---

* The condition $q>(n-3)\rho-\frac{1}{2}(\rho-1)(\rho-2)$ is equivalent only to $h>0$; so that the condition $h>\frac{1}{2}(\rho-1)(\rho-2)$ is unnecessary when the points Q are independent for curves of order $n-3$ through the set R.

$$\tfrac{1}{2}\rho(\rho+3)-\{n\rho-[h+n\rho-\tfrac{1}{2}(\rho+1)(\rho+2)]\},$$

which again is $h-1$. Wherefore, any curve of order n through the set B (and, therefore, through the set $B+R$) meets $f=0$ in points lying on a curve of order ρ. This proves the result enunciated, $f=0$ being any one of the ∞^h curves in question.

Ex. 1. If $n-1$ of the n^2 intersections of two non-singular plane curves of order n lie in line, then all curves of order n through $\frac{1}{2}(n-1)(n+4)$ independent points, chosen from the other intersections, pass through the remaining $\frac{1}{2}(n-2)(n-3)$ of these n^2-n+1 points; and these latter do not lie on a curve of order $n-4$. Further any two curves of order n through the n^2-n+1 points have their remaining intersections in line. (This is the case when $\rho=1, h=2$. Cf. Zeuthen, *Lehrbuch d. abzähl. Geom.* 1914, p. 241; White, *Proc. Camb. Phil. Soc.* XXII, 1924, p. 5.)

Ex. 2. Prove, for $r=1$, and $r=2$, the following result (Bacharach, *loc. cit.* p. 292; after Olivier, *Crelle*, LXX, 1869, p. 159): If curves $U=0$, $V=0$, $W=0$, of order n, have in common a set, O, of $n^2-\frac{1}{2}(n-r)(n-r+3)$ points, and $U'=0$ be the curve of order $n-r$ through the set, A, of remaining intersections of $V=0$, $W=0$, supposed to give $\frac{1}{2}(n-r)(n-r+3)$ independent conditions for $U'=0$; with a similar statement for $V'=0$ through the remaining common points B of $W=0$, $U=0$, and for $W'=0$ from $U=0$, $V=0$; then prove that the curves $U'=0$, $V'=0$, $W'=0$ have in common a set, O', of $\frac{1}{2}(n-r)(n-3r+3)$ points; and, that the remaining intersections, A', of $V'=0$, $W'=0$, in number $\frac{1}{2}(n-r)(n+r-3)$, lie on the curve $U=0$, with a similar statement for $W'=0$ and $U'=0$, and for $U'=0$, $V'=0$. There are thus 8 sets of points, indicated by $O\,(U, V, W)$, $A\,(U', V, W)$, ..., $A'\,(U, V', W')$, ..., $O'\,(U', V', W')$; all intersections of every two of the curves enter in this enumeration, except those forming the three sets $U=U'=0$, $V=V'=0$, $W=W'=0$. The theorem may be considered in connexion with an identity of the form $UU'+VV'+WW'=0$, which can be set up by remarking that the composite curve $UU'=0$ contains all the common points of $V=0$, $W=0$.

Note IV. On canonical forms for the equation of a manifold, or an aggregate of manifolds. It was proved in Chap. II that the equation of a general plane cubic curve can be supposed to be of the form $X^3+Y^3+Z^3+3mXYZ=0$, where X, Y, Z are linear in the original coordinates; this is in accord with the fact that X, Y, Z together involve 9 coefficients, which, with m, are of the same number, 10, as in a general cubic curve. The properties of the curve are investigated very easily with this equation. Similarly many properties of a cubic surface become clear when it is known that its equation may be expressed by the vanishing of the sum of the cubes of five linear functions of the coordinates (which involve together the same number, 20, of constants as in the general equation). Conversely, however, it is not certain that a specified form of equation which appears to involve the right number of coefficients, is as general as the equation which it is sought to represent; for instance, the sum of 5 fourth powers of linear functions of the three coordinates in a plane involves 15 coefficients, which is the same as the number in the equation of a general plane quartic curve; in fact, however, if the sum of 5 fourth powers be arranged in powers of the coordinates, the fifteen coefficients are not independent functions, and the quartic curve so obtained is not the general plane quartic curve. Or again, in space of three dimensions, 5 linear functions $u_1, \ldots, u_5$ involve 15 *ratios of* their coefficients, and three sums of squares of these, say

$$a_1u_1^2+\ldots+a_5u_5^2,\ b_1u_1^2+\ldots+b_5u_5^2,\ c_1u_1^2+\ldots+c_5u_5^2$$

thus involve in all 27 *ratios of* the coefficients involved, which is the same number as enter in the equations of 3 quadric surfaces. But it is not the case that 3 given quadric surfaces can have their equations put into these forms. On the other hand, four given quadric surfaces can have their equations arranged as sums of squares of the same *six* linear functions (involving 38 coefficients, instead of the necessary 36) (cf. Terracini, *Ann. d. Mat.* XXIV, 1915, pp. 1 ff.).

Some reference to the literature dealing with these questions may be added here. Let two homogeneous polynomials of order n in $r+1$ variables $x_0, \ldots, x_r$, say $\phi(x_0, \ldots, x_r), f(x_0, \ldots, x_r)$, be called conjugate when one (and therefore the other) of the two constants $\phi(\xi_0, \xi_1, \ldots, \xi_r)f(x_0, \ldots, x_r)$, $f(\xi_0, \ldots, \xi_r)\,\phi(x_0, \ldots, x_r)$ vanishes, where ξ_i is the operator $\partial/\partial x_i$. Each form, if general, involves $(n+r, r)$ coefficients. Let F be a specified form of order n in the same variables involving this number of constants, $c_0, c_1, \ldots$. Let $\partial F/\partial c_i$ be denoted by $F_i(x_0, \ldots, x_r)$. Then it can be shewn, by examining the determinantal condition for the $(n+r, r)$ coefficients in F to be independent, that the necessary and sufficient condition that this form F should be possible for f, is that it should be impossible for a form ϕ of order n to exist which is conjugate to all the $(n+r, r)$ forms F_i. In particular when, as is often the case, $r+1$ of the parameters c_i enter in F only in a linear function $y, =c_0x_0+\ldots+c_rx_r$, which enters in F, part of the condition expressed is that $\partial F/\partial y$ be conjugate to all the first polars $\partial\phi/\partial x_0, \ldots, \partial\phi/\partial x_r$, of the form ϕ. And a particular case of this arises when F is of the form y^n+F_1, where F_1 does not contain $c_0, \ldots, c_r$; then, this part of the condition, exhausting the reference to $c_0, \ldots, c_r$, is that $\phi=0$ should have a double point at the point whose coordinates are

$(c_0, c_1, \ldots, c_r)$. In particular, if $(r+1)^{-1}(n+r, r)$ be an integer, say h, then f can be expressed as a sum of h nth powers of linear forms provided h points $(c_0, \ldots, c_r)$ exist for which there is no manifold $\phi=0$, of order n, having these as double points. For example, considering the converse theorem, however 5 points be taken in a plane, a (degenerate) quartic curve exists having these as double points; thus a ternary quartic cannot be expressed as a sum of five fourth powers of linear forms. Again, a cubic primal in space of four dimensions contains 35 terms, and the sum of the cubes of seven linear forms contains the same number; but if seven points be taken, the cubic primal defined by the chords of the rational quartic curve through these points has a double point at all of them. Thus a cubic primal in space [4] cannot be expressed as a sum of seven cubes. Reversely, a binary form of odd order $2k-1$ can be expressed as a sum of k powers, there being no binary form of order $2k-1$ with k double factors. Likewise a plane quintic curve can be expressed as a sum of seven fifth powers; for seven points can be taken (three in a line) which cannot be the double points of a quintic curve; and a cubic surface can be expressed by the sum of five cubics, since a cubic surface cannot have five general nodes. For these considerations see Lasker, *Math. Ann.* LVIII, 1904, p. 434, and Wakeford, *Proc. Lond. Math. Soc.* XVIII, 1920, p. 403. More recently, J. Bronowski (*Proc. Camb. Phil. Soc.* XXIX, 1933, pp. 69, 245) has remarked that the possibility of the expression of a primal of order n by h powers of linear forms is that r primals of order n having nodes at $h-1$ given points should have further free intersections; for instance, two plane curves of order n with $\frac{1}{6}(n+1)(n+2)-1$ nodes (when this is integral) have $\frac{1}{3}(n-2)(n-4)$ further intersections; this vanishes only when $n=2$ or $n=4$; thus a curve of order $3k\pm1$, unless $k=1$, is expressible by a sum of $\frac{1}{2}[3k^2+3k+1\pm(2k+1)]$ powers. The question of the expression as a sum of powers of linear forms has been developed from another point of view; for instance, the impossibility of expressing a conic as a sum of two squares is equivalent with the impossibility of drawing a chord to a Veronese surface in space of 5 dimensions (given by coordinates $\xi^2, \eta^2, \zeta^2, \eta\zeta, \zeta\xi, \xi\eta$, where ξ, η, ζ are arbitrary) from an arbitrary point of this space.

Cf. Palatini, *Atti...Torino*, XXXVIII, 1902, p. 43; *Rend. Lincei*, XII, 1903, p. 378; *Atti...Torino*, XLIV, 1909, p. 362; and Terracini, *Rend. Palermo*, XXXI, 1911, p. 392; and Scorza, *Rend. Palermo*, XXV, 1908, p. 193, and XXVII, 1909, p. 148.

Ex. 1. Let ξ, η, ζ, τ and ω be linear functions of the coordinates (x, y, z, t) in ordinary space, $\phi(\xi, \eta, \zeta, \tau)$ a cubic polynomial of any specified form such that the conditions that the first polars of this, regarded as a polynomial in x, y, z, t, should be conjugate to the first polars of a given cubic polynomial $f(x, y, z, t)$, give sixteen independent conditions for the coefficients in ξ, η, ζ, τ; and let $\psi(x, y, z, t)$ be a given quadric polynomial. Then $f(x, y, z, t)$ can be written in the form $\phi(\xi, \eta, \zeta, \tau)+\omega\psi(x, y, z, t)$. For instance $\phi(\xi, \eta, \zeta, \tau)$ may be

$$\eta\zeta\tau+\zeta\xi\tau+\xi\eta\tau+\xi\eta\zeta;$$

but not $\xi^2\eta+\eta^2\zeta+\zeta^2\tau$.

Ex. 2. A general plane-curve of order n can have its equation expressed by the vanishing of a symmetrical determinant of n rows and columns, in which every element is linear in the coordinates. If ξ, η, ζ be linear in the coordinates, a conic and a plane cubic curve are respectively given by the vanishing of the discriminants of the two quadric forms, in u, v, and in u, v, w, $\xi u^2+2\zeta uv+\eta v^2$, $m(\xi u^2+\eta v^2+\zeta w^2)+2(\xi vw+\eta wu+\zeta uv)$; and the result for a plane quartic was obtained by Hesse, *Crelle*, XLIX, 1855, p. 243.

A direct proof is suggested by A. C. Dixon, *Proc. Camb. Phil. Soc.* XI, 1902, pp. 350, 351; and XII, 1904, pp. 449–53. Let $v_{11}=0$ be one of the ∞^{n-1} curves of order $n-1$ which touch the given curve, $f=0$, in $\frac{1}{2}n(n-1)$ points; denote the most general curve of order $n-1$ which passes through the points of contact by $\lambda_1 v_{11}+\lambda_2 v_{12}+\ldots+\lambda_n v_{1n}=0$. There exist then identities of the form $v_{1r}v_{1s}=v_{11}v_{rs}+fw_{rs}$, from which the polynomials v_{rs} $(r, s=2, \ldots, n)$ may be determined. It can then be proved that the minor of v_{rs} in the determinant of n rows and columns $|v_{rs}|$ is of the form $f^{n-2}\beta_{rs}$, wherein β_{rs} will be linear in the coordinates; and then that the given curve $f=0$ is given by $|\beta_{rs}|=0$.

Ex. 3. Using v_i, or w_i, for a homogeneous polynomial of order i in x and z, shew that a ternary quartic in x, y, z, is capable of the form $f=(y+v_1)\phi+v_2^2$, where ϕ is a cubic. Hence shew that the quartic surface $z^2t^2=f$ is birationally equivalent to a cubic surface. Shew also that the sextic curve discussed on p. 95 above is capable of the form $\psi=(yz+w_2)(y^2z^2+yzv_2+v_4)+(yzv_1+v_3)^2$; thence prove that the sextic surface $z^4t^2=\psi$ is birationally reducible to the quartic surface $z^2t^2=f$ cited.

CHAPTER V

THE PERIODS OF ALGEBRAIC INTEGRALS. LOOPS IN A PLANE. RIEMANN SURFACES

As has been indicated, at the present stage of the theory of algebraic geometry, many results find their clearest statement in connexion with the theory of algebraic integrals; just as the first exhaustive investigation of the genus of an algebraic curve was in fact by Abel, for the theory of algebraic integrals (*Oeuvres*, 1881, p. 145). And, notwithstanding the initial feeling which may arise, of the incongruity of the two theories, the geometrical reader will find it desirable to have an understanding of the main results of the latter theory. We therefore give now an account, with the objects, (i) of shewing how to determine the number of periods of an algebraic integral; (ii) of explaining the conception and use of the so-called Riemann surface; (iii) of making connexion with an arithmetic theory of plane curves. All these are helpful towards an extension of the theory of curves to the theory of surfaces; in particular, the geometrical aspect of the theory of Riemann surfaces is intimately related with a wide theory, Analysis Situs, or Topology, which promises to give a descriptive alternative to much involved computation. It must be understood that what we give is, in the space we allot to it, necessarily very incomplete. It will be sufficient if it indicate clearly the nature and bearings of the ideas involved; detailed developments will be found in many other places.

Meaning and number of periods. We suppose the independent complex variable x to be represented, in the known way, upon a Euclidean plane, regarded as closed at infinity by the single point $x=\infty$, like a sphere. Then, taking three simple examples, it is a familiar fact that the integrals

$$\text{(i)},\ \int_c^x \frac{dx}{x};\quad \text{(ii)},\ \int_0^x \frac{dx}{1+x^2};\quad \text{(iii)},\ \int_0^x \frac{dx}{(1-x^2)^{\frac{1}{2}}},\quad (c \neq 0),$$

are not determined by the assignment of their limits of integration. Each is indeterminate by the addition of a certain constant, and therefore by any integer multiple of this constant. Denoting any value of either integral by I, and any integer by k, the general values of the integrals are, respectively, $I+2\pi ik$, $I+k\pi$, $I+2\pi k$. The respective additive constants, $2\pi i$, π, 2π, are obtainable, in the several cases, by a circuit of the upper limit (x), respectively round $x=0$, round one of the points $x=\pm i$, and round both the points $x=\pm 1$. There is an obvious difference between the case (iii), and the cases (i), (ii), in that the integral (iii) does not become infinite as x approaches to $+1$ or -1, while the integral (i) becomes infinite when x approaches to $x=0$, and (ii) when x approaches to $+i$ or $-i$; another difference is that the subject of integration in (iii) is ambiguous in sign. Now consider an integral

$$\text{(iv)}, \int_0^x \frac{dx}{[(1-x^2)(1-e^2x^2)]^{\frac{1}{2}}},$$

where e is a constant, which we may take to be real and positive and <1; this is a generalisation of (iii), but, unlike (iii), does not become infinite for $x=\infty$. Let (x, y) denote any point of the curve whose equation is $y^2=(1-x^2)(1-e^2x^2)$; then we may understand (iv) to denote the integral

$$\int_{(0,1)}^{(x,y)} \frac{dx}{y},$$

taken by a path of pairs of associated complex values of x and y, all representing points of the quartic curve, from the point $x=0$, $y=1$, to the general point given by the upper limit. Two such paths of integration, extended over a continuum of possibly complex points belonging to the curve, from the same initial point to the same final point, may not lead to the same value of the integral. But the difference between the values obtained by two such paths is evidently equal to what is found by taking the integral over a certain closed path; that, namely, which begins from the lower limit, passes by one of these paths to the upper limit, and then returns to the lower limit by a path which is the reverse of the second of the two paths referred to. The determination of the complete aggregate of values of which the integral is capable, for given positions of the lower and upper limits, thus requires an examination of the values obtainable by all possible closed paths of integration, of which every point (possibly complex) belongs to the curve.

More generally, let $f(x, y)=0$ represent any curve; let $R(x, y)$ be any rational function of x and y, supposed to be associated so as to satisfy the equation of the curve; let (a), (x) denote a fixed and a variable point of the curve, respectively. We may then similarly consider the integral $\int_{(a)}^{(x)} R(x, y)\,dx$, of which the path of integration is by points of the curve. If this integral become logarithmically infinite for some positions of the upper limit (x), a closed path about such a point leads to a value for the integral which does not vanish, as in the simple cases (i), (ii), considered at first, wherein the subject of integration was a single-valued function of x only. Suppose that $R(x, y)$ is such that there are no such points of logarithmic infinity; there may still be closed paths, as in the case (iv) considered above, for which the value of the integral is not zero, arising from the circumstance that $R(x, y)$ is not a single-valued function of x only. We prove in fact that, if p be the genus of the curve $f(x, y)=0$, the general value of the integral, when there are no logarithmic infinities, is of the form $I+m_1\Omega_1+\ldots+m_{2p}\Omega_{2p}$, where I is a particular value, $\Omega_1, \ldots, \Omega_{2p}$ are certain constants, *the periods*, depending upon $f(x, y)$ and $R(x, y)$, and $m_1, \ldots, m_{2p}$ are integers; by proper choice of the path these integers may be chosen at will. When the function $R(x, y)$ is such that the integral has logarithmic infinities, its general value will involve also a sum of integral multiples of constants obtained by circuit of these infinities.

Of this fundamental theorem we consider two methods of proof, the former of a natural and elementary kind, by means of the theory of *loops*, the latter with the help of a so-called *Riemann surface*, constructed to represent the (complex and real) points of the curve $f(x, y)=0$. Although the latter is finally of much greater importance, it may at first seem artificial; and our exposition, making appeal, for the sake of brevity, to intuitive notions of topology, is not final.

The method of loops in a plane. For the first method, we suppose (what is shewn in Chap. II not to be an essential limitation) that the curve represented by the equation $f(x, y)=0$ has no multiple points other than double points at which the tangents are distinct; further, the equation being of order m in x and y, we suppose that the terms of order m have m distinct factors, and that the tangents of the curve having equations of the form $x=$ constant, all touch the curve at simple points, and each only once. The values of x for which two values of y are equal are then to be found from $f(x, y)=0$, $\partial f/\partial y=0$. Among these are the values for the double points, say of number δ; the other solutions correspond to points of contact of tangents of the curve with equation of the form $x=$ constant. The number of these latter, say w, is given by $w=m(m-1)-2\delta$, which is $w=2m+2p-2$, since, as we have seen (Chap. IV, p. 77), $p=\frac{1}{2}(m-1)(m-2)-\delta$. Regarding the complex value x as represented upon a plane, for a general value $x=a$ there will be m values of y, say $b_1, \ldots, b_m$; and the values of y, satisfying the equation $f(x, y)=0$, which are in the neighbourhood of the value $y=b_i$, will be given, when x is sufficiently near to a, by an expression of $y-b_i$ as a power series in $x-a$. By continuation of the power series the complete range of values of y satisfying the equation, as x varies, can be computed from the equation itself, unless we come to a value of x for which two of the m values of y are equal. If this be a double point, (x_0, y_0), at which, then, by what has been supposed, neither tangent of the curve has an equation of the form $x=$ constant, there will be two expansions, corresponding to the two branches of the curve, both of $y-y_0$ in powers of $x-x_0$; the values of y to be associated with x, as this passes through $x=x_0$, are thus determined by the values with which the path approaches this point. But two values of y become equal also at a point $x=x_0$, $y=y_0$, when there is a tangent of the curve given by $x=x_0$; the two values of y for a value of x in the neighbourhood of x_0, are then given by two power series expressing $y-y_0$ in powers of $(x-x_0)^{\frac{1}{2}}$, wherein the coefficients of even powers of $(x-x_0)^{\frac{1}{2}}$ are the same, but the coefficients of odd powers of $(x-x_0)^{\frac{1}{2}}$ differ in sign. Either series is then obtainable from the other by a circuit of the complex variable x round the point x_0 of the representing plane. It is the possibility of this which explains, when we are considering an algebraic integral $\int R(x, y)\,dx$, the existence of closed circuits of integration which give non-vanishing values of the integral, other than those arising from points of logarithmic infinity of the integral. In what follows immediately, we exclude the consideration of points of logarithmic infinity of the integral; and we assume that a closed path of integration for x gives a zero value for the integral unless the path surrounds points, by circuit of which a value of y is changed into another value of y (of the kind considered above).

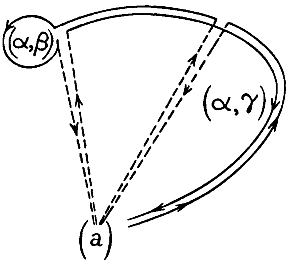
To illustrate the ideas consider first the integral $\int_{(a)}^{(x)} x^{i-1}\,dx/y$, wherein $y^2=(x-c_1)\ldots(x-c_{2p+2})$, the roots $c_1, \ldots, c_{2p+2}$ being all different, and the value of x at the lower limit (a) not being one of these, while i is one of $1, 2, \ldots, p$; thus $(p \geqslant 1)$ the integral does not become infinite for $x=\infty$. The relation between x and y does not satisfy all the limitations which we have supposed above to hold in general; but this is immaterial for our purpose. A simple closed circuit by x round any one of the so-called *branch points* given by $x=c_1, \ldots, x=c_{2p+2}$, starting with a pair of values x_0, y_0 for x, y, evidently leads to the values $x_0, -y_0$. Suppose the value $x=a$ to be arbitrary; consider closed paths of integration for a, each of which begins at $x=a$, with one of the two values of y corresponding thereto from the equation of the curve, either $y=b$ or $y=-b$; then proceeds to the neighbourhood of one of the $2p+2$ branch points; makes a circuit about this, which we think of as lying in the immediate neighbourhood of the branch point, but described in either direction of circulation about this point; and then comes back to $x=a$ along the path of approach. Such a path we call a *loop*. It is supposed that to the value of x belonging to any point of the path there is associated the value of y arising continuously, from the initial value, by means of the equation of the curve. The $2p+2$ possible loops will be considered in a definite order; and it is supposed that no two of them have a point of intersection other than the initial point $x=a$. If we describe first the loop (c_1), and then the loop (c_2), the result of the first is to lead from the values (a, b), for x and y, to the values $(a, -b)$; and the result of the second, beginning with $(a, -b)$, is to lead back to (a, b). The value obtained for the integral $\int x^{i-1}dx/y$ round this pair of loops, by what we have assumed, is independent of the position of $x=a$, and the same as would be obtained by any closed path passing once round the points (c_1), (c_2) but including no other branch points, with a proper initial value for y. The contribution to the value of the integral which arises by the part of a single loop which is in the immediate neighbourhood of a branch point is zero; the effect of this small part is to change the sign of y. The value obtained for the integral by a closed path passing once round the two branch points is not zero, being, as we easily see, the same save for sign, as twice the integral taken by a simple path from $(c_1, 0)$ to $(c_2, 0)$. Now consider any closed circuit which, beginning at $x=a$ with the value $y=b$, comes back to $x=a$ *with the same value* $y=b$. Such a circuit, in virtue of the principle we have assumed, must give the same value for the integral as a composite circuit, of the kind we may denote by $m_1(c_1)+\ldots+m_{2p+2}(c_{2p+2})$, wherein (c_i) denotes the circuit of the loop defined by c_i, and $m_1, \ldots, m_{2p+2}$ are integers; but, in order that, as we have postulated, the final value of y in the circuit may agree with the initial value, the sum of these integers must be even (including zero). It can then be easily seen, if (c_1, c_2) denote such a simple composite circuit about the branch points c_1, c_2 as we have described, in a specified order, beginning and ending with the values a, b for x and y, that the general composite circuit spoken of is equivalent to a circuit which may be expressed in the form $\mu_1(c_1, c_2)+\mu_2(c_2, c_3)+\ldots+\mu_{2p+1}(c_{2p+1}, c_{2p+2})$, wherein $\mu_1, \ldots, \mu_{2p+1}$ are also integers (not necessarily of zero or even sum). In particular, a circuit of this form is that in which every one of $\mu_1, \ldots, \mu_{2p+1}$ is 1; this circuit is evidently equivalent, so far as the value which it gives for the integral is concerned, to a single circuit round $x=\infty$, which would give zero for the value of the integral (there being no branch point at $x=\infty$). Thus, so far as the value of the integral is concerned, any composite circuit which begins and ends with the same pair of associated values for x, y, may be supposed to have the form $\nu_1(c_1, c_2)+\nu_2(c_2, c_3)+\ldots+\nu_{2p}(c_{2p}, c_{2p+1})$, where $\nu_i=\mu_i-\mu_{2p+1}$; and this involves only $2p$ circuits (c_r, c_{r+1}). We thus reach the conclusion that the general value of the integral $\int_{(a)}^{(x)} x^{i-1}dx/y$ is of the form $I+\nu_1\Omega_1+\ldots+\nu_{2p}\Omega_{2p}$, where I is the value of the integral for a particular path arbitrarily chosen, $\Omega_1, \ldots, \Omega_{2p}$ are definite constants, and $\nu_1, \ldots, \nu_{2p}$ are integers. Conversely, by proper choice of path from (a) to (x), we can suppose $\nu_1, \ldots, \nu_{2p}$ to have any assigned integer values. The number of periods of an algebraic integral was known to Galois, who refers to Jacobi and

Abel; see his letter to Auguste Chevalier, 29 May 1832 (*Œuvres mathématiques d'Évariste Galois...*, with an introduction by M. Émile Picard, Paris, 1897, pp. ix, 30).

Consider now more generally the integral $\int R(x, y)\,dx$, supposed to have no logarithmic infinities, the fundamental equation $f(x, y)=0$ being simplified as above, so that there are $2m+2p-2$ simple branch points, all arising for finite values of x. Every one of these has the property that, when x makes a circuit about it in its immediate neighbourhood, a particular pair of the m roots y of the equation $f(x, y)=0$ are changed either into the other, the other $m-2$ roots being unaffected. Suppose that these branch points are considered in a definite order, and a loop, as explained, put about each, all these loops having the same arbitrary initial point, $x=a$, and no two intersecting except at $x=a$. Denote the values of y satisfying the equation $f(a, y)=0$, supposed different, by $y_1, \ldots, y_m$, in any definite order. Every one of the loops must then be associable with a symbol of two numbers chosen from $1, 2, \ldots, m$, say (α, β), or (β, α); the meaning being that, if we start from $x=a$ with one of the two values y_α, y_β for y, and follow the value of y, with the help of the equation $f(x, y)=0$, round the loop, we shall arrive back at $x=a$ with the other of these two values; the direction in which the loop is described is in fact indifferent. If we follow the same loop with an initial value of y other than y_α or y_β, the final value of y will be the same as the initial value. There may be several branch points associated with the same symbol (α, β); but the whole set of numbers $1, 2, \ldots, m$ must occur if we consider the binary symbols for all the loops; and, more generally, it must be possible, starting at $x=a$ with any assigned value of y, say y_i, to make a selection from the loops, so that the circuit of these selected loops, in a proper order, will lead, from the initial value $y=y_i$, to any other assigned value, $y=y_j$, as final value. For, otherwise, there will be a set of values of y, chosen from $y_1, \ldots, y_m$, but not including all these, which are interchanged among themselves by every possible combination of the loops; and this will involve, it is easy to see, that the polynomial in y, $f(x, y)$, can be written as the product of two, or more, polynomials in y all with coefficients rational in x. We assume that this is not so, the curve $f(x, y)=0$ being irreducible.

Not every one of the possible $\frac{1}{2}m(m-1)$ binary symbols, obtainable by taking two numbers from $1, 2, \ldots, m$, will occur, in general, among the binary symbols associated with the loops as originally drawn. But, by suitable combinations of the loops as drawn, we can obtain interchanges among $y_1, \ldots, y_m$ other than those arising for these single loops; and it will be convenient to denote such a combination of single loops which, for instance, interchanges y_β and y_γ, by (β, γ), or (γ, β). Consider in particular the combination

$(\alpha, \gamma)(\alpha, \beta)(\alpha, \gamma)$, where (α, β), (α, γ) are among the original single loops. By the loop (α, γ), the roots $(y_\alpha, y_\beta, y_\gamma)$ are changed, respectively, to $(y_\gamma, y_\beta, y_\alpha)$; by the loop (α, β) these are then changed, respectively, to $(y_\gamma, y_\alpha, y_\beta)$; and then, finally, these are changed, respectively, by the loop (α, γ), to $(y_\alpha, y_\gamma, y_\beta)$; the composite loop thus interchanges y_β and y_γ, but leaves y_α unaltered; it may then be denoted by (β, γ), or (γ, β). As indicated by the diagram, the composite loop, deformed so as to include self destructive portions (indicated by the dotted lines), may be regarded as a loop to enclose the branch point (α, β) which approaches this by a detour about the branch point (α, γ). If we interchange β and γ we may prove in the same way that $(\alpha, \beta)(\alpha, \gamma)(\alpha, \beta) = (\beta, \gamma)$; and the two results (since the repetition of a loop gives no change in the initial value of y) may both be expressed by



$$(\alpha, \beta)(\alpha, \gamma) = (\alpha, \gamma)(\beta, \gamma) = (\beta, \gamma)(\alpha, \beta);$$

or, in words, by saying that the order of two succeeding symbols $(\alpha, \beta)(\alpha, \gamma)$, which have a common number, α, may be altered, by carrying either symbol, unchanged, over the other; if the symbol, over which this transference is made, be modified, by changing the number in it which is common to the two symbols, into the other number which enters in the transferred symbol; thus in the first equation which is written, (α, γ) is carried over (α, β), being put to the left, with the simultaneous change of (α, β) to (γ, β). In applying this, we may for clearness refer to it as the *rule of transference*. It is clear, on the other hand, that a succession of two symbols (α, β), (γ, δ), which have no number in common, has the same effect as the same two symbols in the other order (γ, δ), (α, β), the values of y interchanged by one symbol not being affected by the other; we may suppose this remark included in the rule.

Now write down the whole aggregate of the original w binary symbols, one for each of the w branch points, in order, as they arise from the loops as originally drawn, the symbols written to the right referring to operations carried out first. We desire to shew that, by the rule of transference, the aggregate of operations given thereby can be so modified as to be expressible by the aggregate

$$(m-1, m)(m-1, m) \ldots . (3, 4)(3, 4\}(2, 3)(2, 3)(1, 2) \ldots (1, 2),$$

where there are; first, $2p+2$ symbols $(1, 2)$; then, two symbols $(2, 3)$; then, two symbols $(3, 4)$, and so on; and, finally, two symbols $(m-1, m)$; in all $2p+2+2(m-2)$ symbols. For this, we use the principles: (a), the effect of the whole sequence of w loops, upon any one of the values of y, must be to change this into itself, the whole sequence being clearly equivalent to a circuit about $x=\infty$, at which, by hypothesis, there is no branch point; (*b*), by a suitable selection from the loops it must be possible to change any root y_i to any other root y_j, as already remarked. And we use certain rules, which are consequences of the rule of transference; these, for clearness, may be stated: (i), a pair of consecutive symbols (representative of different loops but) containing the same two numbers, may be simultaneously carried over another symbol immediately consecutive to, or preceding, this pair, without change of the numbers of this symbol; for example $(\alpha, \gamma)(\alpha, \beta)(\alpha, \beta)$ may be replaced by $(\alpha, \beta)(\alpha, \beta)(\alpha, \gamma)$, transferring (α, β) over (α, γ), and then (α, β) over the changed value of (α, γ), which is (β, γ); (ii), in a set of three consecutive symbols such as those just considered, where two consecutive symbols involving the same numbers are preceded (or followed) by a symbol containing a number which occurs in the two like symbols, this number may be replaced, in both the like symbols, by the other number of the single (preceding, or following) symbol, without change in the order of the three symbols; for instance $(\alpha, \gamma)(\alpha, \beta)(\alpha, \beta)$ is the same as $(\alpha,\gamma)(\beta,\gamma)(\beta,\gamma)$; for the former has been seen to be the same as $(\alpha, \beta)(\alpha, \beta)(\alpha, \gamma)$, and in this (α, γ) may be carried over both the symbols (α, β) in turn; we may call this the *rule of attraction*; (iii), in a sequence of three pairs of symbols, such as $(\gamma, \delta)(\gamma, \delta)(\beta, \gamma)(\beta, \gamma)(\alpha, \beta)(\alpha, \beta)$, in which the first pair of symbols involve the same numbers α, β, the second pair involve the same numbers β, γ, of which one (β) occurs in the symbols of the first pair, and the third pair also involve the same numbers γ, δ, of which one (γ) has occurred in the second but not in the first pair, we may replace the numbers α, β, γ, δ by α, γ, β, δ, namely the composite symbol is the same as

$$(\beta,\delta)(\beta,\delta)(\beta,\gamma)(\beta,\gamma)(\alpha,\gamma)(\alpha,\gamma).$$

This may be proved by two applications of the rule of attraction; it may be called the rule of *permutation of numbers*.

With these rules, suppose, in the sequence of w binary symbols to be considered, that the first symbol, that written on the right, is (α, β). Regarding the sequence of symbols from right to left, we may come to other symbols (α, β), and these may all be carried over intervening symbols, in turn, and placed to follow the first symbol, at its left. We thus have a batch of symbols (α, β) at the right. We may suppose the number of these symbols is *even*. For suppose it is odd; then the effect, upon y_α, of this batch of symbols, is to change it into y_β; but, as, by principle (*a*), the complete sequence of the w symbols must change y_α into itself, as we survey the whole sequence to the left of the batch of symbols (α, β), we must come to a symbol involving the number β (and not (α, β)), say the symbol (β, γ); let this be carried over, so as to follow immediately to the left of the batch of symbols (α, β); so far, then, y_α would be changed to y_γ; thus, as we survey the whole sequence further, from right to left, we must find a symbol involving γ, say (γ, δ); let this be carried over any intervening symbols to follow immediately on (β, γ). Proceeding thus, the sequence must be replaceable by one of the form $\ldots(\alpha, \lambda)(\lambda, \kappa)\ldots(\delta, \gamma)(\gamma, \beta)(\beta, \alpha)\ldots(\beta, \alpha)$, with other of the w symbols to the left of (α, λ). But now we can carry the batch $(\lambda, \kappa)\ldots(\gamma, \beta)$ over the symbol (α, λ), until this follows the last (α, β), where it will appear in the form (α, β). We then have, for the whole sequence, a form $\ldots(\lambda, \kappa)\ldots(\delta, \gamma)(\gamma, \beta)(\alpha, \beta)(\beta, \alpha)\ldots(\beta, \alpha)$, with an *even* number of symbols (α, β) on the right. Arguing in the manner thus indicated, we can finally reach the conclusion that the whole sequence of w symbols can be replaced by $[\rho, \sigma][\mu, \rho]\ldots[\beta, \gamma][\alpha, \beta]$, wherein each such parenthesis as $[\alpha, \beta]$ means a sequence of an even number of symbols (α, β), and one number of this parenthesis occurs in the following parenthesis, until we come to the penultimate parenthesis $[\mu, \rho]$, of which the number ρ occurs in the last parenthesis $[\rho, \sigma]$.

But herein, by repeated use of the rule of attraction (ii), we can modify the parentheses until each of them, except the first (written at the right) contains only two binary symbols. For instance, $(\rho, \sigma)(\rho, \sigma)(\rho, \sigma)(\mu, \rho)$ is first, by the application of this rule to the three symbols $(\rho, \sigma)(\rho, \sigma)(\mu, \rho)$, the same as $(\rho, \sigma)(\mu, \sigma)(\mu, \sigma)(\mu, \rho)$, and then, by the application of this rule to the three symbols $(\rho, \sigma)(\mu, \sigma)(\mu, \sigma)$, the same as $(\rho, \sigma)(\mu, \rho)(\mu, \rho)(\mu, \rho)$; whereby, if $[\rho, \sigma]$ contains more than two symbols, two of them are transformed to symbols (μ, ρ), which may be included in the parenthesis $[\mu, \rho]$. Thereby, the sequence of symbols is reduced to consist, first, of a parenthesis of $2p+2$ symbols all of the same form, followed by $m-2$ parentheses each of two like symbols, each parenthesis (except the last) being described by two numbers of which one occurs in the preceding. The numbers occurring must, by principle (*b*), consist of the numbers $1, 2, \ldots, m$, in some order. We can then use the rule of permutation of numbers, (iii), or, more simply, can rename the roots y_i, so as finally to obtain the sequence stated $[m-1, m] \ldots [3, 4] [2, 3] [1, 2]$.

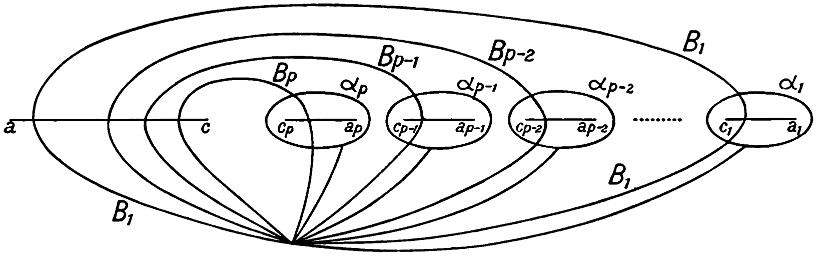
For this theory, and more detail, see Severi, *Algebraische Geometrie*, Leipzig, 1921, pp. 205–210; Lüroth, *Math. Ann.* IV, 1871, p. 181; Clebsch, *Math. Ann.* VI, 1873, p. 216; Bertini, *Rend. Lincei*, III, 1894, p. 106; Clebsch u. Gordan, *Abelsche Functnen*, Leipzig, 1866, pp. 80–90; Weierstrass, *Werke*, IV, 1902, p. 329.

Now consider the application of this sequence of loops to the proof of the number of periods of an algebraic integral. First take the case when the sequence consists only of the $2p+2$ symbols

(12) (12) ... (12) (12). A form of the equation $f(x, y) = 0$ for which this is the case is $y^2 = (x, 1)_{2p+2}$, where $(x, 1)_{2p+2}$, a polynomial of $2p+2$ different linear factors, can be denoted by $u_0 u_1 \dots u_p$, where $u_i = (x - a_i)(x - c_i)$. We have already remarked that every circuit about the branch points which leads to the same value of y as that with which we start out, may be regarded as equivalent to an aggregate, with integer coefficients, of $2p$ circuits, each definable by a pair of loops, which we may denote by $\alpha_1 = (a_1, c_1)$, $\beta_1 = (c_1, a_2)$, ..., $\alpha_i = (a_i, c_i)$, $\beta_i = (c_i, a_{i+1})$, ..., $\alpha_p = (a_p, c_p)$, $\beta_p = (c_p, c)$. The circuit is thus also equivalent to an aggregate, with integer coefficients, of the following $2p$ aggregates, built up from the preceding pairs of loops,

$$\alpha_1,\ B_1 = \beta_1 \beta_2 \dots \beta_p;\ \dots\dots;\ \alpha_i,\ B_i = \beta_i \dots \beta_p;\ \dots\dots;$$
$$\alpha_{p-1},\ B_{p-1} = \beta_{p-1} \beta_p;\ \ \alpha_p,\ B_p = \beta_p;$$

these $2p$ circuits are those represented in the diagram.



The periods of any integral $\int R(x, y)\,dx$, which is without logarithmic infinities, may be taken to be the values obtained by integration round these circuits.

Next, take the case when there are three values of y, and the loops enclosing the branch places have been reduced to the $2p+4$ represented by (23)(23)(12) ... (12). Then an integral, whose closed path of integration starts with the value y_3 for y, and finishes with the same, can only give a value which is not zero if the path enclose both the branch points defining the two loops (23); but even such a path is ineffective, since it can clearly be deformed, over the two branch points (12), until it is a circuit round $x = \infty$, which is not a branch point. An integral path, beginning with y_2, which encloses both the branch points defining the two loops (23), gives in fact the same value as an integral path beginning with y_3 which encloses both these points, described in the opposite direction; this is clear if we replace the closed path by two open paths, passing from one of the two branch places to the other, remembering that the contribution to the integral due to the portion of a loop immediately near a branch place is zero. Thus a closed integration with the initial value y_2, about the two branch points defining the two loops

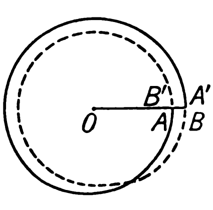
(23), can be deformed to be round $x=\infty$, and gives rise to no period. For paths which give non-vanishing periods, there remain then only paths beginning with the values y_1 or y_2, enclosing the whole or part of the $2p+2$ branch places (12). Of these, a path enclosing all the $2p+2$ points can effectively, we have seen, be deformed, over the branch points (23), to a path round $x=\infty$. We are thus finally reduced to a path such as was considered when only the branch points (12) were present; so that the number of periods is still $2p$.

Consider next the case of $2p+6$ branch places, given by the scheme (34) (34) (23) (23) (12) ... (12). As before, a path enclosing the two branch points (34), which begins with either of the values y_4 or y_3, leads to no period; and effectively such a path needs consideration only when the initial value is y_4. A path, with initial value y_3, which encloses the four branch points (34) (34) (23) (23) may be deformed to be about $x=\infty$, and is also ineffective; a path, with initial value y_3, enclosing only (23) (23) is thus reducible to one enclosing only (34) (34), and is equally ineffective. Similar remarks hold, for circuit of the branch points (34) (34) (23) (23), when the initial value is y_2. We are thus reduced as before to $2p$ circuits about the branch points (12).

And the general case can be treated in the same way, beginning always with the branch points denoted by the symbols at the left of the scheme. The general conclusion is that there are $2p$ periods, obtainable by circuits of the branch points (12), with y_1 or y_2 as initial values.

Riemann surfaces. We pass now to a method of representing the corresponding values of x and y which satisfy the (irreducible) equation $f(x, y)=0$, which adds greatly to clearness of thought. It is in intimate connexion with what has been said, but differs from it by employing, not one plane for the values of x only, but superposed planes with the points of which are associated the values of y as well as of x.

Consider first, as a simple example, the function $y=x^{\frac{1}{2}}$, or the equation $y^2=x$, in particular for values of x near to $x=0$. The values of x being represented upon a plane, there are two values of y for any small value of x; but a circuit by x about $x=0$ changes one of these continuously into the other. Now suppose, in order to convey the idea, we have two equal circular pieces of paper, one laid over the other, both with their centres at $x=0$, say at the point O; let them both be cut, from the centre O, along overlying radii, so that the radius, OA, of the upper sheet, is replaced by two edges of the cut, say OA and OA', while the underlying radius, OB, of the lower sheet, is replaced by the edges OB, OB' of the cut in the lower sheet, OB lying beneath OA, and OB' beneath OA'; next, let the edge OA, of the upper sheet, be joined to the edge OB', of the lower sheet, say by sticking paper; and suppose that it is possible, simultaneously, to join the edge OB, which lies underneath OA, to the edge OA' of the upper sheet, by sticking paper passing through the other sticking paper without any connexion therewith (except at O). Then a circuit of the whole circumference of the upper sheet, beginning at A' and ending at A, may be followed without break of continuity by a circuit of the whole circumference of the lower sheet, beginning at B' and ending at B, from which we can pass without break to A'. Two sheets, so supposed to cross, without connexion (save at one point O), evidently furnish a representation of the values of $x^{\frac{1}{2}}$ about $x=0$, there being a complete (1, 1) correspondence between the points of the two sheets, and the values assumed by y in the neighbourhood of $x=0$.

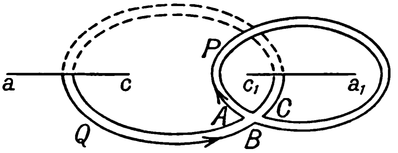


Consider next, for example, the function $y=[x(x-1)]^{\frac{1}{2}}$, or the equation $y^2=x(x-1)$, there being now *two* branch points, one at $x=0$, the other at $x=1$; the factor $(x-1)^{\frac{1}{2}}$ is single valued for values of x in the immediate neighbourhood of $x=0$, and the factor $x^{\frac{1}{2}}$ is single valued for values of x in the immediate neighbourhood of $x=1$. This time, imagine two pieces of paper, one laid over the other, to cover a portion of the plane of x which contains $x=0$ and $x=1$; let a slit be cut in the upper sheet from $x=0$ to $x=1$, and an underlying slit be cut in the lower sheet between the same two points; then, let one edge of the slit in the upper sheet be supposed joined to the edge, of the lower sheet, which lies underneath the other edge of the slit in the upper sheet; and, likewise, the other edge, of the slit in the upper sheet, be joined to the still free edge of the slit in the lower sheet, with the supposition, however, that the two joins can maintain their several identities though interpenetrating. Two such sheets will obviously serve to represent the values of $y=x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}$ in the portion of the plane of x above which the sheets are laid. In this example, we may suppose the two sheets to be each continued to infinity without further intersection, each being closed as if laid over the surface of a sphere. Then we shall have a representation, for all finite and infinite values of x, of the values of y satisfying the equation $y^2=x(x-1)$, by a two-sheeted surface, having two branch points, between which is a *cross line.* In the previous example, $y^2=x$, such a representation is also obtained by supposing a branch point at $x=\infty$, in addition to that at $x=0$, there being a cross line, at which the two sheets cross, extending from $x=0$ to $x=\infty$.

The preceding ideas have a simple application to the representation of the values of y given by the equation $y^2=u_0u_1\ldots u_p$, where $u_i=(x-a_i)(x-c_i)$, all the $2p+2$ values a_i, c_i being different. For this, it is evidently only necessary to think of a two-sheeted surface, with a cross line between the two branch points a, c, a cross line between the two branch points a_1, c_1, and, in general, a cross line between a_i and c_i; to every pair of values (x, y), satisfying the equation, corresponds a point of this surface, and conversely, the pairs (x, y), $(x, -y)$ being represented by the points in the two sheets which lie over the point representing x in the plane below. Instead, then, of a closed curve of integration on the plane of x, which so passes round the critical values of x as to finish with the same value of y as the initial value, we shall consider, on the two-sheeted surface, a closed curve which finishes, *in the same sheet*, at its initial point. A period of an integral $\int R(x, y)\,dx$, which has no logarithmic infinities, will be the result of integration along such a closed curve, unless this gives a zero value. This value will be zero for a small closed curve, lying wholly in one sheet of the surface, in the immediate neighbourhood of a point (as follows from Cauchy's theorem for a single-valued function of a complex variable); it will also be zero for a curve enclosing a region of the double surface which is capable of being divided into *cells* bounded by such small closed curves. Thus the value of the integral, taken round any closed curve on the double surface, may be the sum of the values obtained by integration round several other closed curves. The problem then immediately arises, of finding, if possible, a fundamental set of closed curves on the double surface, such that the integral, taken round any closed curve whatever, is expressible in terms of the values obtained by integration round the fundamental curves. If this can be done, the values of the integral taken round these fundamental curves will be the periods of the integral. The introduction of the multiple-sheeted surface, though simplifying very much the representation of the values of y belonging to a given x, thus raises a new problem, of the topology, or connectivity, of this surface, to be solved independently of the particular integral under consideration. And it is convenient to regard this problem as different from that of the number of independent loops in the plane x, which we have dealt with, though this may furnish suggestions. Consider for example, first, the simple case of the two-sheeted surface associated with the equation

$$y^2 = (x-a)(x-c)(x-a_1)(x-c_1).$$

Let an oval cut be made in the upper sheet, enclosing the branch points c_1, a_1; and let another oval cut be made, from a point on one side of the first oval cut, to the point immediately opposite on the other side, this second cut passing over both the cross lines, so as to lie partly in the lower sheet (as indicated by the dotted lines in the diagram). Then any closed path of integration on the two-sheeted surface may be shewn to be equivalent to the repetition, a certain number of times, of integration along one side of the former oval cut (along APB, in the diagram), taken with the repetition, a certain number of times, of integration along one side of the latter cut (along CQB, in the diagram). Thus, as was found by the method of loops, there are two periods. The double-sheeted surface, in this case, so far as its topology is concerned, is like the surface of an anchor ring; on this, two independent closed circuits can be made, one through the hole, the other round the hole. Or, it is like the surface of a solid constituted by a sphere with a handle attached. More generally, it may be shewn that the two-sheeted surface associated with the equation $y^2 = u_0 u_1 \dots u_p$, where $u_i = (x-a_i)(x-c_i)$ is, in its topology, like the surface of the solid formed by a sphere with p independent handles.



A general equation $f(x, y) = 0$, with m values for y, simplified as explained above, so that the loops to the critical points on the plane of x may be given by a scheme (p. 117, above)

$$(m-1, m)(m-1, m) \dots (2, 3)(2, 3)(1, 2) \dots (1, 2),$$

in which there are $2p+2$ symbols $(1, 2)$, may similarly be represented by a surface of m sheets, consisting of, (i), two sheets, (1) and (2), connected by $p+1$ cross lines, with a branch point at both ends of each cross line; (ii), a sheet (3), connected by a single cross line with the sheet (2); (iii), a sheet (4), connected by a single cross line with the sheet (3); and so on, until we come to the last sheet (m), connected only, by a single cross line, with the preceding sheet ($m-1$). Analogously then to the reduction of the loops in the plane of x, above given, it may be shewn, first, that the presence of the sheet (m) adds nothing to the possibility of drawing independent circuits of integration on the surface, circuits thereon being reducible to circuits on the sheet ($m-1$); then, that circuits on the sheet ($m-1$) may similarly be reduced to circuits on the sheet ($m-2$), and so on. Till, finally, it appears that effectively all period circuits are obtainable by consideration of the sheets (1) and (2); if any periods exist, these sheets are connected by at least two cross lines ($p > 0$). The system of independent period circuits is thus of the same character as for the two-sheeted surface considered above.

Topology of a general Riemann surface. It appears desirable to sketch the relations of what precedes, which is based directly on the fundamental algebraic equation, with a very interesting wider theory; it will be necessary, however, to assume as intuitive many properties that require detailed consideration.

We consider, then, a surface which we suppose to lie entirely in the finite part of space. It consists of real points, some called interior points, and others not interior points; the aggregate of the latter points consitutes the boundary; we shall suppose here that the boundary consists of one or more closed curves, of finite number. An interior point is one which is an interior point of a single small region, of which every point is a point of the surface, having the property that the points of this small region are in (1, 1) continuous correspondence with the points of a simple closed area in a plane; this may, for instance, be a triangle, and in any case has a definite boundary curve to which there corresponds (continuously) a single boundary curve of the region of the surface. The correspondence is to be such that, to any sequence of points of either region which has a limiting point belonging to the region, there corresponds a sequence of points of the other region, with a limiting point corresponding to the former. Later on it is assumed that the correspondence is such that the points of the region considered on the surface correspond to the values of one branch of an algebraic function of the complex variable which is represented by the points of the plane region. The definition does not, for example, allow the vertex of a double cone to be an interior point of a small region of the surface of the cone, but does allow a branch point of a two-sheeted surface, such as we have described, to be such an interior point; though, in this latter case, the boundary curve of the small region on the surface winds twice round the point. The small region of the surface may be called a *cell*; and, with a definite direction of circulation assigned round its boundary curve, it may be called an *oriented cell*. Two oriented cells on the surface may be said to be *adjacent* when they have no interior point in common, but a portion of the boundary curve of one coincides with a portion of the boundary curve of the other, and these two portions are described in opposite directions by the orientations of the two cells. Two oriented cells may be called adjacent, however, also when the common portion of their boundary curves consists only of discrete points (the cells having no interior points in common). We may then have on the surface an aggregate of oriented cells of which each is adjacent to one or more of the cells of the aggregate, and in such a way that the portions of boundary curves, of cells of the aggregate, which belong each only to one cell, form together one or more continuous closed curves, each with a definite direction of description. The simplest case is that of a single simple closed curve forming the boundary of a region entirely filled with adjacent cells, the single closed curve having a definite direction of description which, over every portion, agrees with that of the cell whose boundary curve contains this portion. If C be such a single simple closed curve, we denote its bounding character, which we have described, by writing $C \sim 0$. And, if the boundary of an aggregate of adjacent cells consist of two simple closed curves, each with a definite direction of description, in consonance with that of the portions of boundary of cells from which the closed curve is made up (as explained), these curves being A and B, then we write $A+B\sim 0$; or, if we assign to B the opposite direction, denoting it then by B_1, we write $A \sim B_1$. We shall assume that the whole surface under consideration may be regarded as an aggregate of adjacent cells, and that the portions of boundary curves which are not common to two adjacent cells, form (one or more) boundary curves, each with a definite direction of description, which, for every portion of such a boundary curve, is that derived from the orientation of the cell to which this portion belongs; and further, that such a boundary curve does not cross itself or any other boundary curve. This assumption involves not only that the surface is of one piece, but also that it is consistently orientable (sometimes described as one-sided). A *closed* surface is one for which no boundary curve exists. It will be understood that the relation expressed by $C \sim 0$ may be established by infinitely many different ways of dividing the surface into cells. This relation holds when the closed curve C can be continuously deformed (remaining on the surface) into a small closed curve lying entirely in the immediate neighbourhood of a single point of the surface, but this is not a necessary condition for $C \sim 0$.

It may be worth while to make this last remark clear by considering a simple concrete case. Consider a conical hill, of which the whole surface is accessible, including the summit. Evidently a circuit round the base of the hill is ~ 0, for it can be continuously deformed, up the surface of the hill, to vanish about a point at the summit. But now suppose a tunnel, of circular section, is cut through the hill (from one side of the hill to the other), and the surface of this tunnel (including its roof) is added to the outside surface of the hill. The circuit round the base of the hill (below the tunnel level) cannot now be continuously deformed (remaining on the surface of the hill) to a closed curve at the summit. But it is still ~ 0. For let A, B be points of the circuit at the base, on opposite sides of the hill, and let ASB be a path from A up one side of the hill, to the summit S, and down by the other side to B; denote the same path described in the opposite direction by BSA. The points A, B, of the circuit at the base, divide this into two portions, which we may distinguish by taking a point P in one and a point Q in the other, so that $APBSA$ is a closed circuit, and $BQASB$ is also a closed circuit; the circuit $APBSA$ is deformable, on the surface of the hill, into a circuit λ about one entrance to the tunnel; and the circuit $BQASB$ is deformable into a circuit μ, about the other entrance, in both cases with retention of the direction of description of the circuits. It is easy to see that, if λ, μ be deformed to coincide along a section of the surface of the tunnel in its interior, they will, along this section, be described in opposite directions. Thus the surface of the hill bounded by the circuit $APBQA$ at its base, can be divided into adjacent regions, such that the common portion of the boundary of any two adjacent regions is, as belonging to the two regions, oppositely directed. So that $APBQA$ is ~ 0.

Consider now such a surface as we have described, with or without boundary curves. It may be that every closed curve drawn thereon is ~ 0; this is the case for instance for the complete surface of a sphere, as also for a portion of the surface of a sphere bounded by any simple closed curve drawn thereon. But, on the curved surface of a right circular cylinder, bounded by the circumferences of the two flat ends, closed curves can be drawn, encircling the cylinder, which, not utilising the two bounding circumferences, are not ~ 0. In this case we remark, (i), that if one such closed curve be drawn, say C, any other such curve is $\sim C$; (ii), that, if the surface be cut along a generator, from the circumference at one end to the circumference at the other end (an operation which does not separate the surface into two pieces), then, on the mutilated surface, every closed curve is ~ 0. In what follows we shall assume that, on the surface which we consider, it is possible to find a finite number of simple closed curves $C_1, C_2, \ldots, C_r$, such that every other closed curve is $\sim$ a linear aggregate of these curves, each described a certain number of times; or, in symbols, that, for every closed curve C on the surface, there is a set of positive or negative integers, $m_1, \ldots, m_r$, such that $C \sim m_1C_1 + \ldots + m_rC_r$; the number r, taken as small as may be possible by proper choice of $C_1, \ldots, C_r$, will be called, for the present, the *rank* of the surface. We shall consider how the rank may be modified by the introduction of cuts in the surface, this being, in many cases, a simple way of computing r. The final object is, for the case of a surface which represents the pairs (x, y) satisfying an algebraic equation, $f(x, y) = 0$, to make a connexion between the number r, which gives the number of periods of an algebraic integral associated with this equation, and the genus p of the equation.

The operations of cutting the surface which we employ may be: (i), the making of an infinitesimally small hole, say a *pinhole*, whereby the surface acquires a pinhole boundary besides those it may have had; the hole may be extended to a slit, or be of finite size, so that it does not interfere with other cuts made in the surface; (ii), the making of a cut from a point of a boundary curve to another point of the same, or a different, boundary curve; when the cut, beginning at a point of a boundary curve, has been partially made, the two edges may be counted as new boundary curves, and the cut, instead of ending on a new boundary curve, may end at one edge of the portion of the cut already made. Such a cut is often called a cross-cut (Querschnitt); we shall call it a ***traverse***; (iii), the making of a (infinitesimal) pinhole, followed by a traverse which begins and ends at the boundary of the pinhole; the total cut so obtained will be called an *oval cut*; it is sometimes called a *loop cut*.

We now develop in order a set of twelve results, for which, throughout, we make appeal to intuitive conceptions:

I. A surface with two, or more, simple closed boundary curves, which do not meet, must have $r>0$. For, make a traverse from a point of one boundary curve to a point of another boundary curve. We can evidently pass, on the surface, from a point on one side of this cut to the opposite point on the other side; for instance by following one of the two boundary curves; it appears clear, therefore, that on the original surface, before the cut is made, a closed curve can be drawn which does not constitute by itself the complete boundary of a portion of the surface.

II. If a surface having only one simple closed boundary curve have $r>0$, it is possible to make thereon an oval cut, and it is also possible thereon to make a traverse, without dissevering the surface into two or more separate pieces. For suppose an oval cut to be possible, and to be made, which does dissever the surface; the number of pieces must then be two, one bounded by the original boundary curve of the surface taken with one edge of the oval cut, the other bounded by the other edge of the oval cut; this latter then shews how to draw on the original surface a closed curve forming by itself the complete boundary of a portion of the surface. Thus, if every oval cut, made on the original surface, dissevered the surface, every closed curve on the original surface would be ~ 0; and this is contrary to $r>0$. We can thus make, on the original surface, an oval cut which does not dissever the surface. And this oval cut, with a traverse joining one edge of it to the original boundary line of the surface, constitutes a (looped) traverse which does not dissever the surface.

III. It follows from I and II that a surface with a boundary curve (or curves) which is dissevered into separate pieces by every traverse must have $r=0$. This is a consequence of I for a surface with two or more boundary curves, and of II for a surface with only one boundary curve. It is also true that a surface without boundary which is dissevered by every oval cut must have $r=0$.

IV. For a surface without boundary, the introduction of a pinhole boundary does not alter the value of r. We shall therefore suppose all the surfaces which we consider to have such a boundary, if they have no other. For a surface with one or more boundary curves, the making of a new pinhole boundary increases the value of r by unity.

V. If a surface have $r=0$, it follows from I that it cannot have more than one simple boundary curve; and it follows also from

$r=0$ that every oval cut, as also every traverse, dissevers the surface. For a surface with $r>0$, having one or more simple boundary curves, a traverse, which does not dissever the surface, diminishes r by unity. Such a traverse, in fact, destroys the possibility of completing, into a closed curve, a path which passes, on the surface, from a point on one edge of the traverse into the opposite point of the other edge.

VI. It follows from IV and V that, on a surface with one or more boundary curves, an oval cut, which does not dissever the surface, does not alter the value of r. For such an oval cut may be obtained by making a pinhole boundary, and then a traverse beginning and ending thereat. As said in V, the possibility of such an oval cut involves $r>0$. The boundary curve assumed (when there is but one) may be only a pinhole boundary, of which the oval cut must then be independent. For a surface without boundary the theorem is not true, as we see for instance by considering the surface of an anchor ring.

VII. If a traverse dissever a surface of rank r into two portions, of respective ranks r_1 and r_2, then $r=r_1+r_2$. The possibility of a traverse involves that the given surface has at least one boundary curve. To prove this theorem, let r_1 closed curves be taken on one portion of the dissevered surface, and r_2 closed curves on the other portion, in terms of which all closed curves, of the two portions respectively, can be represented with the connexion $\sim$, as in the definition of the rank. Then we assert that the aggregate of these r_1+r_2 curves would, on the original surface before the traverse was made, be a system in terms of which every closed curve of the surface could be represented; and that there is no aggregate of curves consisting of multiples of curves from the system of r_1 curves, taken with multiples of curves from the system of r_2 curves, which, on the original surface, is ~ 0. The former assertion is equivalent with $r\leqslant r_1+r_2$. In regard to this, it is evidently true that closed curves lying entirely in one of the two portions of the surface are representable by aggregates of curves from the r_1+r_2 curves; but also, a closed curve of the original surface, which crosses the track along which the traverse is subsequently made, can be replaced by an aggregate of curves each lying entirely in one of the two portions of the surface, namely by adding, thereto, parts lying along the track of the traverse, to be described twice over, in opposite directions. The second assertion, equivalent to $r\geqslant r_1+r_2$, is that the r_1+r_2 curves cannot be replaced by a smaller system after removal of the traverse. Now, by hypothesis, the traverse dissevers the surface, and thus renders impossible any path on the surface connecting a point of one of the r_1 curves with a point of any one of the r_2 curves; if then a certain aggregate of curves from the r_1 curves, with a certain aggregate of curves from the r_2 curves, forms the complete boundary of a portion, α, of the original surface, there must be points of the traverse lying in this portion α. But, from such a point, a traverse cannot pass to the boundary of the original surface, without passing over the boundary of the portion α; this boundary is made up, however, of curves from the r_1+r_2 curves, and these are supposed so drawn that none of them meets the traverse. There is thus no aggregate of curves, from the r_1+r_2 curves, which is ~ 0; which is the second assertion. If there were a less number than r_1+r_2 curves, on the original surface, in terms of which the r_1+r_2 specified curves could be expressed, with the connexion $\sim$, these specified curves would be connected in the way now proved impossible.

VIII. If an oval cut dissever a surface of rank r, with one or more boundary curves, into two pieces of ranks r_1 and r_2, then $r+1=r_1+r_2$. For, to make the oval cut, we first introduce a pinhole boundary, thereby increasing the rank to $r+1$ (by IV), and afterwards make a traverse beginning and ending at this point, in this case a dissevering traverse.

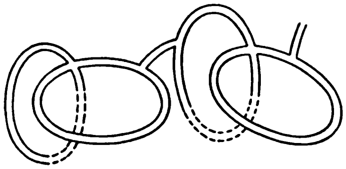
IX. From VII and VIII we infer that, if a surface of rank r, with one or more boundary curves, be dissevered into two portions, of respective ranks r_1 and r_2, by means either of a traverse, or of an oval cut, then $r-1=q+(r_1-1)+(r_2-1)$, where $q=1$ in the case of a traverse, but $q=0$ in the case of an oval cut. If, for a surface with a boundary, for which $r>0$, a traverse be made which does not dissever the surface, the rank, r_1, of the surface with this traverse, is given (V) by $r_1=r-1$; this result is included in the formula we have given if we put $q=1$ and omit the term r_2-1. If an oval cut be made which does not dissever the surface, we have (by VI) $r_1=r$, likewise obtainable from the formula by putting $q=0$, and omitting r_2-1.

X. Hence we can infer, by induction, that if, on a surface of rank r, with one or more boundary curves, there be made q traverses, and also oval cuts of any number (including none), and the result be an aggregate of distinct pieces, with respective ranks $r_1, r_2, \ldots,$ then

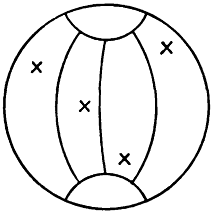
$$r-1=q+\sum_i(r_i-1).$$

This is a capital result.

XI. A particular theorem, useful to us, is that, for a surface with only a single pinhole boundary, the rank r is even. Suppose $r>0$. We can then (by II) make a non-dissevering traverse, beginning and ending at the point boundary, so obtaining (by V) a surface of rank $r-1$. As this has two boundary curves, the edges of the cut, it follows (by I) that $r-1>0$. Hence (by III) we can further make a non-dissevering traverse, passing from one edge to the other of the first traverse, thereby obtaining (by V) a surface of rank $r-2$, with only one boundary curve, constituted by the edges of the two traverses. If $r-2>0$, we can proceed as before, using the single boundary curve as we did the point boundary; then, with two additional traverses, we obtain a surface of one boundary with rank $r-4$. The process can be continued if $r-4>0$. If r were odd, we should reach a single surface of rank unity with one boundary curve, reducible, by a single traverse, to a single surface of zero rank, with two boundary curves; and (by I) such a surface does not exist. Thus r is even; and the process stops with a surface of zero rank, with one boundary curve, after $\frac{1}{2}r$ pairs of traverses have been made. The operation is illustrated by the diagram, where the dotted lines are to indicate the absence of intersection of the traverses.



XII. Now consider a surface constructed to represent the pairs of values of x, y which satisfy an irreducible algebraic equation $f(x, y)=0$, consisting of m sheets laid over the surface of a sphere, connected together at branch points. We need not limit ourselves to the hypothesis of merely simple branch points, in which only two sheets of the surface wind into one another, such as we have described in detail; it is clearly possible, in a similar way, to represent the k values of y, $=x^h$, where $h = 1/k$, when x is small, by means of k sheets, winding into one another about $x=0$, having $(k-1)$ cross lines through this point, over which every sheet preserves its identity. We may then suppose the general branch point, of the surface under consideration, to involve k_i sheets, passing into one another over k_i-1 passage lines, each of which is a passage line, or cross line, at some other branch point. We assume that the surface can be constructed, to represent the m values of y satisfying $f(x, y)=0$, and satisfies the conditions we have explained. It will be closed; but we take an arbitrary pinhole boundary. Putting $w_i=k_i-1$ for each branch point, and $w=\Sigma w_i$, we prove that the rank of the surface is given by $r=w-2m+2$. We can suppose that there are two polar regions, within which no two of the m sheets are connected by branch points. By cutting through all the m sheets, say along arctic and antarctic circles, we can separate these polar regions from the rest of the surface, leaving m pieces about each pole. Supposing the pinhole boundary to be, in one of the m sheets, in one of the (in all) $2m$ closed cuts thus made, these cuts constitute one traverse and $2m-1$ oval cuts. We can suppose,



by slight deformation if necessary, that no branch point of the surface lies immediately under another branch point; and then, if s be the number of branch points, we can make s cuttings, through all the sheets, along meridian curves, from arctic circle to antarctic circle, so separating the branch points, any one of which now lies, in one of the m sheets, between two meridian curves. This is equivalent to making ms further traverses on the surface. Between two such cuttings along meridian curves, enclosing a branch point connecting say k_i sheets, there will be, beside $m-k_i$ simple pieces, all separate, and each of zero rank, a single piece consisting of k_i sheets connected at a branch point; and this single piece also has $r=0$. The total number of pieces, after the cutting, is

$$\alpha, \;=2m+\Sigma(m-k_i+1), \text{ or } 2m+sm-w;$$

the total number of traverses is $q, =1+ms$; and every piece is of zero rank. Whence, by the formula $r-1=q+\Sigma(r_i-1)$, we have $r-1=1+ms-\alpha, =w-2m+1$; as was said.

This simple proof of the value of r is due to Carl Neumann (*Riemann's Theorie*, Leipzig, 1884, p. 169). Riemann's own proof (*Ges. Werke*, 1876, p. 107) is by a contour integration round the boundary of the surface, after it has been dissected, as explained above, by r traverses, into a single surface with zero rank. The result is of importance to us, as establishing the connexion between r, and the genus, p, of the algebraic equation $f(x, y)=0$. For we have previously shewn (Chap. IV, p. 83) that $w=2m+2p-2$. Wherefore $r=2p$. This shews that the number of periods of an algebraic integral, associated with $f(x, y)=0$, which has no logarithmic infinities (which, by the definition, is r), is equal to $2p$, as was previously proved in a different way, under the hypothesis of simple branch points.

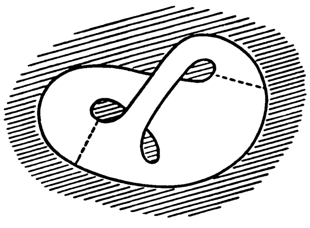
Curve systems on a surface. We proceed now to indicate how the preceding theory of the topology of a Riemann surface is connected with another theory which goes back to Euler and Moebius. Consider in space a set of points, α_0 in number, and a set of lines, α_1 in number, each joining one of the points to another, no two of the lines having an intersection which is not one of the points. We may, in many ways, consider the lines, each used only once, as forming, properly grouped, a set of closed circuits, each with a certain number of *diagonals*, joining pairs of (the given) points on the circuit, this being so done that any of the circuits is connected, to other of the circuits, by one or more of the given lines. The given lines are thus distributed into, portions of circuits, diagonals, and *joins of circuits*. If any of the points is then common to, say, s circuits, we suppose it replaced by s points, of which one is connected to the $s-1$ others by newly introduced lines; this introduces changes in the numbers α_0 and α_1, as well as new lines of the third of the three categories spoken of. After this modification, denote $1-\alpha_0+\alpha_1$ by k. Then the sum of the numbers of lines which are either diagonals or joins of circuits is $k-1$; for the contribution to $\alpha_1-\alpha_0$ by each polygonal circuit is zero, there being as many points as lines entering in the circuit, and each diagonal and each join means a contribution of unity to α_1, with no addition to α_0. Next consider a closed surface, with one pinhole boundary, dissected, as we have described, by q traverses and any number of oval cuts, into pieces of respective ranks r_i. Suppose that, on the undissected surface, in place of these traverses and oval cuts, we draw lines. The points of intersection of these lines, and the lines themselves, then form such a system as we have described; in this system the oval cuts, and the first traverse which was possible (beginning and ending at the point boundary) are represented by polygonal circuits, but the other traverses are represented either by diagonal lines or by joins of circuits. Thus the numbers q and k are the same. If we denote by α_2 the number of pieces into which the surface was dissected, there will be α_2 regions on the undissected surface on which the lines are drawn, with the property that we cannot pass, on this surface, from the interior of one of these regions to the interior of any other, without crossing one of the lines; we call this the number of *faces*. The formula $r-1=q+\Sigma(r_i-1)$ thus gives, for the rank of the closed surface with one point boundary, $r=1+1-\alpha_0+\alpha_1+\Sigma r_i-\alpha_2$, which is $r=2-(\alpha_0-\alpha_1+\alpha_2)+\Sigma r_i$. Of this result a very simple example is given by Euler's well-known formula for the surface of a polyhedron, which has α_0 vertices, α_1 edges, and α_2 faces, where every face has zero rank, and the whole surface of the polyhedron has zero rank ($\Sigma r_i=0$, $r=0$); the formula then becomes $\alpha_0-\alpha_1+\alpha_2=2$.

We may illustrate the results of this chapter by some simple examples.

Ex. 1. If, in the curved portion of the surface of a right circular cylinder, we cut a hole, the remaining part of this curved portion may be dissected into a single piece by two traverses joining the hole to the circumferences of the two ends. For this piece we have $r_1=0$. Putting $q=2$ in the formula $r-1=q+\Sigma(r_i-1)$, we have $r-1=2-1$; thus $r=2$, for the curved portion in which the hole exists. It is indeed easy to see that, if we draw two closed curves on this surface, one encircling the hole, the other encircling the cylinder (not intersecting the former), then any other closed curve on the surface forms by itself, or with one, or both, of these, the complete boundary of a part of the surface.

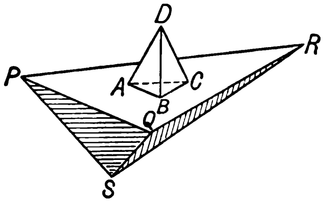
Ex. 2. The part of a plane which is interior to a circle but exterior to k other circles which lie inside the first (not intersecting this or one another) has, likewise, $r=k$.

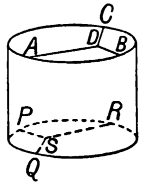
Ex. 3. The surface, of one piece, with one boundary line, and one portion overlying another, illustrated in the diagram, is reduced to a



single piece of rank zero by two traverses (such as those dotted in the diagram). Thus $r=2$.

Ex. 4. The surface of the convex solid formed by cementing one face *ABC*, of a pyramid *ABCD*, to the larger face *PQR* of another pyramid, has evidently the connectivity of the complete surface of a circular cylinder, including its ends (or of the surface of a sphere), namely $r=0$.

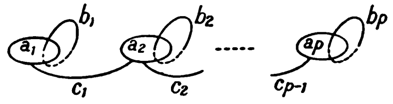




If the lines be drawn on the surface of the cylinder which correspond to the edges in the duplicate pyramid, we may employ the formula $r=2-(\alpha_0-\alpha_1+\alpha_2)+\Sigma r_i$, with $\alpha_0=8$, $\alpha_1=12$, $\alpha_2=7$, $r_i=0$ except for the curved portion of the surface of the cylinder which has rank unity; and this also gives $r=0$.

Ex. 5. For the surface of the solid which is formed by joining each of n spheres (lying externally to one another) to the other $n-1$, by thin circular cylinders (each cylinder meeting in closed curves the two spheres which it joins), we may find the rank by dissection. If we make a cutting along each of the $n(n-1)$ closed curves in which the cylinders meet the spheres, there will result n spheres, each with $n-1$ holes, together with the curved surfaces of $\frac{1}{2}n(n-1)$ cylinders. For each such sphere we have $r_i=n-2$, and for each such curved cylindrical surface we have $r_i=1$. The cuttings (if the pinhole boundary be properly taken) consist of one traverse, and $n(n-1)-1$ oval cuts. Thus, by $r-1=q+\Sigma(r_i-1)$, we have $r-1=1+n(n-2-1)+\frac{1}{2}n(n-1)(1-1)$, or $r=(n-1)(n-2)$. In particular, for $n=2$ the surface is similar to the surface of a sphere; for $n=3$ it is similar to the surface of an anchor ring.

Ex. 6. We have shewn how a Riemann surface can be dissected to a single piece of zero rank. Hence, on the undissected surface a system of



lines can be drawn consisting of p couplets linked in order by $p-1$ other lines, a couplet consisting of two simple closed curves with one intersection. For this system of lines we have $\alpha_0 = p + 2(p-1)$, $= 3p-2$; $\alpha_1 = [3 + 4(p-2) + 3] + p - 1$, $= 5p-3$, since the first and last couplets have 3 edges, the intervening couplets have each 4 edges, and the linking lines are $p-1$ in number; and passage remains possible on the surface between any two points, or $\alpha_2 = 1$. Thus we have, in agreement with $r = 2 - (\alpha_0 - \alpha_1 + \alpha_2) + \Sigma r_i$, the result $\alpha_0 - \alpha_1 + \alpha_2 = 2 - r = 2 - 2p$.

Ex. 7. Similarly a system of lines on the Neumann model, considered in XII above, is indicated by the cuttings, for which $\alpha_0 = 2ms$, $\alpha_1 = 2ms + ms$, $\alpha_2 = 2m + sm - w$; and these give $\alpha_0 - \alpha_1 + \alpha_2 = 2m - w$, which is also $2 - 2p$.

As is briefly noticed in Vol. VI (in connexion with the so-called Zeuthen-Segre invariant I) the expression $\alpha_0 - \alpha_1 + \alpha_2$ is capable of wide generalisation. It is clear from what is said here that, if the value of this be proved to be independent of the system of lines on the Riemann surface by which it is defined, it can be made the basis of the determination of the number of periods of the algebraic integrals.

CHAPTER VI

THE VARIOUS KINDS OF ALGEBRAIC INTEGRALS. RELATIONS AMONG PERIODS

THE present chapter is in the nature of a Note, collecting together briefly details in regard to the characters of the various kinds of algebraic integrals which present themselves for a fundamental algebraic equation $f(x, y)=0$.

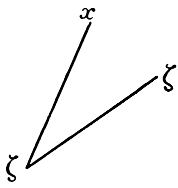
From the point of view in which the equation $f(x, y)=0$ is regarded as representing a plane curve, in the case in which this curve has no multiple points other than double points with distinct tangents, and the lines $x=0$, $y=0$ are in general positions with respect to this curve, we have shewn that, if p be the number of everywhere finite algebraic integrals belonging to the curve, then p is also the number of adjoint polynomials of order $m-3$ (if m be the order of the curve); and further that these polynomials have each $2p-2$ zeros on the curve (other than at the multiple points), no one of which is common to all of them. Also that the number of tangents which can be drawn to the curve from an arbitrary point is $2m+2p-2$. We have also shewn that the theory, for a curve whose multiple points have any complexity, can be reduced to this case, by birational transformation, which will leave the value of p, as defined, unaffected. We have then shewn, for the simple case, by means of the formula $w=2m+2p-2$, using loops of integration, that a general algebraic integral without logarithmic infinities, associated with the curve, has $2p$ linearly independent periods; and we have obtained this same result by use of the Riemann surface. It is very desirable that the notions associated with the theory of the Riemann surface should be familiar; and we regard this point of view as always in the background. But the algebraic theory of the algebraic integrals owes very much to Weierstrass, whose lectures (*Ges. Werke*, IV) should be consulted. When the Riemann surface has been dissected by the loop pairs $a_1, b_1, \ldots, a_p, b_p$, and the linking cuts $c_1, \ldots, c_{p-1}$, as described above (Chap. V), it is simply connected, any closed curve theorem being reducible to a point boundary. On this surface then (as follows from Cauchy's theorem for the contour integral of a single-valued function of a complex variable) every algebraic integral, associated with it, which is without logarithmic infinities, is determined by its initial and final points of integration alone. An algebraic integral having also logarithmic infinities, say at $(z_1), \ldots, (z_k)$, may also be rendered dependent only on the limits of its integration, namely by introducing further cuttings of the surface along curves joining $(z_1), \ldots, (z_k)$ to an arbitrary point of the surface (or in simpler ways, as will be seen).

Consider any algebraic integral $\int R\,dx$, where R is rational in the x, y, which are throughout connected by the rational (irreducible) equation $f(x, y) = 0$. For the neighbourhood of a place at which the local parameter is t, substitution for x and y gives to the expression $R\,dx/dt$ a form

$$R\frac{dx}{dt} = \frac{M}{t^\mu} + \frac{N}{t^\nu} + \ldots + \frac{A}{t} + B + Ct^\rho + Dt^\sigma + \ldots,$$

where $\mu, \nu, \ldots$, in finite number, are positive integers, for which $\mu > \nu > \ldots > 1$, and $\rho, \sigma, \ldots$, in indefinite number, are likewise positive numbers $(\rho < \sigma < \ldots)$. The integral $\int R\,dx$, regarded as a function of its upper limit (x, y), is thus infinite at the place in question like $-M/(\mu-1)t^{\mu-1} - N/(\nu-1)t^{\nu-1} - \ldots + A \log t$, in which $A \log t$ is the logarithmic part, and the other parts may be described as algebraic. We can, however, shew how to construct integrals of the following kinds: (1), An integral which has no place of algebraic infinity (so that the coefficients $M, N, \ldots$ are absent in the expansion about every place), but has a logarithmic infinity at each of two arbitrary places (and not elsewhere). The coefficients A which arise for these two places are then (Chap. III, p. 47) equal and of opposite sign. If the two places be denoted by (ζ) and (γ), and, by division by the constant A, the integral be taken so that the multiplier of the logarithmic term at (ζ) is $+1$, and at (γ) is -1, such integral, integrated from the place (a) to the place (x), will be denoted by $P^{x,a}_{\zeta,\gamma}$, and called an *elementary integral of the third kind.* (2), An integral which has no place of logarithmic infinity, but is algebraically infinite at one arbitrary place, the infinite term in the expansion of the integral, in terms of the parameter, for the neighbourhood of this place, being $-t^{-1}$, there being no negative powers of t of higher than the first order. If this place be (ζ), and the integral be taken from (a) to (x), such integral will be denoted by $E_\zeta^{x,a}$, and called an *elementary integral of the second kind.* (3), We have shewn (Chap. IV, p. 64) how to construct everywhere finite integrals. If p such integrals, linearly independent of one another, taken from (a) to (x), be denoted by $u_1^{x,a}, \ldots, u_p^{x,a}$, the definitions of $P^{x,a}_{\zeta,\gamma}$ and $E_\zeta^{x,a}$ are equally satisfied by $P^{x,a}_{\zeta,\gamma} + U$, $E_\zeta^{x,a} + V$, where U, V denote linear aggregates of $u_1^{x,a}, \ldots, u_p^{x,a}$ with constant coefficients. Conversely, if P, P' be any two functions satisfying the definition of $P^{x,a}_{\zeta,\gamma}$, with the same limits, and infinities, and multipliers, the difference $P' - P$ is necessarily an everywhere finite algebraic integral, and is thus a linear aggregate such as U or V;

and the similar remark is possible for the difference of two elementary integrals $E_\zeta^{x,a}$, with the same pole.



From the integral $E_\zeta^{x,a}$ can be formed, by differentiation in regard to ζ, an integral which is algebraically infinite, at the place (ζ) only, to any integral order. For, let (x) and (ζ') be places, both in the neighbourhood of (ζ)—on a branch of the curve, of which (ζ) is the centre—in the range for which the local parameter for (ζ) is valid; let the passage from (ζ) to (x) be by the value t, and from (ζ) to (ζ') by the parameter τ. We can then suppose (x) so near to (ζ') that we pass from the latter to the former by a parameter t' equal to $t-\tau$. Because (x) is in the neighbourhood of both (ζ) and (ζ'), we have expressions of the forms

$$E_\zeta^{x,a} = -t^{-1}+M+Nt+\dots,\quad E_{\zeta'}^{x,a} = -t'^{-1}+M'+N't'+\dots,$$

leading to

$$E_{\zeta'}^{x,a}-E_\zeta^{x,a} = -(t-\tau)^{-1}+t^{-1}+M'+N't'+\dots-(M+Nt+\dots);$$

if we now suppose (ζ') to approach to (ζ), so that t remains constant but τ diminishes in absolute value, this difference is expressible as a power series in τ, of the form $-\tau t^{-2}-\tau^2 t^{-3}+\dots+\omega$, where ω will ultimately remain finite when (x), later, approaches to (ζ). It follows then that, as τ vanishes, the limit of $(E_{\zeta'}^{x,a}-E_\zeta^{x,a})/\tau$ is an algebraic integral, with (x) as variable place, which becomes infinite, when (x) approaches to (ζ), only like $-t^{-2}$. This limit we denote by $D_\zeta E_\zeta^{x,a}$. We shall see (in the next chapter) how to obtain an explicit expression of this, from the expression of $E_\zeta^{x,a}$, by differentiation under an integral sign. Precisely similar reasoning shews that we can obtain, as $D_\zeta^{k-1}E_\zeta^{x,a}$, an algebraic integral, in (x), which becomes infinite only when (x) approaches to (ζ), like $-(k-1)!/t^k$. And similar reasoning shews that we may obtain $E_\zeta^{x,a}$, by differentiation in regard to ζ, from $P^{x,a}_{\zeta,\gamma}$, namely that $E_\zeta^{x,a}=D_\zeta P^{x,a}_{\zeta,\gamma}$, where the differentiation is in regard to the parameter at (ζ).

It is easy to form an explicit expression for the integral $P^{x,a}_{\zeta,\gamma}$, and convenient to explain the process by regarding the fundamental equation as representing a curve. First, suppose (ζ) and (γ) to be ordinary simple points of this curve, not branch places; let the line joining these points meet the curve again in the (generally) simple points $A_1, \dots, A_{m-2}$. The general adjoint curve of order $m-2$ drawn through the points $A_1, \dots, A_{m-2}$ has other intersections with the curve, beside those at the multiple points, whose number is $2p-2+m-(m-2)$, or $2p$; and this curve contains, homogeneously, $2p-p+1$, or $p+1$ arbitrary coefficients; this follows from Chap. IV,

p. 78. If $L=0$ be the equation of the line joining the points (ζ) and (γ), and ϕ be the general adjoint polynomial of order $m-3$ (containing p arbitrary homogeneous coefficients), it follows that the adjoint curve in question, through $A_1, \ldots, A_{m-2}$, has the equation $L\phi+cU=0$, where c is an arbitrary constant, and $U=0$ is a particular curve satisfying the conditions. Consider now the algebraic integral $\int(L\phi+cU)dx/Lf'(y)$; by the conditions of adjointness satisfied by $L\phi+cU=0$, this integral is everywhere finite save at the points (ζ) and (γ), the other zeros of $L=0$ being compensated by zeros of $L\phi+cU$. At (ζ) and (γ), the integrand having a pole of the first order, the integral is logarithmically infinite; and the constant c can be chosen so that the multiplier of the logarithm is $+1$ at (ζ); it is therefore -1 at (γ), as we have shewn. We see that the integral is of the form $\int cU\,dx/Lf'(y)+\int\phi\,dx/f'(x)$, of which the second part gives the addition we have found above, of p everywhere finite integrals, to a general integral $P_{\zeta,\gamma}^{x,a}$. When (ζ), (γ) are any two distinct places of the curve, a precisely similar construction can be made, if an appropriate birational transformation of the curve be first introduced. In the simple case (and therefore in general) the integral $E_{\zeta}{}^{x,a}$ can be constructed in like manner, if for $L=0$ we take the tangent of the curve at the point (ζ).

Now consider any algebraic integral $\int R(x, y)dx$ whatever. It is clear, if the places at which this integral has algebraic infinities be places (ξ), and the places where it has logarithmic infinities be places (ζ)—both kinds of infinity may enter at the same place—and, if (γ) be an arbitrary place, that we can form an algebraic integral of the form

$$-\int_a^x R(x, y)\,dx+\sum_{\xi}\left[AE_{\xi}{}^{x,a}+A_1D_{\xi}E_{\xi}{}^{x,a}+\ldots+A_{k-1}D_{\xi}{}^{k-1}E_{\xi}{}^{x,a}\right]$$
$$+\sum_{\zeta}BP_{\zeta,\gamma}^{x,a},$$

wherein $A, A_1, \ldots, A_{k-1}, B$ are appropriate constants, which becomes infinite, if at all, only at the single place (γ). This, however, it does not do, since the sum of the values of the constants B, at the places where $\int R(x, y)dx$ has logarithmic infinities, is necessarily zero (Chap. III, p. 47). This integral then must be an aggregate of everywhere finite integrals. In other words, the given integral $\int R(x, y)dx$ is expressible as an aggregate, (i), of integrals $P_{\zeta,\gamma}^{x,a}$, as many as the number of places (ζ) at which $\int R(x, y)dx$ has logarithmic infinities, the place (γ) being arbitrary; (ii), of integrals $E_{\xi}{}^{x,a}$ and their derived integrals $D_{\xi}{}^{i-1}E_{\xi}{}^{x,a}$; (iii), of integrals which are everywhere finite. The aggregate (ii), however, can be much simplified; it can be shewn that, if (c_1), (c_2), ..., (c_p) be quite arbitrary general places of the curve, the aggregate (ii) can be replaced by the sum of an appropriate rational function and a linear aggregate

$$\lambda_1 E_{c_1}{}^{x,a} + \ldots + \lambda_p E_{c_p}{}^{x,a},$$

wherein $\lambda_1, \ldots, \lambda_p$ do not depend on (x); or, alternatively, if (c) be a quite arbitrary place, the aggregate (ii) can be replaced by the sum of an appropriate rational function and a linear aggregate

$$\lambda_1 E_c^{x,a} + \lambda_2 D_c E_c^{x,a} + \ldots + \lambda_p D_c{}^{p-1} E_c^{x,a},$$

where $\lambda_1, \ldots, \lambda_p$ are independent of (x). This is easy to prove; it will be sufficient to take the former statement: Let (z) be any place; there exists (Chap. IV, p. 78) an unique rational function of (x), having poles of the first order at $(c_1), \ldots, (c_p)$ and (z), with coefficient of t^{-1}, in the expansion of this function in terms of the parameter t for (z), when (x) is near to (z), equal to -1, which also vanishes at a further arbitrary place (a). Denote this rational function by $\Psi(x, a; z, c_1, \ldots, c_p)$. From this function, by differentiation with respect to the parameter t at (z), we can form a rational function of (x), with poles of the first order at $(c_1), \ldots, (c_p)$, and a pole at (z) of order k; denote this function by $D_z{}^{k-1}\Psi$; its single infinite term in its expansion near (z) will be $-(k-1)!/t^k$. With the integral $\int R(x, y)\,dx$ we can then form an aggregate of the form

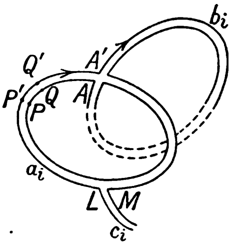
$$-\int_a^x R(x, y)\,dx + \sum_\xi [A\Psi(x, a;\ \xi, c_1, \ldots, c_p)$$
$$+ A_1 D_\xi \Psi(x, a; \xi, c_1, \ldots, c_p) + \ldots + A_{k-1} D_\xi{}^{k-1}\Psi(x, a; \xi, c_1, \ldots, c_p)],$$

wherein $A, \ldots, A_{k-1}$ are appropriate constants, which has poles only at $(c_1), \ldots, (c_p)$, beside possessing the logarithmic infinities which $\int R(x, y)\,dx$ may have. This aggregate is then of the form

$$\lambda_1 E_{c_1}{}^{x,a} + \ldots + \lambda_p E_{c_p}{}^{x,a} + \Sigma B P_{\zeta,\gamma}^{x,a} + \mu_1 u_1{}^{x,a} + \ldots + \mu_p u_p{}^{x,a},$$

where $\lambda_1, \ldots, \lambda_p, B, \mu_1, \ldots, \mu_p$ are certain constants, the summation extends to all the logarithmic infinities of $\int R(x, y)\,dx$, and $u_1{}^{x,a}, \ldots, u_p{}^{x,a}$ are independent everywhere finite integrals. And this is equivalent to the statement we have made. The second statement follows similarly, the places $(c_1), (c_2), \ldots, (c_p)$ being all at one place (c).

It appears from the results we have found that the nature of the infinities and the periods, of any algebraic integral, can be studied by considering integrals $P_{\zeta,\gamma}^{x,a}$, and a sum of integrals of the form $\lambda_1 E_{c_1}{}^{x,a} + \ldots + \lambda_p E_{c_p}{}^{x,a} + \mu_1 u_1{}^{x,a} + \ldots + \mu_p u_p{}^{x,a}$, where (γ), and $(c_1), \ldots, (c_p)$ are arbitrary places of the fundamental curve. The number $2p$, of constant multipliers, in the last integral sum, is the same as the number of periods previously found for an algebraic integral without logarithmic infinities. Some detailed formulae for an integral sum of this form are given in the next chapter. In addition now to the reductions thus found by algebraic methods, we consider further simplifications which can be made if the integrals are considered on a Riemann surface upon which the dissection, which reduces the surface to a simply connected surface, *is supposed to be given.* This dissection is along p pairs of oval loops, $a_1, b_1, \ldots, a_p, b_p$, linked by cuts $c_1, c_2, \ldots, c_{p-1}$. We suppose a definite direction attached to each of $a_1, b_1, \ldots, a_p, b_p$, and speak, correspondingly, of the positive or left edge, and of the negative or right edge, of each cut; *and we suppose that the positive direction of the loop b_i issues from the left edge of the cut a_i.* If P, P' be points, opposite to one another, respectively on the right and left edges of any cut a_i, the element of an algebraic integral $\int R(x, y)\,dx$, in passing from P' to a consecutive point Q' of the same edge, is the same as in passing from P to the consecutive point Q, opposite to Q', of the other edge; hence it can be shewn that the value at P', of the integral $\int R(x, y)\,dx$, which is a single-valued function of its upper limit on the dissected surface, exceeds the value of this integral at P by a quantity which is independent of the position of P on the cut. This quantity may then be defined as that obtained for the integral by a circuit of the positive edge of the cut b_i, *in the negative direction* (from A to A' in the diagram). This quantity will be called a period, and denoted by Ω_i. Similarly a *positive* circuit of the positive edge of the cut a_i gives a period Ω_i'; this is the quantity by which the value of the integral at any point on the positive edge of the cut b_i exceeds its value at the opposite point, on the right edge of this cut. From any point L, on the positive edge of the cut a_i, we can make a complete circuit of the edges of the cuts a_i, b_i, describing in all a path which (reading from left to right) may be represented by $a_i b_i a_i^{-1} b_i^{-1}$, until we reach a point M, just before L. Thus, though the cut c_i is necessary to render the surface simply connected, the values of the algebraic integral at two points opposite to one another on the two edges of this cut are the same.



Consider, now, p linearly independent everywhere finite integrals $u_1^{x,a}, \ldots, u_p^{x,a}$; then, at first without reference to the precise dissection of the Riemann surface we have explained, take $2p$ independent closed circuits on the surface, and denote the periods of the integral $u_i^{x,a}$ for these by $\Omega_{i1}, \ldots, \Omega_{ip}, \Omega'_{i1}, \ldots, \Omega'_{ip}$. Then in the array, of p rows and $2p$ columns $u_i | \Omega_{i1}, \ldots, \Omega_{ip}, \Omega'_{i1}, \ldots, \Omega'_{ip}|$, there is certainly one determinant of p rows and columns which is not zero. Otherwise the rows, in the array, would not be linearly independent, and there would exist an integral $\lambda_1 u_1^{x,a} + \ldots + \lambda_p u_p^{x,a}$, in which $\lambda_1, \ldots, \lambda_p$ are constants, which is single valued on the undissected Riemann surface, as originally constructed to represent the equation $f(x, y)=0$. This function, of (x), would have the analytical character, of expression by powers of the parameter t belonging to the neighbourhood of any point of the surface; and would hence, by the argument previously given (Chap. III, p. 57), be expressible rationally by x and y. There exists, however, no rational function without infinities; thus this function is a constant; as it is zero when (x) is at (a), it is always zero. This, however, is contrary to the fact that the p integrals $u_i^{x,a}$ are linearly independent. Suppose now that $\Omega_{i,k}$ is the constant by which the value of $u_i^{x,a}$ at a point on the left edge of the oval cut a_k, used for dissecting the Riemann surface, exceeds the value of $u_i^{x,a}$ at the opposite point of the right edge of this oval cut a_k (equal to the value obtained for $u_i^{x,a}$ by negative circuit of (x) about the loop b_k). We can then shew in particular that the determinant formed by the first p columns, in the period-array set down, is not zero. For let $U+iV$ be any everywhere finite integral, in which U, V are *real* functions (of the ξ, ξ_1, η, η_1, when $x=\xi+i\xi_1, y=\eta+i\eta_1$); and let the period of $U+iV$ *at* the oval cut a_σ (by which the left-edge value exceeds the right-edge value) be $A_\sigma+iB_\sigma$, where A_σ, B_σ are real; and let the period of $U+iV$ *at* the oval cut b_σ (by which the left-edge value exceeds the right-edge value) be $A_\sigma'+iB_\sigma'$, where A_σ', B_σ' are real. If we take the integral $\int UdV$ along the complete closed circuit $a_\sigma b_\sigma a_\sigma^{-1} b_\sigma^{-1}$ before described, the value obtained is $A_\sigma B_\sigma'-A_\sigma' B_\sigma$; and the complete circuit of the single boundary of the simply connected Riemann surface obtained by the dissection $a_1, b_1, \ldots, a_p, b_p, c_1, \ldots, c_{p-1}$ gives the value $\Sigma(A_\sigma B_\sigma'-A_\sigma' B_\sigma)$, the summation being for σ from 1 to p. By a well-known theorem, however, if $x=\xi+i\xi_1$, this contour integral, which is

$$\int\left(U\frac{\partial V}{\partial \xi}d\xi+U\frac{\partial V}{\partial \xi_1}d\xi_1\right),$$

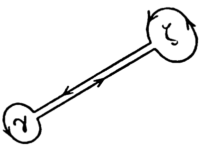
is equal to the double integral

$$\iint\left[\frac{\partial}{\partial \xi}\left(U\frac{\partial V}{\partial \xi_1}\right)-\frac{\partial}{\partial \xi_1}\left(U\frac{\partial V}{\partial \xi}\right)\right]d\xi\, d\xi_1,$$

which is $\iint(U_\xi V_{\xi_1}-U_{\xi_1}V_\xi)d\xi\, d\xi_1$, where $U_\xi=\partial U/\partial\xi$, etc., extended over the whole Riemann surface bounded by the single contour. In virtue of the well-known equations $U_\xi=V_{\xi_1}$, $U_{\xi_1}=-V_\xi$, this is the same as $\iint(U_\xi^2+U_{\xi_1}^2)d\xi\, d\xi_1$. This, however, is definitely greater than zero unless U_ξ, U_{ξ_1} are both everywhere zero, which would involve $V_\xi=0$, $V_{\xi_1}=0$, and hence that the integral $U+iV$ was a constant. A first, important, conclusion, when $U+iV$ is not constant, is, therefore, that $\Sigma(A_\sigma B_\sigma'-A_\sigma' B_\sigma)>0$. This inequality shews that there is no everywhere finite integral for which the periods $A_1+iB_1, \ldots, A_p+iB_p$, at the oval cuts $a_1, \ldots, a_p$ are all zero; nor, indeed, any such integral for which the periods $A_1'+iB_1'$,

$\dots, A_p'+iB_p'$, at the associated oval cuts $b_1, \dots, b_p$ are all zero. And a consequence is that, in the period-array of p rows and $2p$ columns which we have considered, the determinant of the first p columns, referring to the cuts a_σ, is other than zero, as also that of the last p columns; for otherwise, we could form an everywhere finite integral for which every one of A_σ, B_σ was zero. Hence it is possible, by taking suitable linear functions of $u_1^{x,a}, \dots, u_p^{x,a}$, to find p everywhere finite integrals, $v_1^{x,a}, \dots, v_p^{x,a}$, for which the period-array, respectively at the oval cuts a_σ, and the oval cuts b_σ, has the form (with p rows and $2p$ columns) $v_i|\epsilon_{i1}, \dots, \epsilon_{1p}, \tau_{i1}, \dots, \tau_{ip}|$, where $\epsilon_{ii}=1$, $\epsilon_{ij}(j \neq i)=0$. When this is done, the determinant of the last p columns is symmetrical in form, namely $\tau_{ij}=\tau_{ji}$, as we now proceed to prove. The integrals $v_1^{x,a}, \dots, v_p^{x,a}$, evidently unique when the dissection of the Riemann surface is given, are called the *normal elementary integrals of the first kind.* For the proof, consider first any two algebraic integrals whatever, say $I^{x,a}$, $J^{x,a}$, and denote the periods of the former at the oval cuts $a_1, \dots, a_p, b_1, \dots, b_p$ respectively by $\Omega_1, \dots, \Omega_p, \Omega_1', \dots, \Omega_p'$, the similar periods of $J^{x,a}$ being $\Phi_1, \dots, \Phi_p, \Phi_1', \dots, \Phi_p'$. Then take the integral $\int I^{x,a}dJ^{x,a}$ round the complete contour of the simply connected Riemann surface obtained by the $2p+p-1$ cuttings $a_1, \dots, a_p, b_1, \dots, b_p, c_1, \dots, c_{p-1}$. It is easy to prove as before that the value obtained is $\Sigma(\Omega_\sigma\Phi_\sigma'-\Omega_\sigma'\Phi_\sigma)$, where $\sigma=1, \dots, p$. As $I^{x,a}dJ^{x,a}/dx$ is an analytical single-valued function of the complex variable x, on the dissected Riemann surface, it follows, by Cauchy's theorem, that the value of the contour integral is equal to the sum of the values obtained by taking the same integral round the points of the Riemann surface, if any exist, for which the integral is infinite. If, in particular, $I^{x,a}$, $J^{x,a}$ are the two normal elementary integrals of the first kind, $v_i^{x,a}$ and $v_j^{x,a}$, so that $\Omega_\sigma=0$ except for $\Omega_i=1$, and $\Phi_\sigma=0$ save for $\Phi_j=1$, while $\Omega_\sigma'=\tau_{i\sigma}$, $\Phi_\sigma'=\tau_{j\sigma}$, then, as $\Sigma(\Omega_\sigma\Phi_\sigma'-\Omega_\sigma'\Phi_\sigma)=0$, we have $\tau_{ji}-\tau_{ij}=0$.

The same argument establishes another important result: As before, let $P_{\zeta,\gamma}^{x,a}$ denote the elementary integral of the third kind, with logarithmic infinities at (ζ) and (γ); denote the negative of the period of this integral at the oval cut a_ρ by λ_ρ. Then the integral $P_{\zeta,\gamma}^{x,a}+\lambda_1 v_1^{x,a}+\dots+\lambda_p v_p^{x,a}$ has zero periods at every one of the p oval cuts $a_1, \dots, a_p$. There cannot be two algebraic integrals, both logarithmically infinite, as described, at (ζ) and (γ), without periods at the oval cuts $a_1, \dots, a_p$, and both vanishing for $(x)=(a)$, since their difference would be an everywhere finite integral with zero periods at these cuts, and hence would be a constant, equal then to zero. Thus this integral is unique when the oval cuts are given. We shall constantly denote it by $\Pi_{\zeta,\gamma}^{x,a}$, and call it an *elementary normal* *integral of the third kind.* By the argument above, we can prove that the periods of this integral at the oval cuts $b_1, \dots, b_p$ are $2\pi i\, v_1^{x,a}, \dots, 2\pi i\, v_p^{x,a}$. For, in the integral considered above, take $I^{x,a} = v_\alpha^{x,a}$, and $J^{x,a} = \Pi_{\zeta,\gamma}^{x,a}$; then every one of the periods Ω_σ of $v_\alpha^{x,a}$ is zero except Ω_α, which is 1; and all the periods Φ_σ of $\Pi_{\zeta,\gamma}^{x,a}$ are zero; the contour integral thus gives only the result Φ_α', the period of $\Pi_{\zeta,\gamma}^{x,a}$ at the oval cut b_α. We consider then the values obtained by taking the integral round its two points of infinity, which are at (ζ) and (γ). If t be the parameter for the neighbourhood of (ζ), the essential part of the integral $\int v_\alpha^{x,a}\, d\Pi_{\zeta,\gamma}^{x,a}$ for this neighbourhood is $\int v_\alpha^{x,a}\, dt/t$, and, with a change of sign, the essential part at (γ) is of similar form. The value obtained for the two infinities is thus $2\pi i v_\alpha^{\zeta,\gamma}$. Thus $\Phi_\alpha' = 2\pi i v_\alpha^{\zeta,\gamma}$, as was said.



Ex. By applying the contour integration $\int I dJ$, considered above, to the case when I is the elementary normal integral of the third kind with its infinities at (x) and (a), and J is the elementary normal integral of the third kind with its infinities at (z) and (c), prove the identity

$$\Pi_{z,c}^{x,a} = \Pi_{x,a}^{z,c}.$$

The integral $\Pi_{\zeta,\gamma}^{x,a}$ may be used to give a brief proof of a theorem, which we may call *the converse of Abel's theorem.* By the direct theorem, if $(z_1), \dots, (z_k)$ be the zeros, and $(c_1), \dots, (c_k)$ be the poles of any definite rational function associated with the fundamental equation $f(x, y)$, and $u_1^{x,a}, \dots, u_p^{x,a}$ be linearly independent everywhere finite integrals, each of the p expressions $u_\alpha^{z_1,c_1} + \dots + u_\alpha^{z_k,c_k}$ is a sum of integral multiples of the periods of the integral $u_\alpha^{x,a}$, wherein the $2p$ integer multipliers are the same for all the p values of α. On the Riemann surface, dissected by the $2p$ oval cuts a_σ, b_σ, each term $u_\alpha^{z_i,c_i}$ is quite definite, independently of the path of integration from (c_i) to (z_i). Thus, taking the elementary normal integrals of the first kind, we may suppose that $v_\alpha^{z_1,c_1} + \dots + v_\alpha^{z_k,c_k}$ is equal to $m_\alpha + m_1'\tau_{\alpha 1} + \dots + m_p'\tau_{\alpha p}$, for $\alpha = 1, \dots, p$, wherein $m_1, \dots, m_p, m_1', \dots, m_p'$ are definite integers. Conversely, if $(z_1), \dots, (z_k)$ and $(c_1), \dots, (c_k)$ be two sets of each k places, for which, on the dissected Riemann surface, these p equations hold, then there exists a rational function having $(z_1), \dots, (z_k)$ as zeros, and $(c_1), \dots, (c_k)$ as poles. For consider the function $\exp(\phi)$, or e^ϕ, where $\phi = \Pi_{z_1,c_1}^{x,a} + \dots + \Pi_{z_k,c_k}^{x,a} - 2\pi i(m_1' v_1^{x,a} + \dots + m_p' v_p^{x,a})$. This function is analytical on the Riemann surface, being expressible in the neighbourhood of any point for which the parameter is t, by a series of integral powers of t. It is single valued on the dissected surface. It is in fact single valued if the dissection is not made; for its value at any point of the left edge of the oval cut a_ρ is obtained from its value at the opposite point of the right edge by multiplication with e^{μ}, where $\mu = -2\pi i m_\rho'$, so that $e^{\mu} = 1$; and, for two points opposite to one another on the edges of the oval cut b_ρ, the corresponding multiplier is $e^{\mu'}$, where $\mu' = 2\pi i (v_\rho^{z_1, c_1} + \ldots + v_\rho^{z_k, c_k} - m_1' \tau_{1\rho} - \ldots - m_p' \tau_{p\rho})$; namely, by the assumption made at starting, $\mu' = 2\pi i m_\rho$ (since $\tau_{\beta\rho} = \tau_{\rho\beta}$); thus also $e^{\mu'} = 1$. Except at one of the places (z_1), (c_1), ..., (z_k), (c_k), the expression of the function is by a power series in the parameter t; at (z_1), for example, beside a power series, the function ϕ contains a term $\log t$, and the function e^{ϕ} vanishes to the first order. Likewise it vanishes to the first order at $(z_2), \ldots, (z_k)$, and has poles at (c_1), ..., (c_k). Thus, by a proposition proved (p. 57) above, e^{ϕ} is a rational function of (x), with the specified zeros, and poles; as was to be proved.

Ex. 1. By applying the contour theorem for $\int U dV$, as in the text, where $U + iV$ denotes an integral of the first kind $n_1 v_1^{x,a} + \ldots + n_p v_p^{x,a}$, in which $n_1, \ldots, n_p$ are real, prove that, if $\tau_{\alpha,\beta} = \rho_{\alpha.\beta} + i\sigma_{\alpha,\beta}$, where $\rho_{\alpha.\beta}$ and $\sigma_{\alpha,\beta}$ are real, then the quadratic form in $n_1, \ldots, n_p$, $\Sigma\Sigma \sigma_{\alpha,\beta} n_\alpha n_\beta$, is definite and positive, for all real values of $n_1, \ldots, n_p$.

Ex. 2. Denoting the quadratic form $\Sigma\Sigma \tau_{\alpha,\beta} n_\alpha n_\beta$ by τn^2, and the linear form $\Sigma u_\alpha n_\alpha$ by un, where $\alpha, \beta = 1, \ldots, p$, and making a summation with only integer values of $n_1, \ldots, n_p$, each extending from $-\infty$ to $+\infty$, it can be proved from Ex. 1, that the series $\Sigma \exp(\omega)$, where $\omega = 2\pi i un + i\pi \tau n^2$, is absolutely convergent, and thus expresses an integral function of the independent variables $u_1, \ldots, u_p$.

Ex. 3. If in the function $\Theta(u_1, \ldots, u_p)$, of Ex. 2, we replace u_α by $v_\alpha^{x,a} + k_\alpha$, where $v_\alpha^{x,a}$ is the normal elementary integral, it can be shewn, for general values of $k_1, \ldots, k_p$, that the function of (x) so obtained vanishes for p positions of (x), depending on $k_1, k_2, \ldots, k_p$.

Ex. 4. If $u_1^{x,a}, \ldots, u_p^{x,a}$ denote any p linearly independent everywhere finite integrals, it can be shewn, for specified places $(c_1), \ldots, (c_p)$ and specified values of $k_1, \ldots, k_p$, that the p equations

$$u_\alpha^{z_1, c_1} + \ldots + u_\alpha^{z_p, c_p} = k_\alpha, \qquad (\alpha = 1, \ldots, p),$$

can be satisfied by proper choice of the places $(z_1), \ldots, (z_p)$. This is a familiar result when $p = 1$, the coordinates of the place (z_1) being elliptic functions of k_1. In general, the rational symmetric functions of the coordinates of the places $(z_1), \ldots, (z_p)$ are single-valued analytical functions of $k_1, \ldots, k_p$. In the elliptic case, in accordance with the two-period ambiguity of the integral $u^{z,c}$, the elliptic functions have two periods; in the general case the functions are similarly unaltered when each of $k_1, \ldots, k_p$ is increased by an appropriate constant, and there are $2p$ sets of such simultaneous increments which leave the functions unaltered. For particular values of $k_1, \ldots, k_p$, the functions are indeterminate, and the places $(z_1), \ldots, (z_p)$ have not a definite set of positions.

Ex. 5. The functions solving the so-called inversion problem, expressed by the equations in the previous example, can be expressed in terms of such functions as the theta functions referred to in Exx. 2, 3.

Ex. 6. The general theory of (meromorphic) functions of p variables which are unaltered when the variables are increased simultaneously by the elements of any one of $2p$ (or fewer) columns of periods, is of great importance. A preliminary study of the possible reduction of the array of periods adds to the simplicity of the theory.

For the content of Exx. 1–4 here the reader may be referred to the author's *Abel's Theorem* (Cambridge, 1897); and for Exx. 5, 6 to the author's *Multiply-periodic Functions* (Cambridge, 1907); for the reduction of the array of periods, referred to in Ex. 6, a somewhat different formulation is found in Severi, *Rend. Palermo*, XXI, 1906, "Intorno al teorema d'Abel sulle superficie...". Krazer, *Lehrbuch der Thetafunctionen* (Leipzig, 1903), should also be consulted.

CHAPTER VII

THE MODULAR EXPRESSION OF RATIONAL FUNCTIONS AND INTEGRALS

THE present chapter is concerned with a theory of the rational functions and integrals, associated with an (irreducible) algebraic curve $f(x, y)=0$, which began with the arithmetical work of Kronecker and Dedekind* for the theory of integer numbers. Developed in detail the theory gives an alternative to much of what has preceded; what is given here however seems a desirable, if not a necessary, accompaniment of what has already been proved.

We have explained what is meant by a rational function belonging to the curve $f(x, y)=0$; to fix the ideas we recall certain properties of a general kind for such functions. It was seen that, as a rule, there exists no rational function of the first, nor of the second order; there is thus, in general, a least order for which there exists a rational function, associated with the curve. But we cannot expect to be able to construct a function of this least order with its poles taken arbitrarily on the curve; for instance, when the equation $f(x, y)=0$ is of the form $y^2-u=0$, where u is a polynomial in x, though there exists a function, of the form $(x-a)^{-1}$, with two poles, these must be at places for which x has the same value. There is thus another least number, say k, such that, *whatever k places be taken*, a rational function exists with poles, all of the first order, at these k places, or at places chosen from these. Such function will be actually infinite, indeed, at every one of the k places taken; for if, for all sets of k places taken, the function had not k poles from among them, then k has not been taken as small as possible. From these considerations we conclude that, *if $k_1, k_2, \ldots, k_m$ be m positive integers, whose sum is large enough, a rational function can be constructed whose poles are multiple, and at m given arbitrary places, the function being infinite to order k_1, or less, at the first place; to order k_2, or less, at the second place; and so on.* Such a function will not always be actually of the assigned orders of infinity $k_1, k_2, \ldots, k_m$, at these places; for example, for the equation $y^2-u=0$, where u is a polynomial in x of order $2p+2$, with unrepeated linear factors, though, as a rule, a function can be constructed with $p+1$ arbitrary poles, we cannot construct a function with a pole of order p at a place $x=a, y=b$, on the curve, and a pole of the order 1 at the conjugate place $x=a, y=-b$; as the reader may prove $(p>1)$.

Modular expression of a rational function. Now let z denote a rational function which has a pole of the first order at each of m places, $A_1, \ldots, A_m$, and is not else infinite. Consider the aggregate of rational functions existing, which have poles only at these m places, of any possible orders. Denote by $(r_1, r_2, \ldots, r_m)$ such a function, actually infinite at these places respectively to orders $r_1, \ldots, r_m$; these numbers we call, for the present, the indices of the function. It is evidently possible to subtract from this function such a rational polynomial in z, of order r_m, that the difference is not infinite at A_m; for by hypothesis z is infinite at A_m only to the first order. Thus, save for polynomials in z, it is sufficient to consider, among the aggregate of functions $(r_1, r_2, \ldots, r_m)$, only those for which the m-th index, r_m, is zero. Considering then the other indices $r_1, r_2, \ldots, r_{m-1}$, in order, there will be one (or more) of these which is greater than the others, and, if there are several with this greatest value, there will be one, with this value, which occurs first; say this is r_i; thus we have $r_1 < r_i, r_2 < r_i, \ldots, r_{i-1} < r_i$, but $r_i \geqslant r_{i+1}, r_i \geqslant r_{i+2}, \ldots, r_i \geqslant r_{m-1}$, $(1 \leqslant i \leqslant m-1)$; when this is so, let the function $(r_1, r_2, \ldots, r_{m-1}, 0)$ be said to be *of the i-th class*, and to be *of dimension* r_i.

---

* Kronecker, *Crelle*, XCI, XCII, 1882, *Werke*, Bd. II. Dedekind u. Weber, *Crelle*, XCII.

Since it is only when the aggregate order of infinity of a function is great enough, that the function can be constructed with prescribed orders of infinity at prescribed places, it is clear that, for all functions of the i-th class (not mere constants), there is a least dimension occurring among such functions of the i-th class as actually exist. Take now any function of the i-th class which has this least dimension; say this function is

$$(s_1, s_2, \ldots, s_{i-1}, r_i, s_{i+1}, \ldots, s_{m-1}, 0),$$

where

$$s_1 < r_i, s_2 < r_i, \ldots, s_{i-1} < r_i; \quad r_i \geqslant s_{i+1}, \ldots, r_i \geqslant s_{m-1}; \quad r_i > 0, \qquad (i = 1, 2, \ldots, m-1);$$

we may speak of this function as *the chosen reduced function* of the i-th class. If we denote any other existing function (with m-th index equal to zero) which is of the i-th class by

$$(S_1, S_2, \ldots, S_{i-1}, R_i, S_{i+1}, \ldots, S_{m-1}, 0),$$

where

$$S_1 < R_i, S_2 < R_i, \ldots, S_{i-1} < R_i; \quad R_i \geqslant S_{i+1}, \ldots, R_i \geqslant S_{m-1}, R_i \geqslant r_i, \qquad (i = 1, \ldots, m-1),$$

then, by choosing the constant λ suitably, in the difference

$$(S_1, S_2, \ldots, S_{i-1}, R_i, S_{i+1}, \ldots, S_{m-1}, 0) \\ - \lambda z^{R_i - r_i}(s_1, s_2, \ldots, s_{i-1}, r_i, s_{i+1}, \ldots, s_{m-1}, 0),$$

we can secure that, in the function given by this difference, say

$$(T_1, T_2, \ldots, T_{i-1}, R_i', T_{i+1}, \ldots, T_{m-1}, R_i - r_i),$$

we have $R_i' < R_i$, and also that any of the earlier indices T_k, where $k < i$, which certainly does not exceed the greater of S_k and $R_i - (r_i - s_k)$, is less than R_i; while any one of the later indices T_l, where $l > i$, which certainly does not exceed the greater of S_l and $R_i - (r_i - s_l)$, is at most as great as R_i. Thus, in this difference, all the first i indices are less than R_i, and the succeeding indices do not exceed R_i. Then further, forming the difference

$$(T_1, T_2, \ldots, T_{i-1}, R_i', T_{i+1}, \ldots, T_{m-1}, R_i - r_i) - (z, 1)_{R_i - r_i},$$

where the polynomial in z, denoted by $(z, 1)_{R_i - r_i}$, of order $R_i - r_i$, is to be chosen so that this is a function,

$$(S_1', S_2', \ldots, S'_{i-1}, R_i'', S'_{i+1}, \ldots, S'_{m-1}, 0),$$

with last index zero, we see that the index R_i'', which does not exceed the greater of R_i' and $R_i - r_i$, is $< R_i$; further that S_k', when $k < i$, not exceeding the greater of T_k and $R_i - r_i$, is $< R_i$; and that S_l', for $l > i$, not exceeding the greater of T_l and $R_i - r_i$, does not exceed R_i. The formation of this last difference is unnecessary if $R_i = r_i$. In any case, employing the process which is expressed by

$$\begin{aligned}(S_1, S_2, \ldots, S_{i-1}, R_i, S_{i+1}, \ldots, S_{m-1}, 0)\\ = \lambda z^{R_i - r_i}(s_1, \ldots, s_{i-1}, r_i, s_{i+1}, \ldots, s_{m-1}, 0) + (z, 1)_{R_i - r_i}\\ + (S_1', \ldots, S'_{i-1}, R_i'', S'_{i+1}, \ldots, S'_{m-1}, 0),\end{aligned}$$

we obtain a function, $(S_1', S_2', \ldots, S'_{i-1}, R_i'', S'_{i+1}, \ldots, S'_{m-1}, 0)$, of one of two kinds; (i), in which the greatest index is R_i, but, if so, this does not occur in the first i positions, or (ii), in which the greatest index is $< R_i$. In other words, this function obtained, is either of higher class, without change of dimension, or is of less dimension, than the function $(S_1, \ldots, S_{i-1}, R_i, S_{i+1}, \ldots, S_{m-1}, 0)$ from which it was obtained. There is, however, an upper bound for the class, namely $m - 1$; and there is a lower bound for the dimension of a function of specified class. Therefore, if we denote by h_i the chosen reduced function of class i, $(s_1, \ldots, s_{i-1}, r_i, s_{i+1}, \ldots, s_{m-1}, 0)$, for $i = 1, \ldots, (m-1)$, and apply the above process of reduction continually, we shall be able to express any rational function whose poles are all at the places $A_1, \ldots, A_m$, in the form

$$(z, 1)_0 + (z, 1)_1 h_1 + \ldots + (z, 1)_{m-1} h_{m-1},$$

wherein $(z, 1)_i$ is a polynomial in z; and this in such a way that no term arises in this expression, of the form $z^\alpha h_\beta$, whose dimension is higher than that of the function given by this expression.

Fundamental integral functions. Now suppose that, in the fundamental irreducible algebraic equation $f(x, y) = 0$, the highest aggregate order in x and y is n, and that the term in y^n is actually present. As has been remarked, this can always be secured by a preliminary substitution for x, of the form $x + hy$. Then y becomes infinite only for infinite values of x, and may be spoken of as an *integral* function of x. Suppose further that, for $x = \infty$, there are n distinct places, there being no cycle of two, or more, of the n

values of y (or, in another description, no branch place for $x=\infty$). This also is easily secured, as we see by considering the homogeneous equation $F(x_0, x_1, x_2)=0$ corresponding to $f(x, y)=0$, and changing, if necessary, the line denoted by $x_2=0$; for the equation $f(x, y)=0$ this comes to using functions of the forms $x(ax+by+c)^{-1}$, $y(ax+by+c)^{-1}$ in place of x and y.

Then, we apply the preceding theory, of rational functions with poles at m given places, to the case when the function z, there spoken of, is the function x (and thus $m=n$), and the chosen fundamental reduced functions are *integral* functions, becoming infinite only for $x=\infty$. These we denote by $g_1, \ldots, g_{n-1}$; and the theorem proved is that any *integral* rational function, that is one having poles for $x=\infty$ only, is expressible in the form

$$u_0+u_1g_1+\ldots+u_{n-1}g_{n-1},$$

where $u_0, \ldots, u_{n-1}$ are polynomials in x. The dimension of the function g_i will be denoted by σ_i+1; this is the least positive integer such that $x^{-(\sigma_i+1)}g_i$ is not infinite at any of the n places for which $x=\infty$. This number is not zero, or the function g_i, having then no infinities, would be a constant; thus $\sigma_i \geqslant 0$. The sum of the $n-1$ numbers σ_i will be denoted by p (it being proved below that this notation is consistent with previous uses of p); the sum of the dimensions of $g_1, \ldots, g_{n-1}$ is thus $p+n-1$.

From these integral functions $g_1, \ldots, g_{n-1}$, we can obtain such a set of functions $h_1, \ldots, h_{n-1}$ as is given by the general theorem above for the case when $z=(x-a)^{-1}$, where a is any general constant. For let $h_i=(x-a)^{-(\sigma_i+1)}g_i$, so that h_i is infinite only when $x=a$, or $z=\infty$, using z for $(x-a)^{-1}$. Let H be any rational function which is infinite only when $x=a$, or $z=\infty$, and let $\sigma+1$ be its dimension, so that $(x-a)^{\sigma+1}H$ is infinite only when $x=\infty$, and $\sigma+1$ is the least positive integer for which this is so. Then we have an expression $(x-a)^{\sigma+1}H=u_0+u_1g_1+\ldots+u_{n-1}g_{n-1}$, where $u_0, \ldots, u_{n-1}$ are polynomials in x. Let the orders of these polynomials be, respectively, $r_0, r_1, \ldots, r_{n-1}$; then the fact that there occurs no term, in this expression, on the right, which is of higher dimension than that of the function $(x-a)^{\sigma+1}H$, involves that $r_i+\sigma_i+1 \leqslant \sigma+1$. The expression is, however, the same as

$$H=z^{\sigma+1}u_0+\ldots+z^{\sigma+1}u_iz^{-(\sigma_i+1)}h_i+\ldots,$$

while u_i can be written as $z^{-r_i}v_i$, where v_i is a polynomial in z of order r_i. As $\sigma+1-r_i-(\sigma_i+1)$, which we may denote by t_i, is not negative, the product $z^{t_i}v_i$, or w_i, say, is a polynomial in z; thus we have an expression $H=w_0+\ldots+w_ih_i+\ldots$, for any rational function H which is infinite only when $z=\infty$, in terms of the functions h_i; and it is easy to see that the dimension of h_i, in terms of z, is σ_i+1; and that no term occurs, in this expression, on the right, having dimension higher than that of H. Conversely, given a set of functions, $h_1, \ldots, h_{n-1}$, appropriate, in accordance with the general theorem, for the expression of rational functions which are infinite only when $x=a$, we can similarly deduce a set, $g_1, \ldots, g_{n-1}$, appropriate for the expression of all integral functions.

Important properties of the fundamental integral functions. There are certain important properties of the integral functions $g_1, \ldots, g_{n-1}$ which we now develop:

(*a*) There can be no identity of the form

$$U_0+U_1g_1+\ldots+U_{n-1}g_{n-1}=0,$$

where $U_0, \ldots, U_{n-1}$ are rational functions of x only. For as $y, y^2, \ldots, y^{n-1}$ are integral functions of x, each is expressible linearly by $g_1, \ldots, g_{n-1}$, with coefficients which are polynomials in x; if there were such a linear equation connecting $g_1, \ldots, g_{n-1}$ as we have spoken of, then all of $y, y^2, \ldots, y^{n-1}$ would be expressible linearly by at most $n-2$ of the functions $g_1, \ldots, g_{n-1}$, with coefficients rational in x only. There would thus be a linear relation connecting $y, \ldots, y^{n-1}$ with coefficients rational in x. This is contrary to the hypothesis that the original equation, $f(x, y)=0$, is irreducible.

(*b*) Consider a value $x=c$, for which all the n places of $f(x, y)=0$ are distinct; at one of these n places let $g_1=a_1, g_2=a_2, \ldots, g_{n-1}=a_{n-1}$; at another of these let $g_1=b_1, g_2=b_2, \ldots, g_{n-1}=b_{n-1}$. We prove that not all the $n-1$ differences $a_1-b_1, a_2-b_2, \ldots, a_{n-1}-b_{n-1}$ are zero. For choose the constants $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ so that the function $\lambda_1(g_1-a_1)+\ldots+\lambda_{n-1}(g_{n-1}-a_{n-1})$, which vanishes at one of the two places spoken of, vanishes also at the $n-2$ places for which $x=c$, other than these two places. Then, if all the differences $a_1-b_1, \ldots, a_{n-1}-b_{n-1}$ were zero, the function

$$[\lambda_1(g_1-a_1)+\ldots+\lambda_{n-1}(g_{n-1}-a_{n-1})]/(x-c)$$

would not become infinite at any of the n places for which $x=c$; it would thus be an integral function, and, therefore, expressible linearly in terms of $g_1, \ldots, g_{n-1}$ with coefficients which are polynomials in x. We should then have such an identity connecting $g_1, \ldots, g_{n-1}$ as was shewn impossible in (*a*).

(*c*) The result (*b*) is a particular case of a much more general proposition, to which we now proceed. Consider a finite value of x, $x=c$, for which the n corresponding values of y, to take the most general possibility, form several cycles, there being one cycle of r values, $y^{(1)}, \ldots, y^{(r)}$, then another cycle of s values, and so on. There will then be one place expressed by a pair of equations of the forms $x-c=t^r$, $y=A+A_1t+\ldots$, in terms of the local parameter t; then another place expressed by a pair of equations of the forms $x-c=t_1^s$, $y=B+B_1t_1+B_2t_1^2+\ldots$; and so on. For the first place, let the corresponding values, for any one, g_k, of the functions $g_1, \dots, g_{n-1}$, in the neighbourhood of the place, be given by

$$g_k^{(1)} = a_k + a_k' t + \tfrac{1}{2} a_k'' t^2 + \dots, \quad g_k^{(\alpha)} = a_k + a_k' t_\alpha + \tfrac{1}{2} a_k'' t_\alpha^2 + \dots,$$

where $k = 1, \dots, (n-1)$, $\alpha = 2, 3, \dots, r$, and t_α is $\epsilon^{\alpha-1} t$, with

$$\epsilon = \exp(2\pi i/r).$$

For the second place there will be corresponding expressions, with $b_k, b_k', \dots$ in place of $a_k, a_k', \dots$, and s in place of r; and so on for all the places occurring for $x = c$. The theorem to be proved is that a certain determinant A_0, of n rows and columns, formed from the coefficients $a_k, a_k', \dots, b_k, \dots$ in these expansions, is not zero. This determinant is

$$A_0 = \begin{vmatrix} 1, & a_1\;, & a_2\;, & \dots, & a_{n-1} \\ 0, & a_1'\;, & a_2'\;, & \dots, & a'_{n-1} \\ 0, & \tfrac{1}{2}a_1'', & \tfrac{1}{2}a_2'', & \dots, & \tfrac{1}{2}a''_{n-1} \\ \multicolumn{5}{c}{\dotfill} \\ 1, & b_1\;, & b_2\;, & \dots, & b_{n-1} \\ 0, & b_1'\;, & b_2'\;, & \dots, & b'_{n-1} \\ 0, & \tfrac{1}{2}b_1'', & \tfrac{1}{2}b_2'', & \dots, & \tfrac{1}{2}b''_{n-1} \\ \multicolumn{5}{c}{\dotfill} \\ \multicolumn{5}{c}{\dotfill} \end{vmatrix},$$

whose construction is quite simple. The second, third, ... columns are formed from coefficients in the expansions, respectively, of $g_1, g_2, \dots, g_{n-1}$; the first r rows arise from these expansions at the first place considered for $x = c$, containing the coefficients of the powers $t^0, t^1, \dots, t^{r-1}$ of the parameter, at this place; the following s rows similarly arise for the second place; and so on. It is in the nature of the case that $r + s + \dots = n$. If this determinant were zero it would be possible to find constants, $\lambda, \lambda_1, \dots, \lambda_{n-1}$, so that the n expressions, formed from the rows of this determinant,

$$\lambda + \lambda_1 a_1 + \dots + \lambda_{n-1} a_{n-1},\ \lambda_1 a_1' + \dots + \lambda_{n-1} a'_{n-1},\ \dots,$$
$$\lambda_1 a_1^{(r-1)} + \dots + \lambda_{n-1} a_{n-1}^{(r-1)},$$
$$\lambda + \lambda_1 b_1 + \dots + \lambda_{n-1} b_{n-1},\ \lambda_1 b_1' + \dots + \lambda_{n-1} b'_{n-1},\ \dots,$$
$$\lambda_1 b_1^{(s-1)} + \dots + \lambda_{n-1} b_{n-1}^{(s-1)},$$
$$\dots\dots\dots\dots\dots\dots$$

were all zero. This would mean, however, that the function $\lambda + \lambda_1 g_1 + \dots + \lambda_{n-1} g_{n-1}$ vanishes to order r at the first place, to order s at the second place, and so on; and thence that the function $(\lambda + \lambda_1 g_1 + \dots + \lambda_{n-1} g_{n-1})/(x - c)$ is not infinite for $x = c$; this function would then be an integral function, expressible linearly by $g_1, \dots, g_{n-1}$, with coefficients which are polynomials in x. There would then exist an identity such as is proved impossible in (*a*). Particular consequences of this theorem are, first, the theorem (*b*), that if there be no branch place for $x=c$, not every one of the $(n-1)$ differences $a_1-b_1, a_2-b_2, \ldots, a_{n-1}-b_{n-1}$ vanishes; second, that in the expansions of the $(n-1)$ integral functions g_k $(k=1, \ldots, n-1)$ at the first branch place considered at $x=c$, the coefficients of t^m cannot vanish in all these functions, for any m such that $0<m<r$; since otherwise the determinant Δ_0 would have a row of vanishing elements $0, a_1^{(m)}, a_2^{(m)}, \ldots, a_{n-1}^{(m)}$. Thus, for general values of $\lambda, \lambda_1, \ldots, \lambda_{n-1}$, the expansion of the function $\lambda+\lambda_1 g_1+\ldots+\lambda_{n-1}g_{n-1}$, in terms of the parameter t, at a branch place of $f(x, y)=0$ (occurring for a finite value of x, in whose neighbourhood $x=c+t^r$, with y of the form $d+d_1t+\ldots$) contains all powers of t from t to t^{r-1}.

Employing the same notation, we can deduce, from the non-vanishing of the determinant of constants, Δ_0, a general functional theorem. It is convenient to formulate this as an identity connecting matrices. Consider a general (finite) value of x for which the corresponding n places of $f(x, y)=0$ are distinct, let $g_k^{(1)}, \ldots, g_k^{(n)}$ denote the values of the integral function g_k at these n places. Denote by G the matrix of n rows and columns whose i-th row consists of the elements $1, g_1^{(i)}, g_2^{(i)}, \ldots, g_{n-1}^{(i)}$; it is a consequence of what we have shewn that the determinant of this matrix would only vanish if there were branch places for this value of x. Denote also by $\bar{G}$ the matrix formed from G by transposition of rows and columns. Then the product matrix $\bar{G}G$ has for elements in its first row, $n, s_1, \ldots, s_{n-1}$, where $s_k=g_k^{(1)}+\ldots+g_k^{(n)}$, and for elements in its i-th row, $s_{i-1}, s_{i-1,1}, \ldots, s_{i-1,n-1}$, where

$$s_{h,k}=g_h^{(1)}g_k^{(1)}+\ldots+g_h^{(n)}g_k^{(n)},$$

so that $s_{h,k}=s_{k,h}$. Each of these elements is the sum of the n conjugate values, for the same value of x, of a function which only becomes infinite when $x=\infty$; thus each element is expressible as a polynomial in x only. The determinant of the matrix $\bar{G}G$ is thus also a polynomial in x. One consequence of the theorem we shall obtain is that, corresponding to a place where there is a branching of index r (about which the expressions for x and y are of the forms $x-c=t^r$, $y=B+B_1t+\ldots$, in terms of the parameter at the place), there is a factor of the determinant of $\bar{G}G$ equal to $(x-c)^{r-1}$. By what was proved above there is no such vanishing factor in this determinant unless there is a branch place. It will follow, then, that the order of the polynomial in x given by the determinant of $\bar{G}G$, is the sum of the various values of $r-1$ at all the existing branch places occurring for finite values of x. Consider such a branch place, of index r; put $\epsilon=\exp(2\pi i/r)$; let M_s denote the determinant, of s rows and columns, whose (λ, μ)-th element is ϵ^σ, where $\sigma=(\lambda-1)(\mu-1)$, namely

$$M_s=\begin{vmatrix} 1, & 1\ , & \dots, & 1 \\ 1, & \epsilon\ , & \dots, & \epsilon^{s-1} \\ 1, & \epsilon^2\ , & \dots, & \epsilon^{2(s-1)} \\ \dots & \dots & \dots & \dots \\ 1, & \epsilon^{s-1}, & \dots, & \epsilon^{(s-1)^2} \end{vmatrix};$$

denote the cofactor of the ρ-th element of the last column of this determinant by $H_{s,\rho}$, and put $K_{s,\rho}=H_{s,\rho}/M_s$. We evidently have $H_{s,s}=M_{s-1}$, and, hence, $K_{s,s}=M_{s-1}/M_s$. By these definitions, the following sum, of s terms,

$$K_{s,1}+\epsilon^{\rho-1}K_{s,2}+\epsilon^{2(\rho-1)}K_{s,3}+\dots+\epsilon^{(s-1)(\rho-1)}K_{s,s},$$

vanishes for $\rho<s$, and is unity when $\rho=s$. Consider the expansion of the function g_k at the branch place of index r, namely

$$g_k^{(1)}=a_k+a_k't+\tfrac{1}{2}a_k''t^2+\dots,$$

and the other expansions when t is replaced by ϵt, $\epsilon^2 t$, ..., $\epsilon^{s-1}t$, of the forms

$$g_k^{(\lambda)}=a_k+a_k'\epsilon^{\lambda-1}t+\frac{a_k''}{2!}\epsilon^{2(\lambda-1)}t^2+\dots, \qquad (\lambda=2, \dots, s,\ s\leqslant r);$$

we infer that

$$K_{s,1}g_k^{(1)}+K_{s,2}g_k^{(2)}+\dots+K_{s,s}g_k^{(s)}=\frac{1}{(s-1)!}a_k^{(s-1)}t^{s-1}$$
$$+A_k^{(s)}t^s+A_k^{(s+1)}t^{s+1}+\dots,$$

where $A_k^{(s)}$, $A_k^{(s+1)}$, ..., are proper constants. Now, recall the form of the matrix G, and the first r rows of this, which, in a general column, contain $g_k^{(1)}, g_k^{(2)}, \dots, g_k^{(r)}$. By what we have remarked, if we multiply the first two rows of this matrix G respectively by $K_{2,1}$ and $K_{2,2}$, and add the results to form a new second row, this (taking $s=2$ in what was remarked) will consist of elements

$$0,\ \frac{1}{1!}a_1't+(t^2),\ \frac{1}{1!}a_2't+(t^2),\ \dots,\ \frac{1}{1!}a'_{n-1}t+(t^2),$$

where (t^2) denotes, in every case, a power series beginning with a term in t^2. Similarly, taking $s=3$, if we multiply the first three rows of G respectively by $K_{3,1}$, $K_{3,2}$, $K_{3,3}$, and add the results to form a new third row, the elements of this row will be

$$0,\ \frac{1}{2!}a_1''t^2+(t^3),\ \frac{1}{2!}a_2''t^2+(t^3),\ \dots,\ \frac{1}{2!}a''_{n-1}t^2+(t^3),$$

where (t^3) denotes, in every case, a power series beginning with a term in t^3. The process may be continued, the last operation being for $s=r$, and we consider all the first r rows of the matrix G. We may repeat these statements in matrix form by writing

$$\begin{pmatrix} 1 & , & . & . & \ldots, & . \\ K_{2,1}, & K_{2,2}, & . & , & \ldots, & . \\ K_{3,1}, & K_{3,2}, & K_{3,3}, & & \ldots, & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{r,1}, & K_{r,2}, & K_{r,3}, & & \ldots, & K_{r,r} \end{pmatrix} \begin{pmatrix} 1, & g_1^{(1)}, & \ldots, & g_{n-1}^{(1)} \\ 1, & g_1^{(2)}, & \ldots, & g_{n-1}^{(2)} \\ & & & \\ \cdots & \cdots & \cdots & \cdots \\ 1, & g_1^{(r)}, & \ldots, & g_{n-1}^{(r)} \end{pmatrix}$$

$$= \begin{pmatrix} 1, & a_1+(t) & , & \ldots, & a_{n-1}+(t) \\ 0, & a_1' t+(t^2) & , & \ldots, & a'_{n-1}t+(t^2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \dfrac{a_1^{(r-1)}}{(r-1)!}t^{r-1}+(t^r), & & \ldots, & \dfrac{a_{n-1}^{(r-1)}}{(r-1)!}t^{r-1}+(t^r) \end{pmatrix},$$

where the first matrix on the left has r rows and columns, with zeros to the right of the diagonal, the second matrix on the left consists of the first r rows of the matrix G, having r rows and n columns, as has also the matrix on the right. This latter matrix can be written as the product of the diagonal matrix of r rows and columns whose diagonal elements are $1, t, t^2, \ldots, t^{r-1}$, multiplied into a matrix of r rows and n columns, which does not vanish when $t=0$, but reduces to the first r rows occurring in the determinant A_0 above, namely as

$$\begin{pmatrix} 1 & & & & \\ & t & & & \\ & & t^2 & & \\ & & & \ddots & \\ & & & & t^{r-1} \end{pmatrix} \begin{pmatrix} 1, & a_1+(t) & , & \ldots, & a_{n-1}+(t) \\ 0, & a_1'+(t) & , & \ldots, & a'_{n-1}+(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \dfrac{a_1^{(r-1)}}{(r-1)!}+(t), & & \ldots, & \dfrac{a_{n-1}^{(r-1)}}{(r-1)!}+(t) \end{pmatrix},$$

where the unmarked elements of the multiplying matrix are zeros. If there be, for the same value of x, another branch place, with a cycle of r_1 values of y, an exactly similar reduction can be carried out for the r_1 rows of G which follow the first r rows; therein, the appropriate local parameter being t_1, we should use $\epsilon_1=\exp(2\pi i/r_1)$ instead of ϵ, and in place of coefficients $a_k, a_k', a_k'', \ldots$, we should have coefficients $b_k, b_k', b_k'', \ldots$. And so for all other branch places which occur for the same value of x. The final result will be an equation in matrices, which we write $KG=\delta A$, and now explain: Every one of K, δ, A is a matrix of n rows and columns, as is G. The matrix K has, in the first r rows, non-vanishing elements only in the first r columns, these forming the matrix of r rows and columns, with numerical elements $K_{s,\rho}$, which has been made explicit above; in the r_1 rows of K which follow the first r rows, K has elements only in the r_1 columns which follow the r-th column, these being formed from ϵ_1 just as the elements in the first r rows and columns were formed from ϵ; succeeding rows of K similarly break up into batches, corresponding to the other branch places which occur for the value of x considered, there being zero elements in the first $r+r_1$ columns, in the first $r+r_1+r_2$ columns, and so on. The matrix δ is the simple diagonal matrix, whose non-vanishing elements, all in the diagonal, are $1, t, \ldots, t^{r-1}, 1, t_1, \ldots, t_1^{r_1-1}$, etc., in order. Last, the matrix A is one of which every element is a power series, these being series in t in the first r rows, series in t_1 in the succeeding r_1 rows, and so on; and, what is important, these series are such that when all of $t, t_1, \ldots$ are made to vanish, the elements of A reduce to those of the determinant A_0. It was proved that the determinant A_0 is not zero. Further, in discussing the first r rows of the matrix K, we remarked an equation which was written $K_{s,s}=M_{s-1}/M_s$; it follows that the determinant of the matrix K is the product

$$M_1^{-1}.(M_1/M_2)\,.\,\ldots\,.\,(M_{r-1}/M_r).(M_1^{(1)})^{-1}.(M_1^{(1)}/M_2^{(1)})\ldots$$
$$.(M^{(1)}_{r_1-1}/M^{(1)}_{r_1})\,.\,\ldots,$$

where $M_1^{(1)}, M_2^{(1)}, \ldots$ are formed from ϵ_1 just as were $M_1, M_2, \ldots$, from ϵ, and so for succeeding factors. Whence the determinant of the matrix K is $[M_r M_{r_1}^{(1)}\ldots]^{-1}$.

Many consequences follow from the equation $KG=\delta A$, as will be seen. One consequence, already remarked, relates to the determinant of the matrix G. When x is near to the value $x=c$ for which the branch places are those discussed, it follows from $KG=\delta A$ that the determinant of G is the product of a factor which does not vanish when $x=c$ by the factor $t^{1+2+\cdots+(r-1)}.t_1^{1+2+\cdots+(r_1-1)}\ldots$, which is $t^{\frac{1}{2}r(r-1)}t_1^{\frac{1}{2}r_1(r_1-1)}\,.\,\ldots$. We have spoken of the determinant of the matrix $\bar{G}G$; this is the square of the determinant of G. It is a polynomial in x, having, we see, the factor $(x-c)^{(r-1)+(r_1-1)+\cdots}$. We denote the polynomial in x, which is the determinant of $\bar{G}G$, by Δ; for every finite value of x, we have separated Δ into a vanishing factor, and a non-vanishing factor; we thus see that, save for a constant, Δ is the product of all factors $(x-a)^{k-1}$, arising for values $x=a$ at which there is a branch place of index k, there being repetitions of the factor $x-a$, with appropriate exponents, when several branch places occur for the same value $x=a$. Now put $w=\Sigma(r-1)$, the summation extending to all existing branch places (since we have arranged that there shall be no branch places for $x=\infty$); then Δ is a polynomial in x of order w. We have seen, however, (p. 150 above), that we may write G in the form $x^{(\sigma_1+1)+(\sigma_2+1)+\cdots}H$, where H is a matrix, of n rows and columns, with i-th row consisting of $1, h_1^{(i)}, \ldots, h^{(i)}_{n-1}$, in which $h_s=x^{-(\sigma_s+1)}g_s$, and h_s is a function becoming infinite only for $x=0$ (this being the definition of h_s

previously employed), and we can suppose $x=0$ chosen so that no branch place arises for $x=0$. Then, by what has been proved here for G, it follows that the determinant of the matrix H does not vanish for $x=\infty$. Wherefore, Δ is a polynomial in x which becomes infinite for $x=\infty$ to order $2(\sigma_1+1+\sigma_2+1+\ldots)$, that is, to order $2p+2n-2$, if p, as above, be defined as $\sigma_1+\ldots+\sigma_{n-1}$. In other words, the polynomial Δ is of this order. Thus we have the equation $w=2n+2p-2$. It is not, in fact, necessary to choose $x=\infty$ so that no branch places occur for this value. By a similar argument it may be shewn that, if W denote the sum $(r-1)+(r_1-1)+\ldots$, extended to all the branch places at $x=\infty$, of respective indices $r, r_1, \ldots$, then the order of the polynomial Δ is $2n+2p-2-W$. Thus already it is shewn, if we recur to a result previously found (Chap. IV, p. 83, cf. Chap. V, p. 131), that p as defined here agrees with p as used before; but we shall continue to suppose p defined as here.

The importance of the matrix G, or of the polynomial Δ, may be brought out by brief reference to some of its properties. Let $\eta_1, \ldots, \eta_n$ be any n integral functions whatever, subject to the condition that there exists (in virtue of the fundamental equation $f(x, y)=0$) no identity $U_1\eta_1+\ldots+U_n\eta_n=0$, in which $U_1, \ldots, U_n$ are polynomials in x. Denoting by $\eta_k^{(1)}, \ldots, \eta_k^{(n)}$ the n conjugate values of η_k, for a general finite value of x, we may consider the square of the determinant, of n rows and columns, whose general row is $\eta_1^{(i)}, \eta_2^{(i)}, \ldots, \eta_n^{(i)}$. Each of the functions η_k, by the properties of the fundamental integral functions $g_1, \ldots, g_{n-1}$, is expressible linearly in terms of these latter, with coefficients which are polynomials in x (that is homogeneously in terms of $1, g_1, \ldots, g_{n-1}$). Hence it follows that the square of the determinant $|\eta_k^{(i)}|$ is of the form $\nabla^2\Delta$, where Δ is as before, and ∇ is a polynomial in x. It is usual to call the square of the determinant $|\eta_k^{(i)}|$ the discriminant of the system $(\eta_1, \ldots, \eta_n)$; it appears then that Δ, which is a polynomial in x, is the highest common factor of the discriminants of all possible systems of n linearly independent integral functions. In general, though, when $u_1, \ldots, u_n$ are polynomials in x, any function $u_1\eta_1+\ldots+u_n\eta_n$ is an integral function, it is not conversely the case that every integral function can be so expressed; there may exist integral functions whose expression in terms of $\eta_1, \ldots, \eta_n$ is of the form $(u_1\eta_1+\ldots+u_n\eta_n)/D$, where D, like $u_1, \ldots, u_n$, is a polynomial in x. It can be shewn that the only polynomials D that can arise in this way, as denominators in the expression of integral functions by means of $\eta_1, \ldots, \eta_n$, are factors of the particular polynomial ∇, whose square arises in the equation just given for the discriminant of $\eta_1, \ldots, \eta_n$.

The functions reciprocal to the fundamental integral functions. We proceed now to build up, from the fundamental integral functions $g_1, \dots, g_{n-1}$, a set of n important rational functions, which will be denoted by $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$. We recall the expression of the polynomial Δ as a determinant whose elements are the polynomials in x which we denoted by s_k, s_{ij} $(s_0 = n)$; consider the minors of this determinant, which may be briefly represented by such notations as $\partial\Delta/\partial s_k$ and $\partial\Delta/\partial s_{ij}$. These are polynomials in x; the function $\partial\Delta/\partial s_k + g_1 \partial\Delta/\partial s_{k,1} + \dots + g_{n-1}\partial\Delta/\partial s_{k,n-1}$ is thus an integral function; it is this function, divided by Δ, which is the function γ_k. By what has been proved in regard to Δ, this function, for finite values of x, is only infinite at branch places. To find the character of this infinity, and to make clear the simplicity of the matter, it will be best to approach the definition of γ_k in another way, then shewn to be equivalent to that just given. For this we define a matrix Γ, of n rows and columns, of which the general row is represented by $\gamma_0^{(i)}, \gamma_1^{(i)}, \dots, \gamma_{n-1}^{(i)}$, and prove that the elements of a column of this, $\gamma_k^{(1)}, \gamma_k^{(2)}, \dots, \gamma_k^{(n)}$, are the n conjugate values of a single function γ_k, for all the places corresponding to a general finite value of x. The definition in question is that Γ is the inverse of the matrix obtained from G by transposition of its rows and columns, or $\Gamma = (\bar{G})^{-1}$. This definition is equivalent to $\Gamma^{-1} = \bar{G}$, and, therefore, equivalent to either of the two equations $\Gamma\bar{G} = 1$, $\bar{G}\Gamma = 1$. From the former of these we have $\Gamma\bar{G}G = G$, so that $\Gamma = G(\bar{G}G)^{-1}$, and $\bar{G}G$ is the matrix whose general element is $s_{i,j}$, with $s_{0,0} = n$, $s_{0,j} = s_j$, $s_{i,j} = s_{j,i}$, $(i, j = 0, 1, \dots, n-1)$, which we met with earlier (p. 153); from this, combining the i-th row of G with the $(k+1)$-th column of $(\bar{G}G)^{-1}$, we obtain the element $\gamma_k^{(i)}$ in the form

$$M_k + g_1^{(i)} M_{k,1} + \dots + g_{n-1}^{(i)} M_{k,n-1},$$

where $M_{k,h}$ is $\Delta^{-1}\partial\Delta/\partial s_{k,h}$, $M_k = \Delta^{-1}\partial\Delta/\partial s_k$; these are rational in x only; for the n values of i, therefore, the elements $\gamma_k^{(i)}$ are the conjugate values of the function γ_k defined as at first. We have assumed the well-known result that the (i, j) element of the inverse of a matrix $(a_{i,j})$ is obtained by dividing the cofactor of $a_{j,i}$, in the determinant of $(a_{i,j})$, by this determinant itself.

Remark. The definition $\Gamma\bar{G} = 1$ is equivalent with the n^2 equations such as

$$\gamma_0^{(i)} + \gamma_1^{(i)} g_1^{(i)} + \dots + \gamma_{n-1}^{(i)} g_{n-1}^{(i)} = 1,$$

$$\gamma_0^{(i)} + \gamma_1^{(i)} g_1^{(j)} + \dots + \gamma_{n-1}^{(i)} g_{n-1}^{(j)} = 0,$$

for $i, j = 1, \dots, n$, $i \neq j$. These we may abbreviate by

$$\gamma_0 + \gamma_1 g_1 + \dots + \gamma_{n-1} g_{n-1} = 1, \quad \gamma_0 + \gamma_1 g_1' + \dots + \gamma_{n-1} g'_{n-1} = 0,$$

for a value of x for which there is no branch place. And if y, y' be such that $f(x, y) = 0$, $f(x, y') = 0$, the same or different, and $f'(y)$

denote $\partial f/\partial y$, these abbreviated equations are included in the formal equation

$$\gamma_0+\gamma_1 g_1'+\dots+\gamma_{n-1}g'_{n-1}=lim_{y'=y}\,[f(x, y')-f(x, y)]/(y'-y)f'(y).$$

Hence it may be seen that the function γ_k is of the form $\psi_k/f'(y)$, where ψ_k, or $\psi_k(y, x)$, is a polynomial in x and y. And, when the integral functions $g_1, \dots, g_{n-1}$, say $g_1(y, x), \dots, g_{n-1}(y, x)$, are given in explicit form, each as a quotient of a polynomial in x and y divided by a polynomial in x only, the polynomial $\psi_k(y, x)$, and hence γ_k, is determined by the equation

$$\begin{aligned}\psi_0(y, x)+\psi_1(y, x)\,g_1(y', x)+\dots+\psi_{n-1}(y, x)\,g_{n-1}(y', x)&\\ =[f(x, y')-f(x, y)]/(y'-y),&\end{aligned}$$

regarded as an identity in y and y'. All the polynomials $\psi_k(y, x)$, and the numerator polynomials in $g_k(y, x)$, are understood to be reduced by $f(x, y)=0$, so as not to involve any power of y higher than the $(n-1)$-th. For a particular instance, if $g_1, \dots, g_{n-1}$ were $y, y^2, \dots, y^{n-1}$, the polynomials $\psi_0, \dots, \psi_{n-1}$ would be the coefficients of the powers of y' in $[f(x, y')-f(x, y)]/(y'-y)$, that is, ψ_k would be the coefficient of $(y')^k$, and of order $n-1-k$ in y.

The preceding definition of the matrix Γ, if we use the equation, proved above, $KG=\delta A$, enables us to determine the maximum possible orders of infinity of the rational functions γ_k at a branch place; we have seen that (for finite values of x) these functions are only infinite at a branch place. For this equation gives $\bar{G}\bar{K}=\bar{A}\bar{\delta}$, while $\bar{\delta}=\delta$, and hence $(\bar{K})^{-1}(\bar{G})^{-1}=\delta^{-1}(\bar{A})^{-1}$, so that Γ, equal to $(\bar{G})^{-1}$, is equal to $\bar{K}\delta^{-1}(\bar{A})^{-1}$. Herein $\bar{K}$ is a matrix of constant numerical quantities, and $(\bar{A})^{-1}$, the inverse of a matrix, $\bar{A}$, of which the determinant does not vanish at a branch place, as we have seen, is likewise of non-vanishing determinant at the branch place. But δ^{-1} is the diagonal matrix whose elements are $1, t^{-1}, \dots, t^{-(r-1)}, 1, t_1^{-1}, \dots, t_1^{-(r_1-1)}$, etc., wherein $t, t_1, \dots$ are the local parameters for the places arising for the value of x under consideration, these being branch places of respective indices $r, r_1, \dots$. Thus it is clear that, at the first of these branch places, no one of the functions $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ becomes infinite to order higher than the $(r-1)$-th; and so, then, for any other branch place.

This statement is for a finite value of x; we can easily deduce the corresponding fact for $x=\infty$. For this, we consider the rational functions, denoted as before by $h_1, \dots, h_{n-1}$, in terms of which all rational functions, which are infinite only for a finite value of x, say $x=c$, can be linearly expressed. We can suppose c taken so that there are no branch places for $x=c$. From these functions $h_1, \dots, h_{n-1}$ we can deduce functions $\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}$, just as $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ were deduced from $g_1, \dots, g_{n-1}$; as will appear, these functions are very simply connected with $\gamma_0, \dots, \gamma_{n-1}$. Putting z for $(x-c)^{-1}$, the behaviour of $\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}$ for $z=0$ follows from what has been found for $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ for any finite value of x; but $z=0$ when $x=\infty$; thus the behaviour of $\gamma_0, \dots, \gamma_{n-1}$ for $x=\infty$ is easily found. In fact, we have seen above that $h_k=g_k(x-c)^{-(\sigma_k+1)}$; hence, if the matrix H be that defined from $h_1, \dots, h_{n-1}$ just as was G from $g_1, \dots, g_{n-1}$, namely to have for its i-th row the elements $1, h_1^{(i)}, h_2^{(i)}, \dots, h_{n-1}^{(i)}$, we have $H=GC$, where C is the diagonal matrix whose elements in turn are $1, (x-c)^{-(\sigma_1+1)}, (x-c)^{-(\sigma_2+1)}, \dots, (x-c)^{-(\sigma_{n-1}+1)}$. If then the matrix Θ be expressed in terms of the functions $\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}$, to be defined, just as Γ was expressed in terms of the functions $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$, so that the i-th row of Θ is $\vartheta_0^{(i)}, \vartheta_1^{(i)}, \dots, \vartheta_{n-1}^{(i)}$, and $\Theta=(\overline{H})^{-1}$, we have $\Theta=(\overline{G})^{-1}C^{-1}=\Gamma C^{-1}$, and C^{-1} is the diagonal matrix whose elements are $1, (x-c)^{\sigma_1+1}, \dots, (x-c)^{\sigma_{n-1}+1}$. Hence we infer that $\vartheta_0=\gamma_0$ and $\vartheta_k=(x-c)^{\sigma_k+1}\gamma_k$, for $k=1, \dots, n-1$. The equation $\gamma_k=z^{\sigma_k+1}\vartheta_k$ shews that if, near a place for which $z=0$, and $z=t^r$, the function ϑ_k be infinite like $t^{-(r-1)}$, then γ_k vanishes to order $r\sigma_k+1$, which is >0 even if $\sigma_k=0$; for greater ease of statement we have supposed that for $x=\infty$ (or $z=0$), there are no branch places, and then the conclusion is that γ_k vanishes, for $x=\infty$, to order σ_k+1 at least. For γ_0 (which is ϑ_0), the statement is that, at a branch place at $x=\infty$, of index r, the function γ_0 is not infinite to order greater than $r-1$; and is finite if there be no such branch place.

General modular form of the everywhere finite integrals. The reason for introducing the functions γ_k is that they enable us to express the everywhere finite integrals in a simple explicit form. Suppose that $R(x, y)$, or R, is such a rational function of x and y, subject to the fundamental equation $f(x, y)=0$, that the product $R\,dx/dt$, for the neighbourhood of every place (including places at $x=\infty$), when expressed by powers of the parameter t appropriate to that place, involves no negative powers of t. If R be infinite at a place, for which x is finite, which is a branch place of index r, this involves that R cannot be there infinite to order greater than $r-1$. Let g_k be any one of the integral functions $g_1, \dots, g_{n-1}$, and, as before, let $g_k^{(1)}, \dots, g_k^{(n)}$ be the conjugate values of g_k for any general finite value of x; and similarly $R^{(1)}, \dots, R^{(n)}$ be the conjugate values, thereat, of the rational function $R(x, y)$. Then the sum $R^{(1)}g_k^{(1)}+\dots+R^{(n)}g_k^{(n)}$ is expressible as a rational function of x only; and this sum can only be infinite for a finite value of x if $R(x, y)$ be infinite for this value; and, moreover, by the property of R, if such infinity arise for a branch place of index r, the sum can be infinite there at most to order $r-1$. Thus, if Π denote the product of the factors $x-c$, extended to all different finite values of c for which branch places exist, the product $[R^{(1)}g_k^{(1)}+\dots+R^{(n)}g_k^{(n)}]\,\Pi$

is a rational function of x only, not becoming infinite for any finite value of x, and indeed vanishing for every value c of x for which branch places exist. Thus it is a polynomial in x which divides by Π; we denote it by $U_k \Pi$, so that U_k is a polynomial in x. Hence, with the usual notation, by which the n linear functions

$$a_{i1}x_1 + \dots + a_{in}x_n$$

(for $i=1, \dots, n$) are represented by $a\,(x_1, \dots, x_n)$, where a is the matrix (a_{ij}), we have, if we recall the definition of the matrix G, the equation $\bar{G}\,(R^{(1)}, \dots, R^{(n)}) = (U_0, \dots, U_{n-1})$; and hence, from $\Gamma\bar{G}=1$, we have $(R^{(1)}, \dots, R^{(n)}) = \Gamma(U_0, \dots, U_{n-1})$. This is the same as $R\,(x, y) = U_0\gamma_0 + U_1\gamma_1 + \dots + U_{n-1}\gamma_{n-1}$, where $U_0, \dots, U_{n-1}$ are polynomials in x only. Conversely, if we take any rational function of this form, since no one of $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ is infinite at a branch place of index r, occurring for a finite value of x, to higher than the $(r-1)$-th order, this form ensures that $R\,dx/dt$ is finite for the neighbourhood of every finite value of x. To examine, however, the implication of the condition that $R\,dx/dt$ is finite also for $x=\infty$, we proceed as before. We put $(x-c)^{-1}=z$, where c may be chosen so that no branch places occur for $x=c$, and discuss $R\,dx/dt$, as a function of z, for $z=0$. Let $m_0, m_1, \dots, m_{n-1}$ be the respective orders of the polynomials in x, $U_0, \dots, U_{n-1}$; put $V_i = z^{m_i}U_i$, or $(x-c)^{-m_i}U_i$, so that V_i is a polynomial in z of order m_i. Also, as above, let $\gamma_k = z^{\sigma_k+1}\vartheta_k$, $\gamma_0 = \vartheta_0$. Then the preceding form for R, obtained by considering finite values of x, leads, if we introduce σ_0, $=-1$, to

$$R\frac{dx}{dt} = -\left(\sum_{i=0}^{n-1} z^{\sigma_i-1-m_i} V_i \vartheta_i\right)\frac{dz}{dt};$$

the condition for $R\,dx/dt$, at $x=\infty$, requires then, as a necessary and sufficient condition, in order that the right side should, for $z=0$, have the form, as a function of z, above found for $R\,dx/dt$ for finite values of x, that $z^{\sigma_i-1-m_i}V_i$ should be a polynomial in z. This is satisfied if, and only if, V_0 be identically zero, and $m_i \leqslant \sigma_i - 1$. Thus we have proved that there exist everywhere finite integrals, namely $\int[(x, 1)^{\tau_1-1}\gamma_1 + \dots + (x, 1)^{\tau_{n-1}-1}\gamma_{n-1}]\,dx$, where $(x, 1)^k$ is a polynomial in x of order k, and $\tau_i \leqslant \sigma_i$; and no others. When $\sigma_i = 0$, the corresponding term is absent. The number of homogeneously entering arbitrary coefficients in the most general everywhere finite integral is therefore $\sigma_1 + \dots + \sigma_{n-1}$, the number we denoted by p. Such integrals do not exist for $p=0$. This is a second verification that p, as defined here, is the same as p of the earlier chapters. The intimate connexion thus established between the everywhere finite integrals, and the fundamental integral functions $g_1, \dots, g_{n-1}$, is most striking.

Use of the reciprocal functions for algebraic integrals in general. A particular combination. The functions $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, with the integral functions $g_1, \ldots, g_{n-1}$, can also be used to give detailed expressions for rational functions of assigned poles, and for the algebraic integrals which have infinities. To this end, denoting by (x) and (ξ) any two places of $f(x, y)=0$, we study the character of the expression following, which we constantly denote by (x, ξ), namely

$$(x, \xi)=[\gamma_0(x)+\gamma_1(x)g_1(\xi)+\ldots+\gamma_{n-1}(x)g_{n-1}(\xi)](x-\xi)^{-1}dx/dt,$$

where $\gamma_k(x)$ denotes the value of the function γ_k at the place (x), and $g_k(\xi)$ the value of the integral function g_k at the place (ξ). This expression is explicitly a rational function of (ξ). Its value at a place (x) is contemplated as derived by a limiting process; if (x_0) be this place, regarded as the origin of a particular branch, or element, which is given in terms of the parameter t for this branch by forms $x-x_0=t^r$, $y-y_0=$ power series in t, or forms derived from these by putting x^{-1} in place of $x-x_0$, or y^{-1} in place of $y-y_0$, or both, then these forms are to be substituted for x and y in (x, ξ); and the limit found as t vanishes. The notation has already been used. In the first place, we study (x, ξ) regarded as depending on a variable position of (x), for a definite position of (ξ); afterwards we regard (x) as definite, and study (x, ξ) as a rational function of (ξ). We recall that, if (x) be one of the places corresponding to a finite value of x, this being such that the n corresponding places are distinct, there being no branch place, and, if $(x^{(1)})$ be another of these n places, then (p. 158 above)

$$\gamma_0(x)+\gamma_1(x)g_1(x)+\ldots+\gamma_{n-1}(x)g_{n-1}(x)=1,$$
$$\gamma_0(x)+\gamma_1(x)g_1(x^{(1)})+\ldots+\gamma_{n-1}(x)g_{n-1}(x^{(1)})=0.$$

Also, if (x) be in the neighbourhood of a *finite* place (ξ), so that none of $g_1(\xi), \ldots, g_{n-1}(\xi)$ is infinite, and this be a branch place of index r, there being expressions $x-\xi=t^r$, $y-\eta=$ power series in t, then, as before, we write $g_k(x)=a_k+a_k't+\frac{1}{2}a_k''t^2+\ldots$. We can associate with (x) the $(r-1)$ other places, $(x^{(2)}), \ldots, (x^{(r)})$, which, when $t=0$, all coincide with (x) at (ξ), and correspondingly denote (x) by $(x^{(1)})$, so that we have $g_k(x^{(i)})=a_k+a_k't_i+\frac{1}{2}a_k''t_i^2+\ldots$, in which $t_i=\epsilon^{i-1}t$, with $\epsilon=\exp(2\pi i/r)$; it will be sufficient also to use $\gamma_k(x)g_k(x')$ as an abbreviation for $\gamma_0(x)+\gamma_1(x)g_1(x')+\ldots+\gamma_{n-1}(x)g_{n-1}(x')$. Then, by what is recalled above, we have, assuming for simplicity that the other $n-r$ places conjugate to (ξ) are simple, the r equations

$$\gamma_k(x)g_k(x^{(1)})=1, \quad \gamma_k(x)g_k(x^{(2)})=0, \ \ldots, \quad \gamma_k(x)g_k(x^{(r)})=0;$$

hence, if λ be any positive integer,

$$\frac{1}{r}\gamma_k(x)[g_k(x^{(1)})+\epsilon^{-\lambda}g_k(x^{(2)})+\epsilon^{-2\lambda}g_k(x^{(3)})+\ldots+\epsilon^{-(r-1)\lambda}g_k(x^{(r)})]=\frac{1}{r};$$

substituting the expansions in powers of t, this is the same as

$$\gamma_k(x)\left[a_k p_0 + a_k' p_1 t + \frac{a_k''}{2} p_2 t^2 + \dots\right] = \frac{1}{r},$$

where, in virtue of $\sum\limits_{i=0}^{r-1} \epsilon^{im} = r$ or 0, according as the integer m divides by r, or does not, we easily see that $p_\lambda = 1$, $p_{\lambda+r} = 1$, ..., the other p's being zero. Hence we have

$$\gamma_k(x)\left[a_k + \frac{a_k^{(r)}}{r!} t^r + \frac{a_k^{(2r)}}{(2r)!} t^{2r} + \dots\right] = \frac{1}{r},$$

$$\gamma_k(x)\left[a_k' t + \frac{a_k^{(r+1)}}{(r+1)!} t^{r+1} + \dots\right] = \frac{1}{r},$$

and so on, the general equation being

$$\gamma_k(x)\left[\frac{a_k^{(i)}}{i!} t^i + \frac{a_k^{(r+i)}}{(r+i)!} t^{r+i} + \dots\right] = \frac{1}{r}.$$

Now let (x) approach to (ξ). We have shewn that none of the functions $\gamma_k(x)$ contains any higher negative power of t than $t^{-(r-1)}$. Hence we infer that the limits respectively of $\gamma_k(x) a_k$, $\gamma_k(x) a_k'$, $\gamma_k(x) a_k''$, ..., are $\frac{1}{r}$, $\frac{1}{r} t^{-1}$, $\frac{1}{r} 2t^{-2}$, ... With a notation previously employed (p. 138), this is the same as saying that, if (ξ) be a branch place of index r, the limits, respectively, as (x) approaches to (ξ), of the functions

$$\begin{aligned}
&\gamma_0(x) + \gamma_1(x) g_1(\xi) + \dots + \gamma_{n-1}(x) g_{n-1}(\xi),\\
&\gamma_1(x) Dg_1(\xi) + \dots + \gamma_{n-1}(x) Dg_{n-1}(\xi),\\
&\gamma_1(x) D^2 g_1(\xi) + \dots + \gamma_{n-1}(x) D^2 g_{n-1}(\xi), \text{ etc.,}
\end{aligned}$$

are $\frac{1}{r}$, $\frac{1}{r} t^{-1}$, $\frac{2}{r} t^{-2}$, and so on, the second, third, ... of these becoming infinite; here $Dg_k(\xi)$, $D^2 g_k(\xi)$, ... are the values of the differential coefficients, in regard to t, for $t = 0$. The second, third, ... results will be useful below; the first shews that, as (x) approaches to (ξ), the limit of (x, ξ), wherein we put $x - \xi = t^r$, $dx/dt = rt^{r-1}$, is t^{-1}. Namely (x, ξ) has a pole of the first order at (ξ), with multiplier (residue) equal to 1.

Next suppose that beside the branch place at (ξ), of index r, there is, among the $n - r$ conjugate places (that is, for the same value of ξ, if the place (ξ) have $x = \xi$, $y = \eta$), another branch place (ξ_1) of index r_1. We consider the limit of (x, ξ_1) as (x) approaches, not to (ξ_1), but to (ξ). Corresponding to the place (x), in the neighbourhood of (ξ), there will be, near the place (ξ_1), arising for the same value of x, places for which the values of x may be denoted by $x^{(1)}$, $x^{(2)}$, ..., $x^{(r_1)}$, and the value of ξ by ξ_1, so that numerically $x^{(1)} = x$, $\xi_1 = \xi$, and, if we put $x^{(1)} - \xi_1 = t_1^{r_1}$, we have $t_1^{r_1} = t^r$. Using $(x^{(1)})$, $(x^{(2)})$, ... for places near to (ξ_1), we can put $g_k(x^{(1)}) = b_k + b_k' t_1 + \frac{1}{2} b_k'' t_1^2 + \dots$, $g_k(x^{(2)}) = b_k + b_k' \epsilon_1 t_1 + \frac{1}{2} b_k'' \epsilon_1^2 t_1^2 + \dots$, where $\epsilon_1 = \exp(2\pi i/r_1)$. As the place (x) does not coincide with any of the places in the neighbourhood of (ξ_1), here denoted by $(x^{(1)}), \dots, (x^{(r_1)})$, we may suppose the equations denoted by $\gamma_k(x) g_k(x^{(s)}) = 0$, to hold, namely

$$\gamma_k(x)[b_k + b_k' \epsilon_1^{s-1} t_1 + \tfrac{1}{2} b_k'' \epsilon_1^{2(s-1)} t_1^2 + \dots] = 0;$$

adding these r_1 equations we obtain

$$\gamma_k(x)\left[b_k + \frac{1}{(r_1)!} b_k^{(r_1)} t_1^{r_1} + \frac{1}{(2r_1)!} b_k^{(2r_1)} t_1^{2r_1} + \dots\right] = 0,$$

and therein we can replace $t_1^{r_1}$ by t^r. Also, when (x) approaches to (ξ), the function $\gamma_k(x)$ can have no higher negative power of t than $t^{-(r-1)}$. Thus we infer that $\gamma_k(x) b_k = 0$ in the limit; and this is what is expressed by saying that $\gamma_k(x) g_k(\xi_1)$ vanishes as (x) approaches to (ξ). Hence also, $\gamma_k(x) g_k(\xi)$ vanishes as (x) approaches to (ξ_1); namely, expressed in powers of t_1, it must contain at least the first power of t_1 as factor. When this is so, however, $(x-\xi)^{-1} dx/dt$ is infinite only as $r_1 t_1^{-1}$. We conclude, therefore, that (x, ξ) remains finite as (x) approaches to (ξ_1), notwithstanding the vanishing denominator $x - \xi$.

We can now specify the character of the expression (x, ξ), given by

$$(x, \xi) = [\gamma_0(x) + \gamma_1(x) g_1(\xi) + \dots + \gamma_{n-1}(x) g_{n-1}(\xi)] (x-\xi)^{-1} dx/dt,$$

regarded as depending on (x), for all finite positions of (x), the place (ξ) being a given finite place. The expression can become infinite only on account of the vanishing of $x - \xi$, or on account of infinities of the functions $\gamma_0(x), \gamma_1(x), \dots, \gamma_{n-1}(x)$. These latter are, at most, to order $r-1$, at a branch place of index r, and are therefore ineffective, on account of the factor dx/dt. The conclusion then is that only at the single place (ξ) can there be infinity, the function (x, ξ) being there infinite like t^{-1}, if t is the parameter for the neighbourhood of (ξ).

When (x) is at infinity, it follows from what we have proved in regard to the everywhere finite integrals, that each of the products $\gamma_1(x) dx/dt, \dots, \gamma_{n-1}(x) dx/dt$ remains finite. It is also the case that, if u_2 denote a quadratic polynomial in x, the function $\gamma_0(x) u_2^{-1} dx/dt$ remains finite; for, by the substitution $x = c + z^{-1}$, previously employed, this takes the form $-z^2 \vartheta_0 v_2^{-1} . z^{-2} dz/dt$, where, if $u_2 = (x, 1)_2$, then $v_2 = (1, z)_2$, ϑ_0 being the function for $x = c$ previously used; and this is finite for $z = 0$, v_2 not vanishing.

We can thus state the important result: If (c) be any finite place, other than the finite place (ξ), the difference $(x, \xi) - (x, c)$ is not infinite for any position of (x), finite or infinite, other than the two places (ξ) and (c); and the integral, in regard to (x),

$$\int [(x, \xi) - (x, c)] \, dt,$$

where t is the parameter for (x), has no infinities except at (ξ) and (c), being infinite at these respectively like $\log t_1$ and $-\log t_0$, if t_1, t_0 be the respective parameters for the neighbourhoods of these places. An integral, in (x), of this behaviour, can be constructed when (ξ), or (c), is at infinity. It will be sufficient, however, to obtain this by first transforming the infinite places in question to finite places, the resulting changes of the functions $g_k(\xi)$ being simple, as has been explained (p. 160). The discussion of (x, ξ) when (ξ) is at infinity arises below.

It may be remarked, as an incidental consequence of the fact that the limit of $\gamma_0(x)+\gamma_1(x)g_1(\xi)+\ldots+\gamma_{n-1}(x)g_{n-1}(\xi)$, when (x) approaches to the branch place (ξ), of index r, is $1/r$, that *not all of* $\gamma_0(x), \gamma_1(x), \ldots, \gamma_{n-1}(x)$ *can vanish at the same place.* (Cf. p. 71.)

It may also be stated to follow, without great difficulty, from the character we have found for the integral

$$P_{\xi,c}^{x,a} \text{ or } \int_{(a)}^{(x)} [(x, \xi)-(x, c)]\,dt,$$

that the integral, in regard to (x),

$$E_{\xi}^{x,a}, \;=\int_{(a)}^{(x)} D_{\xi}(x, \xi).dt$$

is infinite only at the (finite) place (ξ), and then has a pole of the first order, becoming infinite like $-t_{\xi}^{-1}$, where t_{ξ} is the local parameter for the neighbourhood of the place (ξ). In this, the notation D_{ξ} scarcely needs explanation: When (ξ) is not a branch place, and the function is not infinite at (ξ), D_{ξ} means $d/d\xi$; in general, when (ξ) is a branch place, we take (ξ') in its neighbourhood, and express (x, ξ') in powers of the parameter for the neighbourhood of (ξ); then D_{ξ} means that we take the coefficient of the first positive power of this parameter. The notation has already been employed several times. More generally, an algebraic integral, having only an algebraic infinity, at the branch place (ξ), of index r, being there infinite like $-[A_1 t_{\xi}^{-1}+\ldots+A_{r-1}t_{\xi}^{-(r-1)}+A_r t_{\xi}^{-r}]$, is given by

$$rA_r\int_{(a)}^{(x)} \rho(x, \xi)(x-\xi)^{-2}dx+\int_{(a)}^{(x)}(x-\xi)^{-1}\,U\rho\,(x, \xi)dx,$$

where $$U=A_1D_{\xi}+\ldots+\frac{1}{(r-2)!}A_{r-1}D_{\xi}^{r-1}+\frac{1}{(r-1)!}A_rD_{\xi}^{r},$$

and $\rho(x, \xi)=\gamma_0(x)+\ldots+\gamma_{n-1}(x)g_{n-1}(\xi)$.

Another use of this combination to build a fundamental rational function. We now consider the character of the rational function of (ξ) given by (x, ξ), for finite positions of (x). The function being, save for $(x-\xi)^{-1}$, linear in the integral functions of (ξ), $1, g_1(\xi), \ldots, g_{n-1}(\xi)$, can only become infinite, for finite positions of (ξ), when $\xi=x$. It follows from what has preceded that such infinity only arises when the place (ξ) approaches the place (x); and that, if t_x denote the parameter for the neighbourhood of the place (x), equal to $\xi - x$ when (x) is not a branch place, then, when (ξ) is near to (x), this infinity is that of $-t_x^{-1}$. Another proof of this is indicated below. More intricate is the behaviour of (x, ξ) when (ξ) passes to infinity: Introduce the definition

$$u_{i,m}^{(x)} = \int^{(x)} x^{m-1} \gamma_i(x)\, dx;$$

if, as before, $\sigma_i + 1$ is the dimension of the function $g_i(x)$, this is an everywhere finite integral for integer positive values of m such that $m \leqslant \sigma_i$, as was proved; consider then the function $G_k(\xi)$, of (ξ), for a definite (x), given by

$$G_k(\xi) = \xi^{-1} g_k(\xi)\, du_{k,1}/dt + \xi^{-2} g_k(\xi)\, du_{k,2}/dt + \dots \\ + \xi^{-\sigma_k} g_k(\xi)\, du_{k,\sigma}/dt;$$

this is equal to

$$\gamma_k(x) g_k(\xi)(\xi^{-1} + x\xi^{-2} + \dots + x^{\sigma_k - 1}\xi^{-\sigma_k})\, dx/dt,$$

namely

$$-\gamma_k(x) g_k(\xi)[1 - (x/\xi)^{\sigma_k}]\, (x-\xi)^{-1}\, dx/dt.$$

Hence we can write

$$(x, \xi) = [\gamma_0(x) + \sum_k \gamma_k(x) g_k(\xi)(x/\xi)^{\sigma_k}](x-\xi)^{-1}\, dx/dt - \sum_{k=1}^{n-1} G_k(\xi);$$

here, since $\xi^{-(\sigma_k+1)} g_k(\xi)$ is finite when (ξ) is at infinity, the first part on the right is finite when (ξ) is at infinity. Thus the poles of (x, ξ), as a rational function of (ξ), which arise at infinity, are given by the expression $-\sum_{k=1}^{n-1} G_k(\xi)$.

The alternative proof of the nature of the infinity of (x, ξ), as a function of (ξ), when $\xi = x$, which has been referred to, is, in outline, as follows: When (ξ) is approaching to the branch place (x), of index r, for the neighbourhood of which the parameter is denoted by t, the function $\gamma_0(x) + \gamma_1(x) g_1(\xi) + \dots + \gamma_{n-1}(x) g_{n-1}(\xi)$ can, by what has preceded, be expressed as U/D, where U, D are certain determinants; these we put down, supposing (x) to be not at first actually at the branch place, but to be given by a parameter t_0, appropriate to this branch place, using ϵ for $\exp(2\pi i/r)$; namely U is

$$U = \begin{vmatrix} 1, & g(t) \;\;\;, & \dots \\ 1, & g(\epsilon t_0)\;, & \dots \\ \cdots & \cdots\cdots & \cdots \\ 1, & g(\epsilon^{r-1} t_0), & \dots \\ \cdots & \cdots\cdots & \cdots \end{vmatrix},$$

wherein $g(t)$ denotes a power series of the form $a + a't + \frac{1}{2}a''t^2 + \dots$, and $g(\epsilon^i t_0)$ denotes what this becomes when $\epsilon^i t_0$ is substituted for t;

and D denotes what this becomes when t_0 is put for t. As we have previously seen, the determinant D may be reduced by taking the determinant M_s, of s rows and columns, of which the general (λ, μ)-th element is $\epsilon^{(\lambda-1)(\mu-1)}$; denoting the quotient of the cofactor of $\epsilon^{(m-1)(s-1)}$, in this, by M_s itself, by $K_{s,m}$; then forming the array

$$\begin{pmatrix} 1 & & & \\ K_{2,1}, & K_{2,2} & & \\ \multicolumn{4}{c}{\dotfill} \\ K_{r,1}, & K_{r,2}, & \dots & K_{r,r} \end{pmatrix},$$

and operating on the first r rows of D as if we were multiplying D by the matrix indicated by this array. When this is done, a factor $t_0 t_0^2 \dots t_0^{r-1}$ can be removed, and we are able to infer that

$$D = M_r t_0^{\frac{1}{2}r(r-1)} A_1,$$

where A_1 is a determinant which, for $t_0 = 0$, would become the determinant A_0 considered above (p. 152), there proved not to vanish. A similar process is applicable to the 2nd, 3rd, ..., r-th rows of the determinant U. For this, we begin with the determinant of $s-1$ rows and columns which is the minor, save for sign, of the $(1, s)$-th element of M_s; denoting this determinant by N_s, we reach the equation $U = N_r t_0^{\frac{1}{2}(r-1)(r-2)} A_2$, where A_2 is the determinant

$$A_2 = \begin{vmatrix} 1, & g(t) & , & \dots \\ 1, & g(\epsilon t_0) & , & \dots \\ 0, & a' + \dots & , & \dots \\ \multicolumn{4}{c}{\dotfill} \\ 0, & \dfrac{a^{(r-2)}}{(r-2)!} + \dots, & & \dots \end{vmatrix};$$

if then, in A_2, we put $t_0 = 0$, and then subtract the 2nd, 3rd, ..., r-th rows, respectively multiplied by $1, t, \dots, t^{r-2}$, all from the first row, we obtain (after placing the resulting first row into the r-th row), the result $A_2 = (-1)^{r-1} t^{r-1} A_1$.

We can, however, see at once, by adding all the rows of M_r to the first row, and then dividing the rows by powers of ϵ, that $M_r = r N_r^{\frac{1}{2}r(r-1)}$; and this identity is obtainable also by proving that the inverse of the matrix of which M_r is the determinant is r^{-1} times the matrix obtainable from M_r by changing ϵ into ϵ^{-1}, while $(-1)^{r-1} N_r / M_r$ is the $(r, 1)$-th element of this inverse matrix. Thus finally, the value of U/D is to be obtained from

$$U/D = \frac{(-1)^{r-1}}{r} (t^{r-1}/t_0^{r-1})(-1)^{r-1}, \text{ or } \frac{1}{r}(t/t_0)^{r-1};$$

and hence (x, ξ), or $(U/D)(x-\xi)^{-1} dx/dt$, as (ξ) approaches to (x), is obtainable from $\frac{1}{r}(t/t_0)^{r-1} . r t_0^{r-1} . (x-\xi)^{-1}$; or, now supposing (x) to be actually at the branch place, from $t^{r-1}(x-\xi)^{-1}$. Thus, finally, $(x, \xi) = -t^{-1}$, which is the result we obtained otherwise above. A similar investigation may be made for the approach of (ξ) to a place, other than (x), for which x has the same value as at (x).

We pass now to exhibit the use of the rational function of (ξ) denoted by (x, ξ). For this, we write (x) in place of (ξ), and, denoting by (z) an arbitrary finite place, we consider the rational function of (x), (z, x).

The differential coefficient of the general everywhere finite integral, which we have proved to have the form

$$\Sigma\,(x, 1)^{\sigma_k-1} g_k(x)\,dx/dt,$$

contains p homogeneously entering coefficients. It is thus possible to choose p places, say $(c_1), \ldots, (c_p)$, of such generality that this differential coefficient does not vanish in all of these; and to choose one such differential coefficient uniquely by the conditions that it vanishes in all of $(c_1), \ldots, (c_p)$ except (c_i), but reduces to unity at (c_i). So chosen, let it be denoted by $\omega_i(x)$; and let the corresponding integral, $\int_{(a)}^{(x)} \omega_i(x)\,dt$, be denoted by $V_i^{x,a}$. Then consider the rational function of (x) given by $(z, x) - \lambda_1(c_1, x) - \ldots - \lambda_p(c_p, x)$, where $\lambda_1, \ldots, \lambda_p$ are independent of (x). By what has been proved in this chapter, the poles of this function, for finite positions of (x), occur only at $(z), (c_1), \ldots, (c_p)$, the infinite parts of the function at these places being, respectively, $-t_z^{-1}, \lambda_1 t_{c_1}^{-1}, \ldots, \lambda_p t_{c_p}^{-1}$, where $t_z, t_{c_1}, \ldots, t_{c_p}$ are the local parameters for the neighbourhoods of these places respectively. If, for brevity, we write, temporarily, $\mu_{k,m}(x) = x^{m-1}\gamma_k(x)\,dx/dt$, the infinite part of the function

$$(z, x) - \lambda_1(c_1, x) - \ldots - \lambda_p(c_p, x),$$

when (x) is at infinity, is, by what we have proved,

$$-T(x, z) + \sum_{k=1}^{p} T(x, c_k),$$

where $T\,(x, z)$ denotes

$$\sum_{i=1}^{n-1} [x^{-1} g_i(x)\,\mu_{i,1}(z) + \ldots + x^{-\sigma_i} g_i(x)\,\mu_{i,\sigma_i}(z)],$$

and $T(x, c_k)$ is obtained from this by putting (c_k) for (z); namely, this infinite part is

$$-\sum_{i=1}^{n-1} g_i(x) \sum_{m_i=1}^{\sigma_i} x^{-m_i}[\mu_{i,m_i}(z) - \lambda_1 \mu_{i,m_i}(c_1) - \ldots - \lambda_p \mu_{i,m_i}(c_p)].$$

We can choose $\lambda_1, \ldots, \lambda_p$, independent of (x), so that this infinite part vanishes; we recall that no rational functions of x only, $U_0(x), \ldots, U_{n-1}(x)$, exist, other than all zero, for which there is an identity $U_0(x) + U_1(x)g_1(x) + \ldots + U_{n-1}(x)g_{n-1}(x) = 0$; thus, the necessary, and obviously sufficient, conditions for $\lambda_1, \ldots, \lambda_p$ are that every one of the coefficients $\mu_{i,m_i}(z) - \lambda_1 \mu_{i,m_i}(c_1) - \ldots - \lambda_p \mu_{i,m_i}(c_p)$ should vanish, these conditions, since $\sigma_1 + \ldots + \sigma_{n-1} = p$, being independent. Now, in fact, the equations

$$\mu_{i,m_i}(z) - \omega_1(z)\,\mu_{i,m_i}(c_1) - \ldots - \omega_p(z)\,\mu_{i,m_i}(c_p) = 0$$

are true; for the left side, regarded as a function of (z), is the differential coefficient of an everywhere finite integral, but vanishes when (z) is at any one of the p independent places $(c_1), \ldots, (c_p)$; it is thus identically zero. Thus the values of $\lambda_1, \ldots, \lambda_p$ are $\omega_1(z), \ldots, \omega_p(z)$. We thus have the conclusion that the rational function of (x) given by $(z, x) - \omega_1(z)(c_1, x) - \ldots - \omega_p(z)(c_p, x)$, is not infinite when (x) is at infinity, but is infinite to the first order at each of the $p+1$ general places $(z), (c_1), \ldots, (c_p)$, with respective infinite parts, $-t_z^{-1}, \omega_1(z)t_{c_1}^{-1}, \ldots, \omega_p(z)t_{c_p}^{-1}$. This function we shall denote by $\psi(x, z; c_1, \ldots, c_p)$, or, more often simply by $\psi(x, z)$. It is a function of fundamental importance; and, in case the original curve $f(x, y) = 0$ have a simple form, it is easy to form the explicit expression of the function without recurring to the integral functions $g_1(x), \ldots, g_{n-1}(x)$ which we have employed; or conversely, from such an explicit expression, we can deduce the particular differentials $\omega_1(z), \ldots, \omega_p(z)$, by considering the poles $(c_1), \ldots, (c_p)$. In regard to the places $(c_1), \ldots, (c_p)$ we may add the remark: These places were chosen so that there is no everywhere finite integral whose differential coefficient vanishes at all of them; there can, also, be no rational function having poles of the first order at each of these p places, and no other infinity. For if $U(x)$ were such a function, with infinite parts $a_1 t_{c_1}^{-1}, \ldots, a_p t_{c_p}^{-1}$ at these places, and $V_i^{(x)}$, as before, be the integral $\int^{(x)} \omega_i(x)\,dt$, the application of the general theorem proved (Chap. III, p. 47), expressed by $[U(x)\,dV_i^{(x)}/dt]_{t^{-1}} = 0$, would give the equation $a_1\omega_i(c_1) + \ldots + a_p\omega_i(c_p) = 0$, for all the p values of i; and hence $a_k = 0$ $(k = 1, \ldots, p)$. The function $\psi(x, z)$ is infinite at (z), as well as at $(c_1), \ldots, (c_p)$.

Application to the general theory of rational functions. The Riemann-Roch theorem. We proceed now, independently of what has been said previously, to consider the expression of any rational function, of (x), as a linear aggregate of functions $\psi(x, z)$, with different positions of (z), but always the same places $(c_1), \ldots, (c_p)$. In the first instance we suppose that all the poles of the function to be expressed are of the first order. Let this function have for poles the places $(z_1), \ldots, (z_m)$, supposed all different from $(c_1), \ldots, (c_p)$, its infinite parts at these places being respectively $k_1 t_{z_1}^{-1}, \ldots, k_m t_{z_m}^{-1}$. If $R(x)$ denote the function, the aggregate

$$R(x) + k_1\psi(x, z_1) + \ldots + k_m\psi(x, z_m)$$

is a rational function of (x), with no poles at $(z_1), \dots, (z_m)$, and poles only at $(c_1), \dots, (c_p)$. By the remark just made, it is therefore an absolute constant, say k. By employing once more the theorem $[R(x)dV_i^{(x)}/dt]_{t^{-1}}=0$, or by putting down the infinite parts of $\psi(x, z_1), \dots, \psi(x, z_m)$ at the place (c_i), (from the explicit form of $\psi(x, z_r)$ given above), we see, as $R(x)$ is not infinite at (c_i), that the coefficients (residues) $k_1, \dots, k_m$, at the poles of any existing rational function $R(x)$, must satisfy the p equations

$$k_1\omega_i(z_1)+\dots+k_m\omega_i(z_m)=0,$$

for $i=1, \dots, p$; and, conversely that, if $k_1, \dots, k_m$ satisfy these equations, then there exists a rational function with $(z_1), \dots, (z_m)$ as its poles, of the first order, with the respective infinite parts $k_1t_{z_1}^{-1}, \dots, k_mt_{z_m}^{-1}$. Hence we can deduce a theorem considered before (p. 78) under the name of the Riemann-Roch theorem for functions. For the p linear conditions for $k_1, \dots, k_m$ may not be linearly independent, there being equations of the form

$$c_1\omega_1(z)+\dots+c_p\omega_p(z)=0$$

satisfied for every one of the positions $(z_1), \dots, (z_m)$ of (z); suppose that this is so for $r'+1$ sets of values of $c_1, \dots, c_p$, where $r'+1 \geqslant 0$. Then the p linear equations for $k_1, \dots, k_m$ are equivalent to $p-(r'+1)$, and can be satisfied with $m-p+r'+1$ of $k_1, \dots, k_m$ taken arbitrarily, the other $p-(r'+1)$ of $k_1, \dots, k_m$ being determinate in terms of these. In other words, as the rational function contains a single additive arbitrary constant, k, we can say: *If places* $(z_1), \dots, (z_m)$ *be such that there are* $(r'+1)$ *differential coefficients of everywhere finite integrals which vanish in all these places, then a rational function can be constructed having these places as poles of the first order, and no other poles, and this function contains a number*, $r+1$, *of homogeneously entering arbitrary constants, given by* $r=m-p+r'+1$ (*provided this* $r \geqslant 1$). In particular, when $r'+1=0$, and $m=p+1$, we have $r=1$. The separate functions which thus arise (r in number, beside a constant), each multiplied by an arbitrary constant, are not necessarily, or generally, every one of them infinite in all the m places $(z_1), \dots, (z_m)$; it is possible, however, to choose from $(z_1), \dots, (z_m)$, a set of $m-r$ places, say $(z_1), \dots, (z_{m-r})$, such that, for every one of the remaining r places, say (z'), there exists a set of $(m-r)$ coefficients, $A_1, \dots, A_{m-r}$, some of which may be zero, such that the p equations

$$A_1\omega_i(z_1)+\dots+A_{m-r}\omega_i(z_{m-r})-\omega_i(z')$$

are satisfied $(i=1, \dots, p)$; there exists then a definite rational function, actually infinite at (z'), and infinite at such of $(z_1), \dots, (z_{m-r})$ as really enter in these p equations. The general function obtained is the sum of the r definite functions so existing, each multiplied by an arbitrary constant, with the addition of a further arbitrary constant; the general function is actually infinite at every one of the r places (z'); but there may be places among $(z_1), \ldots, (z_{m-r})$ at which it is not infinite.

As a simple example of this condition, suppose that $m = p+1$, and that $(z_1), \ldots, (z_m)$ consist of a place (z_{p+1}) of general position, together with p places $(z_1), \ldots, (z_p)$ connected by the p equations

$$A_1\omega_i(z_1) + \ldots + A_{p-1}\omega_i(z_{p-1}) - \omega_i(z_p) = 0, \qquad (i = 1, \ldots, p),$$

in which no one of $A_1, \ldots, A_{p-1}$ is zero; there is then a function $\lambda_1\omega_1(z) + \ldots + \lambda_p\omega_p(z)$ which vanishes in all the p places $(z_1), \ldots, (z_p)$, but not in (z_{p+1}). Then $r'+1=0$, and the formula $r = m-p$ gives $r=1$. The set of independent places $(z_1), \ldots, (z_{m-r})$, spoken of above, must then consist of $(z_1), \ldots, (z_{p-1})$ and (z_{p+1}), but the A_p of the general formulation is zero. There is then a rational function with $(z_1), \ldots, (z_{p-1}), (z_p)$ as actual poles, but no function with all the given $(p+1)$ places as poles (cf. Chap. IV, p. 86).

Ex. 1. By considering the rational function expressible by the fraction whose numerator is the differential coefficient of the general everywhere finite integral, and denominator is the differential coefficient of a particular one such integral, prove that such a differential coefficient has $2p-2$ zeros (we have seen, p. 165 above, that there is no zero common to all the p differential coefficients).

Ex. 2. The theorem obtained before (Chap. VI, p. 144), and called the converse of Abel's theorem, may also be proved here, under a certain limiting hypothesis. With the notation used, of $V_i^{x,a}$ for an everywhere finite integral (p. 168), suppose that $(a_1), \ldots, (a_m)$ and $(z_1), \ldots, (z_m)$, are places such that the p equations $V_i^{x_1,a_1} + \ldots + V_i^{x_m,a_m} \equiv 0$ hold, the sign $\equiv$ indicating the possible presence of a linear aggregate of the $2p$ periods of the integral V_i, wherein the multipliers of these periods are the same for all the p equations; make the hypothesis that in the immediate neighbourhoods of $(z_1), \ldots, (z_m)$, respectively, there are other places $(z_1'), \ldots, (z_m')$ also satisfying these equations, $(a_1), \ldots, (a_m)$ remaining unaltered. Then, for proper values of the differentials of the parameters $t_1, \ldots, t_m$ appropriate respectively for the neighbourhoods of $(z_1), \ldots, (z_m)$, we have the p equations $\omega_i(z_1).dt_1 + \ldots + \omega_i(z_m).dt_m = 0$. Hence, if $\mu_1, \ldots, \mu_m$ be finite numbers which are in the ratios of $dt_1, \ldots, dt_m$, and μ be a constant, the rational function $\mu + \mu_1\psi(x, z_1) + \ldots + \mu_m\psi(x, z_m)$ has $(z_1), \ldots, (z_m)$ for poles of the first order, and no other poles. The theorem receives another proof below.

Explicit form of relation for interchange of argument and parameter. We add now some detailed investigations which furnish in particular; (i), a theorem, in explicit algebraic form, for the interchange of the limits and the infinities in an algebraic integral which has two logarithmic infinities. This was one of the achievements of the early theory in the elliptic case $(p=1)$; and, for Riemann's transcendentally defined integral $\Pi_{z,c}^{x,a}$, is true in the form $\Pi_{z,c}^{x,a} = \Pi_{x,a}^{z,c}$ (cf. p. 144, Chap. VI above); (ii), also in explicit algebraic form, a function which may serve as a *prime function* for rational functions of x and y (subject to $f(x, y) = 0$), enabling us to express such a function in factors, each factor vanishing to the first order in one of the poles, or in one of the zeros, of the rational function. The former theorem goes back to Abel (for the hyper-elliptic case, cf. *Œuvres*, 1881, I, p. 49) and to Weierstrass; the prime function to Kronecker (in an entirely different way) and Weierstrass (cf. Weierstrass, *Werke*, IV, p. 253).

Consider the rational function of (z) which is given by

$$R(z)=\psi(z,\xi)\frac{d}{dz}\psi(z,x),$$

where (x), (ξ) are two arbitrary places; and apply to this the fundamental theorem $[R(z)\,dz/dt]_{t^{-1}}=0$, which we may express here by $[\psi(z,\xi)\,d\psi(z,x)/dt]_{t^{-1}}=0$, the parameter t being that by which (z) is expressed in the neighbourhood of a place considered. Infinities arise only when (z) is at (ξ), or at (x), or at one of the places $(c_1), \ldots, (c_p)$ which were used in the definition of the function $\psi(z,\xi)$ or $\psi(z,x)$. To express the result of the theorem we use a notation, already several times employed, of which we repeat the definition here: If, for a rational function $R(z)$, with or without a pole at a place (c), we form the expansion, for the neighbourhood of (c), in terms of the parameter t appropriate to this neighbourhood, the coefficient of t in this expansion is denoted by $D_c R(c)$. Thence; (a), when (z) is near to (ξ), the expansions of $\psi(z,\xi)$, $\psi(z,x)$ give $\psi(z,\xi)=-t^{-1}+P(t)$, $d\psi(z,x)/dt=D_\xi\psi(\xi,x)+P_1(t)$, where $P(t)$, $P_1(t)$ are power series; and the contribution to the summation involved in the theorem is $-D_\xi\psi(\xi,x)$; (b), when (z) is near to (x), the corresponding expansions give $\psi(z,\xi)=\psi(x,\xi)+tD_x\psi(x,\xi)+\ldots$, $d\psi(z,x)/dt=t^{-2}+P_1(t)$; and the resulting contribution is $D_x\psi(x,\xi)$; when (z) is near to (c_i), we have

$$\psi(z,\xi)=t^{-1}\omega_i(\xi)+M+tD_{c_i}\psi(c_i,\xi)+\ldots,$$
$$\psi(z,x)=t^{-1}\omega_i(x)+N+tD_{c_i}\psi(c_i,x)+\ldots,$$

where M, N are independent of t, and the corresponding contribution in the theorem is $\omega_i(\xi)D_{c_i}\psi(c_i,x)-\omega_i(x)D_{c_i}\psi(c_i,\xi)$; thus, altogether, the theorem gives

$$-D_\xi\psi(\xi,x)+D_x\psi(x,\xi)+\sum_{i=1}^{p}[\omega_i(\xi)D_{c_i}\psi(c_i,x)$$
$$-\omega_i(x)D_{c_i}\psi(c_i,\xi)]=0.$$

We can express this result in terms of functions (z, a) defined above, p. 162. For we have $\psi(x,\xi)=(\xi,x)-\sum_k\omega_k(\xi)(c_k,x)$, and hence $D_x\psi(x,\xi)=D_x(\xi,x)-\sum_k \omega_k(\xi).D_x(c_k,x)$; in particular

$$D_{c_i}\psi(c_i,\xi)=D_{c_i}(\xi,c_i)-\sum_{k=1}^{p}\omega_k(\xi)D_{c_i}(c_k,c_i),$$

where, in $D_{c_i}(c_k, c_i)$, for $k=i$, $D_{c_i}(c_i, c_i)$ must be remembered to mean the value of $D_x(c_i, x)$ for $(x)=(c_i)$. Thus we obtain the formula, which we can express in four ways:

(I) The expression

$$D_\xi(x, \xi)+\sum_k \omega_k(x)[D_{c_k}(\xi, c_k)-D_\xi(c_k, \xi)]-\sum_k\sum_i \omega_i(x)\omega_k(\xi)D_{c_i}(c_k, c_i),$$

where $k=1, \ldots, p$ and $i=1, \ldots, p$, is unaltered by the interchange of (x) and (ξ).

(II) Putting

$$U_\xi{}^x=D_\xi(x, \xi)-D_x(\xi, x)+\sum_i\omega_i(x)[D_{c_i}(\xi, c_i)-D_\xi(c_i, \xi)]$$

we have

$$U_\xi{}^x=\sum_k \omega_k(\xi)U_{c_k}{}^x.$$

(III) If, as before, we put $E_z{}^{x,a}=\int_{(a)}^{(x)} D_z(x, z)\,dt_x$, and integrate the result in (I) in regard to (x), we have

$$E_\xi{}^{x,a}=\sum_k\omega_k(\xi)E_{c_k}{}^{x,a}+\psi(x, \xi)-\psi(a, \xi)-\sum_k V_k{}^{x,a}U_{c_k}{}^\xi.$$

Here the function $E_\xi{}^{x,a}$ does not depend on $(c_1), \ldots, (c_p)$. The right side, as a function of (x), becomes infinite when $\psi(x, \xi)$ becomes infinite, or when one of $E_{c_k}{}^{x,a}$ becomes infinite, and these do not depend on (ξ). Thus $E_\xi{}^{x,a}$ becomes infinite at (ξ) like $\psi(x, \xi)$, namely like $-t_\xi{}^{-1}$, and only then. This property of $E_\xi{}^{x,a}$ can be established directly, and hence the equation (III) deduced, as remarked below.

(IV) If, as before, we define $P_{z,k}^{x,a}$ as $\int_{(a)}^{(x)} [(x, z)-(x, k)]\,dt_x$, then, by integration, in regard to (ξ), of the result in (III), we obtain a result which expresses that the function

$$P_{z,k}^{x,a}+\sum_j V_j{}^{x,a}\{E_{c_j}{}^{z,k}-[(c_j, z)-(c_j, k)]\}-\sum_j\sum_i V_i{}^{x,a}V_j{}^{z,k}D_{c_i}(c_j, c_i)$$

is unaltered when we interchange the places (x), (a) respectively with the places (z), (k). Here the summation in regard to j and i is from 1 to p. This is then the generalisation of the formula in the theory of elliptic functions known as the interchange of the argument and parameter.

The direct investigation of the character of the function $E_\xi{}^{x,a}$, referred to under (III), and the consequent direct proof of (III) is as follows: The expression $D_\xi(x, \xi)\,dx/dt$ is $H\,dx/dt$, where

$$H=(x-\xi)^{-1}\sum_i\gamma_i(x)D_\xi g_i(\xi)+[\gamma_0(x)+\sum_i\gamma_i(x)g_i(\xi)]D_\xi(x-\xi)^{-1};$$

now, first, when (ξ) is not a branch place, we have $D_\xi g_i(\xi)=$ the differential coefficient $g_i'(\xi)$, and $\gamma_i(x)=\gamma_i(\xi)+(x-\xi)\gamma_i'(\xi)$, as (x) approaches to (ξ); while, from the fundamental relation $\gamma_0(x)+\Sigma\gamma_i(x)g_i(x)=1$, we have $\gamma_0'(\xi)+\Sigma\gamma_i'(\xi)g_i(\xi)=-\Sigma\gamma_i(\xi)g_i'(\xi)$. Thus we see at once that $H\,dx/dt$, when (x) approaches to (ξ), is the same as $(x-\xi)^{-2}$, and the integral $\int D_\xi(x, \xi)\,dx$ becomes infinite like $-(x-\xi)^{-1}$; and has no other infinity. Next, when (ξ) is a branch place of index r, and (ξ_1) is within the neighbourhood of this, the expansion

$$(x-\xi_1)^{-1}=(x-\xi-t^r)^{-1}=(x-\xi)^{-1}[1+(x-\xi)^{-1}t^r+\ldots]$$

contains no term in t, and $D_\xi(x-\xi)^{-1}=0$; also $\Sigma\,\gamma_i(x_1)D_\xi g_i(\xi)$, which in a previously used notation (p. 152) is $\Sigma\,\gamma_i(x)a_i'$, was proved to be given by $\frac{1}{r}t^{-1}$. Thus, for $D_\xi(x, \xi)\,dx/dt$ the infinite part is $\frac{1}{r}t^{-1}.t^{-r}.rt^{r-1}$ or t^{-2}; and hence the integral $E_\xi{}^{x,a}$ is infinite like $-t^{-1}$; and, as before, has no other infinity.

Now, from the character of the rational function of (x) given by $\psi(x, \xi)$, which was defined as $(\xi, x)-\sum\limits_k \omega_k(\xi)(c_k, x)$, we see that the difference $E_\xi{}^{x,a}-\psi(x, \xi)$, as a function of (x), has poles only at $(c_1), \ldots, (c_p)$; at these also it is infinite, respectively, like $-\omega_1(\xi)t_{c_1}{}^{-1}, \ldots, -\omega_p(\xi)t_{c_p}{}^{-1}$. Whence, the function of (x) defined by

$$K(x)=E_\xi{}^{x,a}-\sum_k \omega_k(\xi)E_{c_k}{}^{x,a}-[\psi(x, \xi)-\psi(a, \xi)]$$

has no infinities. But dK/dx is a rational function of (x). Thus $K(x)$ is an everywhere finite integral, and we may hence write

$$E_\xi{}^{x,a}=\sum_k \omega_k(\xi)E_{c_k}{}^{x,a}+\psi(x, \xi)-\psi(a, \xi)+\sum_k \lambda_k V_k{}^{x,a},$$

where λ_k does not depend on (x). Differentiating this equation in regard to (x), and then putting (x) at (c_i), we find the result given in (III).

Weierstrass's prime function for algebraic functions. We now proceed to another result. The function $P^{x,a}_{z,c}$ is the integral in regard to (x), of a function which is rational in (z). We consider now the integral as a function of (z). If (a), (k) denote arbitrary places, and we put

$$U^{x,a}_{z,k}=\int_{(a)}^{(x)}[(x-z)^{-1}-(x-k)^{-1}]\,dx,$$

and recall that $\gamma_0(z)+\Sigma\gamma_i(z)g_i(z)=1$, and $\gamma_0(k)+\Sigma\gamma_i(k)g_i(k)=1$, we can put $P^{x,a}_{z,k}=U^{x,a}_{z,k}+V$, where

$$V=\int_{(a)}^{(x)}[F(x, z)-F(x, k)]\,dx,$$

and $F(x, z)$, $F(x, k)$ denote functions given by

$$(x-z)\,F(x, z)=\gamma_0(x)-\gamma_0(z)+\Sigma\,[\gamma_i(x)-\gamma_i(z)]g_i(z),$$
$$(x-k)\,F(x, k)=\gamma_0(x)-\gamma_0(k)+\Sigma\,[\gamma_i(x)-\gamma_i(k)]g_i(k).$$

The subject of integration in V remains finite when (x), in the course of integration, passes through the place (z); when x approaches z, but (x) does not approach (z), we know that $P^{x,a}_{z,k}$ remains finite. We infer, then, that $P^{x,a}_{z,k}$, regarded as a function of (z), is single valued except when (z) approaches the path of integration from (a) to (x).

This path we may regard as a barrier for the variation of (z); the function of (z) given by $P^{x,a}_{z,k}$ will have different values at opposite points on the two sides of this barrier; in fact, as $U^{x,a}_{z,k}$ is equal to

$$\log[(x-z)(x-k)^{-1}/(a-z)(a-k)^{-1}],$$

these values differ by $2\pi i$ (cf. Weierstrass, *Werke*, IV, p. 379). We now interchange the places (x), (a) respectively with (z), (k), and infer that the integral

$$\int_{(k)}^{(z)} [\psi(x,z)-\psi(a,z)]\,dz,$$

which is $$P^{z,k}_{x,a} - \sum_i V_i^{z,k}[(c_i,x)-(c_i,a)],$$

regarded as a function of (x), is single valued, save for additive integer multiples of $2\pi i$; also, by its definition, or by the equation (IV) obtained above, this function is infinite when (x) is at (z) like $\log t_z$, and when (x) is at (k) like $-\log t_k$, and also has a pole of the first order at each of the places $(c_1), \ldots, (c_p)$, but is not otherwise infinite. Hence it follows that the function of (x) defined by

$$E(x,a;\, z,k) = \exp\{P^{z,k}_{x,a} - \sum_i V_i^{z,k}[(c_i,x)-(c_i,a)]\}$$

has the following properties:

(i), It is a single-valued function of (x), it being understood that the path of integration for (z), from (k) to (z), does not pass through the place (x); (ii), The function vanishes to the first order when (x) is at (z), and has a pole of the first order when (x) is at (k), being capable of expression by integral powers of the local parameter thereat; (iii), At each of the places $(c_1), \ldots, (c_p)$ the function has an essential singularity.

Now let $(z_1), \ldots, (z_m)$ be the zeros, and $(t_1), \ldots, (t_m)$ be the poles of a rational function $R(x)$, all these being of the first order. Then the quotient

$$Q, = E(x,a;\, z_1,k)\ldots E(x,a;\, z_m,k)/E(x,a;\, t_1,k)\ldots E(x,a;\, t_m,k)$$

is equal to $\exp(W)$, where

$$W = \sum_{r=1}^{m} P^{z_r,t_r}_{x,a} - \sum_{i=1}^{p}\sum_{r=1}^{m} V_i^{z_r,t_r}[(c_i,x)-(c_i,a)];$$

but, by Abel's theorem, if the paths of integration in the integrals $V_i^{z_r,t_r}$ be properly chosen, we can suppose that $\sum_{r=1}^{m} V_i^{z_r,t_r}$ vanishes, for each of the p values of i; while also (see Chap. III, p. 54) we have

$$\sum_{r=1}^{m} P^{z_r,t_r}_{x,a} = \log[R(x)/R(a)].$$

Thus it appears that $R(x)/R(a)$ is expressed by the quotient Q above, in which there is a factor corresponding to every zero and pole. (Cf. Weierstrass, *Werke*, IV, p. 384. The function $\psi(x, x')$ is essentially that which Weierstrass denotes by $H(x, y; x', y')$; *v. loc. cit.* p. 73. For the factors that may arise when the paths of integration involved are arbitrary, *v. loc. cit.* pp. 367, 390). Conversely, suppose that $(z_1), \ldots, (z_m)$ and $(t_1), \ldots, (t_m)$ are such places that, with proper paths of integration, we have the p equations $\sum_r V_i^{z_r, t_r} = 0$, for $i = 1, \ldots, p$. Then the quotient which, for brevity, we have denoted by Q, has $(z_1), \ldots, (z_m)$ for zeros of the first order, has $(t_1), \ldots, (t_m)$ for poles of the first order, and is analytic in (x), being expressible in the neighbourhood of any place by a series of integral powers of the appropriate parameter t. By the definition

$$E(x, a; z, k) = \exp\{P_{x,a}^{z,k} - \sum_i V_i^{z,k}[(c_i, x) - (c_i, a)]\},$$

the quotient has, when (x) is near to (c_i), an essential singularity depending on $\exp\{t_{c_i}^{-1} \sum_r V_i^{z_r, t_r}\}$, which, however, in virtue of the p equations assumed, is ineffective. The quotient is thus a rational function of (x). This furnishes another proof of the converse of Abel's theorem (cf. Weierstrass, *Werke*, IV, p. 418). The proof originally given here, by use of Riemann's elementary normal integral of the third kind (Chap. VI, p. 144), was based on a knowledge of the periods of this function. By what we have proved here in regard to the function $P_{x,a}^{z,k}$, regarded as a function of (x), the periods of the integral $P_{z,k}^{x,a}$, as a function of (x), save for additive multiples of $2\pi i$ when (x) makes a circuit about (z), or (k), can be found from equation (IV) above.

On the whole then, it appears sufficiently that the theory of this chapter can be made the complete basis of a theory of the linear series upon the curve $f(x, y) = 0$, and of the algebraic integrals associated therewith; but, in order that all the algebraic formulae should be explicitly realisable, it is necessary that the fundamental integral functions $g_1, \ldots, g_{n-1}$ should be explicitly known. To determine these is a problem entirely representative of the problem which presents itself in the earlier geometrical theory, of determining explicitly the conditions of adjointness of a polynomial at a singular point of whatever intricacy. For a summary of a procedure for determining $g_1, \ldots, g_{n-1}$, derived from Hensel, reference may be made to the writer's *Abel's Theorem*, Cambridge, 1897, pp. 105–112. A systematic introduction to the theory is given in a volume, *Theorie der algebraischen Functionen*, u.s.w., by Hensel u. Landsberg, Leipzig, 1902, pp. 1–702; and for other points reference may be made to the *Enzyk. d. Math. Wiss.* Bd. II, 3. 5. That the theory may be developed parallel with Weierstrass's earlier theory (but with greater explicitness when $g_1, \ldots, g_{n-1}$ have been computed) is sufficiently indicated here. For an alternative method of dealing with the fundamental difficulty of the multiple points, the papers of J. C. Fields, of which the first is *Theory of the algebraic functions of a complex variable*, Berlin, Mayer u. Müller, 1906, pp. 1–186, may be consulted.

Some indications of the many applications of the theory of this chapter may be added:

Ex. 1. For the case when the fundamental equation is $y^2-(x, 1)_{2p+2}=0$, it is easy to prove that any rational function which becomes infinite only for $x=\infty$ can be expressed as $u+vy$, where u, v are polynomials in x. Thus the integral function g_1 is y, for which $\sigma_1=p$. The equations $\gamma_0+\gamma_1 y=1$, $\gamma_0-\gamma_1 y=0$ (p. 158) determine $\gamma_0=\frac{1}{2}$, $\gamma_1=1/2y$. The everywhere finite integrals are thus $\int(x, 1)_{p-1}dx/2y$. Also, the function (x, ξ) (of p. 162) is given by $(y+\eta)[2y(x-\xi)]^{-1}dx/dt$, if (ξ, η) be the place (ξ). This case is characterised by the existence of a rational function of order 2.

Ex. 2. If there exist a rational function of order 3, say x, and the fundamental algebraic equation $f(x, y)=0$ contain y^3, as highest power of y, the fundamental integral functions g_1, g_2 will be such that the integral functions g_1g_2, g_1^2, g_2^2 are expressible in the respective forms $w+vg_1+ug_2$, $w_1+v_1g_1+u_1g_2$, $w_2+u_2g_1+v_2g_2$, where w_i, u_i, v_i are polynomials in x; herein the dimensions of v and u will not be greater respectively than those of g_2 and g_1; thus, using g_1-u, g_2-v in place of g_1, g_2, respectively, we can suppose $g_1g_2=w$. Forming, from these expressions, the forms for $g_1^2g_2$ and $g_1g_2^2$ (since an identity $U+Vg_1+Wg_2=0$ is impossible, wherein U, V, W are polynomials in x), we can infer $w=u_1u_2$, $w_1=-u_1v_2$, $w_2=-u_2v_1$. Thence, also forming g_1^3, we deduce $g_1^3-v_1g_1^2+u_1v_2g_1-u_1^2u_2=0$; or, if we use y for g_1/u_1, the fundamental equation may be taken to be $u_1y^3-v_1y^2+v_2y-u_2=0$; the two fundamental integral functions are then $g_1=u_1y$ and $g_2=u_2y^{-1}$. Also, in the most general case, from $g_1^2=w_1+v_1g_1+u_1g_2$, etc., we infer that the dimensions of u_1, u_2, v_1, v_2, in terms of σ_1, σ_2, are respectively, $2\sigma_1-\sigma_2+1$, $2\sigma_2-\sigma_1+1$, σ_1+1, σ_2+1. The maximum possible number of arbitrary coefficients (moduli), in the fundamental equation, if we use one constant as a multiplier for y, and three constants in a linear transformation of x, is, therefore, found to be $2(\sigma_1+\sigma_2)+1$, or $2p+1$. For the hyperelliptic case the number was $2p-1$; for the case of the fundamental equation most general for its genus it was $3p-3$ (p. 94 above); in the general case, a rational function can be found (p. 92 above) of order $1+\frac{1}{2}p$, or $1+\frac{1}{2}(p+1)$; and the least value of p for which this is >3 is $p=5$; the existence of a rational function of order 3 involves $3p-3-(2p+1)$ or $p-4$ conditions for the general moduli. Passing now to the computation of the functions γ_0, γ_1, γ_2, conjugate to the integral functions 1, g_1, g_2, we find from $\gamma_0+\gamma_1g_1+\gamma_2g_2=1$, $\gamma_0+\gamma_1g_1'+\gamma_2g_2'=0$, $\gamma_0+\gamma_1g_1''+\gamma_2g_2''=0$ (see p. 158 above) that $\gamma_0=(u_1y^2-v_1y)/f'(y)$, $\gamma_1=y/f'(y)$, $\gamma_2=1/f'(y)$, where $f(x, y)=u_1y^3-v_1y^2+v_2y-u_2$; and hence that the expression (x, ξ) is capable of the form $(u_2-\omega_2)[(x-\xi)(y-\eta)f'(y)]^{-1}dx/dt$, where (ξ, η) is the place (ξ), and ω_2 is the polynomial u_2 when x is ξ. We return to this case again in Ex. 5, to introduce a general theorem.

Ex. 3. Prove that a general plane quartic curve can be transformed so as to be represented by an equation $y^3+my^2+yxu+x^2v=0$, where m is a constant, and u, v are respectively quadratic and cubic polynomials. Prove that, for this equation, $g_1=y$ and $g_2=y(y+m)/x$, with $\sigma_1=1$, $\sigma_2=2$; and that the functions γ_0, γ_1, γ_2 are respectively given by

$$(y^2+my+xu)/f'(y),\quad y/f'(y),\quad x/f'(y).$$

Ex. 4. For the equation of a general quartic curve with a double point at (0, 0), namely $(y, x)_4+(y, x)_3+(y, x)_2=0$, obtain the integral functions g_1, g_2, g_3, the values of σ_1, σ_2, σ_3, and the functions γ_0, γ_1, γ_2, γ_3.

Ex. 5. Suggested by the everywhere finite integral, for the equation in Ex. 2, which is $\int[(x, 1)^{\sigma_1-1}y+(x, 1)^{\sigma_2-1}]dx/f'(y)$, consider the curve of order $2p-2$, in space $[p-1]$, in which the coordinates are

$$(\xi_0, \ldots, \xi_{\sigma_1-1}, \eta_0, \ldots, \eta_{\sigma_2-1}),$$

which is given by

$$\xi_0/yx^{\sigma_1-1}=\xi_1/yx^{\sigma_1-2}=\ldots=\xi_{\sigma_1-1}/y=\eta_0/x^{\sigma_2-1}=\eta_1/x^{\sigma_2-2}=\ldots=\eta_{\sigma_2-1}/1.$$

This is the canonical curve considered in an earlier chapter (p. 81 above). Evidently it lies on the rational ruled surface obtained by joining the general point $(\theta^{\sigma_1-1}, \theta^{\sigma_1-2}, \ldots, 1)$ of the rational normal curve of order σ_1-1 in space $(\xi_0, \xi_1, \ldots, \xi_{\sigma_1-1})$, to the corresponding point $(\theta^{\sigma_2-1}, \ldots, \theta, 1)$ of the rational normal curve of order σ_2-1 in space $(\eta_0, \ldots, \eta_{\sigma_2-1})$; this surface is of order $p-2$, and every generator contains three points of the curve (corresponding to the three values of y in the equations determining the curve). The canonical curve lies on the $\frac{1}{2}(p-2)(p-3)$ quadrics expressed by $\xi_i/\xi_{i+1}=\ldots=\eta_j/\eta_{j+1}=\ldots$ $(i=0, \ldots, \sigma_1-2;\ j=0, \ldots, \sigma_2-2)$, which, however, do not determine this curve, since they all contain the ruled surface; the curve may in fact be determined on the surface as the residual intersection of this with a cubic primal described through $p-4$ generators. (Cf. p. 96 above.) [It will in fact be proved later that an algebraic curve on a ruled surface of order n_1, whose prime section is of genus p_1, which meets each generator k times, and is of order ν, has a genus π such that $k(2p_1-2)=2\pi-2-(2\nu-n_1k)(k-1)$; if herein we put $n_1=p-2$, $p_1=0$, $k=3$, $\nu=2p-2$, we find $\pi=p$.]

Ex. 6. A series $g_1{}^n$ is found in the general case where the everywhere finite integral is of the form $\int[(x, 1)^{\sigma_1-1}\gamma_1+\ldots+(x, 1)^{\sigma_{n-1}-1}\gamma_{n-1}]dx$; for simplicity of statement, we assume every one of $\sigma_1-1, \ldots, \sigma_{n-1}-1$ is $\geqslant 0$, and that $n\leqslant p$. The canonical curve lies then in a space $[p-1]$ wherein the coordinates are $\xi_0, \ldots, \xi_{\sigma_1-1}$; $\eta_0, \ldots, \eta_{\sigma_2-1}, \ldots$; $\zeta_0, \ldots, \zeta_{\sigma_{n-1}-1}$, and is given by

$$\begin{aligned}\xi_0/\gamma_1x^{\sigma_1-1}&=\ldots=\xi_{\sigma_1-1}/\gamma_1=\eta_0/\gamma_2x^{\sigma_2-1}=\ldots\\&=\eta_{\sigma_2-1}/\gamma_2=\ldots=\zeta_0/\gamma_{n-1}x^{\sigma_{n-1}-1}=\ldots=\zeta_{\sigma_{n-1}-1}/\gamma_{n-1}.\end{aligned}$$

Evidently the curve lies on the manifold which is the locus of the space $[n-2]$ which joins the corresponding points

$$[\theta^{\sigma_1-1}, \ldots, \theta, 1],\ [\theta^{\sigma_2-1}, \ldots, \theta, 1], \ldots,\ [\theta^{\sigma_{n-1}-1}, \ldots, \theta, 1],$$

of $n-1$ rational normal curves lying respectively in the spaces

$$(\xi_0, \ldots, \xi_{\sigma_1-1}),\ (\eta_0, \ldots, \eta_{\sigma_2-1}), \ldots,\ (\zeta_0, \ldots, \zeta_{\sigma_{n-1}-1});$$

and the canonical curve is met by each of the spaces $[n-2]$ in n points, corresponding to the possible values of y in the functions $\gamma_1, \gamma_2, \ldots$ which enter in the definitions of $\xi_0, \xi_1, \ldots$. The order of the manifold is $\Sigma(\sigma_i-1)$, or $p-n+1$. We may prove this *ab initio* by remarking that, as the manifold is of dimension $n-1$, the order is the number of the spaces $[n-2]$, each determined by one value of θ, which meet $(n-1)$ given primes of the space $[p-1]$; the values of θ are those which satisfy $n-1$ equations, each of the form $\lambda_1(\theta, 1)_{\sigma_1-1} + \dots + \lambda_{n-1}(\theta, 1)_{\sigma_{n-1}-1} = 0$, in which $\lambda_1, \dots, \lambda_{n-1}$ are the same in all, and the coefficients in the polynomials $(\theta, 1)_{\sigma_i-1}$ are given, and different. By elimination of $\lambda_1, \dots, \lambda_{n-1}$ this leads to an equation of order $p-n+1$ for θ. As follows from the equations $\xi_i/\xi_{i+1} = \eta_j/\eta_{j+1} = \dots = \zeta_k/\zeta_{k+1}$, where there are $p-n+1$ fractions, the manifold (or scroll) of order $n-1$, lies on $\frac{1}{2}(p-n+1)(p-n)$ quadrics; as is known, the canonical curve lies on $\frac{1}{2}(p-2)(p-3)$ quadrics; thus, beside those containing the scroll, there is a number $\frac{1}{2}(n-3)(2p-2-n)$ remaining; this vanishes when $n=3$, as in Ex. 5.

[It may be remarked that if in space $[r]$ we have a scroll of ∞^1 spaces $[s]$, of genus p_1, of order n_1, forming thus a manifold $M^{n_1}_{s+1}$; and thereon a curve of order ν and genus π, meeting each generating $[s]$ in k points; then the number of generating spaces, $2y$, which touch the curve is given by $2y = 2\pi - 2 - k(2p_1 - 2)$, and the number of generating spaces, d, in which $s+1$ of the k points of the curve lying therein lie on a space $[s-1]$ is given by $d = \nu(k-1, s) - n_1(k, s+1) - y(k-2, s-1)$, when (h, l) means $h!/l!(h-l)!$; and we have

$$k(2p_1 - 2) = 2\pi - 2 - \frac{2(k-1)}{s(s+1)}\{(s+1)\nu - n_1 k\} + \frac{2d}{(k-2, s-1)}.$$

This agrees with our case if $n_1 = p-n+1$, $s = n-2$, $k = n$, $\nu = 2p-2$, $\pi = p$, $p_1 = 0$, $y = p-1+n$, $d = 0$. For this formula, and references, see Segre, *Enzykl. Math. Wiss.* III, C. 7, p. 954.]

Ex. 7. It is assumed in Ex. 6 that every one of $\sigma_1 - 1, \dots, \sigma_{n-1} - 1 \geqslant 0$. For an indication of other possibilities we may take the general plane curve of order n without multiple points (which, for $n > 4$, is not general of its genus). For this case, a fundamental set of integral functions is given by $y, y^2, \dots, y^{n-1}$, and we have $\sigma_1 = 0$, $\sigma_2 = 1, \dots, \sigma_{n-1} = n-2$. The scroll in question is then that generated by spaces $[n-3]$, through a point, which contain corresponding points of a line, a conic, ..., a rational normal curve of order $n-3$. Another case is that of the general curve of genus 5, for which again $\sigma_1 = 1$. The scroll is that of planes, through a point in space [4], which contain corresponding points of two lines, pointwise related.

Ex. 8. The matter may be dealt with from the theory of special series considered above (p. 96). We limit ourselves to the consideration of complete series (in the first case of Ex. 7, the series $g_1{}^n$ obtained is not complete), and to the general case, when the curve is general of its genus. As a simple case consider, for $p = 5$, on the canonical curve of order 8 in space [4], which is the complete intersection of three quadrics, the series $g_1{}^4$, in which the sets lie on quadrisecant planes of the curve. The complementary series is then also a $g_1{}^4$. There are through any set A_1 of the series two linearly independent primes of the space [4], determining, suppose, the complementary sets A_1', A_2' on the canonical curve; through A_1' there are two linearly independent primes, one of which, we may suppose, has A_1 for its residual intersection with the curve, the other having a second set, A_2, of the original series $g_1{}^4$. If the primes (A_1, A_1'), (A_2, A_1') be $\phi_1 = 0$, $\phi_2 = 0$, and the primes (A_1, A_2'), (A_2, A_2') be $\psi_1 = 0$, $\psi_2 = 0$, the series $g_1{}^4$ is determined as well by the primes $\lambda_1\phi_1 + \lambda_2\phi_2 = 0$, as by the primes $\mu_1\psi_1 + \mu_2\psi_2 = 0$, in which λ_1, λ_2, μ_1, μ_2 are parameters, and we can suppose ψ_1, ψ_2 multiplied by such constants that $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$. Both the sets A_1, A_2, as well as all other coresidual sets of the series $g_1{}^4 = 0$, thus lie on the quadric $\phi_1\psi_2-\phi_2\psi_1=0$, which contains the plane $\phi_1=0$, $\psi_1=0$ containing the set A_1, and the plane $\phi_2=0$, $\psi_2=0$ containing the set A_2, and similarly the planes of all the sets of the linear series, as well as the planes of all the sets of the complementary series. This quadric is a cone, whose vertex, O, is the point of intersection $\phi_1=\phi_2=\psi_1=\psi_2$; this point O is determined by the particular series g_1^4 considered. There are ∞^1 such series; and a particular set of a particular series is determinable by taking two points P, Q of the canonical curve, and determining a plane through PQ having two other intersections with the curve; then Q may be regarded as varying from set to set of the same series, and P as determining the particular series. There is thus a correspondence between the canonical curve, regarded as the locus of P, and the curve which is the locus of the points O determined by all the series g_1^4. When O is given there are 4 points P, since O lies on a quadrisecant plane of the canonical curve; and when P is given there are 5 positions of O, since 5 quadrisecant planes can be drawn through a given chord; if we assume there are no coincidences in either case, this establishes that the locus of O is a curve of genus 6 (see Vol. VI, Chap. I). That it is of order 10 follows since O is the locus of the vertices of quadric cones containing the complete intersection of three quadrics, say $U=0$, $V=0$, $W=0$; and is therefore the locus for which all determinants of order 3 in the matrix of 3 rows and 5 columns, whose general column is $\partial U/\partial x_i$, $\partial V/\partial x_i$, $\partial W/\partial x_i$, simultaneously vanish. (Cf. p. 110, Ex. 2, above.)

Ex. 9. Now consider more generally a complete special series g_r^n, on the canonical curve in space $[p-1]$, the complementary series being a $g_{r'}^{n'}$, when $n'=2p-2-n$ and r' given by $r=n-p+r'+1$. The sets A_1, A_2, ... of the series g_r^n are the residual sets of intersections, of the canonical curve, with primes passing through the space $[n'-r'-1]$ which contains any set A_j' of the series of sets A_1', A_2', ... of the complementary series $g_{r'}^{n'}$. We may denote the prime which contains the set A_i of g_r^n, and the set A_j' of $g_{r'}^{n'}$, by $[i,j']=0$. The general prime through A_j' is then of the form $\lambda_0[1,j']+\lambda_1[2,j']+\dots+\lambda_r[r+1,j']=0$ and we can, choosing a suitable constant multiplier for every prime $[i,j']$, suppose that the multipliers $\lambda_0,\dots,\lambda_r$ are independent of j. By similar reasoning for the complementary series, it is sufficient to consider only the sets A_j' for which $j=1,2,\dots,r'+1$. Suppose $r'\geqslant r$; then eliminating $\lambda_0,\dots,\lambda_r$ from every $r+1$ of these equations, we see that all the spaces $[n-r-1]$, containing the sets $A_1,\dots,A_{r+1}$, and hence, also, those containing all the sets A_i of g_r^n, lie on the manifold which reduces to zero every determinant of $(r+1)$ rows and columns in the matrix, of $r+1$ rows and $r'+1$ columns,

$$\left\|\begin{matrix}[1,1'], & [1,2'], & \dots, & [1,r'+1]\\ \dots\dots & \dots\dots & \dots\dots & \dots\dots\\ [r+1,1'], & [r+1,2'], & \dots, & [r+1,r'+1]\end{matrix}\right\|.$$

By an algebraical theorem considered below (in Vol. VI) this manifold is of dimension $n-1$, and of order $(r'+1,r)$, where (p,q) is used for $p!/q!(p-q)!$. The manifold evidently contains the space $[\tau-1]$, where $\tau=p-(r+1)(r'+1)$, through which every one of the primes $[i,j']=0$ passes, if $\tau\geqslant 1$; it was shewn (p. 91) that, in general, $\tau\geqslant 0$. The spaces $[n'-r'-1]$, containing the sets A_j' of $g_{r'}^{n'}$, which lie in primes $\mu_0[i,1]+\dots+\mu_{r'}[i,r'+1]=0$, also contain this space $[\tau-1]$. As there are ∞^τ series g_r^n, the spaces $[\tau-1]$ describe a manifold of dimension $2\tau-1$. The general condition $\tau\geqslant 0$ is not necessary for the existence of the manifold; for instance, when the canonical curve is that for the general plane quintic curve, the planes which contain the sets of the series g_2^5 generate a manifold of order 3 and dimension 4, the locus of the planes of the conics lying on a Veronese surface in space [5] (which represents the plane of the quintic curve). The theory applies in particular to the series g_1^n where $n=1+\frac{1}{2}p$, or $n=1+\frac{1}{2}(p+1)$, giving respectively $\tau=0$ and $\tau=1$; in the latter case the manifold is a cone, whose vertex describes a curve.

Ex. 10. Interesting applications of the theory of this chapter arise when, for the curve $f(x, y)=0$, we consider rational functions whose poles are all at one place. The equation may then be transformed to an equation $F(\xi, \eta)=0$, where ξ is the function whose pole at this place has the lowest possible order, say a; and η is the function whose pole has the next lower possible order, r, which is prime to a. Cf. Weierstrass, *Werke*, II, p. 235 (and the writer's *Abel's Theorem*, pp. 34, 93, 99).

CHAPTER VIII

ENUMERATIVE PROPERTIES OF CURVES

Part I. General formulae. In ordinary space of three dimensions, an algebraic surface is the locus represented by a single polynomial equation, $F(x, y, z, t)=0$, homogeneous in the coordinates x, y, z, t. It is generally intended that the polynomial F is incapable of being written as the product of other polynomials, and the surface is then said to be *irreducible*. The *order* of the surface is the number of its intersections with an arbitrary line, and is the order of the polynomial F in x, y, z, t. Unless the contrary is stated we shall suppose the plane $t=0$ to have no special relation with the surface, and shall often replace t by 1, and represent the equation by $F(x, y, z)=0$.

An *algebraic curve* is most naturally regarded as the intersection of algebraic surfaces; two such surfaces meet in a curve. But it is not true conversely that any given algebraic curve is the *complete* intersection of two surfaces; a familiar example to the contrary is the rational cubic curve, which is the part intersection of two quadric surfaces having, beside, a line in common. Nor indeed is it clear that *three* algebraic surfaces can be drawn through a given curve so as to have this as their only common part; they may have, beside, points in common; in fact it will be seen below that a curve may be such that no three surfaces can be drawn through this which do not have also points in common not lying on this curve. But *four* surfaces can be drawn through a curve to have no common curve or point, beside the given curve; and this in infinitely many ways; it is sufficient to take the cones projecting the curve from four points of general position*.

The indeterminateness of the definition of an algebraic curve, as the intersection of surfaces, suggests that it may be desirable to define such a curve in another way. We take the definition as follows: An algebraic curve is a locus of which the (non-homogeneous) coordinates of a point, x, y, z, are rational functions of two parameters ξ, η, which are not independent, but are connected by a rational polynomial irreducible equation $f(\xi, \eta)=0$; these parameters being both expressible rationally in terms of the coordinates, x, y, z, of the corresponding point of the curve. That is, we have equations of the forms $x=U(\xi, \eta)$, $y=V(\xi, \eta)$, $z=W(\xi, \eta)$, $f(\xi, \eta)=0$; $\xi=u(x, y, z)$, $\eta=v(x, y, z)$, where U, V, W, u, v denote rational functions, and $f(\xi, \eta)$ is a rational polynomial. By this definition an algebraic curve in space is in (1, 1) birational correspondence with the irreducible plane curve expressed by $f(\xi, \eta)=0$.

---

* It is similarly true that any algebraic construct, lying in space of r dimensions, may be regarded as the complete intersection of $(r+1)$ primals (loci represented each by a single polynomial equation in the $r+1$ homogeneous coordinates). This remark seems to have been first made by Kronecker, *Werke*, II, 1881, p. 280. A geometrical exposition is given by Segre, "Introduzione...", *Ann. d. Mat.* XXII, 1894, p. 47. An account of Kronecker's point of view is given in the writer's *Multiply-Periodic Functions* (Cambridge, 1907), p. 273; a much more exhaustive account by Molk, *Acta Math.* VI, 1885, p. 159.

That a curve as so defined may be given by the intersection of surfaces is clear enough; a formal proof that a curve defined by the intersection of surfaces is capable of the definition we have given, is contained in the literature referred to in the footnote preceding.

From the definition, from what is known for a plane curve, it appears that all points (x, y, z) of the space curve, which are in the neighbourhood of any point (x_0, y_0, z_0) of the curve, belong to one of a finite number of *branches*; the points of a branch are given, in terms of a parameter t, by equations of the forms

$$x-x_0=A_1t+A_2t^2+\ldots, \quad y-y_0=B_1t+B_2t^2+\ldots,$$
$$z-z_0=C_1t+C_2t^2+\ldots,$$

in which the parameter t is such that its value is unique for every point (x, y, z) of the branch in the neighbourhood of (x_0, y_0, z_0); and the series converge for all sufficiently small values of t. The statement is made supposing all of x_0, y_0, z_0 to be finite, but can easily be modified to cover all cases. Consider now one of these branches in the neighbourhood of (x_0, y_0, z_0), writing, for brevity, x, y, z in place of $x-x_0, y-y_0, z-z_0$. Suppose that not every one of A_1, B_1, C_1 is zero, and, in particular, that A_1 is not zero. Using then, instead of the coordinates y, z, the coordinates $y-B_1A_1^{-1}x$ and $z-C_1A_1^{-1}x$, the expressions take forms

$$x=A_1t+A_2t^2+\ldots, \quad y=Q_2t^2+Q_3t^3+\ldots, \quad z=R_2t^2+R_3t^3+\ldots;$$

suppose here that Q_2 is not zero; then, using, instead of z, the coordinate $z-R_2Q_2^{-1}y$, the expressions are reduced to the forms

$$x=A_1t+A_2t^2+\ldots, \quad y=Q_2t^2+Q_3t^3+\ldots, \quad z=H_3t^3+H_4t^4+\ldots.$$

For a branch expressed by three such equations, we can define a certain line, called the *tangent line* of the branch, and a certain plane, called the *osculating plane* of the branch, at the point considered. First, it is obvious that any plane through the point, of equation $px+qy+rz=0$, meets the branch in one point there; for the power series equation in t

$$p(A_1t+A_2t^2+\ldots)+q(Q_2t^2+Q_3t^3+\ldots)+r(H_3t^3+H_4t^4+\ldots)=0$$

has $t=0$ as a simple root. If, however, $p=0$, the root $t=0$ is double, and every plane $qy+rz=0$ meets the branch in two points coinciding at the origin considered. Thus the line $y=0$, $z=0$ meets the branch twice at the origin. It is this line which is called the *tangent line* at the point. Again, if both p and q be zero, the root $t=0$ is triple, and the plane $z=0$ meets the branch in three points coinciding at the origin. It is this plane which is called the *osculating plane* of the branch. It is easy to shew further that, if the origin be O, and A, B be two other points of the branch, in the neighbourhood of O, the line OA tends to the tangent line as A tends to O, and the plane OAB tends to the osculating plane as A, B tend both to O; and that the plane containing the tangent line and the point A, tends to the osculating plane as A tends to O.

More generally, a branch of the curve may be representable, by proper choice of coordinates, by three equations of the forms

$$x=a_1t^{l+1}+a_2t^{l+2}+\ldots,\quad y=b_1t^{m+2}+b_2t^{m+3}+\ldots,$$
$$z=c_1t^{n+3}+c_2t^{n+4}+\ldots,$$

where the integers l, m, n, which are all positive or zero, are such that $l\leqslant m\leqslant n$ and none of a_1, b_1, c_1 is zero. Unless l, m, n are all zero, in which case these expressions reduce to those considered above, the names tangent line and osculating plane (for $y=0=z$, and $z=0$) are not strictly applicable in the preceding sense *at the origin*. But they are applicable at a point (t) near the origin, and may be obtained, if we wish, by writing $t+\tau$ for t in these power series, and rearranging the series in powers of τ. It is more convenient to use differential coefficients. Suppose for example we seek the osculating plane at the point (t); let the equation of this plane be written $x\eta+y\xi+z+\omega=0$; substitute for x, y, z the above power series in t, and determine ξ, η, ω from the resulting equation and the two equations $x'\eta+y'\xi+z'=0$, $x''\eta+y''\xi+z''=0$, where $x'=dx/dt$, $x''=d^2x/dt^2$, etc. If X, Y, Z denote the minors of x, y, z in the determinant Δ, given by

$$\Delta=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

the result is $\xi=Y/Z$, $\eta=X/Z$, $\omega=-\Delta/Z$.

Now introduce numbers l', m', n' given by $l'=n-m$, $m'=n-l$, $n'=n$, which lead to $l=n'-m'$, $m=n'-l'$, $n=n'$; then l', m', n' are positive or zero, and such that $l'\leqslant m'\leqslant n'$, while the expressions for ξ, η, ω in terms of t may be written $\xi=a_1't^{l'+1}+\ldots$, $\eta=b_1't^{m'+2}+\ldots$, $\omega=c_1't^{n'+3}+\ldots$, where

$$a_1'=-\frac{c_1}{b_1}\frac{(n+3)(m'+2)}{(m+2)(m'-l'+1)},\quad b_1'=\frac{c_1}{a_1}\frac{(n+3)(l'+1)}{(l+1)(m'-l'+1)},$$
$$c_1'=-c_1\frac{(l'+1)(m'+2)}{(l+1)(m+2)};$$

and the reverse formulae are of precisely similar forms, namely

$$a_1 = -\frac{c_1'}{b_1'}\frac{(n'+3)(m+2)}{(m'+2)(m-l+1)}, \quad b_1 = \frac{c_1'}{a_1'}\frac{(n'+3)(l+1)}{(l'+1)(m-l+1)},$$
$$c_1 = -c_1'\frac{(l+1)(m+2)}{(l'+1)(m'+2)},$$

and, as none of a_1, b_1, c_1 is zero (or infinite), also none of a_1', b_1', c_1' is zero or infinite.

A branch of a curve for which the three indices l, m, n are all zero may be called an *ordinary branch*, the name being justified because, as will sufficiently appear, the number of branches of an *algebraic* curve which are not ordinary is necessarily finite; it is for such a branch that we have defined the tangent line and the osculating plane. A branch which is not ordinary at the origin (or centre) of its coordinates, above denoted by $x=0$, $y=0$, $z=0$, is ordinary at points in the immediate neighbourhood of its origin.

The expressions we have just found serve to emphasize the duality which exists for a curve, regarded as defined by its points, and as defined by its osculating planes. An algebraic curve is a locus of points defined by the continuous variation of a parameter; we may dually consider an aggregate of planes similarly defined by a parameter. Just then as a curve has *chords*, each defined by two points of the curve; and the limit of a chord, when one end is the origin of an ordinary branch of the curve, and the other end tends to this, is the definite line called the tangent line; so, the aggregate of planes has *axes*, each defined by the intersection of two of the planes; and there is a similar limit theorem. Similarly, as the osculating plane of the curve is obtainable by passing to a limit, with three points of which one is the centre of an ordinary branch, so, a point is obtained from three planes of the aggregate of planes; and the locus of such points is exactly a curve as originally considered. When the algebraic aggregate of planes is regarded as defining a *developable surface* (of which the planes are the tangent planes), the curve obtained is that which is called the *cuspidal edge* (or the edge of regression) of this surface; any algebraic curve is thus the cuspidal edge of the developable surface formed by its own osculating planes; and the tangent line of the curve is the ultimate position of the line of intersection of two consecutive planes of the developable surface (the *generating line* of the developable).

Further, we notice that, in the formulae above, an ordinary branch, for which all the indices l, m, n are zero, gives expressions for ξ, η, ω for which all the indices l', m', n' are zero, and conversely. Points which are origins of branches which are not ordinary are thus exceptional as well from the dual as from the original point of view.

Of branches which are not ordinary we shall generally consider only three kinds:

(i) Those for which $l=0$, $m=0$, $n=1$. Here the expressions of the coordinates, in terms of the parameter, are of the forms $x=a_1t+\dots$, $y=b_1t^2+\dots$, $z=c_1t^4+\dots$, and the difference from the ordinary case is that the plane $z=0$ has *four*, not *three*, intersections with the branch at the origin. The origin may therefore be called a point at which there is a branch of *stationary osculating plane*. The origin itself may perhaps be called a *stall*. The total number of such branches for the whole curve (which, as we have said, is finite for an algebraic curve) will finally be denoted by κ'. In the dual formulae, for the osculating plane at a point of the branch in the neighbourhood of the origin, the indices l', m', n' are given by $l', =n-m, =1$, $m', =n-l, =1$, $n', =n, =1$.

(ii) The second kind of branch which we commonly recognise, which is not ordinary, is that for which $l=1$, $m=1$, $n=1$. Here the expressions for the coordinates are of the forms $x=a_1t^2+\dots$, $y=b_1t^3+\dots$, $z=c_1t^4+\dots$, and the striking difference from the ordinary case lies in the fact that every plane through the origin meets the branch in *two* points at the origin, and every plane, $qy+rz=0$, through the line $y=0=z$, meets the branch in *three* points at the origin. While, in the ordinary case, y/x^2 has a definite limit at the origin, it is here y^2/x^3 which becomes definite. This is similar to the behaviour of a plane curve at an ordinary cusp; thus we shall also say here that the branch has a *cusp*, or *stationary point*, at the origin. This case is evidently dual to the case (i); the total number of branches of the curve which have a cusp will finally be denoted by κ.

(iii) Last, we may consider branches for which the indices are given by $l=0$, $m=1$, $n=1$; these give also $l'=0$, $m'=1$, $n'=1$, and the singularity is dual to itself. The expressions for the coordinates are of the forms $x=a_1t+\dots$, $y=b_1t^3+\dots$, $z=c_1t^4+\dots$; in this case, a general plane through the origin meets the branch in only one point at the origin, but any plane through the line $y=0$, $z=0$ meets the branch in three points coincident at the origin. This is analogous to a point of inflexion on a plane curve; we shall therefore say that the branch has an *inflexion*, or *stationary tangent line*; and we shall denote the whole number of such branches, arising for the curve, by i.

Ex. 1. If for a branch of a *plane* curve, with expressions in terms of a parameter t given by

$$x=a_1t^{l+1}+\dots,\quad y=b_1t^{m+2}+\dots,\qquad l, m\geqslant 0,\ l\leqslant m,$$

the tangent line at a point in the neighbourhood of the origin be written $px+y+q=0$, prove that

$$p=a_1't^{l'+1}+\dots,\quad q=b_1't^{m'+2}+\dots,\qquad \text{with } l'=m-l,\ m'=m,$$

where $$a_1'=-\frac{b_1}{a_1}\,\frac{m+2}{l+1},\quad a_1=-\frac{b_1'}{a_1'}\,\frac{m'+2}{l'+1},\quad b_1'=b_1\frac{l'+1}{l+1},$$

and $$l', m'\geqslant 0,\quad l'\leqslant m'.$$

Thus, just as $l+1$ is the number of points of the branch which lie on an arbitrary line passing through the origin $x=0$, $y=0$, so $l'+1$ is the number of tangent lines of the branch which pass through an arbitrary point lying on the line $p=0$, $q=0$, that is, the line $y=0$. And, as $m+2$ is the number of points of the branch (at $t=0$) lying on the line $y=0$, so $m'+2$ (equal to $m+2$) is the number of tangents of the branch containing the point $q=0$, that is the point $x=0$, $y=0$.

Similar remarks may be made, from the formulae of the text, for the osculating plane of the curve in space. Cf. Halphen, *Bull. Soc. Math. d. France*, VI, 1878, p. 10.

Ex. 2. The line-coordinates of the tangent line, at the point (t), of the branch of the curve in space expressed by $x=at^{l+1}+\ldots$, $y=bt^{m+2}+\ldots$, $z=ct^{n+3}+\ldots$, if the line-coordinates be computed from

$$\lambda:\mu:\nu:\lambda':\mu':\nu'=x':y':z':yz'-y'z:zx'-z'x:xy'-x'y,$$

where $x'=dx/dt$, $y'=dy/dt$, $z'=dz/dt$, are, in order, only the first terms being written, for λ, μ, ν, $(l+1)a$, $(m+2)bt^{m-l+1}$, $(n+3)ct^{n-l+2}$, and, for λ', μ', ν', $bc(n-m+1)t^{m+n-l+4}$, $-ca(n-l+2)t^{n+3}$, $ab(m-l+1)t^{m+2}$. When $t=0$ these approach to the values $(1, 0, 0,\ 0, 0, 0)$, which we may regard as the line-coordinates of the tangent line of the branch. The points of the branch at which the tangent line meets this particular tangent line are given by the vanishing of a power series in t of which the first term (that in λ') is $bc(n-m+1)t^{m+n-l+4}$. Thus we say that the number of tangent lines of the branch, "consecutive with" the tangent at the origin, which meet this tangent, is $m+n-l+4$, which is also $m'+n'-l'+4$. In particular for an ordinary branch this number is 4.

The number of such tangents which pass through the origin is to be found by considering the equations $\lambda'=0$, $\mu'=0$, $\nu'=0$, and is the highest power of t entering as factor in all these; this, as we see at once, is $m+2$. In particular, for a stationary point, for which $l=m=n=1$, this is 3. The number of such tangents which lie in the plane $z=0$, is similarly found from $\lambda'=0$, $\mu'=0$, $\nu=0$, and is $n-l+2$ or $m'+2$. In particular for a stall, where there is a stationary osculating plane, for which $l=0$, $m=0$, $n=1$, this is also 3.

It may be remarked that an algebraic curve is completely determinate by the power series which express x, y, z on any one branch, all other points and the branches thereat being deducible by "analytical continuation" from these power series. This is a result following from the corresponding theorem for a Riemann surface (above, p. 144).

We assume now that a general plane meets the algebraic curve in a definite number of points, which we call the *order* of the curve. This result may be rigorously deduced from the definition we have adopted for the curve; it is the same as saying that, for the curve $f(\xi,\eta)=0$, the rational function $pU(\xi,\eta)+qV(\xi,\eta)+rW(\xi,\eta)+s$ has a definite number of zeros, independent of precise values of the ratios of p, q, r, s; naturally the number of intersections of the plane with the curve at points which are not ordinary must be properly estimated. Dually, with a similar proper interpretation of the phrase, there is a definite number of osculating planes of the curve passing through an arbitrary general point. This we call the *class* of the curve. We shall denote the order of the curve by n, and its class by n'. Further, the curve has ∞^1 tangent lines, and for a tangent line to meet a given line is one condition; thus an arbitrary general line is met by a definite number of tangent lines of the curve; this we call the *rank* of the curve, and denote by r. The number of tangent lines of the curve which meet a line of special position may of course be different; in particular, from Ex. 2 above, the number of other tangent lines which meet the tangent line of the curve at an ordinary point is $r-4$ (if $r \geqslant 4$). Likewise, the curve has ∞^2 chords, and for a line to pass through a given point two conditions must be satisfied; thus, through an arbitrary general point of the space, there will pass a definite number of chords of the curve; the tangents of the curve generate a locus consisting of ∞^2 points, a surface; and the arbitrary general point will not lie upon a definite surface. Thus the chords of the curve from an arbitrary general point will be proper chords, having two points of intersection with the curve which are distinct; the number of such chords we denote by h; it is sometimes spoken of as the number of *apparent double points* of the curve. Dually, we denote by h' the number of lines of intersection of two distinct osculating planes of the curve, or, as before, the number of *axes* of the curve, which lie in an arbitrary general plane. Another characteristic of the curve arises from what we may call the *nodal curve* of the given curve. As we have remarked, a tangent of the curve is met by a certain number of other tangents; the points of intersection generate another curve, which is what we call the *nodal curve*. The characteristic in question is the order of this nodal curve; it is the number of points, in an arbitrary general plane, at which two distinct tangents of the original curve intersect one another. This number we denote by ν. Dually, we denote by ν' the number of planes, through an arbitrary general point, which contain two distinct intersecting tangent lines of the original curve; this is the number of planes through the point which touch the curve at two points, say, the number of bitangent planes of the curve through the point. The bitangent planes of a curve are, clearly enough, the tangent planes of a developable surface, upon which the original curve is a cuspidal curve, and the nodal curve, of the original curve, is a double curve; we may call this the *bitangent developable* of the original curve; it is to be distinguished from the developable surface of which the tangent planes are the osculating planes of the original curve, sometimes called the *osculating developable* of the original curve. The curve order, n, is different from the branch index, n.

For simplicity of reference we tabulate the notations thus introduced: n is the order, r is the rank, n' is the class;

κ is the number of branches of the curve with cusps, or stationary points, where the expressions in terms of a parameter have the forms $x=at^2+\ldots$, $y=bt^3+\ldots$, $z=ct^4+\ldots$, the indices l, m, n of the general description being all equal to 1;

κ' is the number of branches with stationary osculating plane, the number of stalls, where the expressions have forms $x=at+\ldots$, $y=bt^2+\ldots$, $z=ct^4+\ldots$, the indices l, m, n having the respective values 0, 0, 1;

i is the number of inflexions, with expressions of the forms $x=at+\ldots$, $y=bt^3+\ldots$, $z=ct^4+\ldots$, for which l, m, n have the respective values 0, 1, 1. In all these it is supposed that none of the coefficients a, b, c is zero, and it is provisionally assumed that κ, κ', i are all finite;

h is the number of chords of the curve through an arbitrary general point, and, dually, h' is the number of axes of the curve lying in an arbitrary general plane;

ν is the order of the nodal curve, the locus of points of intersection of two distinct tangents of the original curve; and, dually, ν' is the number of planes through an arbitrary general point which contain two distinct tangents of the original curve.

Evidently, on a curve which is general, regarded as a locus of points—there being ∞^1 points of the curve, and the condition for an osculating plane to be stationary being one-fold—there will as a rule be points of stationary osculating plane, namely κ' will not vanish. Similarly κ will not as a rule be zero for a curve which is general when regarded as an envelope of osculating planes. But, in either case, it is exceptional for i not to be zero; for the condition for a third consecutive point of the curve to lie in the tangent line is two-fold.

We may also allow for the possibility of actual double points of the curve, where the curve crosses itself. Such a point will not generally exist (as requiring 3 conditions to be satisfied by 2 parameters); but we shall be concerned below with curves which arise as the intersection of surfaces, and where two such surfaces touch their curve of intersection has a double point. The number of such actual double points for the curve will be denoted by δ. Dually, δ' will denote the number of planes which are osculating planes of the curve at two distinct points. Further, the number of still less probable lines which are tangent to the curve at two distinct points will be denoted by τ.

Recall now, that, for a plane algebraic curve, of order N, class N', and genus P, whose singularities consist of D double points, K cusps, T double tangents, and I points of inflexion, we have

$$D=\tfrac{1}{2}N(N-7)+N'-3(P-1), \qquad K=2N-N'+2P-2,$$
$$T=\tfrac{1}{2}N'(N'-7)+N-3(P-1), \qquad I=2N'-N+2P-2.$$

These equations lead to

$$N' = N(N-1) - 2D - 3K, \qquad N = N'(N'-1) - 2T - 3I,$$
$$I - K = 3(N' - N),$$
$$P = \tfrac{1}{2}(N-1)(N-2) - D - K, \ = \tfrac{1}{2}(N'-1)(N'-2) - T - I.$$

We use these equations for a plane curve, to make inferences for the curve in space, first projecting this from an arbitrary general point upon an arbitrary general plane, and then by considering, upon an arbitrary general plane, the locus of the point in which this plane is met by the tangents of the given curve.

When we project the given curve in space on to a plane, we obviously obtain a curve of order N, $= n$, and of class N', $= r$. It is also easy to see that we obtain cusps of the plane curve from the stationary points of the curve in space, or that $K = \kappa$. Again, when an osculating plane of the space curve, at an ordinary point of this, passes through the point of projection, we very clearly obtain a point of inflexion on the plane curve; the three coinciding intersections of the space curve with the osculating plane, project into three coinciding intersections of the plane curve with a tangent line. We also obtain a point of inflexion of the plane curve from any existing inflexion of the space curve. Thus we have $I = n' + i$. Likewise, we evidently have a double point on the plane curve for every chord of the space curve which passes through the centre of projection; so that, allowing for actual double points of the space curve, we have $D = h + \delta$. It is also clear that the number of double tangents T, of the plane curve, is $T = \nu' + \tau$.

Next, consider the plane curve, in an arbitrary general plane, which is the locus of the point in which the tangent line, of the curve in space, meets this plane. The order, N, of the plane curve, is evidently the rank, r, of the space curve. The tangent line of the plane curve, obtainable as the limit of a chord, arises from an osculating plane of the space curve, this containing two coincident tangent lines; thus the class N', of the plane curve, is given by $N' = n'$. At a point where the space curve meets the plane of the section, supposed to be an ordinary point of the space curve, two, ultimately coinciding, tangent lines of the space curve will give the same point of the plane curve; this point will thus be a stationary point (cusp) of the plane curve. There will likewise be a cusp on the plane curve arising from the stationary character of the tangent line at any of the existing i inflexions of the space curve. Wherefore $K = n + i$. It may be seen, further, that a point of the space curve which is the origin of a branch with a stationary osculating plane (a stall on the space curve) gives rise to an inflexion on the plane curve, so that $I = \kappa'$. Also, it is clear that there is a double point of the plane curve at every point where the nodal curve of the space curve meets the plane of section—and further, also at the intersection of this plane with any actual double tangent line of the space curve; hence $D=\nu+\tau$. Finally, for the double tangents of the plane curve, we have, easily, $T=h'+\delta'$.

Tabulating these statements, supposing that we have enumerated all the possibilities, we have, for the two plane curves considered, obtained by projection and section:

	Order (N)	Class (N')	Cusps (K)	Inflexions (I)	Double points (D)	Double tangents (T)
By projection	n	r	κ	$n'+i$	$h+\delta$	$\nu'+\tau$
By section	r	n'	$n+i$	κ'	$\nu+\tau$	$h'+\delta'$

Hence, from the Plücker equations recalled above, applied to these curves respectively, we have, for the space curve

$$r=n(n-1)-2(h+\delta)-3\kappa,$$
$$n=r(r-1)-2(\nu'+\tau)-3(n'+i),$$
$$n'+i-\kappa=3(r-n),$$

and

$$n'=r(r-1)-2(\nu+\tau)-3(n+i),$$
$$r=n'(n'-1)-2(h'+\delta')-3\kappa',$$
$$n+i-\kappa'=3(r-n');$$

also, if P, P' be the respective genera of the two plane curves, obtained by projection and section,

$$P=\tfrac{1}{2}(n-1)(n-2)-(h+\delta+\kappa),\ =\tfrac{1}{2}(r-1)(r-2)-(n'+i+\nu'+\tau),$$
$$P'=\tfrac{1}{2}(r-1)(r-2)-(n+i+\nu+\tau),\ =\tfrac{1}{2}(n'-1)(n'-2)-(h'+\delta'+\kappa').$$

The first six equations, however, lead to

$$\kappa-\kappa'=2(n-n'),\quad \nu-\nu'=n'-n,$$
$$h+\delta-(h'+\delta')=\tfrac{1}{2}(n-n')(n+n'-7),$$

from which in particular it appears that $P=P'$. This number is thus independent of the centre from which the projection is made, and of the plane on to which it is made; and is independent of the plane on which the section is taken. We denote this number by p, and call it the genus of the curve in space.

It is at once seen that, if n, n', p and r be supposed known, these equations give

$$i=-(n+n')+2r+2p-2,$$
$$\kappa=2n-r+2p-2,$$
$$\kappa'=2n'-r+2p-2,$$
$$\nu+\tau=\tfrac{1}{2}(r-1)(r-6)+n'-3p,$$
$$\nu'+\tau=\tfrac{1}{2}(r-1)(r-6)+n-3p,$$
$$h+\delta=\tfrac{1}{2}(n-1)(n-6)+r-3p,$$
$$h'+\delta'=\tfrac{1}{2}(n'-1)(n'-6)+r-3p.$$

If τ, δ and δ' be zero, these seven equations determine the other characters i, κ, κ', ν, ν', h, h' in terms of n, n', r and p. In any case all the equations found, and in particular the four forms for p, can be deduced from these seven equations. From these equations also the five following equations can be deduced:

$$n'+2\kappa+i=3(n+2p-2), \quad n+2\kappa'+i=3(n'+2p-2),$$
$$3\kappa+2i+\kappa'=4(n+3p-3), \quad \kappa+2i+3\kappa'=4(n'+3p-3),$$
$$\kappa+2i+\kappa'=2(r+4p-4).$$

Connexion of previous results with general principles. It is an interesting fact that these five equations, and the first three of the preceding seven equations (those which do not involve ν, ν', h or h'), are all illustrations of a single principle. This we propose to explain now, with the warning to the reader that the ideas involved are somewhat less elementary than those so far used in this chapter. We deal first with the second and third of the first seven equations (those giving κ and κ') and the first four of the last five equations; then, in a still less elementary way, with the two remaining equations, namely with the two $n+i+n'=2(r+p-1)$, $\kappa+2i+\kappa'=2(r+4p-4)$.

Referring to Chap. IV, preceding, for the ideas involved, suppose that on a plane curve, of genus p, given by an equation $f(\xi, \eta)=0$, we have a linear series of sets of points, of freedom s; that is, a series of sets given by the intersections of the curve with a system of variable curves whose equation is

$$\lambda\phi(\xi, \eta)+\lambda_1\phi_1(\xi, \eta)+\ldots+\lambda_s\phi_s(\xi, \eta)=0,$$

in which $\phi, \phi_1, \ldots, \phi_s$ are definite rational polynomials, linearly independent on $f(\xi, \eta)=0$; in addition to possible fixed intersections, common to all of $\phi(\xi, \eta)=0, \ldots, \phi_s(\xi, \eta)=0$, $f(\xi, \eta)=0$, these curves have a number, say m, of intersections with $f(\xi, \eta)=0$, which vary when $\lambda, \lambda_1, \ldots, \lambda_s$ vary. There exists then a set of the series of which s points are arbitrarily chosen on $f(\xi, \eta)=0$. In particular, there exists a set with s points all coincident at any arbitrary general point, P, of $f(\xi, \eta)=0$; as P takes its possible ∞^1 positions on this curve, the remaining $m-s$ points of the set vary also; that one of these $m-s$ points should coincide with P is one condition. Thus there is a finite number of positions of P at which there fall $(s+1)$ coincident points of a set of the linear series. The principle which we assume is that the number of such $(s+1)$-fold points is $(s+1)[m+s(p-1)]$. It is understood that, in this total, special points of coincidence may need to be counted more than once; the difficulty in the application of the principle lies, in fact, most often, in this circumstance. The formula is applicable for a partial series, of freedom s_1, where $s_1<s$, chosen from the given linear series, given, say, by a system $\lambda\phi+\lambda_1\phi_1+\ldots+\lambda_{s_1}\phi_{s_1}=0$, and gives the number of (s_1+1)-fold coincidences contained in this partial series; in particular the principle has already been employed

for a pencil ($s_1=1$), in connexion with the definition of the genus p (Chap. IV, p. 84). The proof of the formula is a simple application of a formula of coincidence for correspondence of points on a curve. For clearness we refer to a following chapter (Vol. VI, Chap. I) for this proof; other proofs may be given (cf. Segre, *Ann. d. Mat.* XXII, 1894, p. 85).

Precisely the same principle can be employed for an algebraic curve in space of any number, say r, of dimensions. Such a curve is, by definition, as in the case of a curve in space of three dimensions, in (1, 1) birational correspondence with a plane algebraic curve. On such a curve, in space in which the homogeneous coordinates are $x_0, x_1, \ldots, x_r$, a linear series of sets of points is given by the intersections of the curve with a linear system of loci expressed by a single equation $\lambda\phi+\lambda_1\phi_1+\ldots+\lambda_s\phi_s=0$, wherein $\phi, \ldots, \phi_s$ are definite homogeneous polynomials of the same order in the coordinates $x_0, \ldots, x_r$, and $\lambda, \lambda_1, \ldots, \lambda_s$ are variable parameters. Such a system evidently leads to a linear series on the representative plane curve, and conversely. And, if we define the genus of the space curve as that of the representative plane curve, the same formula remains valid as in the case of the plane curve; more properly, as, in the (1, 1) correspondence, coincidences correspond to coincidences, the formula itself defines the genus.

Of this principle, the formula $r+\kappa=2(n+p-1)$, for the curve of order n and genus p, in the space of three dimensions, is an obvious application. We have only to consider the linear series, of sets of n points, of freedom 1, cut upon the curve by variable planes drawn through an arbitrary general line of the space. Whenever a tangent line of the curve meets this arbitrary axis, there is a coincidence of two of the intersections of the curve with a plane through the axis; and there is a coincidence also for a plane, through the axis, which contains a cusp of the curve. It is assumed that these two cases exhaust all the coincidences which are possible.

The formula $r+\kappa'=2(n'+p-1)$ is the dual of the other, or may be similarly obtained. More generally, representing branches of the curve as before, at points which are not ordinary, by series of the forms $x=at^{l+1}+\ldots$, $y=bt^{m+2}+\ldots$, $z=ct^{k+3}+\ldots$, the same principle leads to the result $r+\lambda=2(n+p-1)$, where λ is the sum of all not-zero values of the index l which arise (the genus p being appropriately defined).

Likewise we can obtain the formula $n'+2\kappa+i=3(n+2p-2)$; for this we consider the series, of sets of n points, of freedom 2, on the curve, which is obtained by variable planes drawn through an arbitrary general point, O. It is only necessary to shew that the left side of the equation is the number of planes through O meeting the curve in sets of which three points coincide. Of such planes, there are clearly n' osculating planes; and there are i planes, each joining O to a tangent line at a point of inflexion of the curve. To complete the formula it must be shewn that the plane joining O to the tangent line at a cusp of the curve counts doubly, as a plane with three coincident intersections; this is most clearly seen by considering the number of three-fold intersections with the curve, of planes from O, which arise for a branch $x=at^{l+1}+\ldots$, $y=bt^{m+2}+\ldots$, $z=ct^{k+3}+\ldots$. This number is found to be $l+m$. Thus, if λ, μ denote respectively the sum of all values of l and m which are not zero, arising at all points of the curve, the complete formula is

$$n'+\lambda+\mu=3(n+2p-2);$$

this includes the original form. For the proof see below, Ex. 5. The dual formula $n+2\kappa'+i=3(n'+2p-2)$ may be treated in a similar way.

Now take the formula $3\kappa+2i+\kappa'=4(n+3p-3)$. Consider the linear series, of freedom 3, of sets of n points, on the curve, determined by a general plane of the space. The right side of the equation gives, by the principle enunciated, the number of such planes meeting the curve in four coincident points. The stationary osculating planes of the curve, in number κ', are evidently such planes; but to complete the formula it is necessary to shew, (a), that through an inflexion of the curve there is one such plane which counts twice, and, (b), that through a cusp there is such a plane counting three times. More generally, in terms of the indices l, m, n for branches of the curve which are not ordinary, the formula to be proved is $\Sigma\,(l+m+n)=4(N+3p-3)$, where N is the order of the curve, and the summation extends to every branch of the curve for which l, m, n are not all zero. For the proof, in this general form, which includes the simpler case, see Ex. 5 below. The dual formula for $\kappa+2i+3\kappa'$ needs no independent treatment.

We pass now to the other two formulae referred to, namely $n+i+n'=2(r+p-1)$, $\kappa+2i+\kappa'=2(r+4p-4)$. These we interpret as applications of the fact that a line in space of three dimensions can be represented by a point in space of five dimensions, lying on a quadric, Ω, of that space (*Principles of Geometry*, Vol. IV, pp. 40 ff.). The tangents of the original curve will thus be represented by the points of an algebraic curve lying on Ω. The order of this curve, which is equal to the number of its intersections with a general prime (or flat four-fold space) of the five-fold space [5], and therefore equal to the number of its intersections with a general tangent prime of Ω, is equal to the number of tangent lines of the original curve which meet a general line of the three-fold space [3]; thus the curve on Ω will be of order r. We have defined the genus of a curve in any space as that of a plane curve with which it is in (1, 1) birational correspondence, incidentally assuming that two plane curves in such correspondence have the same genus, as is known. Whence, the curve on Ω, being in (1, 1) correspondence with the tangents of the original curve, and hence with the curve itself, will be of genus p. For the representation upon the quadric Ω, a point of the original space is regarded as determined by the ∞^2 lines of that space which pass through it; and a (line) coordinate of one of these lines is a linear function of the corresponding line coordinates of three of these lines, of general position; that is, there are six equations of the form $\lambda = al_1 + bl_2 + cl_3$, with a, b, c the same in all. Thus, as the coordinates of a point of the quadric Ω are the line coordinates of the line, in the original space, which it represents, it follows that a point of the original space is represented, in the space [5], by a plane, of which every point lies on the quadric Ω. In particular, the lines through a point of the original curve contain a flat pencil of lines, lying in the osculating plane of the curve at this point, of which pencil the tangent line of the curve is one line. Also, a pencil of lines of the original space is represented, in the space [5], by the points of a line lying on Ω; and, two consecutive tangents of the original curve ultimately belong to the pencil in the osculating plane which they define. Thus, the tangent line of the curve on Ω, at any point of this, lies entirely on the quadric Ω, in the plane on Ω which represents a point of the original curve. There are, in fact, two systems of planes lying entirely on Ω, one system, which we denote as α-planes, representing the points of the original space; but, by similar reasoning, the planes of the original space are also represented by planes on Ω, say β-planes; and, through the tangent line of the curve on Ω, passes, not only the α-plane representing a point of the original curve, but also the β-plane representing the osculating plane of this curve at this point. Indeed, through any line lying entirely on Ω (which represents a pencil of lines of the original space), there passes a single α-plane and a single β-plane; and these two planes together are the complete intersection of Ω with a flat three-fold space, or *solid*, of the space [5]—this solid being the intersection of the polar primes, in regard to Ω, of any two points of the line. Consider now the meaning, for the curve on Ω, of the order, n, of the original curve: An arbitrary plane, of the space [3] of this curve, which meets this curve in n points, is represented by a β-plane on Ω; the condition for a point of the original space to lie on a given plane, is that the α-plane on Ω which represents the point should meet *in a line* the β-plane which represents the plane; in general, the α-planes through the tangent lines of the curve on Ω will not meet an arbitrary β-plane; but there are just n tangent lines for which the corresponding α-plane meets an arbitrary β-plane (and therefore meets this plane in a line, as we easily see. In general an α-plane of Ω does not meet a β-plane; when they do meet, they meet in a line). Thus n is the number of tangent lines of the curve on Ω which meet an arbitrary β-plane. Similarly the class, n', of the original curve, is the number of tangent lines of the curve on Ω which meet an arbitrary α-plane. If then we take an α-plane and a β-plane which have a line in common, $n+n'$ will be the number of intersections, of tangent lines of the curve on Ω, with the solid which is defined by these two planes. Assume now that this is the same as the number of intersections of tangent lines, of the curve on Ω, with any general solid of the space [5]; through a solid, in space [5], there passes a pencil (an ∞^1 aggregate) of primes (spaces of four dimensions); each of these meets the curve on Ω in r points; and, in the series of ∞^1 sets of r points on the curve, there are, by the principle we have already used, $2(r+p-1)$ sets for each of which two of the r points coincide. Such a coincidence arises when a tangent line, of the curve on Ω, meets the solid which is the base of the pencil of primes; and there are $n+n'$ such cases, as we have seen. Such a coincidence, however, can arise in another way: namely, an inflexion of the original curve, where there is a stationary tangent, gives rise to a stationary point of the representative curve on Ω; and the prime joining this point to the base of the pencil of primes meets the curve on Ω in a set of which two points coincide at this stationary point. Thus, with the hypothesis that we have enumerated all cases of such coincidence, we have

$$n+n'+i=2(r+p-1),$$

which is the first of the two equations we desired to interpret.

We proceed now to the other equation referred to. We have represented the tangent lines of a curve in ordinary space by the points of a curve on the quadric Ω in space [5]. More generally, any algebraic system of ∞^1 lines in ordinary space, in (1, 1) correspondence with the points of an algebraic curve, say the generators of a ruled surface, may be represented by the points of a curve on Ω. This will be more general than the curve we have so far considered, its tangent lines not lying on Ω; the condition that this should be so is that every consecutive two of the system of lines in ordinary space, with which we begin, should intersect one another, or that the ruled surface should be a developable surface. In the general case there will be a certain finite number of the tangent lines of the curve on Ω which lie wholly thereon. In the special case we have taken above, of a curve on Ω of which every tangent line lies on Ω, the osculating planes of the curve will not generally lie on Ω; but there will be a certain finite number of these osculating planes which do. The curve being of order r and genus p, this number in fact is $2(r+4p-4)-2K$, where K denotes the number of cusps of the curve. The curve being derived as before from a curve in space [3], we have already remarked that $K=i$, where i is the number of inflexions of the original curve. For the proof of this number cf. Ex. 3, below. Consider now what character of the original curve gives rise to the property that the osculating plane of the representative curve lies entirely on Ω. This requires that all points of the plane, determined by three points of the curve which are tending to coincidence, should lie on Ω; there must then be three consecutive tangents of the original curve ultimately passing through the same point of this curve, or else three such tangents ultimately lying in the same osculating plane of this curve. The former arises for a cusp of the original curve, the latter for a stationary osculating plane. Thus we reach the interpretation of the equation

$$\kappa+\kappa'=2(r+4p-4)-2i$$

which we desired.

Ex. 1. For a so-called rational curve, whose points have coordinates $x:y:z:1=u:v:w:\omega$, wherein u, v, w, ω are general polynomials of order n in a parameter θ (which is thus conversely expressible as a rational function of x, y, z), prove, (i), that the line coordinates of the tangent line involve θ to order $2(n-1)$; (ii), that the equation of the osculating plane involves θ to order $3(n-2)$; (iii), that there is no point of the curve such that every plane through this meets the curve doubly thereat. Whence, for this curve, $n=n$, $r=2(n-1)$, $n'=3(n-2)$, $\kappa=0$, $\delta=0$, and, from $r+\kappa=2(n+p-1)$, also $p=0$. Hence infer

$$i=0,\quad \kappa'=4(n-3),\quad \nu+\tau=2(n-1)(n-3),\quad \nu'+\tau=2(n-2)(n-3),$$
$$h=\tfrac{1}{2}(n-1)(n-2),\quad h'+\delta'=\tfrac{1}{2}(9n^2-53n+80).$$

In particular, for $n=3$, since $\tau=\delta'=0$, as is obvious geometrically, we have $r=4$, $n'=3$, $\kappa=\delta=i=\nu=\nu'=0$, $h=1=h'$; and for $n=4$, assuming $\tau=0$, the equation $\delta'=0$ being again obvious, $r=6$, $n'=6$, $\kappa=\delta=i=0$, $\nu=6$, $\nu'=4$, $h=3$, $h'=6$.

Ex. 2. We have deduced from Plücker's formulae for a plane curve the formulae for a space curve, by projection from a general point and section by a general plane. The characters of the plane curve are modified if either the point of projection, or the plane of section, be specially chosen. Prove the following results (cf. Cayley, *Papers*, VIII, p. 81) for the curve obtained by projection when the centre of projection has the respective positions stated, and the dual results for the curve obtained by section by the planes respectively stated:

For the curve obtained by projection

With the vertex of projection	Order (N)	Class (N')	Double points (D)
General	n	r	$h+\delta$
On a tangent of the curve ...	n	$r-1$	$h-1+\delta$
On the curve	$n-1$	$r-2$	$h-n+2+\delta$
At a double point of the curve	$n-2$	$r-4$	$h-2n+5+\delta$
At a cusp	$n-2$	$r-3$	$h-2n+6+\delta$
On an inflexional tangent ...	n	$r-2$	$h-2+\delta$
At a point of inflexion ...	$n-1$	$r-3$	$h-n+1+\delta$

For the curve obtained by projection (cont.)

With the vertex of projection	Cusps (K)	Double tangents (T)	Inflexions (I)
General	κ	$\nu'+\tau$	$n'+i$
On a tangent of the curve ...	$\kappa+1$	$\nu'-r+4+\tau$	$n'-2+i$
On the curve	κ	$\nu'-2r+8+\tau$	$n'-3+i$
At a double point of the curve	κ	$\nu'-4r+20+\tau$	$n'-6+i$
At a cusp	$\kappa-1$	$\nu'-3r+13+\tau$	$n'-4+i$
On an inflexional tangent ...	$\kappa+2$	$\nu'-2r+9+\tau$	$n'-4+i$
At a point of inflexion ...	$\kappa+1$	$\nu'-3r+14+\tau$	$n'-5+i$

For the curve obtained by section

With plane of section	Order (N)	Class (N')	Double points (D)
General	r	n'	$\nu+\tau$
Containing a tangent	$r-1$	n'	$\nu-r+4+\tau$
An osculating plane	$r-2$	$n'-1$	$\nu-2r+8+\tau$
Touching curve at two points ...	$r-4$	$n'-2$	$\nu-4r+20+\tau$
A stationary osculating plane ...	$r-3$	$n'-2$	$\nu-3r+13+\tau$
Containing an inflexional tangent	$r-2$	n'	$\nu-2r+9+\tau$
Osculating curve at an inflexion	$r-3$	$n'-1$	$\nu-3r+14+\tau$

With plane of section	Cusps (K)	Double tangents (T)	Inflexions (I)
General	$n+i$	$h'+\delta'$	κ'
Containing a tangent	$n-2+i$	$h'-1+\delta'$	$\kappa'+1$
An osculating plane	$n-3+i$	$h'-n'+2+\delta'$	κ'
Touching curve at two points ...	$n-6+i$	$h'-2n'+5+\delta'$	κ'
A stationary osculating plane ...	$n-4+i$	$h'-2n'+6+\delta'$	$\kappa'-1$
Containing an inflexional tangent	$n-4+i$	$h'-2+\delta'$	$\kappa'+2$
Osculating curve at an inflexion	$n-5+i$	$h'-n'+1+\delta'$	$\kappa'+1$

Ex. 3. Prove that if, for a curve of order n and genus p, with κ cusps, lying on a quadric locus Ω in space of any number of dimensions, it be the case that the tangent lines of the curve, the osculating planes, the osculating solids, and so on, and finally the osculating $(h-1)$-folds, all lie entirely on Ω, then the number of the osculating h-folds of the curve which lie on Ω, is $2[n+2h(p-1)]-2\kappa$. The formula holds for $h=1$, in which only the curve and not all its tangent lines, lie on Ω. A proof will be found in the *Proceedings of the Edinburgh Math. Soc.* Vol. I (2nd series), 1927, p. 19.

Ex. 4. In the text, beside ordinary points and double points, we have considered in detail only singular points which are cusps, or inflexions, or stalls. This makes easier the application of the ordinary Plücker formulae for a plane curve, to obtain the equations for a curve in space. But it is more fundamental to employ the general formulae for a branch of the curve, whether in a plane or in space. If we slightly change the notation, and write the expressions for a branch of the space curve in the forms $x=a_1t^{l_1+1}+\dots$, $y=a_2t^{l_2+2}+\dots$, $z=a_3t^{l_3+3}+\dots$, with l_1', l_2', l_3' in the dual formulae for the osculating plane, we shall have, instead of κ, i, κ', respectively, Σl_1, $\Sigma(l_2-l_1)$, or $\Sigma(l_2'-l_1')$, and $\Sigma l_1'$, or $\Sigma(l_3-l_2)$, where the summations extend to all branches of the curve in space for which these are not zero. For a cusp, inflexion or stall these give respectively a contribution of unity. The formula for a space curve, $r+\Sigma l_1=2n+2p-2$, would then follow, as in the text, from the formula for a plane curve, $N'+\Sigma l_1=2n+2p-2$. The generalised form of the plane curve formula $2K+I=3(n+2p-2)$ would be $\Sigma(l_1+l_2)=3(n+2p-2)$, where l_1, l_2 refer to this curve; this leads, as in the text, to the formula

$$n'+\Sigma(l_1+l_2)=3(n+2p-2)$$

for a space curve which projects into the plane curve.

Ex. 5. It is proper, however, to obtain the formulae directly for the curve in space; and, as this is almost as simple for the case of an algebraic curve in space of any number, say r, dimensions, as for a curve in ordinary space, for brevity's sake, we pass at once to this case. Let non-homogeneous coordinates in the space be denoted by $x_1, x_2, \dots, x_r$, these being chosen so that the points of a particular branch of the curve are given by

$$x_1=a_1t^{l_1+1}+\dots,\quad x_2=a_2t^{l_2+2}+\dots,\ \dots\dots,\quad x_r=a_rt^{l_r+r}+\dots,$$

in terms of a parameter t, where $l_1\leqslant l_2\leqslant\dots\leqslant l_r$. That is, by proper choice of the coordinates, about the origin, $(0, 0, \dots, 0)$, of the branch, there is a line, p_1, through the origin, given by the equations $x_2=0=\dots=x_r$; then a plane, p_2, through the line p_1, given by the equations $x_3=0=\dots=x_r$; and, in general, a space p_s, of s dimensions, given by the equations $x_{s+1}=0=\dots=x_r$, which contains the space p_{s-1}. Further, every prime (or space of $r-1$ dimensions) passing through the origin, not containing the line p_1, has l_1+1 coincident intersections with the branch at the origin; then, every prime through the line p_1, but not containing the plane p_2, has l_2+2 coincident intersections with the branch at the origin; and so on. At an ordinary point of the curve, where every one of $l_1, \dots, l_r$ is zero, the line p_1 is the tangent line, the plane p_2 the osculating plane, and so on; and, in general, we may speak of p_1 as the tangent line, p_2 as the osculating plane, and so on, with p_s as the osculating $[s]$ space.

We denote then by n the order of the curve, the total number of intersections of the curve with an arbitrary general prime, by p the genus of the curve. The ∞^1 lines p_1 will generate a surface, V_2; the order of this surface we denote by n_1. Thus n_1 is the number of lines p_1 which meet an arbitrary general space $[r-2]$, of $r-2$ dimensions; and n_1 is also the number of primes, through an arbitrary general space $[r-2]$, for which the n intersections with the curve have a coincidence of two, lying on a tangent line p_1; but when the point of contact of this line p_1 is not ordinary, and this line p_1 has l_2+2 coincident intersections with the curve, we must count the corresponding branch as furnishing l_2+1 such two-fold coincidences. In this general case also, as any line through the origin of the branch has l_1+1 coincident intersections with the branch at the origin, the line p_1 is multiple on the surface V_2, with multiplicity l_2-l_1+1. We speak of the number n_1 as the *first rank* of the curve. If we consider the series of ∞^1 sets of n points on the curve, obtained by the primes through a fixed arbitrary space $[r-2]$, there is, by a formula which has been quoted, a number, given by $2n+2p-2$, of coincidences of two of the points of a set. These coincidences include those, already explained, in number n_1, whose position on the curve depends on the particular space $[r-2]$ chosen for base of the pencil of primes. The number $2n+2p-2$ will, however, also include a contribution l_1 for every branch of the curve for which $l_1>0$; for an arbitrary prime through the origin of such a branch has l_1+1 intersections coinciding at this origin. Thus we have the formula $n_1+\Sigma l_1=2n+2p-2$. More generally, the osculating spaces p_s at all branches of the curve generate a locus V_{s+1}, of dimension $s+1$; the order of this locus we denote by n_s. Thus n_s is the number of spaces p_s which meet an arbitrary space $[r-s-1]$, of $r-s-1$ dimensions; and n_s is also the number of primes, drawn through a general space $[r-s-1]$, of which the n intersections with the curve have a coincidence, on a branch of the curve, of $(s+1)$ points lying on a space p_s. As the primes through a space $[r-s-1]$ are a linear system of freedom s, such a prime can be put through s coincident points of a p_s, which, being given by $x_{s+1}=\ldots=x_r=0$, contains $l_{s+1}+s+1$ coincident points of the branch. Of the primes through the space $[r-s-1]$ there are, however, in general, others, having $(s+1)$ coincident intersections with the curve, on branches whose position on the curve does not depend on the particular $[r-s-1]$ chosen. For, consider a general branch of the curve; and the condition that it be possible, from among the primes passing through a given space $[r-s-1]$, to take a prime, through the origin of this branch and s other neighbouring points of this branch; if such space be determined by $r-s$ points with coordinates, relatively to the origin of the branch considered, given by $\xi_{1k}, \ldots, \xi_{rk}$ $(k=s+1, \ldots, r)$, the condition in question is that the determinant, of r rows and columns,

$$\begin{vmatrix} x_1, & \ldots\ldots, & x_r \\ x_1', & \ldots\ldots, & x_r' \\ \ldots & \ldots & \ldots \\ x_1^{(s-1)}, & \ldots\ldots, & x_r^{(s-1)} \\ \ldots & \ldots & \ldots \\ \xi_{1k}, & \ldots\ldots, & \xi_{rk} \\ \ldots & \ldots & \ldots \end{vmatrix},$$

where $(x_1, \ldots, x_r)$ denotes a point of the branch in the neighbourhood of the origin, and (in the first rows) $x_h^{(\lambda)}$ denotes $d^\lambda x_h/dt^\lambda$, should, on substitution of the power series in t for $x_1, \ldots, x_r$, be divisible by t^{s+1}. It is easy to see that this determinant divides by t^μ with $\mu=l_1+\ldots+l_s+s$; thus, when $l_1+\ldots+l_s=1$, there is just one such prime, with an $(s+1)$-fold intersection; and, when $l_1+\ldots+l_s>1$, we reckon the branch as furnishing $l_1+\ldots+l_s$ such $(s+1)$-fold intersections. Thus, by the general formula of coincidences quoted in the text, we infer the result

$$n_s+\Sigma(l_1+\ldots+l_s)=(s+1)[n+s(p-1)],$$

the summation extending to all branches of the curve. In particular, the $(r-1)$-th rank, most often called the *class* of the curve, is given by

$$n_{r-1}+\Sigma(l_1+\ldots+l_{r-1})=r[n+(r-1)(p-1)],$$

these equal numbers being the number of osculating primes of the curve which pass through an arbitrary general point of the space. The same argument proves that $\Sigma(l_1+\ldots+l_r)=(r+1)[n+r(p-1)]$, these being the number of stalls of the curve, namely the number of branches where an osculating prime contains $(r+1)$ coincident points of the branch, or is *stationary* (for which $l_1+\ldots+l_r=1$), together with contributions for other branches at which $l_1+\ldots+l_r>1$.

The reader may consult Veronese, *Math. Annal.* XIX, 1882, pp. 161 ff., and Segre, *Ann. d. Mat.* XXII, 1894, pp. 86–8, where a proof of the formula for n_s, by *induction*, from the two cases of n_1 and n_2, will be found. The set of formulae can be built up, either from such as

$$n_k+\lambda_k-n_{k-1}=n-2k+2kp, \qquad k=1, \ldots, r,$$

with $n_0=n$, $n_r=0$, where λ_k denotes the sum of the values of l_k for all branches of the curve; or from such as

$$n_k+n_{k-2}+\lambda_k-\lambda_{k-1}=2n_{k-1}+2p-2, \qquad k=1, \ldots, r;$$

and either of these may be obtained directly.

Ex. 6. In order to call attention to the relation of the results of the last example to the well-known theory of Weierstrassian points on a curve (see Chap. IV, p. 90; also the writer's *Abel's Theorem* (1897), pp. 34–46, and pp. 90 ff.), we may briefly consider the case of the canonical curve, of order $2p-2$, in space $[p-1]$, where the (non-homogeneous) coordinates are $x_1, \ldots, x_{p-1}$. By applying Clifford's theorem (above, p. 82) to the linear series, of sets of $2p-2-(l_s+s)$ points, which is determined by the ∞^{p-s-1} primes passing through the osculating space p_{s-1}, at a general branch of the curve, we see that $l_s < s$; in particular $l_1 = 0$, and the curve has no branch at which the origin is not simple. By the Riemann-Roch theorem, we find that the l_s+s intersections of p_{s-1} with the branch determine a linear series of coresidual sets which has freedom l_s; there is thus a rational function, on the curve, which has a (p_s+s)-fold pole at the origin of the branch and no other pole; and in this function there enter l_s arbitrary constants (beside the additive constant). Further, in this case, with $r=p-1$, the number $(r-1)[n+r(p-1)]$ is $(p-1)p(p+1)$, and this is the sum of the values of $l_1+\ldots+l_r$ for all branches of the curve; hence, as $l_s < s$, the number of points of the curve which are not ordinary is certainly $>2p+2$. Also, the least value which $l_1+\ldots+l_r$ can have, at a point which is not ordinary, arises when $l_1=l_2=\ldots=l_{r-1}=0$ and l_r (i.e. l_{p-1}) $=1$; there is then a stall, the osculating prime $x_{p-1}=0$ having p coincident intersections with the branch at the origin. If all the not-ordinary points of the curve are stalls, their number is $(p-1)p(p+1)$, and the s-th rank of the curve is $n_s=(p-1)(s+1)(s+2)$; in particular the class (i.e. the rank n_{p-2}) is $p(p-1)^2$. The simplest case is that of the general plane quartic curve $(p=3)$. The case $p=4$ arises by the intersection of a general quadric surface and a general cubic surface in ordinary space; the sextic curve is then of rank 18, of class 36, and has 60 stalls, where the osculating plane meets the curve in four coincident points.

Part II. Curves which are the complete intersection of two surfaces. It appears necessary to deal in some detail with the case of a curve in ordinary space, which is the complete intersection of two algebraic surfaces. We denote the orders of these surfaces by m and M, so that the order of the curve is given by $n=mM$. Further, we suppose the number, κ, of stationary points, and the number, i, of stationary tangents, to be given. We have expressed all the characters of the curve in terms of four of these, for instance n, n', r, p; if we assume a knowledge of n, κ and i, it will be sufficient to find one other character, and the rest, of those we have recognised, can then be computed. We put down the formulae with the inclusion of the number, δ, of actual double points, and of, τ, actual double tangents, of the curve; it is well known that, where two surfaces touch, at ordinary points of the surfaces, their curve of intersection has a double point; we shall suppose the double points of the curve considered to arise in this way, and similarly the cusps to arise from such contacts, at which the two directions of the curve of intersection coincide; in the most general case δ, κ, i and τ are all zero.

We can give a direct deduction either of h, or r, or n'; of these the first involves perhaps the most elementary considerations. We give, however, also a direct deduction of r, to illustrate a method of subsequent importance; and also an indication of the direct deduction of n'.

We prove that $h=\frac{1}{2}mM(m-1)(M-1)$, assuming the following theorem of elementary algebra: Let

$$f=a_0\lambda^m+(m,\,1)a_1\lambda^{m-1}+(m,\,2)a_2\lambda^{m-2}+\ldots+a_m,$$
$$F=A_0\lambda^M+(M,\,1)A_1\lambda^{M-1}+(M,\,2)A_2\lambda^{M-2}+\ldots+A_M,$$

be two polynomials in the variable λ, of respective orders m and M, written with binomial coefficients $(m,\,1)$, $(m,\,2)$, etc. Writing, momentarily, also p_i for the coefficient $(m,\,i)a_i$, and P_j for $(M,j)A_j$, denote by Δ_k the determinant, of $m+M-2k$ rows and columns,

$$\Delta_k=\begin{vmatrix} p_0, & p_1, & p_2, & . & . \\ 0\ , & p_0, & p_1, & . & . \\ 0\ , & 0\ , & p_0, & . & . \\ . & . & . & . & . \\ P_0, & P_1, & P_2, & . & . \\ 0\ , & P_0, & P_1, & . & . \\ . & . & . & . & . \end{vmatrix},$$

where there is first a set of $M-k$ rows, containing the elements $p_0, p_1, p_2, \ldots$, in order, beginning in turn in the first, second, ... columns, and then a set of $m-k$ rows, containing the elements P_0, P_1, P_2, ..., in order, beginning in turn in the first, second, ... columns, all other elements of the determinant being zero. Then, (i), a necessary and sufficient set of conditions for the polynomials f, F to have common a polynomial factor of order i in λ is the vanishing of the i determinants Δ_0, Δ_1, ..., Δ_{i-1}; while, (ii), the determinant Δ_k is unaltered if for a_r and A_s we put, respectively, the values $\bar{a}_r$, $\bar{A}_s$ given by

$$\bar{a}_r=a_r+(r,\,1)a_{r-1}\theta+(r,\,2)a_{r-2}\theta^2+\ldots+a_0\theta^r,$$
$$\bar{A}_s=A_s+(s,\,1)A_{s-1}\theta+(s,\,2)A_{s-2}\theta^2+\ldots+A_0\theta^s,$$

where θ is an arbitrary parameter. This last fact is equivalent to saying that Δ_k is reduced to zero by the operator

$$a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots+ma_{m-1}\frac{\partial}{\partial a_m}+A_0\frac{\partial}{\partial A_1}$$
$$+2A_1\frac{\partial}{\partial A_2}+\ldots+MA_{M-1}\frac{\partial}{\partial A_M}.$$

For (i), see Scheibner, *Leipziger Bericht.* XL, 1880, p. 1; Kronecker, *Werke*, II, 1881, p. 115; Netto, *Algebra*, I, 1896, p. 157. For (ii), see *Proc. Edinburgh Math. Soc.* II, 1930, p. 55.

Now suppose that $f_x^m=0$, $F_x^M=0$, are, in homogeneous variables x, y, z, t, the equations of two surfaces, of respective orders m and M. Taking a fixed point (ξ), and a point (x), substitute, in the equations of the surfaces, for x, y, z, t respectively, the values $\lambda\xi+x$, $\lambda\eta+y$, $\lambda\zeta+z$, $\lambda\tau+t$. We thus obtain, for the determination of the intersections of the line, joining (ξ) and (x), with the surfaces, the equations

$$\lambda^m f_\xi{}^m+(m,\ 1)\lambda^{m-1}f_\xi{}^{m-1}f_x+\ldots+f_x{}^m=0,$$
$$\lambda^M F_\xi{}^M+(M,\ 1)\lambda^{M-1}F_\xi{}^{M-1}F_x+\ldots+F_x{}^M=0;$$

these, as polynomials in λ, are of the forms considered above, with $a_i=f_\xi{}^{m-i}f_x{}^i$, $A_j=F_\xi{}^{M-j}F_x{}^j$, so that a_i, A_j are of respective orders i and j in (x, y, z, t).

The condition that the line joining (ξ) to (x) should meet the two surfaces in the same point, is that these two equations in λ should have a common root. This, however, is that (x) should satisfy the equation $\Delta_0=0$. Now, when (x) are current coordinates, the equation $\Delta_0=0$ represents a *cone*, with vertex at (ξ); for if, in $(\lambda\xi+x, \ldots, \lambda\tau+t)$, we put $(x+\theta\xi, \ldots, t+\theta\tau)$ for $(x, \ldots, t)$, this will be equivalent to considering the equations in λ which, in the notation used above, are

$$\bar{a}_0\lambda^m+(m,\ 1)\bar{a}_1\lambda^{m-1}+\ldots+\bar{a}_m=0,$$
$$\bar{A}_0\lambda^M+(M,\ 1)\bar{A}_1\lambda^{M-1}+\ldots+\bar{A}_M=0,$$

these being what are obtained from the original two by putting $\lambda+\theta$ in place of λ; the equation, $\bar{\Delta}_0=0$, obtained from these two equations in λ, is, however, we have said, the same as $\Delta_0=0$, whatever θ may be. Wherefore, $\Delta_0=0$, as an equation in (x), represents the cone joining (ξ) to the curve of intersection of the two surfaces. As this equation is, we easily see, isobaric in regard to the suffixes in a_i, A_j, and contains the term $a_0{}^M A_M{}^m$, it is of order mM in (x). By a similar argument, the equation $\Delta_1=0$ represents a cone, with vertex (ξ), of order $(m-1)(M-1)$. The lines common to $\Delta_0=0$, $\Delta_1=0$ are thence the lines from (ξ) which have two intersections with the curve of intersection of the surfaces; these lines are then the double lines of the cone $\Delta_0=0$, and their number is

$$\tfrac{1}{2}mM(m-1)(M-1).$$

If the two surfaces have a point of contact, the cone $\Delta_0=0$ will have a double generator through this point. But, unless the common tangent plane of the two surfaces, at this point, pass through the point (ξ), this generator, having a simple intersection with each surface at this point, will not meet the common curve of the surfaces in a further point, and will not lie on $\Delta_1=0$. Thus, no correction of the number $\frac{1}{2}mM(m-1)(M-1)$ arises for double points (or cusps) of the curve arising from contacts of the two surfaces; we have agreed not to consider other possible double points of the curve (as for instance when one surface passes through an actual conical point of the other surface). Hence we have

$$h=\tfrac{1}{2}mM(m-1)(M-1).$$

Ex. 1. For the quadric surfaces $f_x{}^2=0$, $F_x{}^2=0$, the equation $\Delta_1=0$ is

$$\begin{vmatrix} f_\xi{}^2, & f_\xi f_x \\ F_\xi{}^2, & F_\xi F_x \end{vmatrix}=0,$$

and represents the plane through (ξ), drawn to the line of intersection of the polar planes of (ξ) in regard to the quadric surfaces. This plane meets the quadric surfaces in two conics, in regard to which the polar lines of (ξ) are the same line; two of the common chords of these conics intersect in (ξ), these being the two chords of the quartic curve, of intersection of the two quadric surfaces, which can be drawn from (ξ). These lines are also the generators through (ξ) of a particular quadric surface, $f_x{}^2+kF_x{}^2=0$, which passes through (ξ).

Ex. 2. As an example of the algebraical theorem which has been used, the reader may prove that, if $f=a\lambda^2+b\lambda+c$, $F=p\lambda^3+q\lambda^2+r\lambda+s$, while the constants $P, Q, \dots, B$ are so chosen that

$$(P\lambda^2+Q\lambda+R)f+(A\lambda+B)F=\Delta_0,$$

then, with Δ_1 as before,

$$\begin{vmatrix} A, & B, & 0 \\ 0, & A, & B \\ P, & Q, & R \end{vmatrix}=-\Delta_1{}^4,$$

this being an identity not requiring $\Delta_0=0$.

Ex. 3. As a further example, prove that the invariant Δ_k, of the two polynomials of orders m and M in λ,

$$(p_0\lambda^{m-1}+\dots+p_{m-1})(\lambda-\theta),\quad (q_0\lambda^{M-1}+\dots+q_{M-1})(\lambda-\theta),$$

when multiplied by a certain determinant whose elements depend only on θ, is equal to the invariant Δ_{k-1}, of the polynomials, of orders $m-1$ and $M-1$, $p_0\lambda^{m-1}+\dots+p_{m-1}$, $q_0\lambda^{M-1}+\dots+q_{M-1}$.

Note. That the chords from an arbitrary point to the complete curve intersection of two surfaces of orders m and M, meet this curve on a surface of order $(m-1)(M-1)$, is proved in Salmon, *Geom. of Three Dimensions* (1882, p. 331). That the chords lie on a *cone* of this order, was proved by Valentiner, *Acta Math.* II, 1883, p. 191; and by Noether, *Berlin. Abh.* 1882, p. 27. A converse theorem was given by Halphen, *J. d. l'École polyt.* LII, 1883, p. 106, namely: If, for a curve in space, of order mM, the chords from an arbitrary point lie on a cone of order $(m-1)(M-1)$, and the curve itself does not lie on a surface of order less than the lesser of m and M, then the curve is the complete intersection of two surfaces of orders m and M.

Having obtained the formula $h=\frac{1}{2}mM(m-1)(M-1)$, if we use the formulae, previously found for an algebraic curve of order n,

$$\begin{aligned} n'+i&=3(r-n)+\kappa,\\ r&=n(n-1)-2h-2\delta-3\kappa,\\ p&=\tfrac{1}{2}(n-1)(n-2)-h-\delta-\kappa, \end{aligned}$$

we can deduce, for the curve which is the complete intersection of two surfaces of orders m and M, that $n=mM$ and

$$\begin{aligned} n'+i&=3mM(m+M-3)-6\delta-8\kappa,\\ r&=mM(m+M-2)-2\delta-3\kappa,\\ p-1&=\tfrac{1}{2}mM(m+M-4)-\delta-\kappa, \end{aligned}$$

and thence, using the general formulac which express i, κ', $\nu+\tau$, $\nu'+\tau$, $h+\delta$, $h'+\delta'$ in terms of n, n', r, p, we infer

$$\begin{aligned} i=0, \quad \kappa' &= 2mM(3m+3M-10)-3(4\delta+5\kappa), \\ \nu+\tau &= \tfrac{1}{2}\mu_2(\mu_2-2\rho-4)+\tfrac{1}{2}\rho_2^2+4\delta+\tfrac{11}{2}\kappa, \\ \nu'+\tau &= \tfrac{1}{2}\mu_2(\mu_2-2\rho-10)+\tfrac{1}{2}\rho_2^2+10\delta+\tfrac{27}{2}\kappa+4mM, \\ h'+\delta' &= \tfrac{1}{2}\mu_3(9\mu_3-12\sigma-22)+2\sigma^2+22\delta+28\kappa+\tfrac{5}{2}mM, \end{aligned}$$

where, for brevity, $\mu_2=mM(m+M-2)$, $\mu_3=mM(m+M-3)$, $\rho=2\delta+3\kappa$, $\sigma=3\delta+4\kappa$.

We next develop a direct verification of the formula

$$r=mM(m+M-2)-2\delta-3\kappa.$$

Denote the equations of the two surfaces now by $u=0$, $U=0$; let λ, μ be two arbitrary planes, say

$$a_1x+b_1y+c_1z+d_1t=0, \quad a_2x+b_2y+c_2z+d_2t=0.$$

Consider the locus of a point (x, y, z, t) whose polar planes, taken in regard to the two surfaces, meet in a line which intersects the line (λ, μ). The equation of this locus is $(u, U, \lambda, \mu)=0$, where

$$(u, U, \lambda, \mu)=\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ U_1, & U_2, & U_3, & U_4 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \end{vmatrix},$$

u_1 denoting $\partial u/\partial x$, etc. This surface is of order $m+M-2$. A point common to the surfaces $u=0$, $U=0$, at which both these have definite, different, tangent planes, will be on the surface $(u, U, \lambda, \mu)=0$ if, and only if, the tangent line of the curve (u, U), at this point, meets the line (λ, μ). Thus, if the surfaces $u=0$, $U=0$ have ordinary points with different tangent planes at all points of the curve (u, U), then this curve is of rank $mM(m+M-2)$. The determinant (u, U, λ, μ) vanishes, however, at any point where the surfaces $u=0$, $U=0$ touch, and the tangent lines (or line) of the curve (u, U), at such a contact, do not, as a rule, meet the chosen line (λ, μ). It can be proved that the tangent plane of the surface $(u, U, \lambda, \mu)=0$, at such a contact, is definite in general, and meets both branches of the curve (u, U) at this point, when these are distinct, in a single point; but contains the tangent line of the curve (u, U) at this point, when this curve has a cusp. Such a contact thus diminishes the previous expression for the rank of the curve (u, U) by 2, or by 3, in these two cases. Thus we reach the result above given for the rank, r, of the curve, if we consider, as we have agreed to do, only double points or cusps of the curve

(u, U) which arise at simple contacts of the surfaces. It may be shewn, however, that double points of the curve (u, U) which arise by simple passage of, say, $u=0$, through a conical node of $U=0$, may be supposed to be included in the formula.

It is important to notice also, that, if the curve of intersection of the surfaces $u=0$, $U=0$ break up into two curves, one of order n_1, with δ_1 double points and κ_1 cusps, at ordinary contacts of $u=0$, $U=0$, the other similarly with n_2, δ_2, κ_2, these curves having t points of intersection at ordinary contacts of the surfaces, then the rank of one curve (n_1) is given by $r_1=n_1(m+M-2)-2\delta_1-3\kappa_1-t$; the proof is precisely as in the simpler case above.

We now indicate a direct verification of the equation

$$n'+i=3mM(m+M-3)-6\delta-8\kappa,$$

arising from a formula given by Hesse for the equation of the osculating plane at an ordinary point (x, y, z, t) of the curve (u, U). Using (x', y', z', t') for current coordinates, the equation of this osculating plane is

$$\omega(x'U_1+y'U_2+z'U_3+t'U_4)=\Omega(x'u_1+y'u_2+z'u_3+t'u_4),$$

where, as before, $u_1=\partial u/\partial x$, etc., and ω denotes the determinant

$$\omega=(m-1)^{-2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & U_1 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & U_2 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & U_3 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & U_4 \\ U_1, & U_2, & U_3, & U_4, & 0 \end{vmatrix},$$

wherein $u_{12}=\partial^2 u/\partial x\partial y$, etc., while Ω denotes what ω becomes when m, u, U are replaced by M, U, u. To verify that this is so, we may first verify that the given equation remains of the same form after any homogeneous linear (so-called projective) transformation of the coordinates; then we can suppose the coordinates to be three non-homogeneous coordinates x, y, z, and a fourth, t, equal to 1. By means of Euler's formulae for homogeneous polynomials, it then follows that, on the curve (u, U), the function ω may be replaced by

$$u_{11}(u_2U_3-u_3U_2)^2+\ldots+2u_{23}(u_3U_1-u_1U_3)(u_1U_2-u_2U_1)+\ldots;$$

if then we regard x, y, z on this curve as functions of a parameter, θ, it follows hence that ω may be replaced by

$$u_{11}\left(\frac{dx}{d\theta}\right)^2+\ldots+2u_{23}\frac{dy}{d\theta}\frac{dz}{d\theta}+\ldots,$$

and hence also by

$$u_1\frac{d^2x}{d\theta^2}+u_2\frac{d^2y}{d\theta^2}+u_3\frac{d^2z}{d\theta^2},$$

because $u_1 dx/d\theta + u_2 dy/d\theta + u_3 dz/d\theta$ vanishes on the curve. And a similar replacement is possible also for Ω. Wherefore, the plane given by the equation, which obviously contains the tangent line of the curve (u, U), contains also the point

$$\left(\frac{d^2x}{d\theta^2}, \frac{d^2y}{d\theta^2}, \frac{d^2z}{d\theta^2}, 0\right),$$

and is therefore the osculating plane.

Now, in the original form, the function ω is a polynomial of order $3(m-2)+2(M-1)$ in x, y, z, t; the whole equation is thus of order $3(m+M-3)$ in (x, y, z, t). The ordinary points of the curve (u, U) for which the osculating plane passes through an arbitrary point (x', y', z', t'), are thus upon a surface of this order; the number of such points is thus $3mM(m+M-3)$, save for a correction for points of the curve which are not ordinary. To complete the verification of the formula under discussion it must be shewn that this correction is one at an inflexion of the curve (u, U), six at a double point, and eight at a cusp.

To examine the case of a double point or cusp arising from a simple contact of the surfaces $u=0$, $U=0$, we may take non-homogeneous coordinates x, y, z, with origin at the point of contact, so that the equations of the surfaces are

$$u \equiv z + \tfrac{1}{2}(ax^2 + 2hxy + by^2) + gxz + fyz + \tfrac{1}{2}cz^2 + \ldots = 0,$$

$$U \equiv z + \tfrac{1}{2}(Ax^2 + 2Hxy + By^2) + Gxz + Fyz + \tfrac{1}{2}Cz^2 + \ldots = 0;$$

then, in the equation of the osculating plane (with (x', y', z') as current coordinates), namely

$$\omega[(x'-x)U_1 + (y'-y)U_2 + (z'-z)U_3] \\ = \Omega\,[(x'-x)u_1 + (y'-y)u_2 + (z'-z)u_3],$$

the terms of the lowest order in x, y, z are those multiplying z', and therein the lowest terms are

$$\Delta \equiv (a-A)l^2 + (b-B)m^2 + (c-C)n^2 \\ + 2(f-F)mn + 2(g-G)nl + 2(h-H)lm,$$

where $l = u_2U_3 - u_3U_2$, $m = u_3U_1 - u_1U_3$, $n = u_1U_2 - u_2U_1$.

For a contact of the surfaces in which the curve (u, U) has two branches, if we choose the axes so that the tangent line of one branch is $z=0$, $y=0$, we shall have $A=a$, and this branch will be given, in terms of a parameter θ, by expressions of the forms $x=\theta$, $y=q\theta^2+\ldots$, $z=r\theta^2+\ldots$. Along this branch, then, l, m, n are respectively proportional to θ, θ^2, θ^3, and the terms of lowest order in Δ are the terms in ln, lm, which are proportional to θ^3. The surface arising from the equation of the osculating plane thus meets this branch in three coincident points at the origin; and meets the curve in all in six points.

When, however, the curve has a cusp, the branch of the curve is given by expressions of the forms $x=\theta^2$, $y=q\theta^3+\ldots$, $z=r\theta^4+\ldots$, and l, m, n are ultimately proportional, respectively, to θ^3, θ^4, θ^5. In this case we have $h=H$ beside $a=A$. The lowest terms in Δ are the terms in m^2 and

nl, which are proportional to θ^8; thus there are now eight intersections with the curve at the origin.

A similar examination may be made for an inflexion of the curve; it is obvious however that the plane joining (x', y', z', t') to the inflexional tangent has three intersections with the curve at the inflexion; thus this point counts once. The formula $n'+i=3mM(m+M-3)-6\delta-8\kappa$ is thus verified, when only double points and cusps which arise from simple contacts of the two surfaces are taken into account.

Ex. Prove that the osculating plane at (x, y, z, t) on the curve of intersection of the quadric surfaces whose equations are

$$ax'^2+by'^2+cz'^2+dt'^2=0, \quad Ax'^2+By'^2+Cz'^2+Dt'^2=0,$$

is expressed by

$$l'mnx^3x'+m'nly^3y'+n'lmz^3z'+l'm'n't^3t'=0,$$

where $\quad l=bC-cB, \quad l'=dA-aD,$ etc.

Part III. Curves which are the partial intersection of two surfaces. Consider now the case when the two surfaces $u=0$, $U=0$, meet in a curve of order n_1, itself irreducible and simple on both surfaces, this being only part of the intersection, the residual part being another curve, of order n_2, also irreducible and simple on both surfaces.

We can put down equations by which, when the characters of the first curve (n_1) are known, and the surfaces do not touch at points of the second curve (n_2), the characters of this second curve can be found, as well as the number of intersections of the second curve with the first. Let the usual numbers for the two curves be $r_1, \delta_1, \kappa_1, \ldots$, and $r_2, \delta_2, \kappa_2, \ldots$, and the number of their intersections be t. We clearly have $n_1+n_2=mM$, and it has already been remarked (p. 206), considering the surface $(u, U, \lambda, \mu)=0$, that

$$r_1=n_1(m+M-2)-2\delta_1-3\kappa_1-t, \quad r_2=n_2(m+M-2)-2\delta_2-3\kappa_2-t.$$

By a general property of curves, given in Part I, we have, however,

$$r_1+\kappa_1=2n_1+2p_1-2, \qquad r_2+\kappa_2=2n_2+2p_2-2,$$
$$h_1+p_1+\delta_1+\kappa_1=\tfrac{1}{2}(n_1-1)(n_1-2), \quad h_2+p_2+\delta_2+\kappa_2=\tfrac{1}{2}(n_2-1)(n_2-2);$$

thus we have

$$t=n_1(m+M-n_1-1)+2h_1, \ =n_1(m+M-4)-2\delta_1-2\kappa_1-(2p_1-2),$$

with precisely similar forms for t in terms of $n_2, h_2, \delta_2, \kappa_2, p_2$. Comparing the two forms of t we deduce

$$h_2-h_1=\tfrac{1}{2}(n_2-n_1)(m-1)(M-1),$$
$$r_2-r_1=(n_2-n_1)(m+M-2)-(2\delta_2+3\kappa_2-2\delta_1-3\kappa_1),$$
$$p_2+\delta_2+\kappa_2-(p_1+\delta_1+\kappa_1)=\tfrac{1}{2}(n_2-n_1)(m+M-4),$$

and, for the second curve to be entirely known when the first is known, the values of δ_2 and κ_2 are required.

The equations also give* the results

$$h_1+h_2+n_1n_2-t=\tfrac{1}{2}mM(m-1)(M-1),$$
$$p_1+p_2+t-1=1+\tfrac{1}{2}mM(m+M-4)-(\delta_1+\kappa_1+\delta_2+\kappa_2).$$

Of these equations the former is obvious directly if we recall what was found above (p. 203) for the cones Δ_0 and Δ_1; the cone Δ_0 is formed by the lines from an arbitrary point (ξ, η, ζ, τ) to the points of the curve (or curves) of intersection of the surfaces $u=0$, $U=0$, and the cone Δ_1 by the lines from (ξ, η, ζ, τ) upon each of which two of the intersections with $u=0$ coincide with two of the intersections with $U=0$. The argument used does not assume that the curve (u, U) is irreducible, and is applicable to the case now under consideration. Upon the cone Δ_1 will be the h_1 proper chords from (ξ, η, ζ, τ) to the first curve (n_1), also the h_2 proper chords to the second curve (n_2); but there will also be upon the cone Δ_1 the lines from (ξ, η, ζ, τ) which meet both (n_1) and (n_2) in two different points, whose number is n_1n_2-t (the cones joining (ξ) to the curves being of orders n_1 and n_2). Thus the former of the two equations is clear; it may if desired be replaced by the two equations

$$n_1n_2-t=n_1(m-1)(M-1)-2h_1=n_2(m-1)(M-1)-2h_2.$$

The preceding formulae have been obtained, for simplicity, with the supposition that the curve (n_2) is irreducible. With suitable modifications this condition may be omitted.

Ex. 1. Suppose that, through the curve (n_1), a further surface, of order k is drawn; if the curve (n_2) is not degenerate, it will be met by this surface in points, not on (n_1), whose number is kn_2-t. Various forms for this are

$$\begin{aligned} kn_2-t &= mMk-n_1(m+M+k-4)+2p_1-2+2\delta_1+2\kappa_1,\\ &= mMk-n_1(m+M+k-2)+r_1+2\delta_1+3\kappa_1,\\ &= mMk-n_1(m+M+k)+n_1(n_1+1)-2h_1.\end{aligned}$$

---

* For any two curves, not assumed to form together the complete intersection of two surfaces, it may be proved, by elementary algebra, that the equations

$$p_1=\tfrac{1}{2}(n_1-1)(n_1-2)-h_1-[\delta_1+\tfrac{1}{2}\Sigma s_1(s_1-1)],$$
$$p_2=\tfrac{1}{2}(n_2-1)(n_2-2)-h_2-[\delta_2+\tfrac{1}{2}\Sigma s_2(s_2-1)],$$
$$p=\tfrac{1}{2}(n_1+n_2-1)(n_1+n_2-2)-(h_1+h_2+k)-[\delta_1+\delta_2+\tfrac{1}{2}\Sigma(s_1+s_2)(s_1+s_2-1)],$$
$$n_1n_2=t+k+\Sigma s_1s_2,$$

lead to

$$p=p_1+p_2+t-1,$$

as the expression for the genus of a curve which breaks up into two curves with t intersections. Notice that there is no term, in the definition of p, corresponding to these intersections. Cf. Picard et Simart, *Fonctions algéb.* II, p. 106. See Ex. 11, p. 215, below.

And it can be shewn that a rational curve of order r, in space of r dimensions, taken with p chords of this curve, is a possible form of the degeneration of a curve of order $r+p$, of genus p. Cf. Enriques e Chisini, *Teoria geometrica*, III, p. 396. For the connexion of this result with the theory of connected polygons of lines, see Severi, *Geom. Algeb.* 1921, p. 373 (Anhang G).

These various results, though not arising directly from the text, will be of subsequent interest.

Taking the last form we may say: If three surfaces, of orders m, M, k, have common a curve of order n, with h apparent double points, this curve absorbs a number $n(m+M+k-n-1)+2h$ of the mMk points generally common to three surfaces of these orders. This is often called the *point equivalence* of the curve for these surfaces; it depends, we see, only upon the sum of the orders of these surfaces; and it is assumed that these surfaces have no further common curve.

It may be proved that the formula for the point equivalence of the curve (n) remains true when this curve is itself an aggregate of two curves (ν_1), (ν_2), so that $n=\nu_1+\nu_2$, provided h be replaced by the number of chords to this composite curve from an arbitrary point, that is by $\eta_1+\eta_2+\nu_1\nu_2-\tau$, where η_1, η_2 are the numbers of apparent double points for the component curves, and τ the number of their intersections. Putting

$$\sigma=m+M+k,\quad \beta_1=\nu_1^2+\nu_1-2\eta_1,\quad \beta_2=\nu_2^2+\nu_2-2\eta_2,\quad \beta_{12}=\beta_1+\beta_2+2\tau,$$

this is the same as saying that the equivalences of the curves (ν_1), (ν_2), $(\nu_1+\nu_2)$ are respectively $\nu_1\sigma-\beta_1$, $\nu_2\sigma-\beta_2$, $(\nu_1+\nu_2)\sigma-\beta_{12}$.

Ex. 2. The formula $t=n(m+M-n-1)+2h$, for the number of contacts, on a given curve (n, h), of two surfaces $u=0$, $U=0$, drawn through this curve, can be obtained very simply in the case when the curve is the complete intersection of two surfaces $u_0=0$, $U_0=0$ and it is assumed that u, U are capable of the forms

$$u=Au_0+BU_0,\quad U=Cu_0+DU_0,$$

in which A, B, C, D are polynomials (Salmon, *Higher Algebra*, 1885, p. 297, where the so-called *rank* is what we have denoted by $r+2n+2\delta+3\kappa$).

For then, the complete intersection of $u=0$, $U=0$ consists, beside the given curve (u_0, U_0), of a part lying on the surface

$$\begin{vmatrix} A, & B \\ C, & D \end{vmatrix}=0;$$

if u_0, U_0 be of orders m_0, M_0, this surface is of order $m-m_0+M-M_0$, and meets the given curve, which is of order n, $=m_0M_0$, in $n(M+m)-n(m_0+M_0)$ points; as $h=\frac{1}{2}m_0M_0(m_0-1)(M_0-1)$, this is the same as $n(M+m)-n(n+1)+2h$.

Hence, also, a third surface, of order k, through the curve (u_0, U_0), intersects the residual curve $(u, U)-(u_0, U_0)$, which is of order $mM-m_0M_0$, in a number of points, not on (u_0, U_0), given by

$$k(mM-m_0M_0)-m_0M_0(m+M-m_0-M_0),$$

which, as before, is $mMk-n(m+M+k)+n(n+1)-2h$.

Ex. 3. If, in space of four dimensions, there be drawn, through a curve of order n, three primals of orders m, M, N, and these intersect in a further curve, this last intersects the former in $n(m+M+N)-\beta$ points, where β depends only on the original curve. This can be easily proved, in Salmon's manner, in the case where the original curve is the complete intersection of three primals in terms of which the primals (m), (M), (N) are linearly expressible. It will be proved below (in Vol. VI), that, when the original curve is without multiple points, the value of β is $3n+r$, where r, the rank of the curve, is the number of its tangent lines which meet an arbitrary plane. It follows at once that the point equivalence of the given curve for four primals, of orders m, M, N, k, passing through it, is $n\sigma-\beta$, where $\sigma=m+M+N+k$. It can be shewn, as in Ex. 1, that this formula for the point equivalence remains valid when the original curve is composite, if for β we put the sum of the values of β for the separate components increased by twice the total number of intersections of the components. It follows easily also that, if the curves (n, ν) and (n', ν') be, together, the complete intersection of the three primals (m), (M), (N), then $n(m+M+N)-3n-r=n'(m+M+N)-3n'-r'$.

Ex. 4. As a consequence of Ex. 3, prove that, if ten lines in space of four dimensions have in all fifteen intersections, the point equivalence of these lines, for four cubic primals drawn through them, is 60. Deduce from this that the number of quadric surfaces, in space of three dimensions, which pass through 5 given points and touch 4 given planes is 21 (Salmon, *loc. cit.*).

Ex. 5. It has been stated, at the beginning of this chapter, that there exist curves in space of three dimensions, such that no set of three surfaces exists, passing through the curve, without residual point intersections. We give two examples, one due to Vahlen (*Crelle*, CVIII, 1891, p. 346), of a rational quintic curve, the other, given by Enriques-Chisini (*Teoria geometrica*, III, p. 515), of a quintic curve of genus 1. A quintic curve is evidently not the complete intersection of two surfaces.

A rational quintic curve is in fact obtained by the residual intersection with a quadric surface of a quartic surface drawn through three skew generators of the quadric surface; but this curve has an infinite number of quadrisecants, or transversals meeting it in four points (generators of the quadric surface). There exists, however, a rational quintic curve with only one quadrisecant, and it is this curve which we proceed to consider. It may be defined as the residual intersection of two cubic surfaces which are drawn through a rational cubic curve and through a line which does not meet the cubic curve. For, a cubic surface contains such a rational cubic curve if it contain ten points of this curve, and contains a line if it contain four points of this line; and a cubic surface has twenty coefficients in its equation; we infer then that there are six linearly independent cubic surfaces through the cubic curve and the line. Take two of these cubic surfaces, and the quintic curve which is their remaining intersection. To see that this curve is rational, remark that an arbitrary plane through the line meets these two cubic surfaces, further, in conics, with four intersections, not generally on the line; and of these three are on the cubic curve; so that an arbitrary plane through the line gives one point on the quintic curve; and conversely. The curve is thus expressible by the parameter which determines the plane. Also, the quintic curve meets the line in four points, these, with the point of the curve on a plane through the line, being the five intersections with the plane. The quintic curve does not lie on a quadric surface, since, else, the quadrisecant of the curve would lie thereon, and, therefore, all other generators of the quadric surface of the same system would be quadrisecants of the curve (as we easily see); these generators, and hence the quadric surface, would then be upon every cubic surface containing the curve. Further, the curve will have no other quadrisecant, since this would also lie on all cubic surfaces containing the curve.

The quintic curve has a rank r given by $r=2n+2p-2$, or 8; and the chords from an arbitrary point are in number given by

$$h=\tfrac{1}{2}(n-1)(n-2)=6.$$

Thus the point equivalence of the curve for surfaces of order m, M, k, passing through it, is $5\sigma-\beta$, where $\sigma=m+M+k$, and $\beta=r+2n$, $=n(n+1)-2h$, $=18$. As the curve does not lie on a quadric surface, we can suppose $m=\lambda+3$, $M=\mu+3$, $k=\nu+3$, where λ, μ, ν are not negative; the point equivalence is thus $5(\lambda+\mu+\nu)+27$. If the three surfaces have no further common curve, they intersect then in

$$(\lambda+3)(\mu+3)(\nu+3)-5(\lambda+\mu+\nu)-27$$

further points; this is easily seen to be

$$\lambda\mu\nu+3(\mu\nu+\nu\lambda+\lambda\mu)+4(\lambda+\mu+\nu),$$

and can only be zero if λ, μ, ν be all zero. While, in this case, the three cubic surfaces would have further points in common, since they would all contain the quadrisecant of the curve.

It is not possible then to put three surfaces through the rational quintic curve considered, which shall have no further points in common.

Next consider the quintic curve which is the residual intersection of two cubic surfaces drawn through a rational quartic curve. The formula $p_2-p_1=\frac{1}{2}(n_2-n_1)(m+M-4)$ shews that this quintic curve is of genus 1. Hence it can be proved that this curve does not lie on a quadric surface, on which existing quintic curves are either rational or of genus 2 (see below, Ex. 7). But three surfaces of orders $\lambda+3$, $\mu+3$, $\nu+3$ passing through the quintic curve, if without further common curve, intersect further, we easily see, in $(\lambda+3)(\mu+3)(\nu+3)-5(\lambda+\mu+\nu)-25$ points, which is $\lambda\mu\nu+3(\mu\nu+\nu\lambda+\lambda\mu)+4(\lambda+\mu+\nu)+2$, and is at least 2. It is not possible then to put three surfaces through this elliptic quintic curve to have no further common points.

Ex. 6. Prove that the rational quintic curve of the last Ex., the residual intersection of two cubic surfaces passing through a line and a rational cubic curve, beside meeting the line in 4 points, meets the cubic curve in 8 points. For these three curves, the numbers of chords through an arbitrary point are, respectively, $h_1=0$ (for the line), $h_2=1$ (for the cubic curve), and $h_3=\frac{1}{2}(5-1)(5-2)$ (for the quintic curve), or $h_3=6$. The orders are $n_1=1$, $n_2=3$, $n_3=5$. The numbers of intersections of pairs of the curves are $t_{23}=8$, $t_{31}=4$, $t_{12}=0$. Hence we have an example of the formula

$$h_1+h_2+h_3+n_2n_3+n_3n_1+n_1n_2-t_{23}-t_{31}-t_{12}=\tfrac{1}{2}mM(m+M-2),$$

(for $m=M=3$), which generalises the formula proved in the text for two curves (see below, Ex. 11).

Ex. 7. We may enumerate the simpler cases of existing curves:

A general cubic curve, in space of three dimensions, is the residual intersection of two quadric surfaces having a common generator, in infinitely many ways; and the curve is rational.

A quartic curve is either, (i), the intersection of two general quadric surfaces; it is then of genus 1, and meets every generator of any quadric surface, containing the curve, in two points; or, (ii), the intersection of two quadric surfaces which have a point of contact; then the curve has a double point, and is rational; or, (iii), the residual intersection with a quadric surface of a cubic surface containing two skew generators of the quadric surface; then the curve is rational. Either of the rational quartic curves is obtainable by projection from a rational quartic curve in space of four dimensions.

For quintic curves, two rational curves have been spoken of in Ex. 5; and also a curve of genus 1, the residual intersection of two cubic surfaces through a rational (non-singular) quartic curve. For this last case, prove that the quartic and quintic curves meet in 10 points; and that the quadric surface described through 9 of these points contains the quartic curve entirely, thus meeting the quintic curve further only in the tenth point. A quintic curve of genus 2 is obtainable as the residual intersection of a quadric surface with a cubic surface drawn through one of its generators. Prove that there are no other (non-singular) quintic curves, not lying in a plane.

It will be proved below (Part VI, p. 234) that, in space of three dimensions, no algebraic curve of order n exists with genus greater than $\frac{1}{4}(n-2)^2$ or $\frac{1}{4}(n-1)(n-3)$, according as n is even or odd. Further, Noether has proved that, upon a given surface, the greatest genus possible, for curves of given order, arises for curves which, if not complete intersections of this surface with other surfaces, are part of a complete intersection of which the other part (possibly composite) lies in a plane. When n is odd, there exists a curve upon a quadric surface meeting all generators of one system in $\frac{1}{2}(n-1)$ points, and all generators of the other system in $\frac{1}{2}(n+1)$ points, and this curve has the genus $\frac{1}{4}(n-1)(n-3)$; when n is even, there exists a curve of genus $\frac{1}{4}(n-2)^2$ upon a quadric surface, meeting every generator in $\frac{1}{2}n$ points (see the following example). Conversely, a curve of order n, in space of three dimensions, with the maximum possible genus, necessarily lies on a quadric surface. For curves of order n on a cubic surface, however, the maximum possible genus is $\frac{1}{6}(n-1)(n-2)$, or $\frac{1}{6}(n^2-3n+6)$, according to the form of n.

An enumeration of existing curves is given to a high order by Noether ("Zur Grundlegung u.s.w."), *Berl. Abh.* 1882 (also, *Crelle's Journ.* XCIII, 1882). To order 6 an enumeration is given in Pascal's *Repertorium*, II, 2 (Leipzig, 1922), p. 932. See also Halphen, *J. école polyt.* LII, 1882; and Valentiner, *Acta Math.* II, 1883.

Ex. 8. Consider algebraic curves lying on a quadric surface. Such a curve meets all generators of the surface of one system (a) in the same number of points, say in α points, and all generators of the other system (b) also in the same number, say β, of points, the order of the curve being $\alpha+\beta$. This is easily seen by considering the intersections of the curve with a general plane through a generator. Suppose $\alpha \geqslant \beta$; then the surfaces of lowest order, other than the quadric surface, which contain the curve, are of order α; these are surfaces containing $\alpha-\beta$ of the (a) generators of the surface. The general surface of order α through $\alpha-\beta$ such generators, gives a system of "coresidual" curves on the quadric surface, all of order α; each of these curves meets every (a) generator in α points, and every (b) generator in β points. A definite curve of the system is determined by the prescription of passing through $\alpha+\beta+\alpha\beta$ given points of the quadric surface; a surface of order α passes through the curve and $\alpha-\beta$ (a) generators, arbitrarily chosen. More generally, it may be shewn that, in the equation of a surface of order m, so far as its intersection with the quadric surface is concerned, there are effectively $(m+1)^2$ homogeneous coefficients; thus a surface of order α, through $\alpha-\beta$ (a) generators, gives a residual curve of intersection, with the quadric surface, which is determined by $(\alpha+1)^2-1-(\alpha+1)(\alpha-\beta)$, or $\alpha+\beta+\alpha\beta$ points. A particular curve, of the system of coresidual curves of order $\alpha+\beta$ spoken of, is constituted by β generators of the (a) system, together with α generators of the (b) system; thus the curves of the system may be said to be $\equiv \beta u+\alpha v$, where u, v denote generators respectively of the (a) system and the (b) system; we may also speak of a curve of the system as an (α, β) curve. The genus, p, of the curves of the system is given by $p=(\alpha-1)(\beta-1)$, as may be proved, for example, by projecting the curve into a plane curve with an α-fold and a β-fold multiple point. With this system of curves is associated another system, called the *adjoint system* of this; this is the $(\alpha-2, \beta-2)$ system; the curves of the adjoint system cut, upon a curve of the original system, a series of points, of freedom $p-1$, of which every set consists of $2p-2$ points. Any curve of the system (α, β) meets any curve of the system (α', β') in $\alpha\beta'+\alpha'\beta$ points, as we may see, for example, by taking curves consisting of sets of generators; in particular this is $2p-2$ when $\alpha'=\alpha-2$, $\beta'=\beta-2$; and a curve of the system $(\alpha-2, \beta-2)$ can be described through

$$(\alpha-2+1)(\beta-2+1)-1,$$

or $p-1$ arbitrary points. Thus, on any curve of the (α, β) system, the complete canonical series is determined by the curves of the adjoint system. Further, two curves of the (α, β) system cut in $2\alpha\beta$ points; and, when $\beta=2$, these curves are hyperelliptic. And, it may be noticed, in anticipation of a general result which we reach later, the freedom, $\alpha+\beta+\alpha\beta$, of the system of (α, β) curves, is $n-p+1$, where n, $=2\alpha\beta$, is the number of intersections of any two curves of the system.

For curves on quadric surfaces see Halphen, *Bull. Soc. Math. d. France*, I, 1872, p. 19.

Ex. 9. A theory almost as simple holds for curves upon a cubic surface. But the expression of a system of coresidual curves thereon is not by two elements (as the generators, u, v in the preceding example), but by seven elements, consisting of six skew lines of the surface, together with one rational cubic curve thereon which does not meet any of the six lines. The expression is then of the form $\lambda u-\lambda_1 a_1-\dots-\lambda_6 a_6$, where u denotes the cubic curve, $a_1, \dots, a_6$ denote the lines, and $\lambda, \lambda_1, \dots, \lambda_6$ are integers. Two curves of this system intersect in n points given by

$$n=\lambda^2-\lambda_1{}^2-\dots-\lambda_6{}^2;$$

any curve of the system is of order m given by $m=3\lambda-\lambda_1-\dots-\lambda_6$, and of genus p given by $p=1+\frac{1}{2}(n-m)$, while any particular curve of the system is determined by passing through r points, given by $r=\frac{1}{2}(n+m)$, which is $n-p+1$. More generally, two curves, respectively of the systems $(\lambda, \lambda_1, \dots, \lambda_6)$ and $(\lambda', \lambda_1', \dots, \lambda_6')$, intersect in a number of points given by $\lambda\lambda'-\lambda_1\lambda_1'-\dots-\lambda_6\lambda_6'$. For the system $(\lambda-3, \lambda_1-1, \dots, \lambda_6-1)$, the freedom r', given by $\frac{1}{2}(n'+m')$, is easily seen to be $\frac{1}{2}(n-m)$ or $p-1$, while a curve of this system meets a curve of the original system in $2p-2$ points. This system is the *adjoint system* of the original, determining the complete canonical series on any curve of the original system.

Proofs of these statements are easily obtained by considering the familiar representation of the cubic surface upon a plane, whereon cubic curves through six fundamental points represent the plane sections of the cubic surface, and the lines represent rational cubic curves on the cubic surface. Or, arguing directly on the cubic surface, a surface of order r, so far as its intersection with the cubic surface is concerned, may be proved to have, in its equation, a number of homogeneous coefficients given by $\frac{1}{2}(3r^2+3r+2)$; while, through any curve of order m lying on the cubic surface, there can be drawn a surface whose order is the greatest integer in $2m/3$. For this, and the preceding example, a note in the *Proc. Lond. Math. Soc.* XI, 1912, pp. 286, 290, may be consulted.

Ex. 10. Any curve in space of three dimensions may be defined as the partial intersection of a cone, of equation $f(x, y, t)=0$, and a surface with equation of the form $z\phi(x, y, t)=\psi(x, y, t)$, where f, ϕ, ψ are homogeneous in x, y, t. The cone $\phi(x, y, t)=0$ may be taken arbitrarily, save that it must contain the chords of the curve drawn from the (general) point $(0, 0, 1, 0)$. The common generators of the cones $f(x, y, t)=0$, $\phi(x, y, t)=0$, together with the curve, are the complete intersection of the cone $f(x, y, t)=0$ with the surface $z\phi(x, y, t)-\psi(x, y, t)=0$. In general, the curve not passing through $(0, 0, 1, 0)$, the common generators of $f=0$, $\phi=0$ will also lie on the cone $\psi=0$. The surface $z\phi-\psi=0$, having a point at $(0, 0, 1, 0)$ of multiplicity one less than its order, was called by Cayley a *Monoid* (*Papers*, V, 1862, p. 8).

Ex. 11. As has already been indicated in a simple example (Ex. 6, p. 212), the formula $h_1+h_2+n_1n_2-t=\frac{1}{2}mM(m+M-2)$ can be extended to the case when the curve (u, U) breaks up into any number of curves, its general form being $\Sigma h_i+\Sigma n_in_j-\Sigma t_{ij}=\frac{1}{2}mM(m+M-2)$, where i, j refer to any two different component curves, these having t_{ij} intersections; the direct proof for two curves applies to the general case.

For two component curves, the formula (p. 209)

$$p_1+p_2+\delta_1+\kappa_1+\delta_2+\kappa_2+t-1=1+\tfrac{1}{2}mM(m+M-4)$$

is at once deducible from the formula above for h_1+h_2, using $p_1=\frac{1}{2}(n_1-1)(n_1-2)-h_1-\delta_1-\kappa_1$, etc. This formula may be expressed by saying that, if, on a surface of order m, one surface of order M cuts an irreducible curve of genus p, with δ double points and κ cusps, and another surface, also of order M, cuts a couple of irreducible curves of genera p_1, p_2, with respectively δ_1, κ_1 and δ_2, κ_2 multiple points, and t intersections, then $p-1+\delta+\kappa=p_1-1+\delta_1+\kappa_1+p_2-1+\delta_2+\kappa_2+t$; and, as we see, if, instead of two irreducible curves, we have several, the second member is to be replaced by $\Sigma(p_i-1+\delta_i+\kappa_i)+\Sigma t_{ij}$.

In view of subsequent developments we consider this further, assuming also a familiarity with the notion of the completeness of a linear series upon a curve in space, which follows from its (1, 1) correspondence with a plane curve, for which the notion has been explained (p. 66, above). Suppose that the surfaces $u=0$, $U=0$, drawn through the irreducible curve (n_1), intersect again, not in a single irreducible curve of order (n_2), as on p. 208 above, but in an aggregate of curves, all irreducible, of orders n_{2i}, with multiple points $(\delta_{2i}, \kappa_{2i})$; we may put $n_2=\Sigma(n_{2i})$. Let the component curve (n_{2i}) meet the curve (n_1) in t_i points, and meet the curve (n_{2j}) in t_{ij} points. Using r_{2i}, h_{2i}, p_{2i} for the curve (n_{2i}), as were r_2, h_2, p_2 for the curve (n_2), we prove, as in the simple case,

$$r_{2i}=n_{2i}(m+M-2)-2\delta_{2i}-3\kappa_{2i}-t_i-\sum_j t_{ij},$$

$$r_{2i}+\kappa_{2i}=2n_{2i}+2p_{2i}-2,$$

$$h_{2i}+p_{2i}+\delta_{2i}+\kappa_{2i}=\tfrac{1}{2}(n_{2i}-1)(n_{2i}-2),$$

and hence $t=\Sigma t_i=\Sigma[n_{2i}(m+M-2)-2\delta_{2i}-3\kappa_{2i}-r_{2i}]-2\sum_{i,j} t_{ij}$, which is the same as

$$n_2(m+M-4)-(2p_2-2+2\delta_2+2\kappa_2),$$

if we put $\quad p_2-1+\delta_2+\kappa_2=\Sigma(p_{2i}-1+\delta_{2i}+\kappa_{2i})+\Sigma t_{ij}$.

Also we have, from the preceding equations,

$$n_{2i}(m+M-4)=2\delta_{2i}+2\kappa_{2i}+t_i+\sum_j t_{ij}+2p_{2i}-2.$$

Consider now the number of conditions in order that a surface of order $m+M-4$ should contain all the component curves (n_{2i}). Such a surface will contain the multiple points of each separate component, and its intersections with the other components. We therefore consider first the surfaces of order $m+M-4$ through the multiple points of all the components and through the mutual intersections. Upon the single curve (n_{2i}) such surfaces will have free intersections of number

$$n_{2i}(m+M-4)-2\delta_{2i}-2\kappa_{2i}-\sum_j t_{ij},$$

which is the same as $t_i+2p_{2i}-2$. Thus, if we suppose $t_i>0$, the series determined by these surfaces upon the curve (n_{2i}) is not special, though it may be incomplete; the freedom of this series is then of the form

$t_i+p_{2i}-2-\zeta_i$, with $\zeta_i \geqslant 0$; and the number of conditions for one of the surfaces of order $m+M-4$, under consideration, to contain the curve (n_{2i}) is one more than this freedom. The number of conditions for a surface of order $m+M-4$ to contain all the component curves (n_{2i}) is thus, if we allow for the possibility that the separate conditions are not independent, $\Sigma t_{ij}+\Sigma(\delta_{2i}+\kappa_{2i})-\epsilon+\Sigma(t_i+p_{2i}-1)-\zeta$, with $\epsilon \geqslant 0$, $\zeta \geqslant 0$; and this, with the notation employed above, is $t+p_2-1+\delta_2+\kappa_2-\sigma$, where $\sigma\ (=\epsilon+\zeta)$ is $\geqslant 0$. The number of surfaces of order $m+M-4$ containing the aggregate of component curves (n_{2i}) is then obtained by subtracting this number from $(m+M-1, 3)$.

Ex. 12. It may be proved that an algebraic curve of order ν, without multiple points, lying on a ruled surface of order n (or on a cone, but not through the vertex), and meeting every generator of the surface in k points, touches $(k-1)(2\nu-kn)$ generators, and has a genus π given by $2\pi-2=k(2p-2)+(k-1)(2\nu-nk)$, where p is the genus of the plane section of the ruled surface. This will be proved below (Vol. VI), as a simple application of the theory of correspondence; but a proof of the expression for π can be given which is an application of the ideas we have illustrated in Exx. 8, 9, in dealing with the curves lying on a quadric or a cubic surface. For it can be shewn that such a curve can be regarded as coresidual with an aggregate representable by $kP+(\nu-nk)G$, where P denotes a plane section of the surface, and G a generator. Assuming this fact, it follows that two curves (k, ν), (k', ν') intersect in a number of points given by

$$kk'(P, P)+[k'(\nu-nk)+k(\nu'-nk')](P, G)+(\nu-nk)(\nu'-nk')(G, G),$$

where (P, P), the number of intersections of two plane sections of the surface, is n, and (P, G), the number of intersections of a plane section with a generator, is 1, while, similarly, $(G, G)=0$. The number in question is thus $k\nu'+k'\nu-nkk'$; for example, for a quadric surface, putting $k=\alpha$, $\nu=\alpha+\beta$, in the notation used in Ex. 8, this gives the number, $\alpha\beta'+\alpha'\beta$, there found.

Applying now the formula, for the genus of a composite curve, given above in Ex. 11, namely $1+\Sigma(p_i-1)+\Sigma t_{ij}$, the genus of the composite component kP is $1+k(p-1)+\frac{1}{2}nk(k-1)$, the genus of the composite component $(\nu-nk)G$ is $1+(\nu-nk)(-1)$, and there is a contribution to the genus of the whole curve $kP+(\nu-nk)G$ arising from the intersections of kP and $(\nu-nk)G$, which is $k(\nu-nk)$; the genus of a curve $kP+(\nu-nk)G$, by the application of the same formula, is thus

$$\pi=1+k(p-1)+\tfrac{1}{2}nk(k-1)-(\nu-nk)+k(\nu-nk),$$

so that we have $2\pi-2=k(2p-2)+(k-1)(2\nu-nk)$, as was stated above. In connexion with this Ex. the reader may consult Severi, *Mem. Torino*, LIV, 1903, p. 23.

Two remarks may be added. (1), The formula for the number of intersections of the curves (k, ν), (k', ν'), on the ruled surface, would give, if there existed a curve with $k'=-2$, $\nu'=2p-2-n$, the value $2\pi-2-N$, where N, equal to $2k\nu-nk^2$, is the number of intersections of two curves of the system (k, ν). Let this number N be called the *grade* of the system (k, ν); for the plane sections, $(1, n)$, its value is n. Further let $2\pi-2-N$ be called the *canonical number* of the curves of the system (k, ν). Then the result is that the canonical number of any curve of a system is equal to the number of intersections of the curve with a certain fictitious curve for which $k'=-2$; the order of this fictitious curve (as of any curve on the surface) is equal to the number of its intersections with a plane section, equal then to the canonical number, $2p-2-n$, of the plane sections. The value of k' is the canonical number of a generator. Also, the canonical number of a curve $(k_1+k_2, \nu_1+\nu_2)$ is the sum of the canonical numbers of the curves (k_1, ν_1) and (k_2, ν_2). The second remark, (2), is that the above formula for π, in terms of ν and k, holds equally when the ruled surface is in space of any number of dimensions, and also holds when the curve (ν, k) has double points, in number d, if we replace $2\pi-2$ by $2\pi-2+2d$ (in case the surface be in space of four dimensions, however, such double points of the curve as arise at the so-called accidental double points of the surface (see below, Chap. IV in Vol. VI) are not to be included in d).

Part IV. Linear series upon a curve in space. It is clear, from the definition we have adopted for an algebraic curve, that there is a correspondence between the sets of a linear series, on the curve, and the sets of a linear series on a corresponding plane curve. In this statement it is to be understood that the linear series on the curve is determined by a linear system of surfaces, given by an equation of the form $\lambda\Phi+\lambda_1\Phi_1+\ldots+\lambda_r\Phi_r=0$, in which $\lambda, \ldots, \lambda_r$ are variable parameters; the surfaces $\Phi=0, \ldots, \Phi_r=0$ are supposed to be linearly independent on the curve, and may have intersections with the curve which are common to all. The linear series on the plane curve is similarly determined by a system of curves with an equation of the form $\lambda\phi+\lambda_1\phi_1+\ldots+\lambda_r\phi_r=0$. The substitution, in either of these equations, for the coordinates entering therein, of the values of these coordinates in terms of the coordinates appropriate to the other curve, leads from this equation to the other. The *freedom*, r, of the two corresponding series, is thus the same; and, from the (1, 1) correspondence between the points of the curves, the number of points which vary with the parameters $\lambda, \ldots, \lambda_r$, in the sets of one series, is the same as for the other; this is the number called the *grade* of the series.

There is therefore (from the theory developed for a plane curve in Chap. IV), upon an algebraic curve in space whose genus is p, a *canonical series*, of sets of $2p-2$ points, with freedom $p-1$; and the Riemann-Roch theorem, for complete series, which gives the freedom of the series in terms of the grade and the degree of specialness, holds also for non-plane curves. But there remain questions needing discussion, in regard to the completeness of the series determined on the space curve by surfaces whose construction is prescribed.

A simple example may illustrate this: On the rational quartic curve which is the intersection of two quadric surfaces having a point of contact, a series of sets, of each four points, of freedom 3, is obtained by the intersections of the curve with variable planes. A series of sets, of each four points, is also obtained by the intersections of the curve with variable quadric surfaces prescribed to pass through the double point, O, of the curve, and also to pass through two arbitrary fixed points, A, B, of the curve; of such quadric surfaces there are seven which are linearly independent, of which two may be taken to be surfaces wholly containing the curve. Thus, these quadric surfaces determine a linear series of sets, of each four variable points, on the curve; and the freedom of this series is 4. This series contains the previous series; namely of the five linearly independent quadric surfaces through O, A, B, there are four which break up into the plane OAB and a general plane. Thus, on the quartic curve, the series determined by arbitrary plane sections is *not complete*, being contained in another linear series of the same grade whose freedom is greater. For the rational quartic curve which is the residual intersection of a quadric surface with a cubic surface, we similarly have a linear series, of freedom 3, of sets of 4 points, obtained by plane sections; but this is contained in a series of freedom 4, also of grade 4, obtainable by the intersections with cubic surfaces drawn through the two generators.

It seems on the whole to be briefest and clearest to enunciate and prove immediately the following general theorem*, which covers many cases: let C be a given algebraic curve; through this curve let two surfaces $F=0$, $\Phi=0$, of orders m and M, be put, so that the curve is simple on both these surfaces. These surfaces may intersect in further curves; to make an inclusive statement, suppose that such a further curve, C_j, of intersection of $F=0$ and $\Phi=0$, is j_1-fold on $F=0$ and j_2-fold on $\Phi=0$; and also that a k_1-fold conical point, P_k, of $F=0$, is a k_2-fold conical point on $\Phi=0$, though this will be of importance only when this point is on the curve C. Consider now surfaces, of given order, passing through all such common curves C_j of $F=0$, $\Phi=0$ other than C, these surfaces being such that every such curve C_j is (j_1+j_2-1)-fold thereon, and also such that they meet every branch of the curve C which passes through a common multiple point P_k of $F=0$, $\Phi=0$ in (k_1+k_2-2) points, beside containing every point where the surfaces $F=0$, $\Phi=0$ touch one another. The theorem is that, for every order for which such surfaces are possible, these surfaces determine a complete series on the curve C. In particular, there are surfaces with the specified behaviour which are of order $m+M-4$; and these cut on the curve C (of genus p) the complete canonical series, of freedom $p-1$, of sets of $2p-2$ points.

As an illustration of the theorem, suppose C is the complete intersection of two non-singular surfaces, being simple on both. Then the surfaces of given order through the points of contact of these two surfaces (the double points and cusps of the given curve) cut a complete series on the curve; in particular there are such surfaces, of order $m+M-4$, determining the canonical series. That the surfaces of order $m+M-4$ give a series of sets of $2p-2$ points, was already clear from the formula found above

$$2p-2=mM(m+M-4)-2\delta-2\kappa;$$

but, that these surfaces give a series of freedom $p-1$ is a new fact.

---

* A particular consequence is proved independently below (p. 227).

To prove this independently we should require to evaluate the number of surfaces of order $m+M-4$ which contain the curve, and to consider whether the multiple points of the curve present independent conditions for surfaces of this order not containing the curve. Or (see below, p. 227) to invoke the Riemann-Roch theorem. The general theorem, however, gives also the striking result that a complete series is obtained by surfaces of any order, however low, which can be put through the multiple points of the curve. For a simple instance, on the quartic curve of intersection of two quadric surfaces which touch in one point, a complete series is obtained by the planes through this point; or, more generally, on the complete intersection of two non-singular surfaces which have no point of contact, supposed a simple curve on both surfaces, the series, of freedom 3, determined by arbitrary planes, is complete.

This result leads to the conclusion that the curve is *normal* in the space of three dimensions, that is, cannot be obtained by projection of a curve, of the same order, existing in higher space. It will be found indeed, speaking in general terms, that any manifold, in space of any dimensions, which is a complete intersection, and without multiple parts of dimension one less than its own, is likewise normal (Severi, "Su alcune questioni di postulazione", *Rend. Palermo*, XVII, 1903, § 2).

The general theorem we have stated is given by Noether, *Math. Annal.* VIII, 1875 (*Zur Theorie des eind. Entspr. alg. Geb.* § 7). We give Noether's proof:

Preliminary theorem. Any surface $\Psi=0$, which passes through the complete intersection of two surfaces $F=0$, $\Phi=0$, in such a way that every component curve of this intersection which is j_1-fold on $F=0$ and j_2-fold on $\Phi=0$ is (j_1+j_2-1)-fold on $\Psi=0$, is expressible by taking $\Psi=AF+B\Phi$, where A, B are suitable polynomials in the coordinates. The condition for $\Psi=0$ is that, on a general plane, it shall cut a curve having a (j_1+j_2-1)-ple point at every common point of the sections of $F=0$, $\Phi=0$ by this plane, where these curves have respectively a j_1-ple and a j_2-ple point. No condition for the behaviour of $\Psi=0$ at isolated multiple points of the curve (F, Φ) is postulated. The reader may prefer to pass over the proof of this preliminary theorem (pp. 220–24), to its application for the proof of Noether's general theorem (pp. 224–26).

We first prove* that if $f=0$, $\phi=0$ be two curves in a plane, respectively of orders m, n, and with respectively an h-fold and a k-fold point at an intersection, so that hk intersections of the curves are absorbed at this point, and $\Sigma hk = mn$, then any curve $\psi = 0$ in the plane, which has a $(h+k-1)$ point at such intersection, is given by taking $\psi = uf + v\phi$, where u, v are suitable polynomials. And this we first prove under the hypothesis that the order N, of the curve $\psi = 0$, is so high, that the prescribed conditions at the common points of $f = 0$, $\phi = 0$ are all linearly independent for the equation of $\psi = 0$; the number of these conditions is then $\Sigma(h+k, 2)$, that is $\frac{1}{2}\Sigma(h+k)(h+k-1)$. In this case the most general curve $\psi = 0$ satisfying these conditions contains $(N+2, 2) - \Sigma(h+k, 2)$ homogeneously entering independent coefficients. To shew then that the form $uf + v\phi$ is general enough for the representation of any curve ψ, it is sufficient to shew that, with general curves $u = 0$, $v = 0$, of respective orders $N-m$, $N-n$, chosen to have respectively a $(k-1)$-ple and a $(h-1)$-ple multiple point at the general intersection (f, ϕ) spoken of, for all such intersections, the polynomial $uf + v\phi$ contains as many as $(N+2, 2) - \Sigma(h+k, 2)$ homogeneously entering arbitrary coefficients*. Now, the form u, after the prescribed conditions are satisfied, will contain $(N-m+2, 2) - \Sigma(k, 2)$ arbitrary homogeneous coefficients; or more if the conditions at the intersections (f, ϕ) are not independent. Similarly v will contain $(N-n+2, 2) - \Sigma(h, 2)$, at least. The aggregate $uf + v\phi$ will thence contain a number of homogeneous arbitrary coefficients equal to the sum of the two preceding numbers, diminished by the number of terms which can enter simultaneously in uf and $v\phi$. But, assuming that f, ϕ have no common factor, an identity $u_1 f = v_1 \phi$, where u_1, v_1 denote respectively aggregates of terms in u and v, requires that v_1 divides by f, and hence $N - n \geqslant m$. Conversely, if $N \geqslant m + n$, by taking $v_1 = wf$, and hence v of the form $v_0 + wf$, where the terms $v_0\phi$ are independent of the terms uf, and w is a general polynomial, of order $N - m - n$, the possible reduction does arise; in this case the form $uf + v\phi$ contains, at least, a number of arbitrary homogeneous coefficients given by

---

* The theorem which we prove for plane curves is sufficient for the purpose to which we apply it, the section of a figure in space by an arbitrary plane. For the case when the common multiple points (f, ϕ) are of more intricate kind, reference may be made to Severi, *Algebr. Geom.* (Leipzig, 1921), p. 116; and for the general theorem in higher space, to Bertini, *Geom. d. iperspazi*, 1907, pp. 262 ff. A further list of authorities is given below (p. 224).

---

* That is, we assume that two linear systems of linearly independent curves, $\lambda_0 U_0 + \ldots + \lambda_r U_r = 0$, $\mu_0 V_0 + \ldots + \mu_s V_s = 0$, are identical if they are of the same order, have the same base points (simple or multiple), and the same freedoms (i.e., if $r = s$). In the application made the parameters $\mu_0, \ldots, \mu_s$ on the right are the coefficients entering in the polynomials u and v. We could also consider the theorem by analysing both sides of the equation into the individual monomials $x^i y^j z^k$ which enter; the coefficients on the right would then be *linear functions of* the coefficients in the polynomials u and v (the coefficients in these linear functions being taken from the given coefficients in f and ϕ). When N is so small $(N < m + n)$ that no identity $u_1 f + v_1 \phi = 0$ is possible, these linear functions are independent; and so the identity $\psi = uf + v\phi$ would follow, for proper values of u and v, from shewing that the prescribed conditions for u, v and ψ, at the intersections of $f = 0$, $\phi = 0$, involved that the two sides of the equation contained the same monomials. Similarly in general.

$$(N-m+2,\ 2)+(N-n+2,\ 2)-(N-m-n+2,\ 2)-\Sigma(h,\ 2)-\Sigma(k, 2).$$

Remarking then the identities

$$(N+2,\ 2)-(N-m+2,\ 2)-(N-n+2,\ 2)+(N-m-n+2,\ 2)=mn,$$
$$(h+k,\ 2)-(h,\ 2)-(k,\ 2)=hk,$$

and recalling the hypothesis $\Sigma hk=mn$, the conclusion is that the form $uf+v\phi$ contains enough arbitrary coefficients to represent the form ψ, however general the latter may be, subject to the conditions it is to satisfy. Such general ψ will be a linear aggregate of particular forms ψ satisfying these conditions, and these particular forms will occur in $uf+v\phi$, each multiplied by an arbitrary coefficient. Thus any particular form ψ, for which the curve $\psi=0$ has the prescribed multiple points, is expressible in the form $uf+v\phi$, with u, v as particular curves of the prescribed behaviour.

This is on the hypothesis that ψ is of sufficiently high order. We can, however, deduce from this that the expression in question equally holds whatever be the order of ψ, provided this has the prescribed behaviour at the multiple points $(f,\ \phi)$. For assume that it has been proved that all curves of order M which have a $(h+k-1)$-ple point at a point which is h-ple for $f=0$ and k-ple for $\psi=0$ are expressible in the form $u\phi+v\psi=0$. Let $\psi=0$ be any curve of order $M-1$ with the like behaviour at every common point $(f,\ \phi)$, and $\zeta=0$ be an arbitrary line. Then the composite curve $\zeta\psi=0$ is of order M, and has the like behaviour. By linear change of coordinates, let all the curves be expressed homogeneously in terms of x, y, ζ; and denote ζ by z; in the changed coordinates denote f, ϕ, ψ by $f_1(x, y, z)$, $\phi_1(x, y, z)$, $\psi_1(x, y, z)$. Then, by the assumption made,

$$z\psi_1(x,\ y,\ z)=u(x,\ y,\ z)\,f_1(x,\ y,\ z)+v(x,\ y,\ z)\,\phi_1(x,\ y,\ z),$$

with proper polynomials u and v. Hence we have the identity

$$u(x,\ y,\ 0)f_1(x,\ y,\ 0)+v(x,\ y,\ 0)\,\phi_1(x,\ y,\ 0)=0,$$

and we can suppose that the general line $z=0$ has been chosen so that $f_1(x,\ y,\ 0)$, $\phi_1(x,\ y,\ 0)$ have no common factor. Thus we can infer that $u(x,\ y,\ 0)=\theta(x,\ y)\ \phi_1(x,\ y,\ 0)$, where $\theta(x, y)$ is a homogeneous polynomial in x and y; and hence

$$u(x,\ y,\ z)=\theta(x,\ y)\,\phi_1(x,\ y,\ z)+zu_1(x,\ y,\ z),$$

where u_1 is a polynomial homogeneous in x, y, z. With this, the equation $z\psi_1=uf_1+v\phi_1$ takes the form $z\psi_1=(\theta\phi_1+zu_1)f_1+v\phi_1$, which is $z(\psi_1-u_1f_1)=(\theta f_1+v)\,\phi_1$; and, from this identity it follows that $\psi_1-u_1f_1$ divides by ϕ_1, or we have an equation $\psi_1=u_1f_1+v_1\,\phi_1$, where v_1 is a homogeneous polynomial. We can now return to the original variables x, y, z, and hence have an expression for ψ, in terms of f and ϕ, of the form specified, though ψ is only of order $M-1$. The reduction can be continued in the same way, to any form ψ, whatever its order, which satisfies the condition at the intersections (f, ϕ).

There are remarks, not of importance for our immediate purpose, which may usefully be made here. It follows from what is proved that, for any value of N, the number of independent conditions which are involved, for a (ternary) form ψ of order N, in order that the curve $\psi=0$ should have a multiple point of order $h+k-1$ at every (h, k) common point of the curves $f=0$, $\phi=0$ (supposed to be multiple points of the most general kind), is given by

$$(N+2, 2)-(N-m+2, 2)-(N-n+2, 2)+(N-m-n+2, 2) + \Sigma(h, 2)+\Sigma(k, 2),$$

where m, n are the orders of $f=0$, $\phi=0$; for this is equal to the number of terms in a general polynomial ψ, less the number of independent terms in the form $uf+v\phi$. In this statement, however, the term $(N-m-n+2, 2)$ is to be omitted if $N<m+n$, as we see by recalling the enumeration above made of the terms in $uf+v\phi$; this term vanishes identically if $N=m+n-1$, or $N=m+n-2$. Thus, recalling an identity remarked above, the number of conditions in question is $(h+k, 2)$ when $N>m+n-3$; and, for values of N of the form $N=m+n-3-q$, the number is $(h+k, 2)-(q+2, 2)$. We may easily see that in all cases the number is $\sigma+\gamma$, where σ denotes $\Sigma(h, 2)+\Sigma(k, 2)$, and γ denotes the coefficient of t^N in the expansion, in ascending powers of t, of $(1-t^m)(1-t^n)(1-t)^{-3}$. We may call this number the *postulation* of the (h, k) base points for curves of order N. In particular, if mn points in a plane be the complete intersection of two curves of orders m and n, all curves of order $m+n-3-q$, through $mn-(q+2, 2)$ of these intersections, pass through the remaining $(q+2, 2)$ intersections, in general (q being not greater than the less of $m-3$ and $n-3$). A more exact statement has already been given (Chap. IV, Note III, On the Cayley-Bacharach theorem). A particular consequence is, that curves of order $m+n-4$ passing through all but 3 of the mn intersections of two general curves of orders m and n, pass, as a rule, through these 3 also (but, when these 3 are collinear, it is necessary also to prescribe that the curve of order $m+n-4$ passes through one of them).

Ex. We have (pp. 218, 219) deduced from Noether's theorem two results, (a), that on the complete intersection of two non-singular surfaces of orders m and n, which do not touch (the curve of intersection being simple on both surfaces), general surfaces of order $m+n-4$ give the canonical linear series of freedom $p-1$; (b) that the series cut on the curve by arbitrary planes is complete. It may be interesting to see how the result (b) follows from (a), in virtue of the result just obtained, for plane curves of order $m+n-4$ through the complete intersection of two curves of orders m and n. In fact, by the Riemann-Roch theorem for the curve of order mn, the complete series on the space curve, which contains the sets of mn intersections by general planes of the space, has a freedom $mn-p+i$, where i is the number of canonical sets on the space curve which contain the intersections by an arbitrary plane. By the result (a), this number (i) is the number of surfaces of order $m+n-4$ (not containing the space curve) which pass through the mn plane intersections. The conditions for such a surface to contain these mn points are the conditions for the curve of order $m+n-4$, which is the plane section of the surface, to contain these mn points, whose number, by what is proved above, is $mn-3$. Now all possible surfaces of order $m+n-4$ may be regarded as made up of two linear aggregates, those which wholly contain the space curve, and those which do not; the number in the latter aggregate is one more than the freedom of the series cut on the space curve by surfaces of order $m+n-4$, namely is p. Whence, the number of surfaces of order $m+n-4$, not containing the space curve, which contain the mn intersections with a plane, is $p-(mn-3)$. If we put this for i in the formula $mn-p+i$, we obtain 3 as the freedom of the complete series, on the space curve, defined by the plane sections; which is the freedom of the system of planes.

From the theorem now proved for curves in a plane, we can pass to the theorem for surfaces which we have called the preliminary theorem (p. 219). The proof depends on the obvious remark that, of surfaces (Ψ), through the intersection of the given surfaces (F), (Φ), having the specified relation to these on every arbitrary general plane section, *there is a least possible order.* For instance, as an extreme case, there is no *plane*, containing the common curve of (F) and (Φ) with the appropriate multiplicity, even if this common curve is wholly plane; for this curve, as coinciding with the plane section of the surface (F), which is of order m, must be M-fold on (F), in order that it may be of the order mM; and must similarly be of multiplicity m on (Φ); and it is not then $(m+M-1)$-fold on the plane.

Let $t=0$ be an arbitrary general plane; change the coordinates, by linear transformation, so that the surfaces are represented in terms of the homogeneous coordinates x, y, z, t. We can then suppose that $F=f+tf_1$, $\Phi=\phi+t\phi_1$, $\Psi=\psi+t\psi_1$, where f, ϕ, ψ are homogeneous *in x, y, z only.* In virtue of the supposed relation of (Ψ) to (F) and (Φ), we can then suppose, from the theorem proved above for plane curves, that $\psi=uf+v\phi$, where u, v are polynomials in x, y, z only. Whence $\Psi=uf+v\phi+t\psi_1=uF+v\Phi+t(\psi_1-uf_1-v\phi_1)$, which we may write in the form $\Psi-(uF+v\Phi)=t\Psi_1$, where Ψ_1 is a homogeneous polynomial in x, y, z, t.

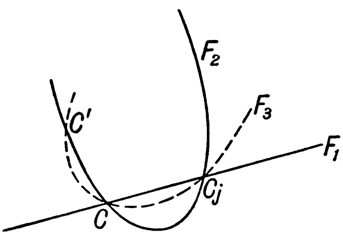
Assuming that the curves (u), (v), on the perfectly general plane $t=0$, have the multiplicities (j_2-1), (j_1-1), respectively, at every typical common point (j_1, j_2) of the curves (f), (ϕ), it follows, from the last equation, that, at every common point of (F), (Φ) on $t=0$, the section of (Ψ_1) by this plane has the behaviour specified for (Ψ) in the theorem under proof.

Whence, applying to (Ψ_1) the argument used for (Ψ), we infer an equation $\Psi_1-(u_1F+v_1\Phi)=t\Psi_2$, where (Ψ_2) is a surface whose section by $t=0$ has the specified relation to the sections of (F) and (Φ). This, however, leads to $\Psi=(u+tu_1)F+(v+tv_1)\Phi+t^2\Psi_2$, and a similar process can be continued. But the surfaces (Ψ_1), (Ψ_2), ... are of diminishing order, and all in the specified relation to (F) and (Φ). Therefore, by the remark made at starting, there is some stage at which the corresponding surface (Ψ_s) is absent, so that we finally have an equation of the form $\Psi = AF + B\Phi$, where A, B are polynomials; and this is the preliminary theorem required.

It is important to remark that the argument does not require the assumption that the surface (Ψ) is irreducible.

The proposition we have obtained belongs to a theory which has a wide literature. We may make reference to *Gergonne's Annales*, XVII, 1827, p. 214; Jacobi, *Ges. Werke*, III, p. 292; Noether, *Math. Ann.* VI, 1872, p. 358; *ibid.* VIII, 1875, § 8; *ibid.* XL, 1891, p. 140; C. A. Scott, *Math. Ann.* LII, 1899, p. 593; Severi, *Rend. Lincei*, XI, 1902; *Rend. Palermo*, XVII, 1903; *Atti...Torino*, XLI, 1906; Bertini, *Geom. d. iperspazi*, 1907, p. 268; Picard-Simart, *Fonct. algéb.* II (Paris, 1906), p. 17; Enriques-Chisini, *Teoria geometrica*, III, p. 530; Castelnuovo-Enriques, *Enzyk. d. math. Wiss.* III, C., p. 645.

Having considered the preliminary theorem (p. 219), we pass now to the proof of the general theorem enunciated on p. 218. Through a given curve, C, are put two surfaces F_1, F_2, of orders N_1, N_2, upon each of which C is simple. We consider a surface, S, passing through the residual curve intersection of F_1 and F_2, so as to have any part of this, which is a curve C_j, of multiplicity j_1 on F_1 and j_2 on F_2, as a (j_1+j_2-1)-ple curve; and, also, so as to meet any branch of C containing a common multiple point of F_1 and F_2, k_1-ple on F_1 and k_2-ple on F_2, in (k_1+k_2-1) points coincident thereat; the surface S, moreover, containing every point of contact of F_1 and F_2. We are to shew, first, that the series, cut on C by the general surface S of this behaviour, is independent of the surfaces F_1, F_2 which are used.



Describe, passing simply through C, another surface, F_3, of order N_3, to contain, beside C, all the residual curves of intersection, C_j, of F_1 and F_2, this curve C_j being j_3-fold on F_3. The surface F_3 may have further undesigned curves of intersection with the surface F_2; denote these by C'. It is supposed that C' is simple on F_2 and F_3.

Consider the surface SF_3, composed of the surface S spoken of together with F_3, in relation to the two surfaces F_1 and F_2; passing simply through C, and, with multiplicity at least j_1+j_2-1, through every other curve C_j common to F_1 and F_2, this surface is capable of expression in the form $SF_3 = TF_1 + AF_2$, where T, A are polynomials. This equation shews, if σ be the order of S, that T is of order $\sigma + N_3 - N_1$, so that if $\sigma = N_1 + N_2 - 4$, T is of order $N_2 + N_3 - 4$; and shews that $T=0$ passes through the general curve C_j with multiplicity $j_1+j_2-1+j_3-j_1$, or j_2+j_3-1; and, further, that $T=0$ passes through the curve (or curves) C' which are common to F_2 and F_3 but are not on F_1; also $T=0$ passes through the points of contact of F_2 and F_3. Also, at any multiple k_2-fold point of F_2 which is k_3-fold on F_3 (and k_1-fold on F_1), any branch of C, which passes through it, meets SF_3 in $(k_1+k_2-2+k_3)$ points, and F_1 in k_1 points, and so meets $T=0$ in (k_2+k_3-2) points.

On the whole, then, $T=0$ is defined for the curve C, with the help of the surfaces F_2 and F_3 through the curve, just as S was defined with the help of the surfaces F_1 and F_2.

We can, however, prove that the surfaces S and T have the same intersections with the curve C. For this curve lies on the surface F_2, whereon, by the preceding identity, we have the identity $SF_3=TF_1$. This shews, as the surface F_3 wholly contains C, that the intersections of the surface T with the curve C lie on the surface S; and, also, as F_1 contains C, that the intersections of the surface S with the curve C lie on the surface T.

The definition of the series on the curve C by the surfaces S, with the use of the surfaces F_1 and F_2, can thus be modified by the substitution, for F_1, of any other surface F_3, (which we have chosen, however, in the argument, so general as to have with F_2, outside the curves (F_1, F_2), only simple intersections); and it has been shewn that if S is of order N_1+N_2-4, then T is of order N_2+N_3-4. Thus, for F_1 and F_2, any two general surfaces through C can be chosen. In particular, we can (as already remarked, Ex. 10, p. 214) suppose these surfaces to be K and L, where K is a cone projecting C from a suitably general point, and L is a surface, of order m, say, having a $(m-1)$-fold point at this point of projection; the equations of these can be supposed to be

$$K \equiv f_n(x_1, x_2, x_3)=0, \quad L \equiv x_0 f_{m-1}(x_1, x_2, x_3)+f_m(x_1, x_2, x_3)=0.$$

Beside C, these surfaces intersect in the lines common to the cones $f_n=0, f_{m-1}=0$, and, if $(1, 0, 0, 0)$ be a general point, not lying on the curve C, such a line must equally lie on the cone $f_m=0$, being r-fold for this if r-fold for $f_{m-1}=0$. The surfaces S will then be surfaces ϕ, cones of vertex $(1, 0, 0, 0)$, passing $(q+r-1)$ times through every q-fold line of f_n which is r-fold for f_{m-1}; so that, from what was proved above, ϕ is of the form Bf_n+Af_{m-1}, where B, A are homogeneous in x_1, x_2, x_3. In particular when, in the notation used above, the surface S was of order N_1+N_2-4, the cone ϕ will be of order $m+n-4$; and its intersections with the cone f_n, other than those on f_{m-1}, will be on A, which is then of order $m+n-4-(m-1)$, or $n-3$; a q-fold generator of f_n which is r-fold for f_{m-1}, is $(q-1)$-fold for this cone $A=0$. The curve C is the complete intersection of the cone f_n and the monoid L, other than the generators (f_n, f_{m-1}), and is met by ϕ on the cone A. A q-fold generator of f_n arises from a q-fold chord of C, drawn from $(1, 0, 0, 0)$; and, in order that there may be q distinct points of C on this generator, f_{m-1} must, we know from the theory of plane curves, have this generator as $(q-1)$-fold at least. If the curve C has an actual q-fold point, this will give a q-fold generator of f_n, which need not be on f_{m-1} (or f_m), but will equally be $(q-1)$-fold on the cone A.

We thus see that the series on the curve C obtained by the surfaces S, described, as explained, through the residual intersection of two arbitrary surfaces F_1, F_2 which are put simply through C, is the series which projects, from an arbitrary general point, into the series obtained, on the projection of C, by *adjoint* curves of this projection (see Chap. IV, preceding, p. 60), and this, by the theory of the plane curve, is a complete series. In particular, the series by surfaces S of order N_1+N_2-4, is the series projecting into that obtained on the plane curve by adjoint curves of order $n-3$; as this latter is the canonical series, the series on C by the surfaces S of order N_1+N_2-4 is equally the canonical series. The series by surfaces S which are of order greater than N_1+N_2-4 will project into series given by plane curves adjoint to the projection of C, of appropriately higher order. If there are surfaces of order less than N_1+N_2-4 which contain the residual curves C_j (or when C is a complete intersection), then there exist adjoint curves of order less than $n-3$ for the projection. Conversely, any complete series on the plane curve corresponds to a series on the curve C obtainable by the rule; for we may take, for the surfaces F_1, F_2, the cone f_n and the monoid $x_0 f_{m-1}+f_m$. This completes the proof.

A very suggestive proof that if the surface S, in the general theorem, give a complete series on the curve C when the surfaces S are of sufficiently high order, then the same is true for existing surfaces S of lower order, is given by Severi, *Rend. Lincei*, XII, 1903, *Sulla deficienza della serie*...No. 2.

Ex. 1. Prove that the $\frac{1}{2}mM(m-1)(M-1)$ chords from an arbitrary general point to the curve of intersection of two surfaces of orders m and M, lie, with mM coplanar points of this curve, upon $4+\frac{1}{2}mM(m+M-6)$ linearly independent cones of order $mM-3$, whose vertex is at the point.

Ex. 2. Prove that if the mnk intersections of three surfaces $f=0$, $\phi=0$, $\psi=0$, of orders m, n, k, are all simple, then any surface through these points has an equation of the form $uf+v\phi+w\psi=0$, where u, v, w are polynomials. Further, that the postulation of these points, for a surface of order N, is the coefficient of t^N in the expansion, in ascending powers of t, of the function $(1-t^m)(1-t^n)(1-t^k)(1-t)^{-4}$, and* is mnk only if $N > m+n+k-4$.

Another proof of the determination of the canonical series on a curve in space. It has been proved that when two curves,

---

* If $\sigma = \sum_{k=1}^{s}(n_k-1)$, $\varpi = \prod_{k=1}^{s} n_k$, it may be proved that $(1-t^{n_1})\dots(1-t^{n_s})(1-t)^{-s-1}$ is of the form $u+\varpi t^{\sigma}(1-t)^{-1}$, where u is an integral polynomial, with positive integer coefficients, of order $\sigma-1$.

C_1, of order n_1, and C_2, of order n_2, are, together, the complete intersection of two non-singular surfaces, of orders m and M, being simple on both these, and the only double points of C_1 or C_2 are simple contacts of the two surfaces, then the canonical series on C_1 is given by the surfaces of order $m+M-4$ through C_2 which contain the double points of C_1. We give now another proof of this result, depending on the Riemann-Roch theorem for the curve C_1, whose validity is a consequence of the (1, 1) correspondence between this curve and a plane curve. Like the foregoing proof it may be assigned to Noether, whose great paper, of 1882, employs this (1, 1) correspondence throughout.

After the detailed considerations of Ex. 11 (p. 215) preceding, it will be unnecessary to suppose that the curve C_2 is irreducible; but we assume, as in Ex. 11, that every component of C_2 has actual intersections with C_1.

Denoting the surfaces through C_1 which intersect further in C_2, by $F=0$, $\Phi=0$, we have proved (p. 219) that all surfaces of order $m+M-4$, containing both C_1 and C_2, have an equation of the form $AF+B\Phi=0$; the number of such surfaces (with $N=m+M-4$) is thus $(N-m+3, 3)+(N-M+3, 3)$, which is the same as

$$(N+3, 3)+(N-m-M+3, 3)-[NmM-\tfrac{1}{2}mM(m+M-4)],$$

or, for $N=m+M-4$, is $(m+M-1, 3)-1-\tfrac{1}{2}mM(m+M-4)$.

Now consider the surfaces of order $m+M-4$ containing C_2, which also contain the multiple points of the curve C_1. The surfaces of this order, conditioned only by containing C_2, are proved in Ex. 11 to be, in number, $(m+M-1, 3)-t-p_2+1-\delta_2-\kappa_2+\sigma$, where $\sigma \geqslant 0$. From the equation

$$2p_1-2=n_1(m+M-4)-2\delta_1-2\kappa_1-t$$

it follows, when the further condition of containing the multiple points of C_1 is imposed, that there arises on C_1 a linear series of $2p_1-2$ points in each set. We prove that the freedom of this series is $p_1-1-\sigma_1$, with $\sigma_1 \geqslant 0$: first, if the series be not special, and be complete, it will have a freedom p_1-2, and will have a less freedom if incomplete; and, if the series be special, it will have a freedom p_1-1, if complete (a set of $2p_1-2$ points not belonging to two canonical sets), but a less freedom if incomplete. Thus, the surfaces of order $m+M-4$, which contain the curve C_2 and the multiple points of C_1, will contain the curve C_1 entirely if put through other $p_1-\sigma_1$ points of this curve, of general position. If the number of conditions, for surfaces of order $m+M-4$ containing C_2, that they should pass through the multiple points of C_1, be denoted by $\delta_1+\kappa_1-\mu$, with $\mu \geqslant 0$, it follows then, that the number of surfaces of order $m+M-4$ containing both curves C_1 and C_2, is

$$[(m+M-1, 3)-t-p_2+1-\delta_2-\kappa_2+\sigma]-[p_1-\sigma_1]-[\delta_1+\kappa_1-\mu].$$

We found, however, that

$$p_1+p_2+t-1+\delta_1+\kappa_1+\delta_2+\kappa_2=1+\tfrac{1}{2}mM(m+M-4);$$

thus, the number of surfaces of order $m+M-4$ containing both curves C_1, C_2, is $(m+M-1, 3)-1-\frac{1}{2}mM(m+M-4)+\sigma+\sigma_1+\mu$; comparing this with the number of such curves which we remarked above, based on the expression $AF+B\Phi$, we infer, therefore, that $\sigma=\sigma_1=\mu=0$.

The argument itself proves that the surfaces of order $m+M-4$ through the curve C_2, even when put through the multiple points of C_1, do not necessarily all contain the curve C_1, for general values of p_1.

From $\sigma_1=0$ it follows that the surfaces of order $m+M-4$, through C_2 and the multiple points of C_1, cut, on this curve C_1, a complete special series, of sets of $2p_1-2$ points, with freedom p_1-1, the canonical series. This is the main theorem we set out to prove.

It is also clear that surfaces of order $m+M-4$, conditioned only by passing through the multiple points of C_1, and the t intersections of this curve with C_2, cut on C_1 a series of the same grade $(2p_1-2)$ as if they entirely contained the curve C_2, with freedom certainly not less than in that hypothesis (that is not less than p_1-1). Hence these surfaces do in fact give on C_1 the same series as that obtained by surfaces wholly containing C_2. For, if this series were not special its freedom would at most be p_1-2; and, being thence special, its freedom is at most p_1-1, so that it coincides with the canonical series obtained by surfaces wholly containing C_2.

From $\sigma=0$, it follows that the numbers ζ_i used in the argument of Ex. 11 above (p. 215) are all zero. Thus, surfaces of order $m+M-4$, drawn through the multiple points of all the components of the curve C_2, when this is composite, and through the mutual intersections of these components (but not their intersections with C_1), cut a complete series on each component, this freedom, in the notation of that example, being $p_{2i}-2+t_i$. In particular, when C_2 is irreducible, surfaces of order $m+M-4$, through the multiple points of C_2, cut on C_2 a complete series of freedom p_2-2+t. From this result, proved when C_1 is irreducible, we infer, by parity of reasoning, that surfaces of order $m+M-4$, through the multiple points of C_1, cut on C_1 a series of freedom $p_1-2+t-\lambda$, this series being complete, and λ zero, when C_2 is irreducible. From this we can prove that, for surfaces of order $m+M-4$ through the multiple points of C_1, the t intersections with C_2 furnish only $t-1-\lambda$ independent conditions, for surfaces of this description to contain them. For denote the number of these conditions by ξ; we have shewn that surfaces of order $m+M-4$, through the multiple points of C_1, cut on this curve a series of freedom $p_1-2+t-\lambda$; those through the multiple points and the t intersections thus cut a series of freedom $p_1-2+t-\lambda-\xi$; but we have shewn that the freedom of this series is p_1-1. Wherefore $\xi=t-1-\lambda$, where λ is zero when C_2 is irreducible, but otherwise is an integer $\geqslant 0$.

But, supposing C_2 is composite, we can go further. Consider the linear aggregate, Ω, of surfaces of order $m+M-4$, which pass through the multiple points of C_1 and through the multiple points of all the components of C_2, which pass also through all the intersections of the components of C_2 with C_1 and with one another. These surfaces cut on C_1 a linear series, of the same grade, $2p_1-2$, as if they contained the curves C_2 entirely, and of freedom not less than the freedom, p_1-1, which would then exist. This series on C_1 is thus special (not being of freedom equal to or less than p_1-2), and is also complete (or its freedom would be less than p_1-1). Thus these surfaces Ω cut the complete canonical series on C_1. Hence, by parity of argument, these surfaces cut the complete canonical series on every component of C_2.

Further, of the linear aggregate of surfaces Ω, those which do not contain the curve C_1 must be p_1 in number (in order to cut thereon a series of freedom p_1-1); and this is in fact the same number as of surfaces Ω, not containing C_1, which contain all the curves C_2, as was shewn. While, similar remarks may be made in regard to any one component of C_2 as are here made in regard to C_1. Form then, from Ω, the aggregate of the surfaces which contain C_2 but do not contain C_1; let this be Ω_1. Form also the aggregate of those which contain C_1 and also every component of C_2 except the curve (n_{2i}); denote this by Ω_{2i}, the number of surfaces in Ω_{2i} being (we have seen) p_{2i}. By the definition, no surface of Ω_{2i} is the same as any surface of Ω_1, or the same as any surface of Ω_{2j}. Finally, form the aggregate Ω_0 of all surfaces Ω which contain both C_1 and C_2; the surfaces of Ω_0, we have seen, are of number, say P, given by

$$P=(m+M-1, 3)-1-\tfrac{1}{2}mM(m+M-4).$$

We have then $p_1+\Sigma p_{2i}+P$ linearly independent surfaces of the aggregate Ω. These are in fact the whole aggregate. For, if there were in Ω, for instance, a surface not containing either the curve (n_{2i}), or the curve (n_{2j}), which is not a linear aggregate of the surfaces already described, such surface would give on the curve (n_{2i})—as also on (n_{2j})—a set belonging to the canonical series on (n_{2i}), in addition to those already obtained, contrary to the fact proved that these define the complete canonical series on (n_{2i}).

Recurring however to Ex. 11 (p. 215), of which we have continued the notation, we have

$$p_1+\Sigma p_{2i}+P=(m+M-1, 3)-[\delta_1+\kappa_1+\Sigma(\delta_{2i}+\kappa_{2i})+\Sigma t_i+\Sigma t_{ij}-s],$$

where s denotes the number of component curves in C_2. As the first member, we have proved, is the number of linearly independent surfaces of the aggregate Ω, *we infer, that the number of independent conditions for a surface of order $m+M-4$ to belong to the aggregate Ω, that is, to contain all the multiple points of C_1 and C_2, and to pass through all the intersections of the $s+1$ curves involved in C_1 and C_2, is $\delta_1+\kappa_1+\Sigma(\delta_{2i}+\kappa_{2i})+\Sigma t_i+\Sigma t_{ij}-s$.* From this it follows that, if the conditions of passing through all the multiple points are independent, then the conditions, for the surfaces of order $m+M-4$, passing through these multiple points, to pass through the $\Sigma t_i+\Sigma t_{ij}$ intersections, are not independent, having a defect equal to or less than the total number of curves into which the complete intersection of the two surfaces (F), (Φ) breaks up. We have already remarked that this is so when $s=1$.

Ex. 1. In illustration of this theory when C_2 is irreducible, we consider two quintic curves, of genus 2, forming together the complete intersection of a quadric surface and a quintic surface.

Take a quadric surface and put through one generator of this a cubic surface. The remaining intersection is a quintic curve, C_1, for which $n_1=5$, and $p_1=2$, the quintic curve being met in two points by a variable plane through the generator. Through C_1 put a quintic surface, giving, by its remaining intersection with the quadric surface, another quintic curve, C_2, for which also, we easily find, $p_2=2$, having with C_1 a number, t, of intersections given by $t=13$. The curves C_1, C_2 meet the generators of the two systems, of the quadric surface, in, respectively, 2 and 3, and in 3 and 2 points.

Now take 12 of the intersections of the two curves C_1, C_2; let O denote one of these; from O take the generator of the quadric surface which meets the curve C_1 in two points A_1, B_1 beside O; similarly let the other generator through O meet C_2 in A_2 and B_2, beside O. By what we have proved (the curves C_1, C_2 being supposed without multiple points), cubic surfaces (of order $m+M-4$ for $m=2$, $M=5$) cut on C_1 a linear series of sets of 15 points with freedom p_1-2+t, or 13; so that a cubic surface drawn through 14 independent points of the curve C_1 entirely contains the curve. Consider then cubic surfaces through the 12 chosen intersections, and through A_1 and B_1; assuming these conditions to be independent, these cubic surfaces will contain the curve C_1; and they will contain $20-14$, or 6, undetermined homogeneous coefficients. They will thus be given by an equation $(a_1x+b_1y+c_1z+d_1t)Q+\lambda_1V_1+\mu_1W_1=0$, where x, y, z, t are the coordinates, $Q=0$ is the quadric surface on which the curves C_1, C_2 lie, $V_1=0$, $W_1=0$ are definite cubic surfaces through the curve C_1, and a_1, b_1, c_1, d_1, λ_1, μ_1 are undetermined constants. In the same way, the cubic surfaces through the 12 chosen intersections of the two curves C_1, C_2, and through the points A_2, B_2, will be given by an equation $(a_2x+b_2y+c_2z+d_2t)Q+\lambda_2V_2+\mu_2W_2=0$, where $V_2=0$, $W_2=0$ are definite cubic surfaces containing the curve C_2.

Whence, the general cubic surface through the 12 chosen intersections, which is of 8 members, has an equation of the form

$$(lx+my+nz+kt)Q+\lambda_1V_1+\mu_1W_1+\lambda_2V_2+\mu_2W_2=0,$$

where every member belongs to a surface containing one, or both, of the two curves. All these cubic surfaces contain the 13th intersection of C_1 and C_2.

Ex. 2. Consider next the case in which the curve C_2 is composed of three skew lines; and the curve C_1 is the sextic intersection of two cubic surfaces through the lines. For the conventional genus of C_2 (see Ex. 11, p. 215), the formula $p_2-1=\Sigma(p_{2i}-1)+\Sigma t_{ij}$ gives $p_2=-2$, while, from

$$t=n_2(m+M-n_2-1)+2h_2, \quad =3(3+3-4)+6, \quad =12,$$

and $\quad t=n_1(m+M-4)-(2p_1-2),$

we have $p_1=1$. Thus the sextic curve is of genus 1, and meets each of the three lines in 4 points, as is also easily seen by considering sections on an arbitrary plane through one of the three lines.

For quadric surfaces (of order $m+M-4$), required to pass through the 12 intersections of the four curves which are the complete intersection of the two cubic surfaces, these 12 points are equivalent only to 9 conditions; and these quadric surfaces (one in number) cut the canonical series (of zero grade) on each of the four curves.

In this case, general quadric surfaces cut on the sextic curve, C_1, a series of sets of 12 points, of freedom 9 (there being no quadric surface through this curve). The complete series of sets of 12 points on C_1 has freedom 11, and is determinable by quartic surfaces through the three lines, as we have proved; there are $35-15$, or 20 such quartic surfaces; of these there are 8, with equation of the form $AF+B\Phi=0$, where (F), (Φ) are the two cubic surfaces and (A), (B) are planes, which contain the sextic curve; the remaining 12 give the complete series in question. Among such quartic surfaces are the 10 which decompose into the quadric surface containing the three lines, together with another quadric surface.

Ex. 3. Next, let C_2 consist of a conic and a line not meeting this, while C_1 is the remaining sextic intersection of two cubic surfaces drawn through these. Then we find C_1 to be of genus 2, being met by the conic in 6 points, and by the line in 4 points.

By the result of the text, the 10 intersections furnish 8 independent conditions for quadric surfaces (of order $m+M-4$) required to pass through them; and this is directly obvious. These quadric surfaces contain both the conic and the line, and break up into the plane of the conic, and a variable plane through the line, which determines the canonical series on the sextic curve.

Part V. Connexion with the theory of algebraic functions.

Consider, as before, an irreducible curve C_1, simple on each of two surfaces of orders m and M, whose further intersection is a curve C_2, not necessarily irreducible, there being no multiple points for either curve, other than contacts of the two surfaces. We have proved that surfaces of given order which pass through C_2, and through the multiple points of C_1, give a complete linear series on C_1.

Such a series, we know, from the theory of linear series on a plane curve, is determined by one of its sets. Hence, we see that, if any number of simple points, say (D), be taken arbitrarily on the curve C_1, and a surface (δ), of any sufficient order, be put through these points, through the multiple points of C_1, and through the curve C_2, and, if the set of remaining intersections of this surface (δ) with C_1 be denoted by (B), then, the most general surface (ν), of the same order, put through (B), the multiple points of C_1, and through the curve C_2, will give, as its remaining intersection with C_1, the most general set (N) of the linear series determined by the set (D) on C_1. And, the equations of these surfaces being $(\delta)=0$, $(\nu)=0$, where (δ), (ν) are homogeneous polynomials in the coordinates, the most general rational function, on the curve C_1, which has the set (D) as simple poles, will be given by the fraction $(\nu)/(\delta)$, where (ν) will involve a number, given by the Riemann-Roch theorem, of linearly entering arbitrary coefficients.

We also have the geometrical theorem that, if two particular surfaces $(\nu)=0$, $(\delta)=0$, of the same order, through the multiple points of C_1 and through the curve C_2, have, for simple intersections with C_1, respectively the sets (D), (B) and (N), (B), of which (B) is a common part; and, if any other surface (δ') be drawn, containing the curve C_2 and the multiple points of C_1, through the set (N), and if this meet the curve C_1 further in a set of points (B'), then there is a surface (ν'), through C_2 and the multiple points of C_1, whose further intersection with the curve C_1 consists of the sets (N) and (B'). This geometrical theorem is equivalent to saying that, on the curve C_1, the rational function $(\nu)/(\delta)$ is expressible also in the form $(\nu')/(\delta')$. Hence the polynomial $(\nu)(\delta')-(\nu')(\delta)$ vanishes on the curve C_1, that is, as a consequence of the equations $u=0$, $U=0$ of the two surfaces which contain C_1. By hypothesis, all the four polynomials (ν), ... vanish on the curve C_2.

We now apply these functional ideas to form an algebraic integral, on the curve C_1, which is everywhere finite thereon. This gives an interpretation of the formula we have found for the genus of this curve.

Let $\lambda=a_1x+b_1y+c_1z+d_1t$, $\mu=a_2x+\ldots+d_2t$ be two arbitrary linear functions, and (u, U, λ, μ), as before, be the Jacobian of these and of the two surfaces $u=0$, $U=0$ which meet in the curves C_1 and C_2. Also let $\Psi=0$ be the equation of any surface of order $m+M-4$, drawn through the multiple points of the curves C_1 and C_2, and through the intersections of the component curves of C_2 with C_1 and with one another. We have proved that the aggregate of these surfaces cuts the canonical series on C_1, and on each component of C_2. Now consider the integral

$$\int \frac{\Psi(\lambda d\mu-\mu d\lambda)}{(u, U, \lambda, \mu)},$$

whose definition, as we have given it, is symmetrical in regard to the (say $s+1$) curves forming the complete intersection of $u=0$, $U=0$. We consider this integral on C_1.

As (u, U, λ, μ) is of order $m+M-2$, the integral is functional, that is, homogeneous of zero order in the current coordinates x, y, z, t. The form (u, U, λ, μ), we have seen, vanishes, at a node of the curve C_1, once on each branch, and three times at a cusp;

moreover, it vanishes at all points of the curve whereat the tangent line meets the line (λ, μ). Using θ as a parameter for the expression of the coordinates on the curve, we shall shew that $\lambda d\mu/d\theta - \mu d\lambda/d\theta$ vanishes once at each cusp, and also vanishes at the points of the curve whereat the tangent meets the line (λ, μ). The polynomial Ψ' vanishes once on each branch of the curve C_1 at a node, and twice at each cusp. This shews that the integral is everywhere finite on the curve.

The zeros of (u, U, λ, μ) at non-singular points of the curve C_1 are what are known as the Jacobian points of the linear series determined on the curve by variable planes through the line (λ, μ); and what we have stated is thus in agreement with the known fact that the Jacobian set, of a linear series of freedom 1, on a curve, is equivalent (coresidual) with a set of the canonical series together with two sets of the linear series.

The verification that $\lambda d\mu/d\theta - \mu d\lambda/d\theta$ vanishes at the cusps of the curve, and also at the points whereat the tangent line meets the line (λ, μ) is simple. This function is a sum of terms such as $(a_1b_2 - a_2b_1)(x\,dy/d\theta - y\,dx/d\theta)$, $(c_1d_2 - c_2d_1)(z\,dt/d\theta - t\,dz/d\theta)$; and it can be proved at once that the integral is unaltered in form by a linear transformation of coordinates. We can thus suppose $t=1$, and take, for the neighbourhood of a cusp, $x=\theta^2$, $y=b\theta^3+\ldots$, $z=c\theta^4+\ldots$; then the binary determinants in

$$\left\| \begin{matrix} x, & y, & z, & t \\ dx/d\theta, & dy/d\theta, & dz/d\theta, & dt/d\theta \end{matrix} \right\|,$$

or

$$\left\| \begin{matrix} \theta^2, & b\theta^3+\ldots, & c\theta^4+\ldots, & 1 \\ 2\theta, & 3b\theta^2+\ldots, & 4c\theta^3+\ldots, & 0 \end{matrix} \right\|,$$

all vanish for $\theta=0$.

Further, the integral is independent of the particular line (λ, μ) chosen. For, in virtue of the equations $u=0$, $U=0$, we have, at any point of the curve, the five equalities

$$\frac{x\,dy/d\theta - y\,dx/d\theta}{u_3U_4 - u_4U_3} = \ldots = \frac{z\,dt/d\theta - t\,dz/d\theta}{u_1U_2 - u_2U_1},$$

where $u_1 = \partial u/\partial x$, etc. Thus the integral may be replaced by

$$\int \frac{\Psi'(x\,dy/d\theta - y\,dx/d\theta)}{u_3U_4 - u_4U_3}\,d\theta,$$

and, at a point of the curve whereat the tangent line meets the line $x=0$, $y=0$, the expression $x\,dy/d\theta - y\,dx/d\theta$ vanishes.

By the argument, this same integral gives the everywhere finite integrals on every component curve of the intersection of $u=0$, $U=0$. The method, moreover, as we shall illustrate below, is applicable to curves in space of more than three dimensions.

Part VI. The greatest genus possible for a curve of given order, in space of any number of dimensions. In the previous discussion of the linear series on a curve, the curve has been defined by the surfaces which pass through it. We may however seek to formulate results for a curve of which only the order and genus are given; and, if the curve be in space of dimension greater than 3, this is often more convenient. Castelnuovo has obtained some remarkable theorems from this point of view, which we proceed to state in part, with proof of some of these, referring the reader to the original paper for further detail (*Rend. Palermo*, VII, 1893: "Sui multipli di una serie lineare...una curva algebrica").

We consider an algebraical curve of order n, and genus p, lying in a space of r (and not fewer) dimensions; in particular, r may be 3. We denote by σ the greatest integer contained in the fraction $(n-2)/(r-1)$, namely

$$\sigma = E\left(\frac{n-2}{r-1}\right), \quad = \frac{n-2-\epsilon}{r-1}, \quad \epsilon = 0, 1, \ldots, (r-2);$$

then we define π by

$$\pi = \sigma[n-r-\tfrac{1}{2}(\sigma-1)(r-1)], \; = \frac{1}{2}\left(\frac{n-2-\epsilon}{r-1}\right)(n-r+1+\epsilon),$$

where ϵ is such that $(n-2-\epsilon)/(r-1)$ is an integer. One of the principal results obtained is that $p \leqslant \pi$.

Considering the linear series cut on the curve by unconditioned general primals of order m, it is proved that, for $m \geqslant \sigma$, this series is not special (namely its sets are not sets of $2p-2$ points belonging to the canonical series, nor contained in such sets). For $m < \sigma$, no statement is made; it may be that a special series is not obtainable by unconditioned primals of any order. Thus, for $m \geqslant \sigma$, if the series be complete, its freedom will be $mn-p$. When the series is not complete, the incompleteness does not increase with m; but, after m has reached a certain value (depending on the multiple points which the curve may have), the incompleteness remains constant as m increases; in fact, for $m > \sigma + \pi - p$, the freedom is $mn-p-d$, where d does not depend on m. This number d is such that $\Sigma(s_i - 1) \leqslant d \leqslant \pi - p$, where s_i is the multiplicity of a general multiple point of the curve.

A corollary is that, when p reaches its extreme, or $p = \pi$, the curve is without multiple points, and primals of order σ, or more, cut a complete non-special series upon the curve.

The extreme π is reached by p for curves of three kinds:

(1) When $n = r + p < 2r$, in which case $\sigma = 1$. Then complete non-special series are cut upon the curve by primes of the space (and by primals of all orders).

(2) When $n = 2r = 2p - 2$, in which case $\sigma = 2$, and the curve is

canonical (see Chap. IV, p. 81). Then quadrics, and primals of higher order, give complete non-special series on the curve; but the primes of the space give complete special series.

(3) The curves without multiple points for which $n > 2r$, which lie on a rational ruled* surface of order $r-1$, and cut every generator of this in $\sigma_1 + 1$ points, where σ_1 is the greatest integer in $(n-1)/(r-1)$; thus $\sigma_1 = \sigma$ unless $(n-1)/(r-1)$ is an integer, but then $\sigma_1 = \sigma + 1$.

For curves in space of three dimensions, the value of π is $\frac{1}{4}(n-2)^2$ or $\frac{1}{4}(n-1)(n-3)$, according as n is even or odd; on such a curve, surfaces of order one less than the greatest integer in $\frac{1}{2}n$ give a non-special series, not necessarily complete; but surfaces of order $\geqslant E(\frac{1}{2}n) + \pi - p$ give a series whose incompleteness, which does not exceed $\pi - p$, is independent of the order of the surface.

Castelnuovo also proves that, for curves without multiple points, in space of any dimensions, the primals of order $n-2$, or more, cut a complete non-special series on the curve; and further, if there be multiple points of which a general multiplicity is s_i, primals (of order $\geqslant n-2$) passing $s_i - 1$ times through every such multiple point, have the same property. When the curve is not rational this remains true with $n-3$ instead of $n-2$. The proof is by projection on to a plane; and the condition for the primals determining the series, in more precise terms, is that they must meet each branch of the given curve, at a multiple point, as does an adjoint curve of the plane curve. For primals of order $k\,(\geqslant n-2)$, so passing through the multiple points, the freedom of the series which they determine on the curve is therefore $nk - \Sigma s_i(s_i - 1) - p$. Wherefore, the freedom of the series cut on the curve, by primals of order k not passing through the multiple points, is $nk - \Sigma s_i(s_i - 1) - p + \nu_k$, where ν_k is the number of independent conditions for the primals to pass through the multiple points.

For instance, consider a curve in ordinary space which has triple points; and first suppose that the three tangents at a particular triple point are not coplanar. For a surface to meet each of the three branches of the curve at this point in two points, it is necessary for the surface to have a double point, which is 4 conditions for the surface; the contribution to the number $\Sigma s_i(s_i - 1) - \nu_k$ from such a triple point is therefore 2. When however we consider a triple point of which the three tangents are in a plane, two intersections of the surface with each branch are secured by supposing the surface to touch the plane, which is only 3 conditions; and the contribution to $\Sigma s_i(s_i - 1) - \nu_k$ is then 3. Thus, the freedom of the series on the curve, cut by unrestricted surfaces of order k, is $nk-p-2t_1-3t_2$, where t_1 is the number of general triple points, and t_2 of coplanar triple points; and the number of surfaces of order k containing the curve is $(k+3, 3)-(kn-p-2t_1-3t_2+1)$. This number, in the form $(k+3, 3)-\nu_k-(kn-\mu-p+1)$, where $\mu=\Sigma s_i(s_i-1)$, we may also reach by considering surfaces through the multiple points, in number $(k+3, 3)-\nu_k$, which have μ intersections in all at these multiple points.

---

* It was proved by Segre that rational ruled surfaces of order $r-1$ necessarily lie in space of r or lower dimensions, but in the latter case are obtainable by projection from such surfaces in space of r dimensions. Such surfaces are therefore said to be normal in a space $[r]$.

Consider now the proof of the extreme π which has been stated for the genus of a curve of order n. A lemma derived from the theory of special series is required: If a linear series, g_R^N, of sets of N points, of freedom R, not necessarily a complete series, be *special*, and we take a particular set of another series g_r^n, in which $n \leqslant R$, then a set of g_R^N which contains $n-r$ of the points of the chosen set of g_r^n, contains all the n points of this set; in other words, a set of g_r^n furnishes at most $n-r$ conditions for a set of g_R^N which is to contain it. For, first, from $n \leqslant R$, there are sets of g_R^N containing any given set of g_r^n, so that, as g_R^N is a special series, so is g_r^n. If this latter be contained in the complete special series g_ρ^n, we have $\rho=n-(p-i)$, where i is the specialness of a set of g_ρ^n, or g_r^n, namely the number of unrestricted canonical sets (of $2p-2$ points) of which this set of g_r^n forms part; as there are p independent canonical sets, the set of g_r^n furnishes $p-i$, or $n-\rho$, conditions for a canonical set which is to contain it; if $r=\rho-\delta$, the number $n-\rho$ is $n-r-\delta$, which is at most $n-r$. Now the sets of g_R^N, which is a special series, are presumably more restricted than the sets of the complete canonical series; and the necessary conditions for the more restricted sets to contain a set of g_r^n are evidently not more, but may well be less, than for the general canonical sets. Thus, the number of conditions for a set of the special series g_R^N to contain a particular set of g_r^n is at most $n-r$. This is the lemma in question. The argument may be stated in terms of the canonical curve, in space $[p-1]$, as in Chap. IV preceding: The linear space which determines a set of g_R^N, contains the linear space which determines a set of g_r^n, if it contains points of the latter space sufficient to determine this.

Consider now the sets of n points, of the curve under consideration, determined by primes of the space $[r]$ in which the curve lies; these are sets of a linear series g_r^n. And consider the linear series, of sets of kn points, cut upon the curve by unrestricted primals of order k; these sets belong to a linear series g_R^{kn}. Let ν_k be the number, taken as small as possible, of points of a prime section of the series g_r^n, such that any set, of the series cut on the curve by the primals of order k, which contains any chosen ν_k points of a prime section, contains all the n points of this section. It is possible then to find $\nu_k - 1$ points, of the prime set, such that a primal of order k containing these, does not necessarily contain all the n points of this set. This number ν_k is such that $\nu_k \geqslant k(r-1)+1$, if the number on the right is less than n, namely if $k<(n-1)/(r-1)$. For, if $k(r-1)+1<n$, we can choose, from the n points of a prime set, a number k, of sets of each $r-1$ points, still leaving at least 2 points of the set unchosen; and the k primes, each determined by one of these sets of $r-1$ points, form together a degenerate primal of order k which does not contain the whole set of n points, in general. Thus, at least $k(r-1)+1$ points of the prime set are necessary in order that a primal, of order k, passing through them, should entirely contain the set. Next, let r_k be the freedom of the series cut on the curve by primals of order k (not necessarily a complete linear series). Then the series cut on the curve by primals of order k which contain the points of a given prime section, will have freedom $r_k - \nu_k$; such primals may break up into the prime containing the base points, taken with a general primal of order $k-1$, or may have greater freedom than in that case; hence $r_k - \nu_k \geqslant r_{k-1}$, or $r_k - r_{k-1} \geqslant \nu_k$, which, if $k<(n-1)/(r-1)$, involves, we have seen, $r_k - r_{k-1} \geqslant k(r-1)+1$. We may put down this inequality for constantly diminishing values of k, until we come to

$$r_2 - r_1 \geqslant 2(r-1)+1, \text{ and } r_1, =r, =(r-1)+1;$$

hence, by addition,

$$r_k \geqslant \tfrac{1}{2}(r-1)k(k+1)+k, \qquad k<(n-1)/(r-1).$$

The greatest integer less than $(n-1)/(r-1)$ is evidently

$$\sigma = E\left(\frac{n-2}{r-1}\right), \quad = \frac{n-2-\epsilon}{r-1}, \quad \epsilon = 0, 1, \ldots, (r-2),$$

and as this is $\geqslant (n-r)/(r-1)$, we have $\nu_\sigma \geqslant \sigma(r-1)+1, \geqslant n-r+1$. This number is $>n-r$; hence, recurring to the lemma with which we began, we infer that the sets of the series $g_{r_\sigma}^{\sigma n}$ are not special. Wherefore $r_\sigma = \sigma n - p - d$, where d is the incompleteness of this series, which is $\geqslant 0$. This shews then that $p \leqslant \sigma n - r_\sigma$. We found however that $r_\sigma \geqslant \tfrac{1}{2}(r-1)\sigma(\sigma+1)+\sigma$. Therefore $p \leqslant \pi$, where

$$\pi = \sigma n - [\tfrac{1}{2}(r-1)\sigma(\sigma+1)+\sigma], \quad = \sigma[n-r-\tfrac{1}{2}(r-1)(\sigma-1)].$$

This is the result we desired to prove. For the other results we have quoted we refer to Castelnuovo's paper.

Ex. 1. The formula given above in Ex. 12, p. 216, leads, for a curve of order n, on a rational ruled surface of order $r-1$, which cuts each generator in k points, to the genus $\pi = (k-1)[n-r-\tfrac{1}{2}(r-1)(k-2)]$; this has the same value when k is replaced by h, given by

$$h+k-1 = 2(n-1)/(r-1).$$

Ex. 2. For a curve ${}^{p}c^{r+p}[r]$, of order n, equal to $r+p$, of genus p, lying in space of r dimensions, prove that, if $p<\tfrac{1}{2}r(r-1)$, the curve lies on

$\frac{1}{2}r(r-1)-p$ quadrics at least. And if the curve lies on exactly $\frac{1}{2}r(r-1)-p$ quadrics, then a complete series is determined on the curve by the quadrics of the space $[r]$.

On the sextic curve of genus 3 in space of three dimensions, obtained as the remaining intersection of a quadric surface with a quartic surface through 2 skew generators, the planes of the space, as also the cubic surfaces, cut a complete series; but quadric surfaces cut a series of incompleteness 1.

Ex. 3. For the canonical curve, of order $2p-2$ and genus p in space of $p-1$ dimensions, say the curve ${}^{p}c^{2p-2}[p-1]$, we have σ, equal to the greatest integer in $(2p-4)/(p-2)$, equal to 2, and

$$\pi=\sigma[n-r-\tfrac{1}{2}(r-1)(\sigma-1)]=2[p-1-\tfrac{1}{2}(p-2)]=p.$$

On this curve quadrics cut a non-special complete series, of freedom $2(2p-2)-p$, or $3p-4$. The number of quadrics containing the curve is therefore $\frac{1}{2}p(p+1)-(3p-3)$, or $\frac{1}{2}(p-2)(p-3)$. The number of primals of order μ containing the curve is $(\mu>1)$, $(p-1+\mu,\ \mu)-(2\mu-1)(p-1)$ (Noether, *Math. Annal.* XVII, 1880, p. 263, and *Math. Annal.* XXVI, 1886, p. 143. Expounded in the writer's *Abel's Theorem*, 1897, p. 154).

The curve is the complete intersection of the $\frac{1}{2}(p-2)(p-3)$ quadrics, for general values of p, unless the curve contain a special linear series $g_1{}^3$; but for $p=6$, if the curve contain a series $g_2{}^5$, there is also a residual intersection. See pp. 96, 178, above (Enriques-Chisini, III, p. 106, or *Rend....Bologna*, XXIII, 1919). When the curve has a series $g_1{}^3$, the quadrics have, as residual intersection, a rational ruled surface of order $p-2$; when (for $p=6$) there is a $g_2{}^5$, all the quadrics contain a Veronese surface in addition to the curve.

Ex. 4. For quintic curves in ordinary space, $n=5$, $r=3$, the integer $E[(n-2)/(r-1)]$ is 1, and unconditioned surfaces of every order cut a non-special series upon the curve.

Ex. 5. In space [4], the value of the extreme genus π, for curves of order n, is $\frac{1}{6}(n-2)(n-3)$, or $\frac{1}{6}(n-1)(n-4)$, according to the form of n (*mod.* 3). For example, when $n=6$, the greatest possible genus is 2. But in space [5], for $n=6$, the greatest genus is 1.

Ex. 6. In our derivation of the canonical series upon a curve C_1, lying on the surfaces $u=0$, $U=0$, by means of surfaces, Ψ, of order $m+M-4$, passing through the residual intersection C_2 of $u=0$, $U=0$, or through the t intersections of C_2 with C_1, we have not imposed the natural condition that $u=0$, $U=0$ should be the surfaces of least order containing the curve; nor have we explicitly supposed that the surfaces Ψ are irreducible.

Consider, for instance, the sextic curve of genus 4 which is the complete intersection of a quadric surface $u=0$, and a cubic surface $U=0$. There are 5 linearly independent cubic surfaces through this curve, with equation of the form $(ax+by+cz+dt)u+U=0$. Take two of these $u'=0$, $U'=0$, where $u'=xu+U$, $U'=yu+U$. These meet further in a plane elliptic cubic curve, C_2, with equations $x-y=0$, $U=0$, which meets the sextic curve, C_1, in six points lying on the conic $x-y=0$, $u=0$. The surfaces of order $3+3-4$, that is the quadric surfaces through the curve C_2, break up into the plane $x-y=0$ and an arbitrary plane determining the canonical series on C_1. The quadric surfaces through the six points (C_1, C_2) are given by an equation $\Psi'=0$, where, with arbitrary parameters p, q, r, s, k, $\Psi'=(x-y)(px+qy+rz+st)+ku$, and the surfaces $\Psi'=0$ likewise determine the canonical series on the curve C_1. Putting $\Psi'=(x-y)\,\Psi+ku$, we can easily verify that, on the curve C_1, we have

$\Psi'/(u', U', \lambda, \mu) = \Psi/(u, U, \lambda, \mu)$, so that the everywhere finite integrals on C_1 can be determined by use of $u'=0$, $U'=0$.

Ex. 7. The everywhere finite integrals on the curve ${}^5c^8[4]$, of order 8 and genus 5, which is the complete intersection of three quadrics in space of four dimensions, say $u=0$, $U=0$, $V=0$, are given by

$$\int \Psi(\lambda d\mu - \mu d\lambda)/(u, U, V, \lambda, \mu),$$

where λ, μ are arbitrary linear forms, and the denominator is the Jacobian determinant of 5 rows and columns.

Ex. 8. Consider the rational ruled cubic surface in space of four dimensions, which is obtainable by joining corresponding points of a conic and a line not meeting the conic. The curve which is the intersection of this surface with a general quadric is a sextic, of genus 2, meeting any conic of the ruled surface in 4 points. Thus primes through this conic meet the curve in the pairs of points of the canonical series on this curve ${}^2c^6$; these pairs are therefore the intersections of the curve with the generators. If a general quadric be drawn through one of these pairs, to meet the curve in 10 other points, the general quadric through these 10 points will also determine the canonical series on the curve. The curve in fact lies on 4 quadrics, and the 10 points furnish $15-4-2$, or 9, independent conditions for quadrics through them.

Now consider the canonical curve of order 10, and genus 6, in space of 5 dimensions, say ${}^6c^{10}[5]$. It is the intersection of a general quadric with a Del Pezzo surface ${}^2\psi^5$ (Bath, *Proc. Camb. Phil. Soc.* XXIV, 1928, p. 208). It can be shewn that three general quadrics (of the 5, linearly independent, which contain the surface), drawn through the Del Pezzo surface, meet again in a surface lying wholly in a space [4], namely in a ruled cubic surface therein; we have only to consider the section of the figure by a prime, or linear [4] space. Hence, four quadrics through the ${}^6c^{10}[5]$ meet again in a ${}^2c^6[4]$, meeting the former curve in 10 coprimal points. And, the canonical series on the ${}^6c^{10}[5]$, which (as it is a canonical curve) is given by the primes of the space [5], may also be given by the primals of order $(2+2+2+2-6)$, that is, by the quadrics, passing through these 10 intersections. The ${}^6c^{10}[5]$ lies on 6 quadrics, and the 10 points are equivalent only to $21-6-6$, or 9, conditions for quadrics passing through them.

If $\Psi=0$ be the general quadric through these 10 points, the everywhere finite integrals on the ${}^6c^{10}[5]$ are given by

$$\int \Psi(\lambda d\mu - \mu d\lambda)/(u, U, V, W, \lambda, \mu),$$

where the denominator denotes the Jacobian of two arbitrary primes, λ, μ, and the four quadrics from which the 10 base points of Ψ are determined. This integral also gives the everywhere finite integrals on the residual curve ${}^2c^6[4]$.

Ex. 9. A general theorem (Severi, *Rend. Palermo*, XVII, 1903, § 3) is that, if, through an irreducible curve C_1, in space $[r]$, of order n_1 and genus p_1, without multiple points, there be drawn $(r-1)$ primals, of orders $m_1, m_2, \ldots, m_{r-1}$, meeting again in an irreducible non-singular curve C_2, then the canonical series $g_{p_1-1}^{2p_1-2}$ on C_1 is determined by primals through C_2, of order $k-2$, where k is the sum of the $r-1$ numbers m_i-1; the whole number of linearly independent such primals is thus the sum of p_1 and the number of such primals which also contain C_1. Thus primals of order $>k-2$, through C_2, cut a non-special series on C_1.

A slightly more general theorem would however seem to be true; namely, that primals of order $k-2$, through the points of intersection of the curves C_1 and C_2, determine the canonical series on both C_1 and C_2;

and, further, if $\Psi=0$ be the general primal of this character, while $u_1=0, \ldots, u_{r-1}=0$ are the primals whose complete intersection consists of the curves C_1 and C_2, then the everywhere finite integrals of both curves are given by the formula $\int \Psi(\lambda d\mu-\mu d\lambda)/(u_1, u_2, \ldots, u_{r-1}, \lambda, \mu)$, where the notation is as before.

As in the case of ordinary space, the t intersections of C_1 and C_2 are equivalent only to $t-1$ conditions for the primals Ψ, of order $k-2$, required to pass through them. This follows from the fact that the postulation, $\chi(k-2)$, of the complete intersection (C_1+C_2) of $u_1=0, \ldots, u_{r-1}=0$, for primals of order $k-2$, is such that $\chi(k-2)=p_1+p_2+t-1$, where p_2 is the genus of C_2. If n_2 be the order of C_2, we have, in fact,

$$t=n_1k-(2n_1+2p_1-2)=n_2k-(2n_2+2p_2-2),$$

and
$$\chi(k-2)=1+\tfrac{1}{2}m_1 \ldots m_{r-1}(k-2).$$

The number $\chi(k-2)$ is the coefficient of x^{k-2} in the ascending expansion of $(1-x^{m_1})\ldots(1-x^{m_{r-1}})(1-x)^{-r-1}$. More generally, the postulation of the complete intersection of h primals, of orders $m_1, \ldots, m_h$, this intersection being supposed to be without multiple parts, for primals of order l, is the coefficient of x^l in the ascending expansion of

$$(1-x^{m_1})\ldots(1-x^{m_h})(1-x)^{-r-1}.$$

It appears probable that the formula given above for the everywhere finite integrals remains valid, as in ordinary space, when the curve C_2 is reducible, provided $\Psi=0$ be put through the mutual intersections of its components, as well as through their intersections with C_1.

Ex. 10. In particular, for the canonical curve ${}^pc^{2p-2}[p-1]$, it appears that the everywhere finite integrals are given by

$$\int \Psi(\lambda d\mu-\mu d\lambda)/(u_1, \ldots, u_{p-2}, \lambda, \mu),$$

where $u_1=0, \ldots, u_{p-2}=0$ are $p-2$ of the $\frac{1}{2}(p-2)(p-3)$ quadrics containing the curve $(p>4)$, and $\Psi=0$ is a primal of order $p-4$ passing through the $2(p-1)(p-5)$ intersections of the canonical curve with the residual intersection of $u_1=0, \ldots, u_{p-2}=0$. In fact, by a result quoted above (Ex. 3) there are $(2p-9)(p-1)$ primals of order $p-4$ which do not contain the curve. (Cf. Klein, *Math. Annal.* XXXVI, 1889, § 3.)

The theorem assumes that the canonical curve can be defined as the complete intersection of quadrics (cf. Ex. 3).

Ex. 11. When, for $p=6$, there exists, on the canonical curve, a $g_2{}^5$, this curve is the intersection of a Veronese surface with a cubic primal drawn through one of the conics of the surface. Obtain an expression for the everywhere finite integrals in this case.

Ex. 12. Prove Noether's theorem that, on any surface in ordinary space, the greatest genus, for curves of given order, arises for curves which form a complete section of the surface when taken with the whole, or part, of a plane section of the surface (Noether, "Grundzüge, u.s.w.", *Berlin. Abh.* 1882, § 6).

Ex. 13. The order of an irreducible algebraic curve which lies in a space of r dimensions (and not in space of lower dimensions) must be greater than $r-1$. If it have the order r, it is a rational curve, without multiple points. If it have the order $r+1$, it is elliptic or rational; in the latter case, the series cut upon the curve by primes of the space is incomplete, being contained in a series given by primals with equation of the form $u(\lambda_0x_0+\lambda_1x_1+\ldots+\lambda_rx_r)+U=0$, where $x_0, x_1, \ldots, x_r$ are the coordinates, u and U are polynomials in these, and $\lambda_0, \ldots, \lambda_r$ are variable parameters. The curve is then obtainable by projection from a curve of order $r+1$ in space $[r+1]$, for which the coordinates of a point are given by

$$\frac{\chi_0}{ux_0}=\frac{\chi_1}{ux_1}=\dots=\frac{\chi_r}{ux_r}=\frac{\chi_{r+1}}{U}.$$

A curve of order n, in any space, which is not a projection of a curve of order n in higher space, is said to be *normal* in its own space.

Prove that a plane curve, of order n, with double points, is normal in the plane when the number of its double points is less than $n-2$, or its genus exceeds $\frac{1}{2}(n-2)(n-3)$; but is not normal when the number of its double points exceeds $\frac{1}{2}(n-2)(n-3)$, or its genus is less than $n-2$. Consider the case when the genus is $n-2$.

Ex. 14. For an irreducible algebraic manifold of dimension k, lying in space $[r]$, but not in lower space, we may speak of $r-k$ as the *complementary dimension of the manifold*. The order, n, of the manifold is the number of its intersections with a linear space $[r-k]$. It can be shewn that the order of the manifold must exceed the complementary dimension of the manifold. For any n points lie in a space of dimension $<n$; and a space $[r-k]$ thus meets the manifold in points lying in a space of dimension ρ with $\rho<n$ and $\rho\leqslant r-k$. If we assume that there is a space for which $\rho=r-k$, we thus have $r-k<n$. Thus, in particular, the order of a surface must be at least $r-1$; it can be shewn that, when the order of the surface is $r-1$, the surface is a rational ruled surface (normal in the space), there being one exception to this, however, namely when $r=5$, and the surface is the Veronese surface. In general, a manifold of dimension k and order $r-k+1$ can be regarded as the aggregate of ∞^1 linear spaces each of dimension $k-1$, there being again an exception to this, namely when $k=r-3$ and the manifold is a cone projecting the Veronese surface from a linear space $[r-6]$ (or the Veronese surface itself). For these results, cf. Clifford, *Collected Papers*, p. 305; Del Pezzo, *Rend. Palermo*, I, 1887, p. 241 (proving that a surface of order r in space $[r]$ is rational); and Bertini, *Geom. d. iperspazi*, 1907, p. 193.

INDEX

The references are to the pages

Abel's great paper, involving investigation of genus of a curve, 59; theorem for a sum of algebraic integrals, 51, 54; converse of, 144, 171, 176; theorem of interchange of argument and parameter, 172

Adjoint polynomial for expression of rational function, 60, 62; adjoint ϕ-curves may have common point not on original curve, 72; may have common part, 72

Albanese, elimination of multiple points of a plane curve, 31

Alexander, on Castelnuovo's reduction of a Cremona transformation, 45

Algebraic series of sets of points, 3

Babbage, on a curve defined by a special series, 98

Bacharach, and Cayley, theorem for intersection of plane curves, 99

Bath, Canonical curve of genus six on Del Pezzo surface, 239

Benoist, *see* Clebsch, Lindemann-Benoist

Bertini, Cremona transformation of pure adjoint system, 72; theorem as to reducible linear system of curves, 72; equivalence of points of a set of a linear series, 77; postulation of the complete intersection of primals, 101; on the method of loops for periods, 119; on lowest order of a manifold of given dimension in given space, 241

Berzolari, on Weierstrass points, in *Enzykl. Math. Wiss.*, 90; article in Pascal's *Repertorium*, 213

Birational transformation of a curve in general terms, 51

Branch of a curve, 35, 184

Branch place on a Riemann surface, 122

Brill and Noether, Report on theory of algebraic functions, 45, 78; fundamental paper, 78

Brill, on number of special series, 92

Bronowski, on canonical forms, 109

Canonical form for the equation of a manifold, 107

Canonical series on a curve, 77, 79, 82; canonical curve, representing non-hyperelliptic curve, 81; canonical series determined from Jacobian series, 84; canonical series on curve in space, 226, 239

Castelnuovo, reduction of Cremona transformation to quadratic transformations, 45; on fundamental curve as part of adjoint system, 72; and Enriques, number of conditions which a surface can satisfy, 96; memoir on curves and surfaces, 224; maximum genus for curve of given order, 234

Cayley, sextactic points, 21; intersection of branches of a curve, 32; integration of a differential equation involving elliptic integrals, 56; and Bacharach, theorem for intersection of plane curves, 99; formulae for a curve in ordinary space, 191; deduced from general principles, 192; and Zeuthen, formulae for curve by projection and section, 197; use of term monoid, 214

Cell, in triangulation of a surface, 125

Chisini, *see* Enriques-Chisini

Clebsch, Lindemann-Benoist, theory of ternary cubic form, 21; and Gordan, treatise on Abelian theory, 78; on method of loops for integrals, 119

Clifford, theorem for special series, 32, 80, 82; extended, for general curve, 90; suggestion for minimum order of manifold of given dimension in given space, 241

Complementary special series, 82

Complete linear series, 65

Conditions, number which can be satisfied by a curve, 95, 96

Contact ϕ-curves, 95

Coresidual sets on a curve, 65; theorem of coresiduation, 67

Cross line, on a Riemann surface, 122; cross cut (Querschnitt), 127

Cubic curve, theorem of coresiduation, 19; investigation of points of inflexion, three methods, 19, 20; sextactic points, 21; Salmon's theorem of relation with Hessian, 22; covariant expression of finite integral thereon, 22

Curve, linear system of curves contained in a given system, 25; with two infinitely near triple points, representation in space, 34; curve systems on a surface, topological characters, 132; enumerative properties of curves in ordinary space, 182; definition of stall, cusp, inflexion, 186; notation used, 189; in higher space, enumerative formulae, 199; in ordinary space, which are complete intersections, 201; which are partial intersections, 208; expressed as intersection of manifolds, 182, 211; simplest cases, 212; on a quadric surface, 213; on a cubic surface, 214; curve intersection which is composite, 215; curve on a ruled surface, 216; on a manifold generated by linear spaces, 179

Cycle of values of one coordinate on a curve, 36

Dedekind, and Weber, modular treatment of integral algebraic functions, 59, 147

Dickson, equation for inflexions of plane cubic curve, 21

Difference of two complete linear series, 67

Dimension of an integral algebraic function, 150

Discriminant of a system of integral functions, 157

Dixon, equation of plane curve by symmetrical determinant, 110

Double planes which are rational, 110

Elliptic curve, 11, 13, 14, 15, 17, 18; Halphen's theorem of linear pencil, 23

Elliptic functions, character of, order of, 16

Enriques and Castelnuovo, conditions for a surface, 96; memoir on curves and surfaces, 224

Enriques-Chisini, *Teoria geometrica*, sextactic points, 22; multiple points, 45; number of moduli, 95; degeneration of a curve in space, 209; curve requiring four surfaces to define it, 211; on special series, 238

Enriques, quadrics containing the canonical curve, 96

Equivalence of a curve for surfaces through it, 210

Equivalent sets of points on a curve, 65

Euler's identity for a polyhedron, 133

Fields, theory of algebraic functions, 177

Finite integral, expression of, 64, 160

Freedom of a linear series, 3, 24

Fricke, *see* Klein-Fricke

Function, rational, *see* Rational

Fundamental theorem for residues of a rational function, 47

Galois, group invariants, 67; periods of algebraic integral, 116

Gap theorem, due to Weierstrass, 86, 89

Genus of a curve, 59, 77; definable by minimum order of rational function with arbitrary poles, 82; definable by winding index, 83, 86; by modular expression of rational function, 157, 161; of a composite curve, 209, 215; maximum possible for curve of given order, 234

Gergonne, on intersections of curves, 224

Gordan, *see* Clebsch and Gordan

Grade of a linear series, 3, 24, 78

Halphen, a pencil of elliptic curves, 23; theory of branches of a curve, 187; chords to curve of intersection of two surfaces, 204; memoir on curves, 213; curves on a quadric surface, 214; treatise on elliptic functions, 23

Hensel, on fundamental integral functions, 176; Hensel-Landsberg, theory of modular expression for algebraic functions, 176

Hesse, plane quartic curve expressed by a symmetrical determinant, 110; equation of osculating plane of curve in space, 206

Hessian of a pencil of plane cubic curves, 21

Hurwitz, minimum number of Weierstrass points, 90
Hyperelliptic curve, 27, 28; in standard form, analysis of multiple point, 33, 45; represented on a ruled surface, 33; characterised by existence of rational function of order two, 50, 79; reducibility of adjoint curve, 72; fundamental integrals for, 177

Independence of multiple points of a curve for adjoint curves, 72
Index of an algebraic series, 4
Inflexions of a plane cubic curve, 21
Integral, everywhere finite, expression of, 64, 160; periods, 111, 141; various kinds of, 136; elementary of the third kind, 137, 164; of the second kind, 137; elementary normal of the third kind, 143; periods, 141, 144; interchange of argument and parameter, 144, 171; modular expression of, 147; everywhere finite, used to transform a curve, 178; on curves in space, 231
Integral functions, fundamental system for, 149; properties of system, 151; fundamental matrix relation for, 155
Integration of total differential equations, 56
Interchange of argument and parameter, 144, 171
Inversion theorem, Jacobi's, 145
Involution on a rational curve is rational, 4

Jacobi, on intersection of curves, 224
Jacobian series of a linear series, 83, 85

Klein-Fricke, quartic curve with maximum number of collineations, 32; definition of canonical curve, 240
Krazer, treatise on theta functions, 146
Kronecker, memoir on algebraic functions, 67; modular treatment of integral functions, 147; elimination of multiple points, 157; prime function for algebraic functions, 172; algebraic definition of a manifold, 182; elimination from two binary forms, 202

Lasker, criterion for a canonical form, 109
Laurent's theorem, history of, 42
Lindemann, edition of Clebsch's lectures, 21
Linear series, 24, 59; complete, 65; sum and difference, 67; notation for, 78; on a curve in space, 217
Loops on a plane, for periods of algebraic integrals, 111, 113; loop cut on a Riemann surface, 128
Lüroth, on method of loops for periods, 119; reducibility of linear systems, 72

Macaulay, postulation of a manifold, 101
Manifold, expressed by intersection of primals, 182
Milne, on contact adjoint curves, 95
Mittag-Leffler, history of Laurent's theorem, 42
Modular theory of rational functions and integrals, 147
Moduli of a curve, in general, 94
Moebius, curve system on a surface, 132
Molk, exposition of Kronecker's methods, 182
Monogeneity of algebraic function, 44
Multiple points, elimination of, 24; maximum number for plane curve of given order, 10; minimum number for curve of general moduli, 93

Netto, equation for inflexions of plane cubic curve, 21; on an algebraic theorem, 202
Neumann, on Riemann's theory, 78, 132
Neutral set of points, 25
Newton's polygon for the analysis of a multiple point, 46
Noether, procedure for description of a multiple point, 45, 46; theorem for plane curves, for theorem of coresiduation, 67; and Brill, fundamental paper on theory of linear series, 78; Report on algebraic functions, 78; on number of conditions a curve can satisfy, 96; on composite curves, 100; and Severi and Bertini, theorem for postulation of a manifold, 101; chords to curve of intersection of two sur-

Noether (*cont.*)
faces, 204, 219, 224; memoir on theory of curves, 213; theorem for canonical series on a curve, 218; greatest genus of a curve on a surface, 240
Normal curve defined, 81; conditions for, 98

Olivier, theorem for the intersection of curves, 107
Order of a rational function on a curve, 46, 49
Oriented cell on a Riemann surface, 125
Oval cut on a Riemann surface, 123, 128

Palatini, on canonical forms, 109
Parametric representation of a branch of a curve, 39; deducible from non-singular space representation, 44
Pascal's *Repertorium*, article by Berzolari, 213
Periodic function, multiply, 146
Periods of algebraic integrals, 111, 136, 141; for everywhere finite integrals, array of, 141
Pezzo, Del, surface defined, 97, mentioned, 96, 98; paper referred to, 241
Picard-Simart, genus of a composite curve, 209, 215
Pick, covariant form for elliptic integral, 22
Pinhole boundary, on a Riemann surface, 127
Place, a point of a curve as centre of a branch, 35
Pliable linear system of curves, defining a simple linear series, 25
Postulation of a manifold for primals, 101
Prime function, Weierstrass's, for algebraic functions, 174
Puiseux, expression of branch of curve by a parameter, 43
Pure adjoint system of a curve, 72

Quadrics containing the canonical curve, 96, 238
Quartic plane curve, 13, 177

Rank, in topological theory of a surface, 127
Rational curve, 2, 5, 6, 7, 8, 13, 79, 197
Rational function, order of, 46, 49; fundamental theorem for residues of, 47; explicit expression of, 57; by adjoint polynomials, 63; minimum order of, 82, 92; condition for poles of, 86; expression by integrals, 139; fundamental integral, 140; expression, modular, 147
Rational series of sets of points, 2
Reciprocal functions from fundamental integral functions, 157, 164, 169
Reduction of linear system by contained system, 26, 29, 30
Riemann surface, 121; topology of, 124, 134; fundamental formula, 132; example, 134
Riemann-Roch formula, for series and functions, 68, 78, 170; Riemann on moduli, 94; normal elementary integrals, 143; relations for periods, 145
Roch, with Riemann, formula, 68, 78, 170
Room, number of conditions which a Veronese, or a Del Pezzo surface, can satisfy, 96

Salmon-Cayley, formulae for a curve in ordinary space, 191; deduced from general principles, 192
Salmon, chords to curve of intersection of two surfaces, 204; equivalence of a curve of intersection, 210
Scheibner, elimination from two binary forms, 202
Scorza, on canonical forms, 109
Scott, reduction of a certain curve, 32
Segre, sextactic points, 22; representation of a hyperelliptic curve on a ruled surface, 33; on Noether reduction of a multiple point, 45, 46; on minimum number of Weierstrass points, 90; Segre and Zeuthen, invariant of a surface, 135; relation for a curve lying on a scroll, 179; algebraic definition of a manifold, 182; formulae for curve in higher space, 200; on space in which rational ruled surface is normal, 235
Series, linear, on a curve in space, 217
Severi, treatise on algebraic geometry (1927), 31; Severi-Löffler, Algebraic

Severi (*cont.*)
geometry (1921), on number of moduli of a curve, 95; on loops for periods of algebraic integral, 119; on reduction of period system for multiply-periodic function, 146; degenerate forms of a general curve, 209; completeness of series on curve intersection, 226; canonical series on a curve in space, 219, 239
Sextactic points on plane cubic curve, 21
Simple linear series, 77, 83, 79
Special set and linear series on a curve, 77, 78, 81, 90, 92, 96, 97, 180, 236, 177, 238
Stall, point of stationary osculating plane of a curve, 186
Stationary point (cusp) of a curve, 186
Steiner, inflexions of a plane cubic curve, 21
Stiff linear system of curves defining a non-simple linear series, 25
Sum of two complete linear series, 67
Surfaces, Riemann, 121; topology of, 124, 134; fundamental formula, 130; example, 134
System of curves, topological characters, 132

Tact invariant of a curve, 86
Terracini, canonical forms for equation of quadrics, 108, 109
Theta function, construction of, 145; zeros of Riemann's, 145
Todd, cubic curves satisfying twelve conditions, 98; rational quartic curves, 98
Topology of a Riemann surface, 124; fundamental formula, 130
Transformation of a curve by a linear system, 26, 29, 83, 93; of non-hyperelliptic curve, 82
Traverse, on a Riemann surface, 127

Ursell, conics and quadrics satisfying eight and nine conditions, 98

Vahlen, curve requiring four surfaces for its definition, 211
Valentiner, memoir on curves in space, 213; chords to curve of intersection of two surfaces, 204
Veronese, formulae for curve in higher space, 200; Veronese surface, 96, 98

Wakeford, a criterion for a canonical form, 109
Weber, inflexions of a plane cubic curve, 21; modular treatment of algebraic functions, 147
Weierstrass, preparatory theorem, 36; on Laurent's theorem, 42; gap theorem, 86, 89; on periods of algebraic integrals, 119; prime function for algebraic functions, 174; normal form for a plane curve, 181; Weierstrass points on a curve in higher space, 201
Welchman, rational quartic curve, 98
White, on rational quartic curve, 98; a theorem for plane curves, 107
Winding index at a branch place, 44, 83, 156

Zeuthen, Text-book on enumerative geometry, the postulation of a particular set of points, 107; Zeuthen-Segre invariant, 135; and Cayley, formulae for curve in space, 197

CAMBRIDGE: PRINTED BY WALTER LEWIS, M.A., AT THE UNIVERSITY PRESS

PRINCIPLES OF GEOMETRY

LONDON
Cambridge University Press
FETTER LANE

NEW YORK · TORONTO
BOMBAY · CALCUTTA · MADRAS
Macmillan

TOKYO
Maruzen Company Ltd

All rights reserved

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., LL.D., F.R.S.
LOWNDEAN PROFESSOR, AND FELLOW OF ST JOHN'S COLLEGE,
IN THE UNIVERSITY

VOLUME VI

INTRODUCTION TO THE THEORY OF ALGEBRAIC SURFACES AND HIGHER LOCI

CAMBRIDGE
AT THE UNIVERSITY PRESS
1933

PRINTED IN GREAT BRITAIN

PREFACE

THE origin and final purpose of this volume, and the preceding, have been stated in the preface to the latter. It may be useful to describe in outline the contents of the present volume. The first chapter deals with the theory of correspondence, mainly of points on one or two curves, with inclusion of the treatment by transcendental methods, and the connection with the theory of defective integrals. The second chapter attempts an exposition of Schubert's remarkable ideas, which are as interesting logically as geometrically, and of the extension of the theory of correspondence to aggregates of any dimension. The third chapter is in part a reminder of theorems which belong to plane geometry, and in part a sketch of general theorems for rational surfaces. In the fourth chapter the elementary preliminary properties of surfaces in ordinary space, and in space of four dimensions, are dealt with. Chapter V is that which is concerned with the most interesting and the most novel ideas of the volume. For this reason it is written in a tentative introductory manner, and will best have served its purpose if it leaves the reader convinced of the importance of the theory involved, and with a desire to follow it further. The next chapter develops in detail the theory of the intersections of manifolds in space of four dimensions. The last chapter collects together various particular theorems, and some easy applications of foregoing theory. Only want of space has led to the exclusion from the volume of many other results which are of interest.

I should like to give expression to my sense of how much this, and preceding volumes, owe to those who have been students with me during their composition; my experience has been of a remarkable and unremitting keenness in the prosecution of the matters treated. Without this encouragement and co-operation, I might not have persevered in the formulation of the ideas, especially in these last two volumes. But besides this personal reference, I would add that it is clear that the purely geometrical and descriptive aspects of the subject are felt by many of our students to offer a discipline which is a welcome complement to others which are open to them.

And, I would repeat, readers of these volumes are under much obligation to the staff of the University Press for the trouble and attention with which the printing has been executed.

H. F. B.

5 *October* 1933

TABLE OF CONTENTS

CHAPTER I. ALGEBRAIC CORRESPONDENCE

PAGES

Part I. Elementary methods . . . . . . 1—46

Function expressing a correspondence with valency . . 5
Set of coincidences in the correspondence; number of coincidences 8, 9
Coincidences of $r+1$ points in a linear system of freedom r . 10
Valency of direct lateral correspondence 11
Bitangents of a curve from theory of correspondence . . 13, 14
Ruled surface by joins of corresponding points of two curves 15
Transformation of canonical series; Zeuthen's formula . 19
Analytic treatment of correspondence between two curves . 20—24
Formula for a curve on a ruled surface 25
Torsal lines, genus of double curve and number of triple points, on a ruled surface . . . . 26—28
Common chords of two curves, trisecants and quadrisecants of a single curve . . . . 28—34
Sets common to an involution and a linear series on a curve 35—37
Condition that sets of an involution belong to a linear series 37, 38
de Jonquières' formulae for contacts of curves . . . 39—43
Linear spaces incident with a curve in higher space . . 44—46

Part II. Transcendental methods. . . . . . 46—59

Relations for valency matrix in general . . . . 46—53
Valency matrix is numerical when the curve has general moduli 53
Coincidences in a general correspondence . . . . 54—56
Representation of correspondence by a curve, on the product of two curves . . . . 57—59

Part III. Correspondence and defective integrals . 59—68

Least number of period columns for a defective system of integrals 62
Theorem of complementary systems of defective integrals . 62—65
Application to theory of correspondence . . . . 65—68

CHAPTER II. SCHUBERT'S CALCULUS. MULTIPLE CORRESPONDENCE

Part I. Schubert's methods. Preliminary, as to notations. Characters of a manifold . . . 69—86

Fundamental conditions for linear spaces. The characters of a manifold . . . . 69—77
The calculus of conditions applied to a line. Examples of more general results . . . . 77—86

PAGES

Part II. The problem of multiple tangents of a manifold . . . . 86—92

Part III. Correspondence of points of two manifolds 92—105

Schubert's notation; application to two and three dimensions 96—98

General formula, and examples . . . . 99—105

Part IV. Pairs of corresponding linear spaces . . 105—108

Appendix. Some enumerative formulae . . . . 108—111

CHAPTER III. TRANSFORMATIONS AND INVOLUTIONS FOR THE MOST PART IN A PLANE

Cremona transformations in a plane . . . . . 112—120

Bertini's four types of involution in a plane . . . 121—130

The four involutions and rational double planes . . . 130, 131

Involutions of sets of more than two points . . . 131—137

Surface representing a plane involution cannot contain an irrational pencil 138, 139

The rationality of the representative surface of a plane involution . . . . 139—141

Number of cyclical sets in a correspondence . . . 142—145

Note. Surfaces with a pencil of rational curves . . . 145—147

CHAPTER IV. PRELIMINARY PROPERTIES OF SURFACES IN THREE AND FOUR DIMENSIONS

Elementary properties for general surface. Jacobian of a net of curves thereon . . . . 148—156

The surface with a double curve. Salmon's formulae . . 157—169

Surfaces in space of four dimensions . . . . . . 169—175

Note I. In regard to pinch points . . . . . . 176—180

Note II. Some formulae given by Noether 180, 181

CHAPTER V. INTRODUCTION TO THE THEORY OF THE INVARIANTS OF BIRATIONAL TRANSFORMATION OF A SURFACE, PARTICULARLY IN SPACE OF THREE DIMENSIONS

Preliminary; a general survey . . . . 182—185

Definition and illustration of the invariant I . . . 185—189

Definition of I in terms of an irrational pencil of curves . 189—191

Introductory definition of the invariant ω . . . . 191—198

Effect of isolated nodal points of the surface upon the definition of the invariants . . . . 198—200

PAGES
Applications to easy cases . . . . 201—206
Sketch of the proof of the invariance of I . . . . 206—214
The invariance of ω in a birational transformation . . 214, 215
The canonical system of curves of a surface . . . 215—217
The modification of the canonical system in a birational transformation . . . . 217—221
Return to the definition of the invariant ω . . . 221—224
The class of immersion and the canonical number of a curve on a surface 225
The computed characters of an exceptional curve . . 225, 226
The modification of the invariant ω in a birational transformation 226
Note I. Examination of fifteen examples cited by Noether 226—232
Note II. The adjoint surfaces of a given surface; the exceptional curves . . . . 232—237
Note III. The birational transformation of the Kummer and Weddle surfaces . . . . 237—240
Note IV. Miscellaneous examples 240, 241

CHAPTER VI. SURFACES AND PRIMALS IN FOUR DIMENSIONS. FORMULAE FOR INTERSECTIONS

The chord curve and trisecant curve for a surface in four dimensions. Rational surfaces . . . . . . 242—247
Intersections of loci in space of four dimensions, introductory 247, 248
Residual intersection of three primals with a common curve 248—251
Residual intersection of two surfaces with a common curve 251—255
Residual intersection of a primal and a surface having a common curve . . . . 255—257
Residual intersection of two primals having a surface in common . . . . 257—263
The postulation of a surface for primals containing it . . 263—266
Residual intersection of three primals having a surface in common . . . . 266—268
Various examples . . . . 268—270

CHAPTER VII. ILLUSTRATIVE EXAMPLES AND PARTICULAR THEOREMS

Particular examples of intersections . . . . . . 271—282
The surface representing pairs of points of one or two curves 282—294
Note on the multiple correspondence of two surfaces . . 294, 295
On complete sections of a non-singular primal . . . 295—298
Miscellaneous theorems and examples . . . . . 298—300

INDEX . . . . 301—308

CHAPTER I

ALGEBRAIC CORRESPONDENCE

Part I. Elementary methods. There is said to be an (s, r) correspondence between points P, Q of a single given curve when, if P is assigned, anywhere on the curve, there is determined a set, $Q_1, Q_2, \ldots, Q_r$, of r points, these being the r positions of Q corresponding to P, and, when Q is assigned, there is similarly determined a set $P_1, \ldots, P_s$, of s points, these being the s positions of P corresponding to Q. In particular, the set of points P so determined where Q is at any one of $Q_1, \ldots, Q_r$, say Q_i, is a set $P_1, \ldots, P_s$ of which P is one; and, when Q is at another of $Q_1, \ldots, Q_r$, say Q_j, the set so determined may be a different set, $P_1', \ldots, P_s'$, but, among these, P is still one. The operation by which we pass from P to $Q_1, \ldots, Q_r$ is called the *forward*, or *direct*, operation; it may be denoted by a symbol, say T, so that we may write $Q = TP$, but it must be borne in mind that TP has r significations. The operation by which we pass from Q to $P_1, \ldots, P_s$ is called the *reverse* operation, and may be denoted by T_1 (and, later, by T^{-1}), and we may write $P = T_1 Q$. It is supposed, in what follows, unless the contrary be stated, that the set $Q_1, \ldots, Q_r$ is determined from P by algebraical processes, so that any rational symmetrical function of the coordinates of $Q_1, \ldots, Q_r$, say $(z_1, t_1), \ldots, (z_r, t_r)$, is a rational function (in virtue of the equation of the curve) of the coordinates (x, y) of P. In particular there are two equations, satisfied respectively by $z_1, \ldots, z_r$ and by $t_1, \ldots, t_r$, say

$$u_0 z^r + u_1 z^{r-1} + \ldots + u_r = 0, \quad v_0 t^r + v_1 t^{r-1} + \ldots + v_r = 0,$$

in which $u_0, \ldots, u_r, v_0, \ldots, v_r$ are rational polynomials in x and y. These equations must be consistent with one another in virtue of the equation of the curve, $f(z, t) = 0$; and, if the coordinates be chosen with generality, it should be possible, by means of $f(z, t) = 0$, to obtain from the second equation a rational expression for t in terms of x, y, z. The relations are then expressed by two equations, say $u(x, y, z) = 0$, $t = R(x, y, z)$, where $u = 0$ denotes the first equation, and R is rational in x, y, z, the equation $f(z, t) = 0$, or

$$f[z, R(x, y, z)] = 0,$$

being satisfied in virtue of $f(x, y) = 0$ and $u = 0$. Thus the correspondence is given by $u = 0$, $t = R$, $f(x, y) = 0$, which involve $f(z, t) = 0$. And from these similar equations are determinable,

$$w(z, t, x) = 0, \quad y = S(z, t, x), \quad f(z, t) = 0.$$

A particularly simple case is that for a rational curve. Then any two points (x), (z), of the curve, can be characterised by parameters, say θ and ϕ. The expression of the correspondence is then by a single equation, which we suppose irreducible, say $R(\theta, \phi)=0$, where R is a rational polynomial both in θ and ϕ, of respective orders s and r in these. A special example is given, for the correspondence of the points $(\theta^2, \theta, 1)$, $(\phi^2, \phi, 1)$ of the conic $x-y^2=0$, by the equation

$$\theta^2(a\phi^2+h\phi+g)+\theta(h\phi^2+b\phi+f)+g\phi^2+f\phi+c=0;$$

this expresses that the two points (z_1, t_1), (z_2, t_2) of the conic, which correspond to the point (x, y) thereon, are the intersections of the conic with the polar of (x, y) taken in regard to another fixed conic.

More generally, if (x, y), (z, t) be points of a curve $f(x, y)=0$, or $f(z, t)=0$, connected by a relation $R(x, y, z, t)=0$, where R is a rational (non-homogeneous) polynomial in x, y, z, t, then, to a point (x, y) of the curve, there correspond the intersections with $f(z, t)=0$ of the curve $R(x, y, z, t)=0$, in which (z, t) are current coordinates, and to a point (z, t) the intersections with $f(x, y)=0$ of the curve $R(x, y, z, t)=0$, in which (x, y) are current coordinates. Such a relation $R(x, y, z, t)=0$ is necessarily of the form

$$u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0,$$

where u_i, v_j are rational polynomials; this equation expresses that the points (z, t) which correspond to a given (x, y), are points of a set belonging to the linear series, on the curve $f(z, t)=0$, obtained by its intersections with the curves $\lambda_0 v_0(z, t)+\ldots+\lambda_k v_k(z, t)=0$, namely, that set for which $\lambda_0, \ldots, \lambda_k$ are in the ratios of

$$u_0(x, y), \ldots, u_k(x, y).$$

Similarly, the points (x, y) which correspond to a given (z, t) are points of a set of a certain linear series. We have given a special example, of a $(2, 2)$ correspondence on a conic. Another very obvious example is that between the point of contact (x, y), of a tangent at any point of a curve $f(x, y)=0$, and the points (z, t), other than the point of contact, in which the tangent meets the curve. The correspondence is then expressed by an equation of the form $f_1(x, y).z+f_2(x, y).t+f_3(x, y)=0$, in which f_1, f_2, f_3 are the partial derivatives of f. In this case, if $f=0$ be of order n, the $(n-2)$ points (z, t) which correspond to the point (x, y), when taken with the point (x, y) itself, counted twice, are all the points of a particular set of the linear series determined by $\lambda z+\mu t+\nu=0$, where λ, μ, ν are parameters. And so in general, for the curve in (z, t) expressed (with given x, y) by the equation,

$$u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0,$$

it may happen that, beside fixed zeros common to all of

$$v_0(z, t)=0, \ldots, v_k(z, t)=0,$$

(which we ignore), there is a certain number of intersections, with the fundamental curve $f(z, t)=0$, which coincide with (x, y). In such case, if this number be denoted by γ, and the set of points (z, t), or Q, which correspond to (x, y), or P, be denoted by (Q), the composite set $\gamma P+(Q)$ forms a set of a certain linear series; and then, if (Q') be the points Q which correspond to another point (P'), the two sets $\gamma P+(Q)$ and $\gamma P'+(Q')$ belong to the same linear series, or, in the phraseology of Vol. v, Chap. iv, are *equivalent*, which we write as $\gamma P+(Q)\equiv\gamma P'+(Q')$. It is not to be assumed, however, though the set (Q) is determined from γ coincident points P, when the correspondence is established, that the complete linear series to which the composite set $\gamma P+(Q)$ belongs, is of freedom γ. We have indeed had above a simple example on a conic, in which the sets belong to a series of freedom 2, though γ is zero. And it will appear below that there are correspondences for which γ is negative, the equivalence above written being then understood to mean $-\gamma P'+(Q)\equiv -\gamma P+(Q')$. The number γ, in correspondences of the kind now considered, is called the *valency*.

A correspondence which is entirely defined by a single equation of the form $u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0$, has, clearly, the same valency as its reverse. For, if the left side, regarded as a function of (z, t), vanishes to order γ when (z, t) approaches to (x, y), regarded as a fixed point, then it is equally true that, regarded as a function of (x, y), it vanishes to order γ when the point (x, y) approaches to the point (z, t), regarded as fixed.

Conversely, we may define a correspondence, of indices (s, r), with (positive or zero) valency γ, between the places (x), (z) of the curve, by the properties: (1), that, when (x) is given, there is determined, algebraically, a set of r places $(z_1), \ldots, (z_r)$, any rational symmetrical function of these latter being expressible rationally by the coordinates x, y, of the point (x); and (2), that, when (z), one of $(z_1), \ldots, (z_r)$, is given, there is a similar determination of the set, $(x_1), \ldots, (x_s)$, of places, which correspond to (z) in the reverse correspondence; with (3), the further property, that, in the direct correspondence, the composite set consisting of $(z_1), \ldots, (z_r)$, and the place (x) taken γ times, are the points of a set of a linear series, which is the same for all positions of (x). For, with these hypotheses, we can construct a function, rational in the coordinates of both the places (x) and (z), which, as a function of (z), vanishes at $(z_1), \ldots, (z_r)$ as well as γ times at (x), and, as a function of (x), vanishes at $(x_1), \ldots, (x_s)$ as well as γ times at (z). This function, by its vanishing, expresses then both the direct and the reverse correspondence.

To prove this, we argue as follows: the meaning of the hypothesis (3)—if $(c_1), \ldots, (c_r)$ be the positions of $(z_1), \ldots, (z_r)$ corresponding to a position (a) of (x)—is, that there exists a rational function of (z), on the curve, having, for its zeros, the place (x), taken γ times, and the places $(z_1), \ldots, (z_r)$, which correspond to (x), taken each once, and having, for its poles, the place (a), taken γ times, and the places $(c_1), \ldots, (c_r)$, which correspond to (a), each taken once. Such a function is definite, save for a constant multiplier; dividing the function by its value at an arbitrary place (c) of the curve, we may then agree that it reduces to unity at (c). So determined, let the function be denoted by

$$R_c\left(z;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right).$$

The actual expression of such a function is to be found, by, first, forming the most general rational function of (z) which has $(a)^\gamma$, $(c_1), \ldots, (c_r)$ as poles, and, then, limiting the arbitrary constants, which enter linearly in such a function, by the condition that $(x)^\gamma$, $(z_1), \ldots, (z_r)$ are its zeros. This limitation is by linear equations; some, relating to the zeros $(x)^\gamma$, being explicitly rational in (x), the others, relating to the zeros $(z_1), \ldots, (z_r)$, being symmetrical in these, and, therefore, by the initial definition, also ultimately rational in (x). The function of (z) thus formed, depending on (x), is thus rational in (x).

Let us now consider this as a function of (x), depending on (z). As such, it vanishes to order γ when (x) approaches to (z). To the place (z) there correspond, by hypothesis, in the reverse correspondence, places $(x_1), \ldots, (x_s)$. When (z) is at any one of the places $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct correspondence, then (x) will be at one of the places $(x_1), \ldots, (x_s)$, which correspond to this position of (z) in the reverse correspondence. Thus, conversely, when (x) is at one of $(x_1), \ldots, (x_s)$, then (z) will be at one of $(z_1), \ldots, (z_r)$; and the function of (x), now under consideration, will vanish to the first order. The function will, we assume, have no other zeros than those mentioned. A like statement is true for the poles: the function of (z), considered above, has a pole of order γ when (z) is at (a), and has also a pole, of the first order, when (z) is at any one of the places $(c_1), \ldots, (c_r)$, which correspond to (a) in the direct correspondence; and this function is of the form

$$R\left(z;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right) \div R\left(c;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right),$$

and is infinite to the first order when (c) is at any one of the places $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct transformation; this arises when, and only when, (x) is at any one of the places

$(a_1), \dots, (a_s)$, which correspond to (c) in the reverse transformation; the quotient is also infinite, to order γ, when (x) is at (c); and is not otherwise infinite; and it reduces to unity when (x) is at (a), and, therefore, $(z_1), \dots, (z_r)$ are, respectively, at $(c_1), \dots, (c_r)$; also, the *infinities* of the numerator and denominator, of the quotient just written do not depend on the position of (x). Thus the function, regarded as depending on (x), may equally be represented by

$$R_a\left(x; \begin{matrix} z^\gamma, x_1, \dots, x_s \\ c^\gamma, a_1, \dots, a_s \end{matrix}\right).$$

Thus we have the result: *For an existing (s, r) correspondence, between the places (x), (z) of the given curve, assumed to be of valency γ (zero or positive) in the direct correspondence from (x) to $(z_1), \dots, (z_r)$, and such that any rational symmetric function of $(z_1), \dots, (z_r)$ is rational in (x), while to (z) there correspond $(x_1), \dots, (x_s)$ in the reverse correspondence, if (a), (c) be two arbitrary places, we can construct a function $\phi(x, z; a, c)$, rational in regard to both (x) and (z), and in regard to both (a) and (c); this function, regarded as a function of (z), is zero to order γ at the place (x), and, to the first order, at each of the places $(z_1), \dots, (z_r)$ which correspond to (x) in the direct correspondence; it is also infinite to order γ at the place (a), and, to the first order, at each of the places $(c_1), \dots, (c_r)$, which correspond to (a) in the direct correspondence. This function of (z) expresses, by its zeros, the direct correspondence. But, regarded as a function of (x), it expresses the reverse correspondence also, being zero to order γ at the place (z), and, to the first order, at the places $(x_1), \dots, (x_s)$, which correspond to (z) in the reverse correspondence, beside having a pole of order γ at the place (c), and poles, of the first order, at the places $(a_1), \dots, (a_s)$, which correspond to (c) in the reverse correspondence. Thus the reverse correspondence is equally of valency γ.*

A simple illustration is that remarked above, where the direct correspondence leads from a place (x) to the $(n-2)$ places $(z_1), \dots, (z_{n-2})$, in which the curve is met by the tangent of the curve at the place (x). The reverse correspondence then leads from a place (z), of the curve, to the $(n'-2)$ places $(x_1), \dots, (x_{n'-2})$, which are the points of contact of the tangents to the curve drawn from (z). The equation of the tangent at (x), in which (z) is used as current coordinate, is equally the equation, with (x) as current coordinate, of the first polar of a point (z). The fact that the valency of the reverse correspondence, like that of the direct correspondence, is 2, is the fact that the first polar, of a point (z) of the curve, touches the curve at this point.

There are two general definitions in regard to correspondence which it is convenient to employ: (a), If we have two correspondences, in which there correspond to P, in the direct correspondences, respectively the sets (Q) and (R), then we shall regard the composite set, consisting of the sets (Q), (R) together, say the set $(Q)+(R)$, as being in correspondence with P, directly, in a correspondence which we call the *sum* of the two given correspondences. It can then be seen that the reverse of this sum correspondence is likewise the sum of the reverses of the two original correspondences. For the correspondence reverse to T, which we denote by T_1, is defined by the fact that T_1, applied to any one of the points TP, gives rise to a set of which P is one point. Now consider $(T+U)\,P$, which is defined as the aggregate of the sets $(Q)=TP$ and $(R)=UP$. The operation $(T+U)_1$, reverse to $(T+U)$, is that which, applied to any one of the set TP, or to any one of the set UP, gives a set including P. This is satisfied only if $(T+U)_1=T_1+U_1$. In symbols,

$$\begin{aligned}(T_1+U_1)(T+U)P&=(T_1+U_1)(TP+UP)\\&=T_1TP+T_1UP+U_1TP+U_1UP,\end{aligned}$$

and P is among the latter aggregate. Beside this definition of the sum of two correspondences, is, (*b*), that of the *product* of two correspondences: If to P, in a direct correspondence T, corresponds the set (Q), of which Q_i denotes every element in turn; and to Q_i, in another direct correspondence U (independent of i), corresponds a set $(R)_i$, the set (Q) consisting of q points, and each set $(R)_i$ consisting of r points, then we regard the aggregate of rq points forming all the sets $(R)_i$, as corresponding to P in a direct correspondence, which we call the product of T by U, and denote by UT, the operation T being that first applied. It is easy to see that the reverse correspondence $(UT)_1$, is T_1U_1.

For correspondences with valency, as considered above, the valency of $T+U$ is the sum of the valencies of T and U. For, if, in a notation employed already,

$$(Q)+\gamma P\equiv(Q')+\gamma P',\quad (R)+\delta P\equiv(R')+\delta P',$$

then also

$$(Q)+(R)+(\gamma+\delta)\,P\equiv(Q')+(R')+(\gamma+\delta)P'.$$

For the product of two correspondences, however, the valency of UT is the negative product of the valencies of T and U. For, from $\Sigma Q_i+\gamma P\equiv\Sigma Q_i'+\gamma P'$, $(R)_i+\delta Q_i\equiv(R')_i+\delta Q_i'$, of which the latter gives $\Sigma(R)_i+\delta\Sigma Q_i\equiv\Sigma(R')_i+\delta\Sigma Q_i'$, we have

$$\Sigma(R)_i-\delta\gamma P\equiv\Sigma(R')_i-\delta\gamma P'.$$

Thus, also, the (generally different) correspondences UT and TU have the same valency, which is also the valency of U_1T, TU_1, UT_1, T_1U, U_1T_1, T_1U_1.

A simple correspondence is obtained by considering, on the fundamental curve, any linear series of sets of points which has freedom unity. Any point P of the curve then determines the other points (Q), of the set to which P belongs; and there is thus a correspondence of valency unity, in which the points corresponding to P are these other points (Q), of the set of the linear series determined by P. In this case, since a set of the series is equally determined by any one of its points, the reverse correspondence is the same as the original. Further, by what has been said, the product of any two such simple correspondences, determined by any two linear series of freedom unity, is a correspondence whose valency is negative unity. By the sum and product of such simple correspondences we can thus set up a correspondence of any arbitrary valency, positive, zero or negative.

The chief utility of the theory of correspondence, in geometry, arises by the consideration of the *united* points, or *coincidences*; these are the positions of P in which one of the points Q, corresponding thereto, coincides with P.

Consider first the easy case of a rational curve: let a correspondence, between two points represented by parameters θ, ϕ, be given by an equation $(\theta, \phi)=0$, of order s in θ and order r in ϕ. The coincidences arise then corresponding to the roots of the equation $(\theta, \theta)=0$, and are $s+r$ in number, provided the original relation contains the term $\theta^s\phi^r$, that is, provided the parameters are so chosen that $\theta=\infty$ is not a coincidence. Further, the set of points forming the coincidences is equivalent with the aggregate of the two sets (Q), $(Q)_1$, which correspond to a point P in the direct and reverse correspondences; indeed, on a rational curve, any two sets of the same number of points are equivalent with one another. The same equivalence, of the set of coincidences with the aggregate of the sets corresponding to any point in the direct and reverse correspondences, holds on any curve, for the coincidences in a correspondence which is of valency zero.

The proof of this remark, as to the coincidences for a correspondence, on any curve, which is of valency zero, arises from the expression of the correspondence by the vanishing of a polynomial $\Sigma u_i(x, y)v_i(z, t)$, the numerator of the function $\phi(x, z;\, a, c)$ investigated above for any correspondence. This, as a polynomial in (z, t), has, when the valency is zero, no zeros coinciding with (x, y). The remark made is the same as the statement that the zeros of $\Sigma u_i(x, y)v_i(x, y)$ are a set equivalent to the aggregate of the set (Q), given by $\Sigma u_i(x_0, y_0)v_i(z, t)=0$ with $f(z, t)=0$, and the set $(Q)_1$, given by $\Sigma u_i(x, y)v_i(x_0, y_0)=0$ with $f(x, y)=0$. The theorem is contained in what follows, but is necessary as a lemma.

In general, however, the set of coincidences (U), of a correspondence of valency γ, in which the direct and reverse sets corresponding to a point P are (Q) and $(Q)_1$, upon a curve on which a canonical set, of $2p-2$ points, is denoted by (K), satisfies the equivalence $(U)\equiv(Q)+(Q)_1+\gamma[2P+(K)]$. To prove this important result, remark, first, that, if true for each of two correspondences, it is true for the correspondence which is their sum; for the coincidences in the sum are evidently the aggregate of those in the separate correspondences. Then, next, the formula is clearly true for the simple correspondence, of valency unity, in which P and (Q) together, as also P and $(Q)_1$ together, are a set of a linear series of freedom unity, on the curve. For, in this case, it becomes the fundamental result $(J)\equiv(L)+(M)+(K)$, in which (L) and (M) are any two sets of the linear series, and (J) is the Jacobian set of the series (Vol. v, p. 84). From these facts it follows that the formula is true for any correspondence, of positive valency γ, which can be obtained as the sum of γ such simple correspondences. Being assumed to be true, in virtue of the lemma preceding, for any correspondence of zero valency, it is therefore true, for any *general* correspondence, of negative valency $-\gamma$, not necessarily assumed to be obtained by composition of the simple correspondences referred to. Hence, also by the lemma, it is true for any general correspondence of positive valency γ.

We may also deduce this result for the coincidences directly, by use of the function $\phi(x, z; a, c)$ obtained above. Suppose the valency to be positive. Consider the function of (z) which is expressible, in the notation employed (p. 4, above), in either of the forms

$$(z-x)^{-\gamma}R_c\left(z;\begin{matrix}x^\gamma, z_1, \ldots, z_r\\ a^\gamma, c_1, \ldots, c_r\end{matrix}\right),\quad (z-x)^{-\gamma}R_a\left(x;\begin{matrix}z^\gamma, x_1, \ldots, x_s\\ c^\gamma, a_1, \ldots, a_s\end{matrix}\right),$$

(x) being a general place; and therein suppose (z) to approach to coincidence with (x). For simplicity of statement, suppose that the curve, $f(x, y)=0$, of order n, has n distinct places at $x=\infty$. Putting, in terms of the representative parameter θ for the neighbourhood of (x), $z-x=\theta(dz/dx)+\frac{1}{2}\theta^2(d^2z/dx^2)+\ldots$, where (dz/dx), etc., are the values at (x), a factor θ^γ, in R_c, will cancel a factor $\theta^{-\gamma}$ in $(z-x)^{-\gamma}$. Further, as this function R_c vanishes to the first order when (z) is at any of the places, $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct correspondence, the function of (x) obtained by putting (z) at (x) will vanish when (x) is at any of the places $(z_1), \ldots, (z_r)$; also, as we see with the help of the second form of the function, this function of (x) will have poles of order γ both at (a) and (c), and poles of the first order at each of $(c_1), \ldots, (c_r), (a_1), \ldots, (a_s)$. The function (dz/dx), regarded as a function of (x), vanishes to order $k-1$ when (x) approaches a branch place of index k (where there is a cycle of k values); in all, then, (dz/dx) has $\Sigma(k-1)$, or, say, w zeros, at such places—whose aggregate we may denote by (w); and it is known that $w=2n+2p-2$; the function of $x,(dz/dx)$, is also infinite to the second order at each of the n places at $x=\infty$. We may denote the aggregate of these n places by (∞).

On the whole, then, when (z) coincides with (x) we obtain a function of (x) whose zeros and poles are those represented by the symbols, in the numerator and denominator, respectively, in the fractional scheme

$$\frac{(U)(\infty)^{2\gamma}}{(w)^\gamma(a^\gamma)(c^\gamma)(c_1)\dots(c_r)(a_1)\dots(a_s)}.$$

In making this deduction, it is assumed that the function of (x) which is the factor of θ^γ in R_c, when (z) approaches to (x), gives only those zeros and poles which we have stated. From this scheme, if ν be the number of coincidences of the correspondence, equating the numbers of zeros and poles of this rational function, we hence deduce $\nu+2\gamma n=(2n+2p-2)\gamma+2\gamma+r+s$, or $\nu=r+s+2\gamma p$.

Further, if u denote any everywhere finite integral of the curve, the rational function of (x), given by du/dx, has zeros and poles which are indicated by the scheme

$$\frac{(K)(\infty)^2}{(w)},$$

where (K) denotes a canonical set. Dividing then the previous function of (x) by $(du/dx)^\gamma$, we obtain a rational function whose zeros and poles are those denoted by the scheme

$$\frac{(U)}{(K)^\gamma(a^\gamma)(c^\gamma)(c_1)\dots(c_r)(a_1)\dots(a_s)},$$

so that we have the equivalence $(U)\equiv(Q)+(Q)_1+\gamma[2P+(K)]$ originally stated. A function of the character of $\phi(x, z;\ a, c)$ can be constructed when the valency γ is negative, and then a similar argument leads to the same equivalence for this case.

Remark. A correspondence, such as the simple correspondences we have constructed from a linear series of freedom unity, in which the distinction between the direct and the reverse correspondence is lost, or say $T_1=T$, is called *symmetrical.* The number of distinct points of coincidence of such a correspondence is $\frac{1}{2}\nu$, namely is $r+p\gamma$.

Ex. 1. To illustrate the formation of the function $\phi(x,z;\ a,c)$, and the limit of $(z-x)^{-\gamma}\phi(x,z;\ a,c)$ when (z) approaches to (x), we may consider the simple example of the correspondence, on a plane cubic curve $f(x,y)=0$, in which, to (x) corresponds the point (z), in which the tangent at (x) meets the curve again, while, to (z) correspond the four points $(x_1),\dots,(x_4)$, in which the tangents drawn from (z) touch the curve. Then $\gamma=2$. Denoting the function $(z-x)\partial f/\partial x+(t-y)\partial f/\partial y$, or, say, $(z-x)f_1+(t-y)f_2$, by $T_{x,z}$, the function $\phi(x,z;\ a,c)$ is

$$T_{x,z}T_{a,c}/T_{a,z}T_{x,c}.$$

If, further, H denote the Hessian form $|\partial^2 F/\partial\xi_i\partial\xi_j|$, where

$$F=\xi_3^3 f(\xi_1\xi_3^{-1},\ \xi_2\xi_3^{-1}),$$

we easily compute that $H=-4\xi_3^7(f_1^2f_{22}-2f_1f_2f_{12}+f_2^2f_{11})$, and, when (z) approaches to (x),

$$(z-x)^{-2}T_{x,z}=-\tfrac{1}{2}f_2^{-2}(f_1^2f_{22}-2f_1f_2f_{12}+f_2^2f_{11}).$$

Thus, putting $\xi_3=1$, $\xi_1=x$, $\xi_2=y$, $(z-x)^{-2}\phi(x, z;\, a, c)$ becomes

$$[\tfrac{1}{8}H/(\partial f/\partial y)^2][T_{a,c}/T_{a,x}T_{x,c}].$$

Here, the Hessian H vanishes in the $1+4+2.1.2$, or 9, inflexions of the curve, which are the coincidences of the correspondence, while $T_{a,x}$ vanishes twice at (a), and once at its tangential (c_1), and $T_{x,c}$ vanishes twice at (c), and once at the four points of contact of tangents from (c); while $\partial f/\partial y$ vanishes once at each of the six points of contact, (w), of tangents parallel to $x=0$, and, being of the second order, is infinite to the second order at the three places (∞). The function which expresses the equivalence of the inflexions with the aggregate $(a^2)(c^2)(c_1)(a_1)\ldots(a_4)$ is then $H/T_{a,x}T_{x,c}$.

Ex. 2. Considering, upon a curve of genus p, a linear series g_r^n, of sets of n points, of freedom r, and the sets of this series in which r points are taken coincident, there is, between each such point of coincidence, P, and the $n-r$ remaining points (Q) of the set determined by P, a correspondence of valency r; in this, there correspond to P the $n-r$ points (Q), and to Q correspond, reversely, a certain number, say ξ, of points P. Let the number of coincidences of a point Q with P be denoted by $x_{n,r}$; this is the number of sets of the series g_r^n for which $(r+1)$ points of the set are coincident. The sets from g_r^n of which a given point Q is one point, form a linear series g_{r-1}^{n-1}; in this are $x_{n-1,\,r-1}$ sets which contain r coincident points; this is then the number, ξ, required in the correspondence (P, Q), of points P which correspond reversely to Q. Thus, using the formula found above for the number of coincidences in a correspondence, we see that $x_{n,r}=x_{n-1,r-1}+n-r+2pr$; if we put

$$y_{n,r}=x_{n,r}-(r+1)[n+r(p-1)],$$

this equation is the same as $y_{n,r}=y_{n-1,r-1}$. But

$$y_{n,1}=2n+2p-2-2[n+p-1]=0;$$

thus, in general, $y_{n,r}=0$, and the required number $x_{n,\,r}$ is given by

$$x_{n,r}=(r+1)[n+r(p-1)].$$

This formula gives the total number of cases; in any particular application it is necessary to consider whether (when r is not a prime) the r-fold coincidences can arise by repetitions of r_1-fold coincidences, in which r_1 is a factor of r. Further, it is to be remarked that it is not assumed that the series g_r^n is complete; if, for example, we had $r=n-p$, then $x_{n,r}$ would reduce to $(r+1)^2p$.

Ex. 3. If we have, upon the curve, two correspondences, (s, r, γ) and (s', r', γ'), of respective valencies γ and γ', in which there correspond to P the respective sets (Q) and (Q'), of r and r' points, then the number of times it happens that, for the same P, a point Q coincides with a point Q', is $rs'+r's-2p\gamma\gamma'$.

Ex. 4. Two linear series, $g_1{}^h$, $g_1{}^k$, have a pair of points in a set of one series coinciding with a pair of points in a set of the other series, in $(h-1)(k-1)-p$ ways.

Ex. 5. On a general plane quartic curve, a direct correspondence from P to three points Q, is given by conics, drawn through three fixed points A, B, C of the quartic curve, which touch the curve at P. Shew that the reverse correspondence, from Q to (P), is of index 12, and of valency 2, being given by quintic curves, with double points at A, B, C, which touch the quartic curve at Q. If $f(x, y, z)=0$ represent the quartic curve, referred to A, B, C as triangle of coordinates, the coordinates of P, Q being respectively (x, y, z) and (ξ, η, ζ), prove that the correspondence is given by

$$\eta\zeta x^2 \frac{\partial f}{\partial x} + \zeta\xi y^2 \frac{\partial f}{\partial y} + \xi\eta z^2 \frac{\partial f}{\partial z} = 0.$$

Generalise this to the case of an (r^2p, p) correspondence, on a plane curve of order m, with multiple points (i), when (Q) is determined from P by adjoint curves of order $m-2$, with a number of fixed points on the curve given by $\frac{1}{2}(m-2)(m+1)-\frac{1}{2}\Sigma i(i-1)-r$, which have r-pointic contact at P with the fundamental curve. Obtain the curves, with r-ple intersection at Q, which determine the corresponding r^2p points (P).

There is a particular theorem, sometimes of use, which it is convenient to explain here, while the notation is familiar.

The direct correspondence, from (x) to $(z_1), \ldots, (z_r)$, enables us to establish a correspondence between, say, (z_1), and the other points of the set $(z_1), \ldots, (z_r)$. For this, when (z_1) is given, we should pass, reversely, from (z_1) to $(x_1), \ldots, (x_s)$, and then, directly, from any one of these, say (x_i), to a set consisting of (z_1) and $(r-1)$ places, say, $(z_{i,1}), \ldots, (z_{i,r-1})$. There exists, therefore, a direct correspondence of (z_1) with $s(r-1)$ places $(z_{i,j})$. Evidently this is a symmetrical correspondence, the reverse set consisting also of $s(r-1)$ places. We proceed to find the valency of this correspondence.

As $(x_i)^\gamma, (z_1), (z_{i,1}), \ldots, (z_{i,r-1})$ are the zeros of a rational function, so are the $s(r+\gamma)$ places

$$(x_1)^\gamma, \ldots, (x_s)^\gamma, (z_1)^s, (z_{1,1}), \ldots, (z_{1,r-1}), \ldots\ldots, (z_{s,1}), \ldots, (z_{s,r-1});$$

but so also are $(z_1)^\gamma, (x_1), \ldots, (x_s)$, and hence also $(z_1)^{\gamma^2}, (x_1)^\gamma, \ldots, (x_s)^\gamma$. Thus the places, obtained by taking the latter set from the former, namely $(z_1)^{s-\gamma^2}, (z_{1,1}), \ldots, (z_{1,r-1}), \ldots\ldots, (z_{s,1}), \ldots, (z_{s,r-1})$, are the zeros of a rational function; this shews that, in the symmetrical correspondence spoken of, of indices $s(r-1)$, $s(r-1)$, between (z_1) and $(z_{i,j})$, the valency is $s-\gamma^2$. The number of distinct coincidences, of (z_1) with a $(z_{i,j})$ is therefore $s(r-1)+p(s-\gamma^2)$.

We may call this correspondence the *direct lateral correspondence* derived from the original correspondence T. If it be denoted by X, and the correspondence on the curve which leaves any point P unaltered, say the *identical* correspondence, be denoted by I, the symbolical representation of the process we have followed (if T_1 be the reverse of T), is $TT_1 = sI + X$. The valency of X is thus obtainable by subtracting from the valency of TT_1 (which is $-\gamma^2$), s times the valency of I. This identical correspondence, expressing that the point P, and its corresponding point Q, are identical, or $P - Q \equiv 0$, is, however, of valency -1. Thus the valency of X is $-\gamma^2 - (-s)$, or $s - \gamma^2$, as we have found.

More generally, denoting by α_i, β_i the indices, by γ_i the valency, and by u_i the number of coincidences of a correspondence, suppose we have several correspondences, from P to the set (Q), from P to the set (R), and so on, which are such that there exists the equivalence

$$kP + k_1(Q) + k_2(R) + \ldots \equiv 0;$$

as we have $(Q) + \gamma_1 P \equiv 0$, $(R) + \gamma_2 P \equiv 0$, etc., this is merely the same as $k = k_1\gamma_1 + k_2\gamma_2 + \ldots$. Then we have

$$k_1(u_1 - \alpha_1 - \beta_1) + k_2(u_2 - \alpha_2 - \beta_2) + \ldots = 2kp,$$

as was remarked by Cayley (*Papers*, VI, p. 12).

An application of the formula for the valency of the direct lateral correspondence is to find the number, due to Brill, given in Ex. 3 preceding, for the number of pairs of points which correspond to one another in *two* correspondences. For this number is the number of coincidences in the direct lateral correspondence of the sum of the two given correspondences, less the numbers of coincidences in the direct lateral correspondences of both the given correspondences—the difference obtained to be halved, on account of the symmetrical character of the resulting correspondence.

We now proceed to give applications of the theory of correspondence of increasing difficulty, which for clearness we tabulate as Exx. 1–15 (pp. 12–46).

Ex. 1. Apply the formula to obtain the *general* number, $2(n'-3)(n-4) + 2(n'-12)p$, of osculating planes of a curve of order n, and class n', and genus p, in ordinary space, which touch the curve again. (Here $s = n'-3$, $r = n-3$, $\gamma = 3$.)

Ex. 2. A natural application of the formula is to the bitangents of a plane curve, of order n. Between the point of contact of a tangent of the curve, and the remaining intersections of this line with the curve, is a direct correspondence (s, r) for which $r = n-2$; if n' be the class of the curve, the number of proper tangents from an arbitrary point, we may take, in the reverse correspondence, $s = n'-2$, the points (x), reversely corresponding to (z), not being supposed to include the cusps of the curve, if such exist; denote the number of cusps by κ. Then the formula, $2s(r-1) + 2p(s-\gamma^2)$, for the number of coincidences in the direct lateral correspondence, each counted twice, gives, for *twice* the number of bitangents of the curve, $2(n'-2)(n-3)+2p(n'-6)-\kappa(n'-3)$, the correction $\kappa(n'-3)$ being for the $n'-3$ proper tangents to the curve from each of the cusps. If however we allow the cusps, as points of contact of tangents to the curve from (z), we must take $s=n'-2+\kappa$; then twice the number of bitangents is given by

$$2(n'-2+\kappa)(n-3)+2p(n'-6+\kappa)-2\kappa(n'-3)-(\kappa^2-\kappa),$$

where there is correction, (1), for tangents drawn from a cusp, (2), for lines from a cusp to other cusps.

The second result agrees with the first, and with the number $2\delta+(n'-n)(n'+n-9)$, where δ is the number of double points of the curve.

Ex. 3. For the particular case of a plane quartic curve, if (z_1), (z_2) be the two further intersections of the tangent at a point (x), the correspondence, from (x) to (z_1) and (z_2), having $r=2$, $s=10$, $\gamma=2$, there are $s(r-1)+p(s-\gamma^2)$, or $10+3.6$, or 28 coincidences, which is the number of bitangents of the curve. If the equation of the curve be written, in the usual symbolical way, as $a_x{}^4=0$, the correspondence between (z_1) and (z_2) is at once found to be expressible by $9(a_{z_1}{}^2a_{z_2}{}^2)^2-16a_{z_1}{}^3a_{z_2}.a_{z_1}a_{z_2}{}^3=0$; this is the condition that the chord from (z_1) to (z_2) touches the curve. The points of contact, (x), of the bitangents of the quartic curve, thus lie on a curve obtainable by finding the limit, when (z) approaches to (x), of

$$[9(a_z{}^2a_x{}^2)^2-16a_z{}^3a_x.a_za_x{}^3](z-x)^{-6}.$$

From this, or directly, it follows, that, if (ζ) be any point of a bitangent of which one point of contact is (x), then

$$[2(a_xa_\zeta{}^3)^2-3a_x{}^2a_\zeta{}^2.a_\zeta{}^4](\zeta-x)^{-6}=0.$$

Using (x, y) and (ζ, σ) for the non-homogeneous coordinates of (x) and (ζ), and eliminating σ by means of the equation of the tangent of the curve at (x) (which is of the form $u\zeta+v\sigma+w=0$, where u, v, w are definite functions of (x)), the factor $(\zeta-x)^{-6}$ will identically disappear. In other words, if the expression here multiplying $(\zeta-x)^{-6}$ be denoted by $\phi(x, y, \zeta, \sigma)$, the function $v^6\phi(x, y, \zeta, \sigma)$ divides by $(\zeta-x)^6$ in virtue of $u\zeta+v\sigma+w=0$. The quotient will then be of order, in x, y, equal to $6.3+2-6$, or 14. This gives a curve of order 14, containing all the points of contact of the 28 bitangents. Cf. also Salmon's *Higher Plane Curves*; and Dersch, *Math. Ann.* VII. (Below, p. 298.)

In particular shew, for the quartic curve $x^4+y^4+z^4=0$, that the curve of order 14 may be taken to be $x^2y^2z^2(x^8+y^8+z^8)=0$. The 28 bitangents, in this case, consist of 16, each with an equation of the form $\theta x+\phi y+z=0$, where $\theta^4=1$, $\phi^4=1$, together with 12 others each meeting the curve in four coincident points, these 12 bitangents being $(y^4+z^4)(z^4+x^4)(x^4+y^4)=0$.

Ex. 4. It is known that there are sets of six pairs, of the bitangents of a plane quartic curve, such that the eight points of contact, of any two of these pairs, lie upon a conic. The points of contact of the 28 bitangents of the curve lie on the degenerate curve of order 14 which is composed of seven such conics, properly taken (and in various ways).

This we may most easily see by regarding the quartic curve as the projection of the profile of a cubic surface, as seen from any point of this surface. It is well known that, if a, b' be two intersecting lines of a cubic surface, meeting another line, c, of the surface, and a', b be two other intersecting lines, also intersecting c, the four bitangents of the quartic curve which arise by projection of a, b', a', b have their points of contact on a conic (*Proc. Lond. Math. Soc.* IX, 1911, p. 163). Hence, if f, g, h be three coplanar lines of the surface, and f_1, f_1' be intersecting lines which meet f, other than g or h, the other such pairs of intersecting lines which meet f being f_2, f_2'; f_3, f_3' and f_4, f_4', then we obtain one such conic as contemplated by projection of f_1, f_1' and f_2, f_2', and another such conic by projection of f_3, f_3' and f_4, f_4'. Proceeding similarly with the lines g and h, we thus obtain six conics in the plane, from f, g and h. Another conic contains the points of contact of the bitangents which arise by projection of f, g, h themselves; and this conic contains the points of contact of the bitangent arising from the neighbourhood of the point of the surface from which the projection is made. The whole 27 lines of the surface are given by f, g, h and the couples f_i, f_i'; g_j, g_j'; h_k, h_k'. Thus the 56 points of contact of the 28 bitangents lie on the seven conics described.

The suggestion that the 56 points of contact lie on 7 conics was made to the writer by Dr P. Du Val, who obtained another solution than that just given. We may for instance take six conics containing the points of contact of the bitangents projecting the 6 quadruples of lines (in Schläfli's notation)

$$(a_1, b_4; c_{23}, c_{56}), (a_2, b_5; c_{31}, c_{64}), (a_3, b_6; c_{12}, c_{45}),$$
$$(a_4, b_2; c_{15}, c_{36}), (a_5, b_3; c_{14}, c_{26}), (a_6, b_1; c_{24}, c_{35});$$

here, each quadruple consists of two pairs of intersecting lines, having a common transversal line (these lines being, respectively, $c_{14}, c_{25}, c_{36}, c_{24}, c_{35}, c_{16}$). With c_{34}, c_{16}, c_{25}, which, on projection, lead to a seventh conic, these 6 quadruples exhaust the 27 lines of the cubic surface.

It may be interesting to remark, in this connexion, on the fact that the 27 lines of the cubic surface are its complete intersection with a surface of order 9 (noted in Salmon's *Solid Geometry*, 1882, p. 510). One particular surface of order 9, with this property, consists of the two quadric surfaces which contain respectively the six lines $a_1, a_2, a_3, b_4, b_5, b_6$ and the six lines $a_4, a_5, a_6, b_1, b_2, b_3$ together with 5 planes each containing 3 of the fifteen lines c_{ij}; such a set of 5 planes can be chosen in 6 ways. (In a phraseology due to Sylvester, we can arrange the 15 duads (i, j) in 5 synthemes in 6 ways; see Vol. II, p. 220.) But in fact (as follows from a result remarked by Dr H. S. M. Coxeter, *Phil. Trans.* CCXXIX (1931), p. 419) the 27 lines lie in 9 tritangent planes, as, for instance, those given by the columns in the scheme

$$\begin{array}{ccccccccc} c_{12}, & a_1, & b_6, & c_{36}, & c_{15}, & c_{26}, & c_{45}, & b_5, & a_2, \\ c_{34}, & c_{13}, & a_4, & c_{25}, & b_1, & b_2, & c_{16}, & a_3, & c_{24}, \\ c_{56}, & b_3, & c_{46}, & c_{14}, & a_5, & a_6, & c_{23}, & c_{35}, & b_4. \end{array}$$

Ex. 5. Suppose that a correspondence exists between one algebraic curve in space, C, of order n, and another curve C', of order n', whereby, to each point P, of the curve C, there correspond α' points P', of the curve C' (so that the rational symmetric functions of the coordinates of the points P' are expressible as rational functions of the coordinates of P), and to each point, P', of C', there correspond α points, P, of C (of which the rational symmetric functions are rational in P'). It may be required to find the order of the ruled surface generated by the line PP', joining a pair of corresponding points of the two curves.

First, suppose the curves to be so situated that corresponding points of the two curves never coincide. Take an arbitrary fixed line, l, which we may suppose to be given by $x=0$, $y=0$. To any point P, of the curve C, correspond α' points of C'; the α' planes, joining the line l to these points P', meet the curve C in $n\alpha'$ points, (Q), which are thus determined when P is assigned. Reversely, a point Q, of the curve C, joined to l, determines n' points of the curve C', and hence $n'\alpha$ points (P) of the curve C. We thus have a correspondence (P, Q), on the curve C, of indices $(n'\alpha, n\alpha')$. And this correspondence is of valency zero; for the product of the α' planes joining l to the points P', of the curve C', which correspond to P, is of the form $(x-ym_1)\ldots(x-ym_{\alpha'})=0$, and hence of the form $u_0x^{\alpha'}+u_1x^{\alpha'-1}y+\ldots+u_{\alpha'}y^{\alpha'}=0$, where $u_0, \ldots, u_{\alpha'}$ are rational polynomials in the coordinates of P. The points (Q) arise from P by the intersections of these planes with the curve C, and hence, for all positions of P, are sets of the same linear series on C; and P is not a member of the general set of this series. Thus the valency of the correspondence (P, Q) is zero.

Hence the number of coincidences is $n'\alpha+n\alpha'$. And such a coincidence, of Q with P, arises when, and only when, the line PP' meets l. As this is an arbitrary line, the number $n'\alpha+n\alpha'$ is the order of the ruled surface generated by the line PP'.

The problem thus illustrates the theory of correspondence of points on a curve which is not necessarily rational. But it can be otherwise treated, as involving a correspondence in the rational pencil of all planes through the arbitrary fixed line l. For we may regard two planes, (ξ) and (ξ'), of this pencil, as corresponding when they contain, respectively, points P, P', of the two curves C, C', which correspond to one another. Then, any plane (ξ) meets C in n points, to which there correspond $n\alpha'$ points of C', each of which gives rise to a plane ξ'; similarly, to any plane ξ' correspond $n'\alpha$ planes ξ. There are thus $n\alpha'+n'\alpha$ coincidences of corresponding planes; and, if no coincidence of corresponding points P, P' occurs, such can only arise when the line PP' meets the line l. But we see also that if there be i coincidences of corresponding points P, P', the number $n\alpha'+n'\alpha$ includes the i planes joining l to these; and the order of the ruled surface is then only $n\alpha'+n'\alpha-i$.

A corollary from the result is that, if two curves in the same plane are in (α, α') correspondence, and there be no coincidence of corresponding points, then the lines joining corresponding points envelop a curve of class $n\alpha'+n'\alpha$.

Ex. 6. If there be a correspondence, of indices α, α', and valency γ, between two points, P and P', of a *single* curve C, in space, of order n and genus p, we may similarly seek the order of the ruled surface formed by the lines PP' which join corresponding points.

In this case, as in the second solution in Ex. 5, we may similarly consider a correspondence between pairs of planes, ξ and ξ', drawn through an arbitrary fixed line l—two planes corresponding when they respectively contain corresponding points, P and P', of the curve. Then, a plane ξ gives rise to n points P, and hence to $n\alpha'$ points P', and hence to $n\alpha'$ planes ξ'; and, reversely, a plane ξ' gives $n\alpha$ planes ξ. The coincidences of corresponding planes of the pencil arise, both when the line PP' meets the line l, and also when P' coincides with P. The number of the latter coincidences is $\alpha+\alpha'+2\gamma p$; the number of lines PP' which meet l, or the desired order of the ruled surface, is thus

$$n(\alpha+\alpha')-(\alpha+\alpha'+2\gamma p), \text{ or } (n-1)\alpha+(n-1)\alpha'-2\gamma p.$$

This result is also obtainable by the theory of correspondence on the curve C. To any point P correspond a set (Q), of $(n-1)\alpha'$ points, on this curve, obtained as the residual intersections of the α' planes which join the line l to the α' points P' which correspond to P. Reversely, a point Q, joined to l, gives $n-1$ points P', and hence gives a set (P) of $(n-1)\alpha$ points. We thus have a correspondence (P, Q), of indices $(n-1)\alpha$, $(n-1)\alpha'$, derived from the original. This derived correspondence is clearly what we have described as the product of the two correspondences, (1), the original correspondence between P and P'; (2), the correspondence between P' and Q, established by means of the line l. The latter is of valency unity; the product correspondence is thus of valency $-\gamma$. We therefore obtain the same number as before for the number of coincidences.

As a particular application, calling *torsal* a chord of a curve in ordinary space when the tangents of the curve at its extremities meet one another, prove that the torsal chords of a curve of order n and rank r, without cusps, form a ruled surface whose order, in general, is $(n-3)r$.

Ex. 7. We have used the notion of an (α', α) correspondence between two curves (C', C). The existence of such a correspondence, however, implies relations between the curves. Consider the particular hypothesis of a $(1, \alpha)$ correspondence, each point P' of C' corresponding to α points P of C, any one of which corresponds to the same point P'. For instance, between the circle whose equation is $x'^2+y'^2=1$, and the quartic curve expressed by $x^4+y^4=1$, there exists such a correspondence, given by $x^2=x'$, $y^2=y'$, whereby, to any point of the circle correspond 4 points of the quartic curve, or $\alpha=4$. In general, with such a $(1, \alpha)$ correspondence between P' on the curve C', and P on the curve C, as the point P' moves on the curve C', we obtain sets of α points (P) on C; and any point P of such a set, through its determination of the point P' of the curve C', determines all the other points (P) of that set. We say then that there exists, on the curve C, an involution $I_1{}^\alpha$, of sets of α points, of freedom 1; and we say that this involution is of genus p', this being the genus of the curve C' described by the point P' which represents a set of the involution on C. We may also say that there is a symmetrical correspondence, of which both indices are of value $\alpha-1$, between P and the other points of the set (P); but these sets (P) do not all belong to the same linear series on C, unless $p'=0$; so that this symmetrical correspondence is not one of those of the kind hitherto considered, which have a valency. When the sets of the involution $I_1{}^\alpha$ do belong to a linear series, it is a familiar fact that the number of coincidences of two points of a set (P) is $2\alpha+2p-2$, these coincidences forming the Jacobian set of the linear series. We proceed now to shew that, in general, for an involution of sets of α points, on a curve of genus p, of which the sets correspond to the points of another curve of genus p', there are d coincidences of two points of a set, where $d=2\alpha+2p-2-2\alpha p'$.

The proof we give assumes the result for the simpler case of a linear series. Consider a linear series, of sets of k points, of freedom 1, on the (supposed plane) curve C'. The points of any set of this series will be given by the intersections of C' with a curve of equation $\lambda\phi(x', y')+\psi(x', y')=0$, where the parameter λ varies from set to set. By hypothesis, there are α points on C corresponding to (x', y'), and, if (x, y) be any one of these, then x' and y' are both rationally expressible in terms of x and y, by use of the equations of the two curves; conversely, any rational symmetric function of the α points (x, y) is rational in (x', y'). To the points of the set, on C', given by $\lambda\phi(x', y')+\psi(x', y')=0$, correspond therefore, on C, the points of a set given by an equation $\lambda\Phi(x, y)+\Psi(x, y)=0$, namely, also a linear series, of freedom 1, of sets of $k\alpha$ points. Now let G' denote any set of the series on C', and J' the Jacobian set of this linear series, the points of coincidence of pairs of points in sets of this series, while K' denotes a canonical set on C', when $p'>0$. Then we have the equivalence, on the curve C', expressed by $K'\equiv J'-2G'$, which remains conventionally true when $p'=0$ (or $p'=1$), if we omit K'. Also let tK', tJ', tG' denote the sets on the curve C which arise by taking the α points which correspond to every point of the sets K', J', G', respectively. As *any* linear series on C' gives rise, in this way, by the argument above, to a linear series on C, we infer that, on C, $tK'\equiv tJ'-2tG'$.

Take now the linear series on C given by $\lambda\Phi(x, y)+\Psi(x, y)=0$, of which a set consists of k subsets, each of α points, each such subset arising from one point of a set of the original linear series on C'. A coincidence in a set of this series on C may arise, (1), from coincidence of two of the α points of a subset; if d be the number of coincidences of a pair of points in all the sets of the involution I_1^{α}, the coincidences of this kind will be d in number; their aggregate may be denoted by D; but, (2), a coincidence, in fact α coincidences, in the linear series on C, may arise, in which two of the subsets entirely coincide, in consequence of a coincidence of two of the k points of a set of the original series on C'; the aggregate of all such coincidences is exactly that denoted by tJ'. The canonical series on C is expressible by the coincidences in the linear series thereon; namely, if G denote a set of this linear series, we have

$$K\equiv(D+tJ')-2G, \quad \text{or} \quad K\equiv D+tJ'-2tG'.$$

Hence we have $K\equiv tK'+D$; *or, the canonical series on the curve C is defined by sets equivalent to,* (1), *the aggregate of the sets of α points arising by direct transformation of the points of a set of the canonical series on the representative curve C', with,* (2), *the addition of the set of coincidences in the involution I_1^{α} which is represented by the curve C'.* This leads to the numerical equation $2p-2=\alpha(2p'-2)+d$, which gives the value of d originally stated. When $p'=0$, the general equivalence is $K\equiv D-2I$, where I denotes a set of the involution I_1^{α}. For in this case all these sets belong to the same linear series. (This is clear because, on the rational curve C', the parameter θ is rational in (x', y'), and hence rational in the (x, y) of a set of the curve C which corresponds to (x', y'); so that the involution on C is given by an equation $\theta U(x, y)+V(x, y)=0$.)

The numerical equation $d=2\alpha+2p-2-2\alpha p'$ was remarked by Zeuthen, to whom also is due the result to which we now proceed*. Suppose that there is an (α', α) algebraic correspondence between the curves C', C; we shew that it is possible to define a curve Γ, such that (C', Γ) are in $(1, \alpha)$ correspondence, while (Γ, C) are in $(\alpha', 1)$ correspondence. To any point of C' correspond then α points of Γ, each of which corresponds to a single point of C; and to any point of C correspond α' points of Γ, each of which corresponds to a single point of C'. Thus, a coincidence of two points of a set of α' points on C' which correspond to the same point of C, requires a coincidence of two points of a set of α' points on Γ which correspond to the same point of C. Let d' be the number of such coincidences which exist; they result from d' coincidences in the $(\alpha', 1)$ correspondence of Γ and C. Similarly, let d be the number of existing coincidences of two points of a set of α points on C which corresponds to a point of C'; these result from d coincidences in the $(1, \alpha)$ correspondence of C' and Γ. If D' be the set of coincidences on Γ in the first case referred to, and D the set of coincidences on Γ in the second case, we have $tK \equiv \Lambda - D'$, $t'K' \equiv \Lambda - D$, where K, K' denote canonical sets respectively on C and C', and t refers to the $(1, \alpha')$ correspondence (C, Γ), and t' to the $(1, \alpha)$ correspondence (C', Γ), while Λ is a canonical set on Γ. From these equivalences we infer that, on Γ, $tK - D \equiv t'K' - D'$; this leads to the numerical equation

$$\alpha' (2p-2)-d=\alpha (2p'-2)-d',$$

including the former result for a $(1, \alpha)$ correspondence.

A simple way of defining a curve Γ, desired for the argument, is to suppose the curves C', C to lie in different spaces, which both lie in another space but have no point common. (Two spaces $[r]$, $[r']$, of respective dimensions r, r', will not generally intersect when they lie in a space $[r+r'+1]$.) Then form a ruled surface by joining every point of the curve C' to all the α points of the curve C, to which it corresponds, and, also, every point of C to all the α' points of C', to which it corresponds; to each point of C' will then correspond α generators of the ruled surface, and to each point of C will correspond α' generators. A section of this ruled surface by a prime space of the space containing both C' and C will then give such a curve Γ as desired. If the curves C', C be plane curves in the same plane, we may also obtain such a curve Γ by joining one arbitrary point O, of the plane, to every point of C, and the corresponding point of C' to another arbitrary point O', and taking the locus of the intersection of the corresponding joining lines.

---

* See the exhaustive investigation in Zeuthen, *Lehrbuch d. abzähl. Geometrie*, 1914, pp. 104–7.

Ex. 7*a*. We may consider the $(\nu, 1)$ correspondence of two (plane) curves, $f=0, f'=0$, in a more direct analytical way, and so obtain additional results. Let $x'=g(x, y, z)$, $y'=h(x, y, z)$, $z'=k(x, y, z)$, be polynomials of order r in the coordinates x, y, z, which transform the curve $f'(x', y', z')=0$, of order n', into the curve $f(x, y, z)=0$, of order n; there will then be an identity $f'(x', y', z')=Mf(x, y, z)$, where M is a polynomial in x, y, z. If α, β, γ be arbitrary numbers, this leads, on $f=0$, to the equations

$$M\left(\alpha\frac{\partial f}{\partial x}+\beta\frac{\partial f}{\partial y}+\gamma\frac{\partial f}{\partial z}\right)=\alpha\left(\frac{\partial f'}{\partial x'}\cdot\frac{\partial x'}{\partial x}+\frac{\partial f'}{\partial y'}\cdot\frac{\partial y'}{\partial x}+\frac{\partial f'}{\partial z'}\cdot\frac{\partial z'}{\partial x}\right)+\ldots+\ldots$$

$$=\alpha'\frac{\partial f'}{\partial x'}+\beta'\frac{\partial f'}{\partial y'}+\gamma'\frac{\partial f'}{\partial z'},$$

where α', β', γ' are given by

$$\alpha'=\alpha\frac{\partial x'}{\partial x}+\beta\frac{\partial x'}{\partial y}+\gamma\frac{\partial x'}{\partial z}, \text{ etc.,}$$

or, say, to
$$M\left(\alpha\frac{\partial f}{\partial x}\right)=\left(\alpha'\frac{\partial f'}{\partial x'}\right).$$

Hence, also, if $x_1'=\partial x'/\partial x$, $x_2'=\partial x'/\partial y$, $x_3'=\partial x'/\partial z$, etc., we find

$$\begin{vmatrix}\alpha', & x', & dx'\\ \beta', & y', & dy'\\ \gamma', & z', & dz'\end{vmatrix}=\frac{1}{r}\begin{vmatrix}x_1', & x_2', & x_3'\\ y_1', & y_2', & y_3'\\ z_1', & z_2', & z_3'\end{vmatrix}\begin{vmatrix}\alpha, & x, & dx\\ \beta, & y, & dy\\ \gamma, & z, & dz\end{vmatrix},$$

which, if J denote the Jacobian determinant $\partial(x', y', z')/\partial(x, y, z)$, we may write

$$(\alpha', x', dx')=\frac{1}{r}J.(\alpha, x, dx).$$

Wherefore, if $\phi'(x', y', z')$ be any polynomial in x', y', z', we have

$$\frac{\phi'(x', y', z').(\alpha', x', dx')}{\left(\alpha'\frac{\partial f'}{\partial x'}\right)}=\frac{\phi'(g, h, k)\,J}{rM}\cdot\frac{(\alpha, x, dx)}{\left(\alpha\frac{\partial f}{\partial x}\right)},$$

of which the former member is expressed by x', y', z', and the latter member by x, y, z.

Now it is a familiar fact, easy to establish, that the fraction

$$(\alpha, x, dx)\Big/\left(\alpha\frac{\partial f}{\partial x}\right)$$

has a value, on $f=0$, which is independent of α, β, γ; and similarly

$$(\alpha', x', dx')\Big/\left(\alpha'\frac{\partial f'}{\partial x'}\right)$$

on $f'=0$; also, it is known that, if $\phi'(x', y', z')$ be an adjoint polynomial of order $n'-3$ for the curve $f'=0$, the left side is the differential of an everywhere finite integral for the curve $f'=0$, and that, conversely, every such integral is capable of this form. And it is obvious that an everywhere finite integral for $f'=0$ is transformed into an everywhere finite integral. Thus, for the curve $f=0$, there is an adjoint polynomial $\phi(x, y, z)$, of order $n-3$, such that, on $f=0$, we have

$$\frac{\phi'(g, h, k).J}{M} = r\phi(x, y, z).$$

This identity involves certain consequences. For it shews that any zero of M on $f=0$, that is, any intersection of the curve $M=0$ with $f=0$, is a zero either of $\phi'(g, h, k)$ or of J; and we can shew that the latter is the right alternative. Indeed, the identity holds for all the p' adjoint polynomials $\phi'(x', y', z')$ of $f'=0$, and these are known not to have a common zero on $f'=0$; whence the p' polynomials in x, y, z, expressed by $\phi'(g, h, k)$, have no common zero on $f=0$; a zero of M, not being a common zero of these $\phi'(g, h, k)$, must then be a zero of J (of at least equal order of multiplicity). Other zeros of J are obvious; for any point (x, y, z), of $f=0$, at which there is a coincidence of two of the points of the set of ν points on $f=0$ which correspond to a single point (x', y', z') of $f'=0$ (in virtue of the equations $x'=g$, $y'=h$, $z'=k$), is a zero, on $f=0$, of the polynomial J. Indeed, from $\rho x'=g(x, y, z)$, etc., $\sigma x'=g(x+dx, y+dy, z+dz)$, etc., we deduce $x_1' dx+x_2' dy+x_3' dz=0$, and so on. Conversely, from $f'=Mf$, if $J=0$, we deduce an equation of the form

$$M(af_1+bf_2+cf_3)=0,$$

and can hence infer that a zero of J, when not a zero of M, is such a point of coincidence, of two points of a set on $f=0$, in general. Thus we have the consequence that, on $f=0$, the zeros of J consist of the zeros of M together with the set of coincidences of the correspondence.

We infer, therefore, that these points of coincidence on $f=0$ are zeros of all the particular adjoint polynomials, of order $n-3$, for $f=0$, which arise, as here, by direct transformation of the p' existing polynomials ϕ' of $f'=0$ (and $p \geqslant p'$).

Now let the p' everywhere finite integrals of $f'=0$ be denoted by $w_1, \ldots, w_{p'}$; and let the everywhere finite integrals of $f=0$ obtainable as here by direct transformation of these be denoted by $u_1, \ldots, u_{p'}$; further, let u_i $(i>p')$ be one of the $p-p'$ other everywhere finite integrals of $f=0$. Also, let $(x_1), \ldots, (x_\nu)$ be the places of $f=0$ corresponding to a place (x') of $f'=0$; and, in particular, when (a') is a fixed place of $f'=0$, let $(a_1), \ldots, (a_\nu)$ be the corresponding places of $f=0$. Then, from the algebraic character of the correspondence, it is clear that the sum $u_i^{x_1, a_1}+\ldots+u_i^{x_\nu, a_\nu}$ is equal to an everywhere finite integral of $f'=0$ (its differential being symmetric in $(x_1), \ldots, (x_\nu)$). This integral of $f'=0$ is thus of the form (where $c_{i,1}, \ldots, c_{i,p'}$ are constants)

$$c_{i,1}w_1^{x',a'}+\ldots+c_{i,p'}w_{p'}^{x',a'};$$

and, as we have seen, this transforms directly into the sum, on $f=0$,

$$c_{i,1}u_1^{x,a}+\ldots+c_{i,p'}u_{p'}^{x,a},$$

where, for (x) and (a), respectively, we may take any one of the points corresponding to (x') and (a'). If then we take an integral on $f=0$ defined by

$$v_i^{x,a}=u_i^{x,a}-\frac{1}{\nu}(c_{i,1}u_1^{x,a}+\ldots+c_{i,p'}u_{p'}^{x,a}), \qquad i=p'+1, \ldots, p,$$

we have $v_i^{x_1,a_1}+\ldots+v_i^{x_\nu,a_\nu}=0$; thus, the $p-p'$ integrals v_i have the property that the sum of the values of any one of these, at the ν places corresponding to a point of $f'=0$, is independent of this point. The p integrals of $f=0$ consist of these $p-p'$ integrals, together with the p' integrals which arise by direct transformation of the integrals of $f'=0$. The fact that, when $p'=0$, the set $(x_1), \ldots, (x_\nu)$ belongs to a linear series on $f=0$, which we have already proved, arises again here, by the converse of Abel's Theorem (Vol. v, pp. 146, 176).

We suppose that neither of the curves $f=0, f'=0$ is hyperelliptic; then these can be represented by the canonical curves, say γ and γ', of respective orders $2p-2$, $2p'-2$, lying in spaces Π, Π', of dimensions $p-1$, $p'-1$, of which we suppose the space Π' to be contained in Π. For coordinates in Π' we may use $z_1, \ldots, z_{p'}$; and, for coordinates in Π, we may use $x_1, \ldots, x_{p'}, x_{p'+1}, \ldots, x_p$, of which $x_1, \ldots, x_{p'}$ are proportional, on the canonical curve, to those ϕ-polynomials of $f=0$ which arise by direct transformation of the ϕ-polynomials of $f'=0$. Thus we have, on the canonical curves,

$$\frac{x_1}{z_1}=\ldots=\frac{x_{p'}}{z_{p'}}.$$

This shews that the canonical curve γ', in the space Π', is obtainable by projection from the canonical curve γ, in the space Π, the centre of projection being the space Σ, of dimension $p-p'-1$, which is given by $x_1=0, \ldots, x_{p'}=0$. By the argument given, this space Σ contains the d points of the curve γ which are the coincidences of the correspondence between the two curves. Through Σ, and an arbitrary point P of the curve γ, passes a space $[p-p']$, which meets the space Π', of dimension $p'-1$, in one point (because two spaces, of dimensions $p-p'$ and $p'-1$, both lying in a space Π

of dimension $p-1$, meet in a point). This point is then on the curve γ', and is the point P' corresponding to P. Through Σ, and an arbitrary point P' of γ', again, passes a space $[p-p']$, which, therefore, intersects the curve γ in the ν points (P) which correspond to P', beside containing the d points of coincidence which lie in Σ. The space, say Ω, through Σ and $p'-1$ points of γ', depending in all on $p-p'+p'-1$, or $p-1$ points, is a prime, or space $[p-2]$, of the space Π, and contains the prime of the space Π' determined by the $p'-1$ points of γ'; this prime space Ω meets γ in a canonical set on γ. This set consists then of the d points of coincidence which lie on Σ, together with $2p'-2$ sets of ν points, each arising from one of the $2p'-2$ points, of γ', lying in the prime space of Π' through which Ω passes. We thus have geometrical interpretations of the equations $K \equiv tK'+D$, $2p-2=\nu(2p'-2)+d$; and the curve γ has the geometrical property that every space $[p-p']$ through the space Σ, and one point of the curve γ', is a multiple secant of γ, meeting it in ν points. These $[p-p']$, though ∞^1 in aggregate, are not all the spaces of a rational pencil through Σ, but form an aggregate of genus p'; and p' is necessarily $\geqslant 3$, since $f'=0$ is supposed not to be hyperelliptic. Further, the total aggregate of p primes, of the space Π, consists of the particular p' primes we have considered, passing through the space Σ, together with $p-p'$ others, independent of these. The former correspond to everywhere finite integrals of the curve γ which are reducible to such integrals for the curve γ'; these integrals have thus only $2p'$ independent columns of periods. It will later be shewn to follow that the remaining $p-p'$ integrals of γ have also a defective system of periods, reducing to only $2(p-p')$ columns at most (p. 62, below).

Repeating, in slightly different form, what we have proved: The sets of ν points, of an involution I_1^{ν}, of genus p', on a canonical curve γ, of genus p, are obtainable by projection of a canonical curve γ', of genus p', from a space $[p-p'-1]$, which contains the points of coincidence, on γ, of pairs of points of a set of the involution. There are canonical sets on γ, each of which contains these points of coincidence, together with the projection of a canonical set of γ'.

Ex. 7*b*. A simple example may be given (not fully representative, however, as $M=1$). Let the curves $f'=0$ and $f=0$ be given respectively by $x'^4+y'^4=z'^4$, $x^8+y^8=z^8$, with $x'=x^2$, $y'=y^2$, $z'=z^2$, so that $p'=3$, $\nu=4$, $p=\frac{1}{2}(8-1)(8-2)=21$, and $d=24$, the points of coincidence being such as $x=0$, $y=z(1)^{\frac{1}{8}}$. Then $J=xyz$. The coordinates for the canonical curve of $f=0$ may be expressed by

$$\frac{x_1}{x^3yz}=\frac{x_2}{xy^3z}=\frac{x_3}{xyz^3}=\frac{x_i'}{u_i}, \qquad i=1, \ldots, 18,$$

every generator of the surface in ν points, then the generators determine on this curve an involution I_1^ν, of genus p', and index 1. The number, d, of points of simple contact of the curve with a generator, is therefore, by what is proved above, given by

$$d = 2\nu + 2p - 2 - 2\nu p'.$$

This number d can however be evaluated in terms of N (cf. also Ex. 12, p. 216 of Chap. VIII, Vol. V). For consider a pencil of primes, that is, planes passing through an arbitrary general line when the ruled surface is in ordinary space, and primes through an arbitrary general space $[r-2]$, when the ruled surface is in space $[r]$; let two primes of the pencil be regarded as corresponding when they meet a proper generator in two points which are among the ν intersections of this generator with the curve in question. Any prime of the pencil meets the curve in n points, and each of the $(\nu-1)$ other points of the curve, on any one of the generators through these n points, gives rise to a corresponding prime. The correspondence between the primes of the pencil, which is obviously symmetrical, is thus of indices $n(\nu-1)$, $n(\nu-1)$. Two corresponding primes coincide, (1), when they both pass through a point where the curve touches a generator, that is, in d cases; or (2), when the generator lies in both primes; in this case there are $\frac{1}{2}\nu(\nu-1)$ pairs of points of the curve lying in this generator, and there are N generators which meet the space which is the base of the pencil of primes; so that $N\nu(\nu-1)$ coincidences are obtained. Hence we have $d + N\nu(\nu-1) = n(\nu-1) + n(\nu-1)$, or* $d = 2n(\nu-1) - N\nu(\nu-1)$. Equating this to the former value of d we have

$$\nu + p - 1 - \nu p' = n(\nu-1) - \tfrac{1}{2}N\nu(\nu-1).$$

This may be regarded as determining the genus, p, of a curve of order n, lying on a ruled surface of order N, whose prime section is of genus p', when the curve meets each generator in ν points; and agrees with the result found on p. 216 of Vol. V. The formula may be written however $(n-1)(\nu-1) - p + \nu p' = \frac{1}{2}N\nu(\nu-1)$. Here the right side evidently is the number of pairs of points of the curve which belong, both to one set of the involution I_1^ν, and also to one set of the linear series g_1^n, which is determined on the curve by the primes of the pencil considered. Thus, for this involution, on the curve, determined by the generators, and this linear series, the number of pairs belonging to a set of both, is $(n-1)(\nu-1) - p + \nu p'$, where p' is the genus of the involution. When $p' = 0$ this reduces to the number (p. 11) of pairs common to two linear series g_1^ν, g_1^n.

---

* This value of d must however be diminished by 2δ, if the curve have δ double points at points of the ruled surface where there is a unique generator (cf. Segre, *Math. Ann.* XXXIV, 1889, p. 3).

That this formula remains valid when the sets of I_1^ν do not lie on lines, generating a ruled surface upon which the given curve (n, p) lies, may perhaps be regarded as evident; also, another proof is given below, in Ex. 12. But a proof, that no limitation of the involution I_1^ν is involved in the assumption, is obtained by recurring to the figure considered above (Ex. 7*a*), of the case of two canonical curves. In that figure, the sets of I_1^ν were obtained by spaces $[p-p']$, passing through a fixed base space Σ, of dimension $p-p'-1$, and a point of the curve γ', in space $[p'-1]$. If, through this last space, there be drawn a space $[p']$, lying in the space $[p-1]$ which contains the curves γ and γ', this space $[p']$ will be met in a line by every one of the spaces $[p-p']$ determining a set of the involution on γ, and this line will contain the point of the curve γ' which fixes the set of the involution.

Ex. 9. We can use the principle of correspondence to determine, for a ruled surface in ordinary space: (*a*), the number of pairs of consecutive generators which intersect one another (that is, the number of torsal generators, or of pinch points of the double curve of the surface); or, (*b*), the genus of the double curve of the surface; or, (*c*), the number of triple points of this double curve. It is assumed that the surface has no cuspidal curve, and that the double curve is irreducible.

For (*a*), we may proceed as follows: let the plane sections of the ruled surface be of order n and genus p. Take an arbitrary line, and, through it, two arbitrary plane sections, α and β. To any point, P, of the line correspond $m=2n+2p-2$ points of contact, T, of tangents from P to the section in one plane, α; and the generator through T meets the plane section in the other plane, β, in a point U; let the tangent line of the section β meet the line in a point Q. We thus have an obviously symmetrical correspondence of points P, Q of this line. When P is at one of the n points in which the line meets the surface, the point T coincides with this P, and the tangent plane at U, which is to give Q, contains PU; so that Q coincides with this P. Similarly, when Q is at this point, so is P. If P be on a torsal generator, the tangent plane is the same at all points of this, and P, Q coincide; and conversely. The number of torsal generators is thus $2m-2n$, or $2\,(n+2p-2)$. Schubert gives two determinations of this number; for one of these see his paper, *Math. Ann.* XVII. The other is noticed in the following Chapter, dealing with Schubert's methods (p. 102).

A clear deduction of the number of torsal lines, and of the number of generators of the ruled surface which touch the double curve, is obtained also by the representation, referred to in Vol. v, p. 195, by which the generators correspond to points of a curve γ, of order n and genus p, lying on a quadric Ω in space of five dimensions. To the $n-2$ generators which meet a generator h correspond then $(n-2)$ points (Q), of the curve γ, lying on the tangent prime of Ω at a point P of the curve. The lines PQ lie on Ω, and there is a correspondence of indices $(n-2,\ n-2)$, and valency 2, between P and Q. The torsal generators correspond then to the coincidences of this correspondence, and are in number $2(n-2)+4p$, as before*. For the double curve to touch a generator h, two of the $n-2$ generators which meet h must coincide. The number of contacts is, therefore, the number of coincidences of the direct lateral correspondence derived from the correspondence (P, Q) on the curve γ. For this correspondence $s=r=n-2$ and $\gamma=2$; and from the formula (above, p. 11) $2s(r-1)+2p(s-\gamma^2)$ we thence derive, for the number, ζ, of contacts of the double curve with a generator, $\zeta=2(n-2)(n-3)+2p(n-6)$.

(*b*) Now let P be the genus of the double curve of the ruled surface; and consider the $(n-2,\ 2)$ correspondence between the double curve, and a plane section, wherein, to any point, L, of the double curve, correspond the two points, M, of the plane section in which this is met by the generators through L, and, to any point M of the plane section, correspond the $n-2$ intersections, L, of the double curve with the generator through M. The coincidences, in this correspondence, which lie on the double curve, are the ζ contacts of this curve with generators; the coincidences on the plane section are the η intersections of this with the torsal generators (where $\eta=2n+4p-4$). By Zeuthen's formula, obtained above, we therefore have $2(2P-2)-\zeta=(n-2)(2p-2)-\eta$, leading to

$$2P-2=(n-5)(n+2p-2).$$

We may also notice that the rank $(2b+2P-2)$ of the double curve, of order b, the number of its tangents which meet an arbitrary line, is equal to ζ, the number of its tangents which are generators of the surface. It may perhaps also be remarked (see below, p. 32) that, for a curve of order n and genus p, for which the numbers of cusps, inflexions, double points and double tangents are respectively κ, i, δ, τ, the number of the tangents of the curve which meet the curve again is $2(n-2)(n-3)+2p(n-6)-(n\kappa+2i+4\delta+4\tau)$.

(*c*) To determine the number of triple points of the double curve, consider the correspondence of pairs of planes of a pencil, drawn through an arbitrary axis, l, determined by the condition that two of these planes correspond when they meet a generator in two points of the double curve. Any plane through l meets the double curve in $b\ (=\frac{1}{2}(n-1)(n-2)-p)$ points, through each of which pass two generators, each meeting the double curve in $n-3$ other points; the indices of the correspondence are therefore $2b\ (n-3)$, $2b\ (n-3)$. The coincidences of corresponding planes arise, either when the two points of the same generator which determine a pair of corresponding planes, coincide with one another, or when a generator meets the line l. The former arises in two ways, either because the double curve touches a generator (ζ cases), or because there are three generators passing through a point of the double curve; then the curve, and the surface, have a triple point; and, as each branch of the double curve is then met by both the other branches, there are 6 coincidences of pairs of corresponding planes, so that, if t be the number of triple points, the number of coincidences so arising is $6t$. For the latter, when a generator meets the line l, each of the $\frac{1}{2}(n-2)(n-3)$ pairs chosen from the $n-2$ intersections of the generator with the double curve, gives rise to a double coincidence of pairs of corresponding planes. On the whole then

---

* If η be the number of torsal generators, and b, $=\frac{1}{2}(n-1)(n-2)-p$, be the order of the double curve, we have $8b+2\eta=4n(n-2)$, in accordance with Salmon's remark (*Solid Geometry*, 1882, p. 591) that the Hessian of a ruled surface (of order $4n(n-2)$) meets the surface in the double curve (which is 4-ple on the Hessian), and in the torsal generators (which are double on the Hessian).

Zeuthen (*Lehrbuch d. abzähl. Geometrie*, 1914, p. 219) obtains the number of torsal generators by allowing p of the generators of the surface to become double, so that the plane section becomes rational.

$$4b\ (n-3)=\zeta+6t+n\ (n-2)(n-3),$$

which leads to

$$t=(n-4)[\tfrac{1}{6}(n-2)(n-3)-p],\ =(n-4)[h-\tfrac{1}{3}n\ (n-2)],$$

if h be the number of apparent double points of the double curve.

It has been assumed that the double curve is irreducible, and that the surface has no cuspidal curve; in general, consult W. L. Edge, *Ruled Surfaces* (Cambridge).

Ex. 10. We prove now that, for two curves, in ordinary space, of orders n_1, n_2, with respectively h_1, h_2 apparent double points, and i mutual intersections, the number of common chords is

$$h_1h_2+\tfrac{1}{4}n_1n_2(n_1-1)(n_2-1)-i(n_1-1)(n_2-1)+\tfrac{1}{2}i(i-1).$$

This result follows from a known result (Vol. IV, p. 51), that the number of common lines of two algebraic congruences of lines, of orders p_1, p_2 and classes q_1, q_2, is $p_1p_2+q_1q_2$ (the order and class of a congruence being the numbers of its lines respectively through an arbitrary point and in an arbitrary plane). This result was stated by Halphen (*Compt. rend.* 1872, p. 41), and is interesting historically as being one of the first of such linear formulae (cf. Schubert, *Abzähl. Geom.* pp. 62, 337).

The chords of the first curve being a congruence of order $p_1=h_1$, and class $q_1=\frac{1}{2}n_1(n_1-1)$, the general formula for the number of common chords is $h_1h_2+\frac{1}{4}n_1n_2(n_1-1)(n_2-1)$. This however includes the joins of every two intersections of the curves, in number

$\frac{1}{2}i(i-1)$; it also includes, for each intersection, the common generators of the two cones projecting the curves from this intersection, of which the number, beside those to another intersection, is $(n_1-1)(n_2-1)-(i-1)$. The total number of proper common chords of the two curves, not passing through any common point, is thus

$$h_1h_2+\tfrac{1}{4}n_1n_2(n_1-1)(n_2-1)-\tfrac{1}{2}i(i-1)-i[(n_1-1)(n_2-1)-(i-1)],$$

agreeing with the statement made.

But an examination of the problem by the method of the theory of correspondence is interesting. From a point P, of the curve (n_1), draw the h_2 chords of the curve (n_2), and let P' be one of the $2h_2$ extremities of these chords; from P' draw the h_1 chords of the original curve (n_1), and let Q be one of the $2h_1$ extremities of these. Thus from P are obtained $4h_1h_2$ points Q, of the curve (n_1), and we regard these as being in correspondence. To pass back from Q to P, we take one of the $n_2(n_1-1)$ common generators of the cones which project the two curves from Q, denoting by P' the extremity, lying on (n_2), of one such chord; and then take one of the $n_1(n_2-1)$ common generators of the cones which project the two curves from P', denoting by P the extremity, lying on (n_1), of one such chord. There are thus $n_2(n_1-1).n_1(n_2-1)$ positions of P corresponding reversely to Q. The relation between P and Q is evidently an algebraic correspondence. We shall prove that this is of zero valency, the $4h_1h_2$ positions of Q, for an arbitrary P, being a set of a linear series, taken by themselves. The number of coincidences, of P and Q, is thus $4h_1h_2+n_1n_2(n_1-1)(n_2-1)$; each such coincidence involves that PP' is a common chord of the two curves; and, conversely, a common chord gives rise to four such coincidences. Thus, after correction for the common points of the two curves, we reach the same result as before.

To complete this proof, we shew that, if the h chords be drawn from an arbitrary point O, to a non-singular curve of order n and genus p, then the $2h$ extremities of these chords, as O varies, are sets of the same linear series on the curve. They are in fact equivalent with the sets of residual intersections, with the curve, of surfaces of order $n-3$, drawn through any chosen canonical set (of $2p-2$ points) on the curve. For, consider the projection of the curve to a plane, from O. The canonical series on the plane curve is obtained by the curves of order $n-3$ through its double points; and, for general positions of O, the canonical sets on the plane curve are in $(1, 1)$ correspondence with the canonical sets on the space curve. There is thus a cone of order $n-3$, with vertex at O, which contains any particular canonical set on the space curve, and also contains the h chords to the curve from the point O. This proves the statement made. In the application to the problem in hand, the set of $4h_1h_2$ points Q is made up of $2h_2$ sets each by itself belonging to a linear series when P varies. The aggregate set of points Q is thus one of a linear series.

Ex. 10*a*. Consider the quintic curve which is the residual intersection of a quadric surface with a cubic surface through one generator; this curve is of genus 2, and meets the generator in 3 points. From any arbitrary point O, there can be drawn 4 chords of this curve; also the plane joining O to the fixed generator has two points A, B on the curve (beside 3 on the generator); the four chords from O, and the two lines OA, OB, lie on a quadric cone, with vertex at O (as is proved in Vol. v, Chapter VIII). Now let P be a point of the quintic curve, and l, m be two fixed arbitrary lines; let the transversal from P meet these lines in P' and H', respectively; from P' draw the four chords to the quintic curve, meeting this in Q_1, Q_1'; Q_2, Q_2'; Q_3, Q_3'; Q_4, Q_4'; also, from H', draw the four chords K_1, K_1'; ...; K_4, K_4'. By what has been proved in Ex. 10, the sixteen points (Q), (K) are a set of a linear series, as P varies on the curve.

Ex. 10*b*. For the number of chords of a curve (n_1, h_1) which meet two arbitrary lines, the formula of Ex. 10 (putting $h_2 = 1$, $n_2 = 2$), gives $h_1 + \frac{1}{2}n_1(n_1 - 1)$. When the lines intersect this is obvious. This is then the order of the ruled surface generated by chords of the curve which meet an arbitrary line; in terms of the genus of the (non-singular) curve, it is $(n_1 - 1)^2 - p_1$.

Ex. 10*c*. The chords of a curve (n) having simple intersections with two other curves (n_1), (n_2), are in number

$$n_1n_2\,[h + \tfrac{1}{2}n(n-1)] - (n-1)\,[n_1i_2 + n_2i_1] - hi_{12} + i_1i_2,$$

where i_1, i_2 are the respective numbers of intersections of the curves (n_1), (n_2) with the first curve (n), and i_{12} the number of their mutual intersections.

Ex. 10*d*. The number of lines having simple distinct intersections with each of four curves (n_1), (n_2), (n_3), (n_4), is

$$2n_1n_2n_3n_4 - \Sigma n_pn_qi_{rs} + \Sigma i_{pq}i_{rs},$$

where the curves (n_r), (n_s) have i_{rs} intersections, and p, q, r, s are the numbers 1, 2, 3, 4 in some order.

Ex. 10*e*. For a curve, of order n, class n', and genus p, with k-ple points, of separated tangents, the number of chords each of which lies in the osculating planes of both extremities of the chord, is

$$(n-3,\ 2) + (n'-3,\ 2) - \Sigma(k,\ 2) - 12p,$$

where (λ, μ) means

$$\lambda\,(\lambda-1)\,\ldots\,(\lambda-\mu+1)/\mu\,!.$$

For example, for a rational quartic curve, there are 3 such chords.

References to the literature for these results are given by Berzolari, *Pascal's Repertorium*, II, 2, 1922, p. 906.

Ex. 11. We consider now the chords having three, or four, intersections with a curve, the trisecants and quadrisecants. Of the former there is in general an infinite number, determining a ruled surface; of the latter a definite finite number.

The trisecant lines which can be drawn, to a curve of order n, and genus p, without multiple points, from any point P of itself, each meeting the curve in two other points, are the same in number as the double points of the curve, of order $n-1$, obtained by projection of the given curve, from P, upon a plane. Assuming that the plane curve is of genus p, this number is $\frac{1}{2}(n-2)(n-3)-p$, which is the same as $h-(n-2)$, if h be the number of apparent double points of the original curve. The points Q, in number $(n-2)(n-3)-2p$, or say ν, where these trisecants from P again meet the curve, are in a correspondence (P, Q) with P, of indices ν, ν. We prove that this correspondence is of valency $n-4$. Take a particular canonical set (of $2p-2$ points) on the plane curve obtained by projection of the given curve from P; let (A) be the set of $2p-2$ points on the space curve projecting from P into this particular canonical set; there is then a cone, of order $n-4$, of vertex P, containing the set (A) and the trisecant chords from P to the space curve (as follows by the definition of canonical sets on the plane curve); but, conversely, if (A_0) be any canonical set of the space curve, these project from a point P of the curve into a canonical set of the resulting plane curve. There is thus a system of cones, of order $n-4$, containing the set (A_0), with vertices at points P of the space curve, each containing the trisecants of the curve drawn from the point P. The points (Q), which are the other intersections of these trisecants with the curve, thus lie with P, taken $(n-4)$ times over, in a set of a linear series upon the curve.

But we found (p. 16, above) that the order of the ruled surface generated by the lines joining points P, Q of the curve, in (s, r) correspondence, of valency γ, is $(n-1)(s+r)-2\gamma p$. Thus, as, in the present case, each trisecant arises from the correspondence in six ways, the trisecants generate a surface of order

$$\tfrac{1}{6}\{(n-1)[2(n-1)(n-3)-4p]-2(n-4)p\}$$

which is $(n-2)[\frac{1}{3}(n-1)(n-3)-p]$. If the curve have a double point, the ruled surface, as so found, will include the cone, of order $n-2$, projecting the curve from this point; and it is the same for a cusp. The residual ruled surface of proper trisecants is then of order $(n-2)[\frac{1}{3}(n-1)(n-3)-p-\delta-\kappa]$, if the curve have δ double points, and κ cusps. This is the same as the number, t_3, given by $t_3=(n-2)[h-\frac{1}{6}n(n-1)]$, where h is the number of apparent double points of the curve. And this form includes the simpler case.

The tangents of the curve which meet the curve again arise by the coincidences of the correspondence (P, Q), and are thus, in general, of number $2\nu+2(n-4)p$, namely

$$2(n-2)(n-3)+2(n-6)p;$$

but, if the curve have i inflexions, κ cusps, δ actual double points, and τ actual double tangents, we must in fact subtract from this the number $2i+n\kappa+4\delta+4\tau$. If r denote the rank of the curve, the resulting number, ξ, may also be written

$$\xi=(n-4)r+4h-2n(n-3)-2i-4\tau.$$

Save for the term -4τ, this corrected result is given by Zeuthen, *Ann. d. Mat.* III, Milan, 1869–70; and, for historical reasons, it may be interesting to give an indication of Zeuthen's reasoning. Of a pencil of planes through an arbitrary axis, let two be regarded as corresponding when they meet the curve in two points of a trisecant; we thus have a correspondence, between pairs of planes of this pencil, whose indices are $2nt$, $2nt$, where t is the number, found above, of trisecants which can be drawn to the curve from a point of itself. There are coincidences in this correspondence, (1), for a tangent meeting the curve again (say ξ cases); (2), for each inflexion (counting twice); (3), for every proper trisecant of the curve which meets the axis of the pencil of planes (counting six times, giving therefore $6t_3$, in the notation above); (4), for each double point of the curve (counting $2n-8$ times); (5), for each cusp (counting $3n-12$ times); (6), for each double tangent (counting four times). Thus we have $4nt=\xi+2i+6t_3+(n-4)(2\delta+3\kappa)+4\tau$; substituting

$$t=h-(n-2),\quad t_3=(n-2)[h-\tfrac{1}{6}n(n-1)],\quad 2\delta+3\kappa=n(n-1)-2h-r,$$

we find for ξ the value put down.

We have above deduced the number $h-(n-2)$ of trisecants to the curve, from an arbitrary point P of itself, with the assumption that the plane curve obtained by projection of the given curve is of genus p. Zeuthen deduces this result without this assumption: taking two arbitrary points A, B of the space, and two points Q, Q' of the curve, and excluding the cases when the planes APQ, BPQ' are coincident, the points Q, P, Q' are in line when the plane APQ contains Q', and the plane BPQ' contains Q. By an ingenious argument, equivalent to using the product of two correspondences, it is shewn that the number of cases in which this happens is $(n-2)^2-p$.

Ex. 11*a*. We pass now to the problem of finding the number of quadrisecants of a curve. Let A_1, A_2 be two arbitrary fixed points, not lying on the curve. Let P_1 denote one of the $n-3$ intersections, with the curve, of the plane joining A_1 to a trisecant of the curve (not on the trisecant), and P_2 one of the $n-3$ intersections, with the curve (other than on the trisecant), of the plane joining A_2 to the same trisecant. When P_1 is any point of the curve, the line A_1P_1 is met by a definite number of trisecants of the curve, and each of these, joined by a plane to A_2, gives $(n-3)$ points P_2; the number of trisecants which meet A_1P_1, excluding those through P_1, is t_3-t (where, as above, $t_3=(n-2)h-\frac{1}{6}n(n-1)(n-2)$ and $t=h-(n-2)$). Thus the points P_1, P_2 of the curve are in correspondence, with indices both equal to $(n-3)(t_3-t)$. We shew that this correspondence is of valency k_{12}, where $k_{12}=n-4-t$. Let P_3, P_4 be two of the intersections, with the curve, of a trisecant (not through P_1) which meets A_1P_1. There is a correspondence from P_1 to P_3, say (P_1, P_3); the sum of this, and the correspondence (P_1, P_2) under examination, is a correspondence leading from P_1 to all the points of the curve which lie on the planes joining A_2 to the trisecants (not through P_1) which meet A_1P_1; this aggregate of planes does not contain P_1, and meets the curve in a set of a linear series. Assuming this, we infer $k_{13}+k_{12}=0$, where k_{13} is the valency of the correspondence (P_1, P_3). Again, when P_3 is given, the planes joining A_1 to the trisecants of the curve through P_3, evidently give, as the aggregate of their intersections with the curve, the points arising from P_3 by the sum correspondence $(P_3, P_1)+(P_3, P_4)$; but they give a linear series on the curve in which P_3 arises t times in each set $(t=h-n+2)$, that is, a correspondence of valency t; thus, with a similar notation, $k_{31}+k_{34}=h-(n-2)$. We have already shewn (p. 31) that $k_{34}=n-4$; hence

$$k_{12}=-k_{13}=-k_{31}=n-4-(h-n+2), \quad =2(n-3)-h,$$

as stated.

Now consider the coincidences in the correspondence (P_1, P_2); these arise in two ways: (1), If a trisecant of the curve meet the line A_1A_2, the planes from A_1 and A_2 to this trisecant coincide in one plane, and the $n-3$ intersections of this plane with the curve, not on the trisecant, are coincidences; thus we have $(n-3)t_3$ coincidences; (2), If a trisecant have a fourth intersection with the curve, the planes joining A_1 and A_2 to this have intersections with the curve, other than at the three points which define the trisecant, which coincide at this fourth point. The quadrisecant thus gives 4 coincidences of P_1 and P_2. Assuming that there is no other possibility, if t_4 denote the total number of quadrisecants of the curve, the whole number of coincidences is thus $(n-3)t_3+4t_4$. The correspondence has been seen to be of equal indices $(n-3)(t_3-t)$, and of valency $n-4-t$. Thus we have

$$4t_4=2(n-3)(t_3-t)+2p(n-4-t)-(n-3)t_3,$$

and $t_4 = (h-2n+6, 2) - (n-3, 4)$, where (λ, μ) is the binomial $\lambda(\lambda-1)\ldots(\lambda-\mu+1)/\mu!$. This is the same as

$$\tfrac{1}{2}h\,(h-4n+11) - \tfrac{1}{24}n\,(n-2)\,(n-3)\,(n-13)$$

or, the curve having no multiple points, as

$$(p, 2) - p\,(n-3, 2) + \tfrac{1}{3}\,(n-2, 2)\,(n-3, 2).$$

The above argument is very slightly modified from Zeuthen (*Lehrbuch*, 1914, p. 257), who also considers the case of multiple points in *Ann. d. Mat.* III, 1869–70. The number t_4 was obtained by Salmon (*Solid Geometry*, 1882, p. 435), Cayley (*Papers*, V, p. 171), Brill (*Math. Ann.* VI, 1873, p. 52), and by others (Pascal, *Repertorium*, II, 2, 1922, p. 905). In Enriques-Chisini, *Teoria Geometrica*, II, p. 303, the result is found by a functional method due to Cayley; and, *ibid.* III, p. 473, by a method indicated in the following Ex. 11*b*.

Ex. 11*b*. Let Q be a point, other than P, of a trisecant drawn to a curve in space from a point P of the curve; let R be a point, other than Q, of a trisecant, other than PQ, drawn to the curve from Q. Denote the correspondences, from P to Q, and from P to R, by (P, Q), (P, R). If I denote the identical correspondence, it may be shewn that $(P, Q)^2 = (P, Q) + (P, R) + 2tI$ (where $t = h-n+2$). We have proved that the valency of (P, Q) is $n-4$; the correspondence (P, R) is symmetrical, of indices both equal to $4t\,(t-1)$; by the identity remarked its valency is $2t-(n-4)-(n-4)^2$, which is $2\,(n-3)-2p$. The coincidences of this correspondence (P, R) are in number $24t_4 + 2\zeta$, where ζ is the number of planes which touch the curve at two points whose join is a trisecant. When ζ is found, we can thence find t_4. It may be shewn that

$$\zeta = -4t^2 + 2t\,(n^2-14) + 18\,(n-2, 3).$$

For this proof compare Zeuthen, *Lehrbuch*, 1914, p. 259.

Ex. 11*c*. Consider three curves in space, whose orders, genera, numbers of apparent double points, and of cusps are n_i, p_i, h_i, κ_i, and the ruled surface generated by a line meeting simply each of these curves. Prove that the genus, P, of the plane section of this ruled surface is given by $2P-2 = 4n_1n_2n_3 + \Sigma n_j n_k\,(2p_i-2)$, where i, j, k are 1, 2, 3 in some order.

Also that the genus, P_0, of the plane section of the ruled surface generated by the trisecants of a curve (n, p, h, κ), is given by $2P_0-2 = \tfrac{1}{3}\,(6h-n^2-n)\,(n-4) + (h-2n+6)\,(2p-2)$. The formulae for the order, rank, etc., of a ruled surface generated by the transversals of three curves, by the chords of a curve which meet another curve, and by the trisecants of a single curve, with many references to the literature, are given in Pascal's *Repertorium*, II, 2, 1922, pp. 907–8.

Ex. 12. Suppose that on a curve we have an involution of sets of ν points, of freedom unity and index 1 (so that only one set of the involution contains a given point of the curve); we denote this by I_1^ν, a single set being denoted by I. Suppose also there exists on the curve a linear series, of sets of n points, of freedom r, say g_r^n which is *simple* (so that sets of the series which have one point in common, of general position on the curve, do not thereby have other points in common). Suppose further that $\nu > r$, and that any r points of a set I of the involution form independent conditions for a set of g_r^n, and so determine this set. We proceed to find how many times it happens that the set of the linear series, so determined by r points of a set I, has another point contained in this set I; so that $r+1$ points are common to a set I and a set of g_r^n. We have already considered the case when $r=1$, in Ex. 8.

If the linear series g_r^n be given by a system of curves

$$\lambda_0\phi_0 + \ldots + \lambda_r\phi_r = 0,$$

it is well known (see Vol. v, p. 26) that, by equations of the forms $X_0/\phi_0 = \ldots = X_r/\phi_r$, the curve can be placed in (1, 1) birational correspondence with a curve of order n in space of r dimensions; the sets of the linear series are then represented by the prime sections of this curve in space $[r]$. And there will be an involution I_1^ν upon this curve similar to that on the original curve. Our problem is then to find the number of sets I of this involution which have $r+1$ coprimal points (where $\nu > r$). Denote this number by $Z_{n,r}$; also denote by d the number of coincidences of a pair of points belonging to the same set of the involution. For brevity, let ρ denote the binomial coefficient $(\nu-2, r-1)$.

Take a point P, on the curve in the space $[r]$. This determines a set I, of ν points of the curve, belonging to I_1^ν; a prime can be put through every r points of the $\nu-1$ points of this set, other than P; and there will be $(\nu-1, r)$ such primes. Each of these primes meets the curve in $n-r$ points, Q, beside those through which it is described. The aggregate of these $(\nu-1, r)$ sets of $n-r$ points Q is determined by P; we consider the correspondence between P and a point of this aggregate. This correspondence (P, Q) we denote by T. With this we consider the correspondence between P and a point P', other than P, of the set of ν points I determined by P; this correspondence (P, P') we denote by U. The primes which determine the aggregate of points Q will use every point P' a certain number of times; namely as many times as there are sets of $r-1$ points in the set I, beside P and P' (which with P' will make up a set of r points determining such a prime); thus each point P' will be used $(\nu-2, r-1)$, or ρ times. Whence we see that the sum correspondence denoted by $T+\rho U$ is of zero valency, the points corresponding to P in this being the complete intersection of the curve with $(\nu-1, r)$ primes. The correspondence U between P and one of the $\nu-1$ points P' is obviously symmetrical. Consider the correspondence reverse to T: the primes through a point Q of the curve determine a linear series g^{n-1}_{r-1} thereon; to get back to a point P it is necessary to find the sets of $I_1{}^{\nu}$ which contain r points of a set of this g^{n-1}_{r-1}. If we assume that the number $Z_{n,r}$, when the involution $I_1{}^{\nu}$ is given, is a function only of n and r, the number of sets of $I_1{}^{\nu}$ which contain r points of the linear series g^{n-1}_{r-1}, is $Z_{n-1,r-1}$; and when one such set, I, is found, P may be any one of the $\nu-r$ points of this set I, other than the r points belonging to a set of g^{n-1}_{r-1}. The reverse index of the correspondence T is thus $(\nu-r)Z_{n-1,r-1}$. The direct and reverse indices of the correspondence $T+\rho U$ are thus

$$(n-r)(\nu-1, r)+\rho\,(\nu-1), \text{ and } (\nu-r)\,Z_{n-1,r-1}+\rho(\nu-1);$$

we have remarked (p. 6) that the reverse of a sum of correspondences is the sum of their reverses. Now consider the coincidences in the correspondence $T+\rho U$. These arise, (1), when a point Q coincides with P; then there are $(r+1)$ points of a set I which are coprimal, and, therefore, $(r+1)Z_{n,r}$ coincidences of the correspondence; (2), when a point P' coincides with P; and this gives ρd coincidences of the correspondence $T+\rho U$. Thus we have

$$(r+1)\,Z_{n,r}+\rho d=(n-r)(\nu-1, r)+(\nu-r)\,Z_{n-1,r-1}+2\rho(\nu-1);$$

since $(\nu-1, r)=(\nu-1)\rho/r$, this may be put into the form

$$(r+1)Z_{n,r}-(\nu-r)Z_{n-1,r-1}=\rho\left[\frac{(n+r)(\nu-1)}{r}-d\right],$$

and we easily compute that, if we put

$$c_{n,r}=\rho\left[\frac{n(\nu-1)}{r}-\tfrac{1}{2}d\right],\ =(\nu-2, r-1)\left[\frac{n(\nu-1)}{r}-\tfrac{1}{2}d\right],$$

then this is the same as

$$(r+1)\,(Z_{n,r}-c_{n,r})=(\nu-r)\,(Z_{n-1,r-1}-c_{n-1,r-1}).$$

To solve this equation, we prove that $Z_{m,1}=m(\nu-1)-\frac{1}{2}d, =c_{m,1}$. Recurring to the original plane curve, this is established by the argument we have used in general, though the result has already been obtained above, in Ex. 8. The number $Z_{m,1}$ is that of the sets of $I_1{}^{\nu}$ which contain two points of a set of a linear series $g_1{}^{m}$. Consider a set I, of $I_1{}^{\nu}$, determined by a point P; let $P_0{}'$ be another point of this set; let Q_0 be one of the points, other than $P_0{}'$, of the set of $g_1{}^{m}$ determined by $P_0{}'$; consider the sum correspondence $(P, Q_0)+(P, P_0{}')$. For the correspondence reverse to (P, Q_0): there is a set of I_1^ν determined by every one of the $m-1$ points, other than Q_0, belonging to the set of g_1^m which contains Q_0; and there are thus $(m-1)(\nu-1)$ points P corresponding reversely to Q_0. Thus, as in the case $r>1$, we have the equation

$$2Z_{m,1}+d=(m-1)(\nu-1)+(\nu-1)(m-1)+2(\nu-1), \; =2m(\nu-1),$$

as desired.

Wherefore, supposing $m=n-r$, the general difference equation leads to $Z_{n,r}=c_{n,r}$, namely

$$Z_{n,r}=(\nu-2, r-1)\left[\frac{n}{r}(\nu-1)-\tfrac{1}{2}d\right].$$

This result may be generalised. See Segre, *Ann. d. Mat.* XXII, 1894, p. 99, and the references given in Severi, *Trattato di geom. algeb.* I, 1, 1926, p. 253, to whom the proof here given is due.

We have supposed the involution I_1^ν to be of index 1. If this be of index ν', the result is at once obtainable from that found by replacing n by $n\nu'$. Or a direct proof, identical with that given, may be followed.

A consequence of the formula

$$Z_{n,r}=(\nu-2, r-1)\left[\frac{n\nu'}{r}(\nu-1)-\tfrac{1}{2}d\right],$$

for the case of an irreducible involution I_1^ν, of freedom 1, of sets of ν points, but with index ν', is remarked by Castelnuovo (*Rend. Lincei*, XV, 1906, p. 337), namely:

The number of coincidences of pairs of points of a set of such an involution, cannot exceed $\nu'(2\nu+2p-2)$; and, if d is as great as this, all the sets of ν points, of the involution, are sets of the same linear series. We give a sketch of the proof of this.

With $\nu-1$ points of a set I of such an involution, and p arbitrary points of the curve, define a linear series, of sets of $\nu-1+p$ points of freedom $\nu-1$, $g_{\nu-1}^{\nu-1+p}$. The number of sets of ν points, common to the linear series and the involution, is then $Z_{\nu+p-1,\nu-1}$, say Z, given by $Z=\nu'(\nu+p-1)-\frac{1}{2}d$, and, as this cannot be negative, we have the necessary upper bound for d. Suppose, however, if possible, that $d=2\nu'(\nu+p-1)$. Then $Z=0$, and there are, in a general linear series $g_{\nu-1}^{\nu-1+p}$, no sets which, having $\nu-1$ points taken in a set of the involution, contain the remaining point. The argument by which we found $Z_{n,r}$, however, as in all similar cases, assumes that this number is finite. Thus if we construct a linear series $g_{\nu-1}^{\nu-1+p}$ with the property that, a set of this determined by $\nu-1$ points of a particular set of the involution contains all points of this set, then the conclusion will be that the same is true for every set of the involution; namely, that all the sets of the involution belong to the same linear series. To construct such a linear series, first form the linear series $g_\nu^{\nu+p}$ of which one set consists of a particular set I of the involution taken with p arbitrary points of the curve, say $A_1, \ldots, A_p$; then take the linear series $g_{\nu-1}^{\nu-1+p}$ formed by all the sets of $g_\nu^{\nu+p}$ which contain the point A_i (other than this). As, by what has been said, every set of the involution consists of points belonging to one set of $g_{\nu-1}^{\nu-1+p}$, we infer that the sets of $g_\nu^{\nu+p}$ which contain sets of the involution all contain A_i; hence, by parity of reasoning, all these sets of $g_\nu^{\nu+p}$ contain $A_1, A_2, \ldots, A_p$. Thus the involution consists of sets of the linear series obtained by the sets of $g_\nu^{\nu+p}$ which contain $A_1, \ldots, A_p$.

We may remark that, for general values of n and r, when $d=\nu'(2\nu+2p-2)$, the value of $Z_{n,r}$ is

$$\nu'(\nu-1, r)\left[n-r-\frac{rp}{\nu-1}\right],$$

which, when $\nu=r+2$, is $\nu'\,[(n-r)(r+1)-rp]$.

Castelnuovo has proved further, as a Corollary, due to Severi (*Ann. d. Mat.* XII, 1906, p. 55), that, if the sets of $\nu'\nu$ points, consisting of all the sets of the involution I_1^{ν} which contain a point P of the curve (including this point itself reckoned ν' times), belong to a linear series, so do the separate sets of ν points of the involution.

Ex. 13. For the developable surface formed by the tangents of a curve, beside the cuspidal curve which is constituted by the original curve, there is a double curve, the *nodal curve*, of order ν, such that $p=\frac{1}{2}(r-1)(r-2)-n-\nu-\tau-i$ (see p. 191 of Vol. V). This double curve has triple points whose number is

$$\tfrac{1}{3}\left[-r^3+13r^2-42r+8n'+\nu(3r-26)-2\tau\right].$$

Cf. Zeuthen, *Ann. d. Mat.* III, 1869–70; Pascal, *Repertorium*, II, 2, 1922, p. 899. The triple tangent planes of the curve are of number obtainable from this by replacing n' by n, and ν by the class ν' of the developable surface formed by the planes which have two contacts with the curve. For example, for the sextic intersection of a quadric and a cubic surface, we have $n=6$, $r=18$, $\nu'=96$, $\tau=0$; and the number of triple tangent planes is 120. (See also, below, Ex. 14*b*.)

Find the genus of the double curve of the developable surface formed by the tangent lines of the curve.

Many particular results will be found in Zeuthen's paper referred to, and in the Pascal. See also Cayley's paper, *Papers*, V, p. 172.

Ex. 14. The curves which determine the sets of a linear series g_r^n, of freedom r, upon a given plane curve, contain r variable parameters. We expect then that there is a determinate number of such curves having $r+1$ coincident intersections with the given curve, if the point of contact is not assigned—and we have found this number (p. 10, above). The conditions to be satisfied may be formulated algebraically, by expressing that the general curve giving a set of the linear series, and the r further curves obtained by differentiation of this, have a common point on the given curve. Similarly, the conditions that, among the intersections of a variable line with a given plane curve, there should be two points of contact, are two in number; we expect then a finite number of bitangents of the curve; though this problem, as formulated algebraically, will have as solutions the lines joining two cusps of the curve, and the tangents to the curve from the cusps. Somewhat similarly, if we consider an algebraic curve in space of four dimensions, since the conditions for a line to meet such a curve are two in number, and there are ∞^2 chords of the curve, we expect to find a finite number of chords meeting the curve in a third point, that is, of trisecants of the curve. Many such problems have been solved by the principle of correspondence. We give here first, with an indication of the proof, a very general formula due to de Jonquières. This determines, in a given linear series g_r^n upon a given plane curve, the number of sets in which there are several coincidences, or contacts, at unassigned points of the curve, of such character that the total number of conditions required for these coincidences is r. And we add, without proof, some formulae for problems of incidence such as that referred to, of the number of trisecants of a curve in space [4].

We consider then, first, upon a given plane curve, a given linear series g_r^n, not necessarily complete, of which the sets have no points in common. We consider the possibility of a set, of n points, of this linear series, consisting of α_0 simple points, of α_1 points each of which is a coincidence of r_1+1 points, of α_2 points at each of which there is a coincidence of r_2+1 points, and so on, finally of α_ρ points at each of which there is a coincidence of $r_\rho+1$ points; so that $\alpha_0+\alpha_1(r_1+1)+\ldots+\alpha_\rho(r_\rho+1)=n$. For a coincidence of r_i+1 points at an unassigned point of the curve, r_i conditions are necessary. We suppose the total number of conditions imposed for the set to be r; that is $\alpha_1 r_1+\ldots+\alpha_\rho r_\rho=r$. We put $k=n-r$, and so have $\alpha_0+\alpha_1+\ldots+\alpha_\rho=k$, this being the number of distinct points in the set. We expect that the linear series contains a finite number of sets of the character described; our object is to find this number. If p be the genus of the curve, we put p_i, k_i for the binomial coefficients, $p_i=p!/i!(p-i)!$, $k_i=k!/i!(k-i)!$, and denote by s_m the coefficient of t^m in the ascending expansion of $(1+r_1t)^{\alpha_1}(1+r_2t)^{\alpha_2}\ldots(1+r_\rho t)^{\alpha_\rho}$, so that s_m is the sum of the products, m together, without repetitions, of

$$r_1^{(1)}, r_1^{(2)}, \dots, r_1^{(\alpha_1)};\ r_2^{(1)}, r_2^{(2)}, \dots, r_2^{(\alpha_2)};\ \dots\dots\ r_\rho^{(1)}, r_\rho^{(2)}, \dots, r_\rho^{(\alpha_\rho)},$$

wherein

$$r_1^{(1)} = r_1^{(2)} = \dots = r_1^{(\alpha_1)} = r_1,\ \dots,\ r_\rho^{(1)} = r_\rho^{(2)} = \dots = r_\rho^{(\alpha_\rho)} = r_\rho.$$

Thus s_m is zero when $m > \alpha_1 + \dots + \alpha_\rho$, or $m > k - \alpha_0$. We also use two symbols, M and f, the former as a multiplier, the latter a series of $1 + \alpha_1 + \dots + \alpha_\rho$ terms, defined by

$$M(0^{\alpha_0}, r_1^{\alpha_1}, \dots, r_\rho^{\alpha_\rho}) = (k!/\alpha_0!\,\alpha_1! \dots \alpha_\rho!)\,(1 + r_1)^{\alpha_1} \dots (1 + r_\rho)^{\alpha_\rho},$$

$$f(0^{\alpha_0}, r_1^{\alpha_1}, \dots, r_\rho^{\alpha_\rho}) = 1 + (p_1/k_1)\,s_1 + (p_2/k_2)\,s_2 + \dots + (p_{k-\alpha_0}/k_{k-\alpha_0})\,s_{k-\alpha_0},$$

and denote the product of these by ψ, so that

$$\psi(0^{\alpha_0}, r_1^{\alpha_1}, \dots, r_\rho^{\alpha_\rho}) = M(0^{\alpha_0}, r_1^{\alpha_1}, \dots, r_\rho^{\alpha_\rho}) . f(0^{\alpha_0}, r_1^{\alpha_1}, \dots, r_\rho^{\alpha_\rho}).$$

The result is that the number of sets of g_r^n satisfying the prescribed conditions is this number ψ.

To explain the notation, first take simple examples: (1), For the number of sets of g_r^n which have a single coincidence of $r+1$ points, we have $\alpha_0 = n - r - 1$, $\rho = 1$, $\alpha_1 = 1$, $r_1 = r$, $k = n - r$, and

$$M = [(n-r)!/(n-r-1)!]\,(1+r),\ = (n-r)(1+r),$$

$$f = 1 + [p/(n-r)]\,r,\ = 1 + rp/(n-r),$$

so that $\psi = Mf = (r+1)(n-r+rp)$, as was formerly found (Ex. 2, p. 10); (2), For the number of lines meeting a plane curve of order n in two points of coincidence each of two points, and in $n-4$ simple points, we have $\alpha_0 = n-4$, $\alpha_1 = 2$, $r_1 = 1$, $k = n-2$, $\rho = 1$, and

$$M = 2^2\,(n-2)!/(n-4)!\,2!,\ = 2\,(n-2)(n-3),$$

so that

$$f = 1 + 2p/(n-2) + p_2/(n-2)_2,$$

$$\psi = 2\,(n-2)(n-3) + 4\,(n-3)\,p + 2p\,(p-1).$$

This can be verified to be the same as

$$\tau + \tfrac{1}{2}\kappa\,(\kappa - 1) + \kappa\,[n\,(n-1) - 2\delta - 3\kappa - 3],$$

where δ, κ, τ denote the number of double points, cusps, and double tangents of the curve; so that ψ is the number of proper double tangents, together with the number of joins of a pair of cusps, and the number of tangents to the curve drawn from the cusps. As a third example, (3), consider how many sets are contained in the canonical series g_{p-1}^{2p-2} which consist of $p-1$ coincidences of two points. For this case, $\alpha_0=0$, $\alpha_1=p-1$, $r_1=1$, $\rho=1$, $k=p-1$ and $n=2p-2$; thus

$$M=[(p-1)!/(p-1)!]\,2^{p-1},$$

$$f=1+[p_1/(p-1)_1]\,(p-1)_1+[p_2/(p-1)_2]\,(p-1)_2+\dots$$
$$+[p_{p-1}/(p-1)_{p-1}]\,(p-1)_{p-1},$$

or $$f=1+p_1+p_2+\dots+p_{p-1}=(1+1)^p-1,\ =2^p-1;$$

thus the number is $2^{p-1}(2^p-1)$, as is well known.

To prove the general formula we proceed by induction; we assume that it is proved for linear series g_s^n, of freedom $s<r$, and thence deduce it for freedom r. This will be sufficient because the result is known for series of freedom 1. For, then, the only possibility is that for which $\rho=1$, with $\alpha_1=1$, $r_1=0$; and the number of Jacobian points of a linear series g_1^n is known.

Take a definite point, P, of the fundamental curve; and consider the sets of the series g_r^n which have r_1 coincident points at P. The variable intersections then belong to a series, of sets of $n-r_1$ points, with freedom $r-r_1$, $g_{r-r_1}^{n-r_1}$. In this series consider the sets which have, save at P, precisely the same coincidences as those sought, namely have (α_1-1) points of (r_1+1)-fold coincidence, α_2 points of (r_2+1)-fold coincidence, and so on, finally α_ρ points of $(r_\rho+1)$-fold coincidence, and therefore (α_0+1) points of simple intersection; such a simple intersection we denote by Q. By the assumption made, the number of such sets is $\psi(0^{\alpha_0+1}, r_1^{\alpha_1-1}, r_2^{\alpha_2}, \dots, r_\rho^{\alpha_\rho})$. The simple points Q, of these sets, are determined when P is assigned, and the coincidences in question are fulfilled. We can therefore regard the points Q as arising from P by a correspondence on the curve; and a coincidence in this correspondence, when one of the points Q coincides with P, will give rise to a set of the original series g_r^n with the coincidences required; conversely, every such set may be supposed to arise in this way. We desire then to obtain the index of the reverse correspondence, from Q to P, and the valency of this correspondence. Now, to pass from Q to P, we should consider all sets of the original series g_r^n having Q as a given point, namely a series g_{r-1}^{n-1}; and therein the sets having one r_1-fold coincidence, at P, with (α_1-1) points of (r_1+1)-fold coincidence, α_2 points of (r_2+1)-fold coincidence, and so on, finally α_ρ points of $(r_\rho+1)$-fold coincidence, and therefore α_0 simple points. As the freedom of this series is $r-1$, the number of such sets, by the assumption made, is $\psi(0^{\alpha_0}, r_1-1, r_1^{\alpha_1-1}, r_2^{\alpha_2}, \dots, r_\rho^{\alpha_\rho})$; this is then the reverse index of the correspondence (P, Q). As there are (α_0+1) points Q in each of the contemplated sets of the series $g_{r-r_1}^{n-r_1}$, the direct index is $(\alpha_0+1)\,\psi(0^{\alpha_0+1}, r_1^{\alpha_1-1}, r_2^{\alpha_2}, \dots, r_\rho^{\alpha_\rho})$. But the determination of the valency of the correspondence (P, Q) is less easy. For this vital step we refer the reader to Zeuthen, *Lehrbuch*, 1914, p. 246, and, especially, Torelli, *Rend. Palermo*, XXI, 1906, p. 58 (see also, de Jonquières, *Crelle*, LXVI, 1866, p. 289; Brill, *Math. Ann.* VI, 1873, p. 47, and Cayley, *Papers*, VII, p. 41). This valency is, in fact, γ, given by $\gamma = r_1 \psi_{p-1}(0^{\alpha_0}, r_1^{\alpha_1-1}, r_2^{\alpha_2}, \ldots, r_\rho^{\alpha_\rho})$, where ψ_{p-1} is formed in accordance with the definition above given for ψ, except in one respect, namely that $p-1$ is substituted for p in the series f; another form for γ is given by

$$p\gamma = \frac{\alpha_1}{r_1+1}\psi(0^{\alpha_0}, r_1^{\alpha_1}, \ldots, r_\rho^{\alpha_\rho}) - (\alpha_0+1)\,\psi(0^{\alpha_0+1}, r_1^{\alpha_1-1}, r_2^{\alpha_2}, \ldots, r_\rho^{\alpha_\rho}).$$

If Z be the number required, of sets of the series g_r^n, which have the specified coincidences (α_1 of (r_1+1)-fold coincidence, ..., α_ρ of $(r_\rho+1)$-fold coincidence, and α_0 simple points), then the number of coincidences of the correspondence (P, Q) will be $\alpha_1 Z$, since P may be at any one of the α_1 points of (r_1+1)-fold coincidence. Thus, to deduce the formula stated for Z, it is necessary to prove that the sum of the two indices above, with $2p\gamma$, is equal to

$$\alpha_1\psi(0^{\alpha_0}, r_1^{\alpha_1}, \ldots, r_\rho^{\alpha_\rho}).$$

If we denote the portions of these symbols which relate to (α_1-1) points of (r_1+1)-fold coincidence, α_2 points of (r_2+1)-fold coincidence, ..., α_ρ points of $(r_\rho+1)$-fold coincidence, simply by S, the necessary equation is

$$\alpha_1\psi(0^{\alpha_0}, r_1, S)$$
$$= (\alpha_0+1)\,\psi(0^{\alpha_0+1}, S) + \psi(0^{\alpha_0}, r_1-1, S) + 2pr_1\,\psi_{p-1}(0^{\alpha_0}, S),$$

of which the algebraic verification is not difficult.

The result stated in regard to the valency is so remarkable that it is perhaps worth a separate statement in more general terms. The sets of a series of given grade (n) and freedom (r), are to satisfy certain conditions of coincidence, of prescribed multiplicity at each of a prescribed number of points; the number of conditions imposed is to be equal to the freedom of the series, a coincidence of multiplicity r_i+1 being counted as requiring r_i conditions for the sets sought. There will be a certain number, α_0, of simple points in each such set, depending on the grade of the series; these impose no condition. Consider apart, for distinctness, one of the coincidences, of multiplicity r_1+1, say at P_1, speaking of the remaining coincidences, with their prescribed character, as imposing conditions S. Now, take, in the sets of the original linear series, those which satisfy the conditions S, but have, at an assigned point P, a coincidence of multiplicity r_1; this imposes as many (r_1) conditions as did the ascription of a coincidence of multiplicity r_1+1 at an unassigned point P_1; there will then be a finite number of such sets; but in each of these there will be α_0+1 simple points, say Q. We consider the correspondence between P and all the sets of α_0+1 points Q which exist satisfying, beside the coincidence at the assigned point P, the conditions S at the other unassigned points. A coincidence in this correspondence, in which a point Q coincides with P, gives a set satisfying the original conditions, of (r_1+1)-fold coincidence at P, beside the conditions S.

Then all these points Q arising from P, taken with P counted $r_1\psi_{p-1}$ times, form a set of a linear series on the original curve; where ψ_{p-1} is the number, on a curve of genus $p-1$, of sets (in a linear series $g_{r-r_1}^{n-r_1-1}$), whose coincidences are those of the conditions S, and have beside α_0 simple points; that is, of sets characterised precisely by the coincidences of the original sets to be enumerated, other than the (r_1+1)-fold coincidence at P_1.

Ex. 14*a*. If on the canonical curve of order $2p-2$ in space of $p-1$ dimensions, we consider primes touching the curve in $(p-2)$ points, the two remaining intersections of such a prime with the curve are in symmetrical correspondence, with indices both equal to $2^{2p-4}(2p-2)-2^{p-2}(2^{p-1}-1)$, and the valency of this correspondence is equal to $2^{p-2}(2^{p-1}-1)$. It can be proved in fact that, if $q_i=q!/i!(q-i)!$, then

$$1+\frac{p_1}{(p-1)_1}(p-2)_1+\frac{p_2}{(p-1)_2}(p-2)_2+\ldots+\frac{p_{p-2}}{(p-1)_{p-2}}(p-2)_{p-2}$$
$$=\frac{2^{p-1}p-(2^p-1)}{p-1}.$$

A direct proof, by the theory of correspondence, for the number, $2^{p-1}(2^p-1)$, of primes having $(p-1)$ simple contacts with the canonical curve, is given by Welchman, ***Proc. Camb. Phil. Soc.*** XXVI, 1930, p. 453.

Ex. 14*b*. The general formula for the number of planes having 3 contacts with a curve of order n and genus p, and $n-6$ simple intersections, is found to be

$$8\,[(n-3,\,3)+(n-4,\,2)\,p+(n-5)\,(p,\,2)+(p,\,3)],$$

where $(\lambda,\,\mu)$ denotes the binomial coefficient. Identify this with the result given in Ex. 13 (p. 38) above, explaining the discrepancy (as in the case of the double tangents of a plane curve on p. 40). Either formula, as in all these cases, supposes that the curve has only a finite number of such triply touching planes; for example, a curve, lying on a developable surface, if it meets each generator in three points, has every tangent plane of the developable as a triple tangent plane.

Ex. 15. Closely connected with the problem considered in Ex. 14 is that of determining the number of linear spaces which satisfy conditions of incidence (or contact) with one, or more, curves; it is supposed, of course, that the number of conditions is equal to the number of parameters on which such a linear space depends. We do not enter into a complete account; but we quote the result for one case.

Consider, in space $[r]$, a curve of order n, and genus p, with $n \geqslant p+r$. The number of conditions for a space of dimension k, say $[k]$, in this space, to meet a line, is $r-k-1$; we assume that this is also the number of conditions necessary for such a space to meet an algebraic curve. The number of conditions necessary for the space to have i intersections with the curve is then $i(r-k-1)$; and a space $[k]$, in space $[r]$, depends on $(k+1)(r-k)$ parameters. These two numbers will be equal if $i(r-k-1)=(k+1)(r-k)$. We suppose this equality to hold. It requires $i > k+1$; we denote $i-k-1$ by j; we also put $m=n-p-r$, and $l=m-j$; and also $s=r-k-1$. Thus the numbers j, m, l, s are perfectly definite.

We take all possible, zero or positive, integers $\lambda, \rho_0, \rho_1, \ldots, \rho_s$, whose sum is p, supposing $\lambda < p$; we denote the product of the squares of the differences of the numbers $\rho_0, \rho_1+1, \ldots, \rho_s+s$, by Δ; also we denote the number

$$\frac{(m+\rho_t+t)!}{(\rho_t+t)!\,(j+t)!\,(l+\rho_t+t)!},$$

for $t=0, 1, \ldots, s$, by n_t; and the product $n_0 n_1 \ldots n_s$ by $n(\rho_0 \ldots \rho_s)$. With these notations, the number of spaces $[k]$ which have i intersections with the curve, is given by

$$p!\sum_{\lambda=0}\frac{(-s)^\lambda}{\lambda!}\Sigma\Delta\,.\,n(\rho_0 \ldots \rho_s), \qquad 0 \leqslant \rho_0 \leqslant \rho_1 \leqslant \ldots \leqslant \rho_s.$$

For this formula we refer to Giambelli, *Mem....Torino*, LIX, 1908, p. 433, quoted by Segre, *Enzykl. Math. Wiss.* III, C 7, p. 888.

To test the meaning of the formula, apply it to the case of the trisecant lines of a curve in space of four dimensions. We thus obtain (writing μ, ν, σ for ρ_0, ρ_1, ρ_2)

$$p!\Sigma\frac{(-2)^\lambda}{\lambda!}(\nu+1-\mu)^2(\sigma+2-\mu)^2(\sigma+1-\nu)^2$$
$$\times\frac{(n-p-4+\mu)(n-p-4+\nu+1)(n-p-4+\sigma+2)}{\mu!\,(\nu+1)!\,(\sigma+2)!\;1!\;2!\;3!}$$

for all values of $\lambda, \mu, \nu, \sigma$ subject to

$$\lambda+\mu+\nu+\sigma=p,\ 0 \leqslant \lambda < p,\ 0 \leqslant \mu \leqslant \nu \leqslant \sigma,$$

for which $n-p-4+\mu > 0$. For example, if $n=10$ and $p=6$, there are seven terms in the sum, corresponding to the values of λ, μ, ν, σ given by

$$(3, 1, 1, 1), (2, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 2), (0, 1, 1, 4), (0, 1, 2, 3), (0, 2, 2, 2),$$

and computation leads to 20, as the number of trisecant chords of the curve.

Simpler formulae were given for particular cases by Castelnuovo (*Rend. Palermo*, III, 1889, p. 27; *Rend. Lincei*, V, 1889, p. 130):

(a), In space $[r]$, the number of primes having r contacts with a curve of order n and genus p is the coefficient of t^r in the ascending expansion of $2^r(1+t)^{n-r-p}(1+2t)^p$. Or, again, in space of an even number of dimensions, with spaces $[k]$ for which $k=\frac{1}{2}r-1$, to meet the curve in the number i, $=(k+2)$, of points which follows from the formula $(k+1)(r-k)=i(r-k-1)$, the number of such spaces is the coefficient of t^{k+2} in the ascending expansion of

$$(1+t)^{n-k-1}[1-t^2/(1+t)^2]^p.$$

Thus the number of trisecants of a curve (n, p) in space $[4]$ is $(n-4)[\frac{1}{6}(n-2)(n-3)-p]$, which is $\frac{1}{3}(n-4)d$, where d is the number of accidental double points of a ruled surface of order n with prime section of genus p, in space $[4]$, as will be seen in a subsequent chapter. For $n=10$, $p=6$, this gives the number 20 found from Giambelli's formula. Or again, the number of quadrisecant planes of a curve (n, p) in space $[6]$ is

$$(n-3, 4)-p(n-5, 2)+(p, 2),$$

where (λ, μ) is the binomial coefficient. A similar, but slightly more complicated formula gives, without requiring r to be even, the number of spaces $[r-2]$ which have $2(r-1)$ intersections with a curve (n, p), in the form

$$\sum_{\lambda=0} \frac{(-1)^\lambda}{r-\lambda}(n-r-\lambda, r-\lambda-1)(n-r-\lambda+1, r-\lambda-1)(p, \lambda),$$

which, for example, leads to the result found, in Ex. 11a (p. 32 above), for the number of quadrisecants of a curve in space $[3]$.

(b), When the curve is in space $[r]$, with $r=n-p$, the spaces $[k]$ with i intersections, where $i(r-k-1)=(k+1)(r-k)$, are of number

$$\frac{p!}{(p-i)!}\,\frac{\{r-k-1\}\{i-k-2\}}{\{r+i-2k-2\}},$$

where $\{m\}$ denotes $1!\,2!\ldots m!$, with $\{0\}=1$. This again gives 20 for $r=4$, $n=10$, $p=6$, $k=1$, $i=3$.

For more general results cf. Severi, *Mem. Torino*, L, 1901, p. 81, LI, 1902, p. 103, where formulae are given for the number of spaces

$[k]$ which meet a curve of order n and genus p in space $[r]$ with ν_1-point intersection at one point, ν_2-point intersection at another point, and so on, finally with ν_t-point intersection at a t-th point, subject to $(r-k)\Sigma\nu_i - t = (r-k)(k+1)$. See also Schubert, *Mitt. d. math. Ges. Hamburg*, III, 1891, p. 12.

Part II. Transcendental methods. The application of Riemann's methods to the theory of algebraic correspondence leads to new results, as was first shewn by Hurwitz, *Math. Ann.* XXVIII, 1887. In particular it is possible to shew that, on a single curve of general moduli, for its genus, every such correspondence is of the kind we have considered hitherto, having a valency; but that, on a curve of particular moduli, there may exist a correspondence in which, instead of the single number, the valency, there arises a matrix of numbers. We explain the matter so as to include both possibilities.

Consider a curve C, of genus p, whose normal integrals of the first kind are denoted by $u_1, \ldots, u_p$. Denote the periods of u_i by $(1)_{ij}, \rho_{ij}$, where $(1)_{ij} = 0$ unless when $j = i$, and $(1)_{ii} = 1$. Thus we speak of these integrals as having the period scheme $(1, \rho)$, where 1 denotes the unit matrix of type (p, p), and ρ denotes the matrix (ρ_{ij}), also of type (p, p). These periods are those corresponding to *crossings* of period ovals on the associated Riemann surface, these ovals being denoted by (γ, γ'); thus in particular ρ_{ij} is the excess of the value of u_i on the left of γ_j' over its value on the right of this oval (see Chap. VI of Vol. V, p. 141). Consider also a curve D, of genus q, with normal integrals of the first kind denoted by $v_1, \ldots, v_q$, whose period scheme is $(1, \sigma)$, over ovals, denoted by (δ, δ'), on the associated Riemann surface. Let an elementary normal integral, of the curve C, of the *third* kind be denoted by $\Pi_{z,c}^{x,a}$, as in the chapter referred to; and a similar integral of the curve D by $\Phi_{z,c}^{x,a}$. For definiteness, we suppose $p \leqslant q$.

Now suppose there exists an algebraic (s, r) correspondence between these curves, whereby to the place (x), of C, correspond the places $(z_1), \ldots, (z_r)$ of the curve D, it being the case that any rational symmetric function of $(z_1), \ldots, (z_r)$ is expressible rationally by (x); while to the place (z), of D, correspond the places $(x_1), \ldots, (x_s)$ of C, with a like property. We denote particular positions of $(x), (x_1), \ldots, (x_s)$ by $(a), (a_1), \ldots, (a_s)$; and particular positions of $(z), (z_1), \ldots, (z_r)$ by $(c), (c_1), \ldots, (c_r)$. From these assumptions it follows that the sum of the values of v_j at the places $(z_1), \ldots, (z_r)$, of D, depends uniquely on the place (x) of C, and is an everywhere finite integral thereon. Hence there exist q equivalences of the forms

$$v_j^{z_1, c_1} + \ldots + v_j^{z_r, c_r} \equiv M_{j,1} u_1^{x,a} + \ldots + M_{j,p} u_p^{x,a}, \quad j = 1, 2, \ldots, q,$$

where the coefficients $M_{j,k}$ are constants; and it will follow that there are similar equivalences

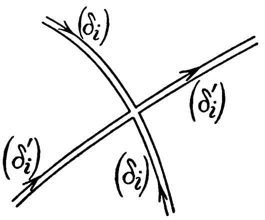
$$u_i^{x_1, a_1} + \ldots + u_i^{x_s, a_s} \equiv N_{i,1} v_1^{z,c} + \ldots + N_{i,q} v_q^{z,c}, \quad i = 1, 2, \ldots, p;$$

here the integrations are on the respective Riemann surfaces, dissected along the ovals (γ, γ') and (δ, δ'), and the signs $\equiv$ denote the omission of *definite* constant aggregates of periods, all the integrals being single valued on the respective dissected Riemann surfaces.

The constants $M_{j,k}$, $N_{i,l}$, which enter here form two matrices, respectively of types (q, p) and (p, q). We proceed to express these matrices in terms of the relations which arise, in virtue of the correspondence, between the period ovals on the two Riemann surfaces representing the curves C and D. Denote any algebraic integral on the curve C by I; and, on the curve D, by J. When (x) describes a closed curve on the Riemann surface C, which we may suppose not to cross any of the $2p$ period ovals (γ, γ'), the set $(z_1), \ldots, (z_r)$ will vary on the surface D, and come back to their original positions, with (possibly) interchange among themselves. We can suppose these paths of $(z_1), \ldots, (z_r)$ also taken so as not to cross the period ovals (δ, δ'), a passage of, for example, δ_i, being avoided by a detour along δ_i'. In virtue of such an identity as

$$J^{z_2, k_1} + J^{z_1, k_2} = J^{z_2, z_1} + J^{z_1, k_1} + J^{z_1, z_2} + J^{z_2, k_2},$$

we can suppose that the aggregate of the paths described by $(z_1), \ldots, (z_r)$ is equivalent with an aggregate of closed paths. Now any closed path, on the surface D, dissected along the $2q$ period loops, (the $q-1$ connecting traverses being ineffective, see Vol. v, p. 141) is equivalent, so far as its effect in adding a period increment, to an algebraic integral, is concerned, to a certain number of circuits of the positive (left) rims of the $2q$ period ovals. And a circuit, in the negative direction, of the left rim of the period oval δ_i', is equivalent to a crossing of the associated δ_i; while a circuit, in the positive direction, of the positive rim of δ_i is equivalent to a crossing of the associated δ_i'. Hence we may express the paths on D, arising by the correspondence, corresponding to circuits on C, by saying that the crossing of an oval γ_i by (x) gives rise, for an integral sum $J^{z_1, c_1} + \ldots + J^{z_r, c_r}$ on D, to a certain number of crossings of the $2q$ oval cuts made along (δ, δ'); and, precisely, that it is equivalent to



$$\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{q,i}; \alpha'_{1,i}, \alpha'_{2,i}, \ldots, \alpha'_{q,i}, \qquad i = 1, \ldots, p,$$

crossings of the respective oval cuts

$$\delta_1, \delta_2, \ldots, \delta_q; \delta_1', \delta_2', \ldots, \delta_q';$$

and, similarly, that the crossing of γ_i' by (x) is equivalent to, respectively,

$$\beta_{1,i}, \beta_{2,i}, \dots, \beta_{q,i}; \beta'_{1,i}, \beta'_{2,i}, \dots, \beta'_{q,i}$$

crossings of the same oval cuts. The crossing is intended to be from the right (or negative) rim to the positive rim, in each case, corresponding to the ascription of the periods; a crossing in the other direction will be called a recrossing. Later we shall modify these symbols, expressing crossings by circuits, of associated ovals.

We thus have four matrices, say α, α', β, β', each of q rows and p columns, of which all elements, such as $\alpha_{j,i}$, etc., are integers, positive or negative or zero; and, in accordance with what is said above, the crossing, by (x), on the surface C, of the loop γ_i, corresponds to the crossings, for the integral sum on D, of the $2q$ period ovals δ, δ', which are given by taking the column (i) of the matrices α, α'; while, when (x) crosses γ_i', the corresponding crossings on D are similarly defined by the i-th column of the matrices β, β'. There will be a similar set of matrices when we employ the reverse transformation to pass from D to C; the values of these will be expressed in what follows.

The coefficients $M_{j,k}$ in the equivalences

$$v_j^{z_1, c_1} + \dots + v_j^{z_r, c_r} \equiv M_{j,1} u_1^{x,a} + \dots + M_{j,p} u_p^{x,a}, \qquad j = 1, \dots, q,$$

can be evaluated at once in terms of the matrices α, α', β, β'. For, first, when (x) crosses γ_i, the right side in this suffers the increment $M_{j,i}$, while the left side has the increment

$$(1)_{j,1}\alpha_{1,i} + \dots + (1)_{j,q}\alpha_{q,i} + \sigma_{j,1}\alpha'_{1,i} + \dots + \sigma_{j,q}\alpha'_{q,i},$$

where $(1)_{j,k}$ is the period of v_j for crossing of δ_k, $(=1$ or zero), and $\sigma_{j,k}$ the period for crossing δ_k'. Thus we have

$$M_{j,i} = \alpha_{j,i} + \sigma_{j,1}\alpha'_{1,i} + \dots + \sigma_{j,q}\alpha'_{q,i}, \qquad \begin{cases} i = 1, \dots, p, \\ j = 1, \dots, q, \end{cases}$$

which we write, in matrix notation,

$$M = \alpha + \sigma\alpha', \qquad (A),$$

M being, like α, α', of type (q, p), while σ is of type (q, q). Similarly, when (x) crosses γ_i', the right side of the equivalence under consideration obtains the increment

$$M_{j,1}\rho_{1,i} + \dots + M_{j,p}\rho_{p,i},$$

and the left side has the increment

$$(1)_{j,1}\beta_{1,i} + \dots + (1)_{j,q}\beta_{q,i} + \sigma_{j,1}\beta'_{1,i} + \dots + \sigma_{j,q}\beta'_{q,i};$$

thus, with the notation of matrices,

$$M\rho = \beta + \sigma\beta', \qquad (B).$$

Eliminating M from the equations (A), (B), we have

$$(a+\sigma\alpha')\,\rho-\beta-\sigma\beta'=0,$$

which is equivalent to

$$(1,\sigma)\begin{pmatrix}-\ \beta, & \alpha\\ -\beta', & \alpha'\end{pmatrix}\begin{pmatrix}1\\ \rho\end{pmatrix}=0,$$

where the three matrices involved are respectively of types $(q, 2q)$, $(2q, 2p)$, $(2p, p)$, the symbol 1 denoting, in the first and last of these, the unit matrix respectively of types (q, q) and (p, p). Similarly the two relations (A), (B) are both expressible by the single equation

$$M(1,\rho)=(1,\sigma)\begin{pmatrix}\alpha\,, & \beta\\ \alpha', & \beta'\end{pmatrix}, \qquad (C),$$

wherein the types of the four matrices are in turn (q, p), $(p, 2p)$, $(q, 2q)$, $(2q, 2p)$.

It may happen that the sets $(z_1), \dots, (z_r)$, for different points (x), all belong to a linear series on the surface D; in this case, by Abel's Theorem, the matrix M is zero; and also, $v_j^{z_1, c_1}, \dots, v_j^{z_r, c_r}$ being single valued on the dissected Riemann surface, every matrix α, α', β, β' consists of zeros. It will follow from the succeeding theory that, in this case, also $(x_1), \dots, (x_s)$ are a set of a linear series on the surface C; and this is clear in an elementary way, the correspondence being then representable by a single polynomial equation $\phi\,(x, z)=0$, as can be easily proved.

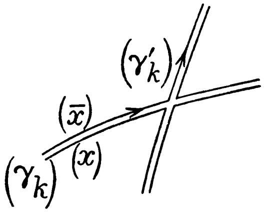
We now consider the sum $\Phi_{z,c}^{z_1,c_1}+\dots+\Phi_{z,c}^{z_r,c_r}$, on the surface D, wherein (z), (c) are arbitrary positions, and $(z_1), \dots, (z_r)$ correspond as before to a place (x) of the surface C, while $(c_1), \dots, (c_r)$ correspond to a place (a) of this surface; the function $\Phi_{z,c}^{\zeta,\gamma}$, as before said, is the elementary normal integral of the third kind. The sum itself we may, for brevity, denote by $(\Phi_{z,c}^{z_\nu,c_\nu})$.

This sum, being symmetrical with respect to $(z_1), \dots, (z_r)$, and also in regard to $(c_1), \dots, (c_r)$, may be regarded as a function of the places (x), (a) of the surface C, as well as of the independent places (z), (c) of D.

When one of the places $(z_1), \dots, (z_r)$, of D, which correspond to (x) on C, is in the neighbourhood of (z), the function is logarithmically infinite, with multiplier $+1$. This will happen if, and only if, (x) be in the neighbourhood of one of the places $(x_1), \dots, (x_s)$, of C, which correspond to the place (z) of D; and, from the analytical character of the correspondence between C and D, the function, regarded as a function of (x) on C, will have the like logarithmic behaviour in this case. There is similarly logarithmic infinity, with multiplier -1, when one of $(z_1), \dots, (z_r)$, on D, is in the neighbourhood of (c); and, therefore, when (x) is in the neighbourhood of one of the places $(a_1), \ldots, (a_s)$ of C which correspond to (c). The function depends on (x) only through $(z_1), \ldots, (z_r)$, and hence has no other infinities than those enumerated. Moreover, the function is single valued (and analytical) in (x), upon the surface C, dissected along the period ovals γ, γ'. Now, on this surface C, make the $(p-1)$ traverses, connecting the p pairs of ovals (γ_i, γ_i'), so as to obtain a single boundary of the surface C, with a direction of description of this consistent with that employed for γ_i and γ_i' (see Vol. v, pp. 131, 141). This being done, and the surface thereby rendered simply connected, consider the integral $\int u_m^{x,a} d\,(\Phi_{z,c}^{z_\nu, c_\nu})$, taken, as a function of (x), along the single boundary explained. The result will be equal to the sum of the values obtained by taking the same integral in turn round the infinities of the function; this sum is

$$2\pi i\,(u_m^{x_1, a_1} + \ldots + u_m^{x_s, a_s}),$$

where the paths of integration are on the dissected surface (and supposed not to intersect one another). Consider the value obtained by taking the integral round the boundary. This is the sum of the values obtained for the p pairs (γ_k, γ_k'), the $(p-1)$ traverses furnishing no contribution. For the pair γ_k, γ_k', the whole path may be considered by taking together opposite rims of the cut γ_k, and also taking together opposite rims of the cut γ_k'. For clearness of description, if (x) denote a point on the negative (right) side of γ_k or γ_k', the opposite point, on the positive (left) side, may be denoted by $(\bar{x})$. The two sides of γ_k thus give a contribution



$$\int u_m^{\bar{x}, a} d_{\gamma_k} (\Phi_{z,c}^{\bar{z}_\nu, c_\nu}) - \int u_m^{x, a} d_{\gamma_k} (\Phi_{z,c}^{z_\nu, c_\nu}),$$

where, in *both* integrals, d_{γ_k} corresponds to an increment of (x) along the positive direction of γ_k, and $(\bar{z}_1), \ldots, (\bar{z}_r)$ are the points of the surface D corresponding to $(\bar{x})$. But we may put $u_m^{\bar{x}, a} = u_m^{x, a} + H_{m,k}$, where $H_{m,k}$, the period of u_m, is independent of the position of (x) on the rim of γ_k. Thus, from the two integrals, the expression under the integral sign is $u_m^{x,a}\, d_{\gamma_k} (\Phi_{z,c}^{\bar{z}_\nu, z_\nu}) + H_{m,k} d_{\gamma_k} (\Phi_{z,c}^{\bar{z}_\nu, c_\nu})$. Herein, however, in the sum $(\Phi_{z,c}^{\bar{z}_\nu, z_\nu})$, the integration, on D, from the set $(z_1), \ldots, (z_r)$ to the set $(\bar{z}_1), \ldots, (\bar{z}_r)$, is obtainable (as in the preceding work), with the integers from the matrices α, α', in terms of crossings of the period ovals (δ, δ') on D, using the k-th columns of these matrices, and the periods of the function (of ζ) $\Phi_{z,c}^{\zeta, \gamma}$; on D. These periods all vanish at the ovals (δ); for the ovals (δ') they are $2\pi i v_1^{z,c}, \ldots, 2\pi i v_q^{z,c}$. Thus the sum is $2\pi i \, (\alpha'_{1,k} v_1^{z,c} + \ldots + \alpha'_{q,k} v_q^{z,c})$. This is independent of (x), so that $\int u_m^{x,a} d_{\gamma_k} (\Phi_{z,c}^{\bar{z}_\nu, z_\nu}) = 0$. We have then to consider $H_{m,k} \int d_{\gamma_k} (\Phi_{z,c}^{\bar{z}_\nu, c_\nu})$.

Now the complete circuit, by (x), of the positive rim of the oval γ_k, which is equivalent to a crossing of γ_k', will correspond to an aggregate of circuits on the surface D, whose expression, in terms of crossings of the period ovals δ, δ', is given by the k-th columns of the matrices β, β'. Wherefore the integral in question is given by $2\pi i H_{m,k} (\beta'_{1,k} v_1^{z,c} + \ldots + \beta'_{q,k} v_q^{z,c})$. Here, in fact, $H_{m.k} = 0$ unless $k = m$; and then $= 1$.

We may similarly examine the contribution to the integral $\int u_m d (\Phi_{z,c}^{z_\nu, c_\nu})$ which arises from the two rims of the period oval γ_k'. There is the difference that the complete circuit by (x), of the positive rim of the oval γ_k', is equivalent to a recrossing, instead of a crossing, of the oval γ_k. The result is therefore

$$-2\pi i K_{m,k} (\alpha'_{1,k} v_1^{z,c} + \ldots + \alpha'_{q,k} v_q^{z,c}),$$

where $K_{m,k}$, the period of $u_m^{x,a}$ for a crossing of γ_k', is $\rho_{m,k}$.

Adding together the results for the p pairs (γ_k, γ_k'), we thus have

$$2\pi i \sum_{k=1}^{p} \sum_{j=1}^{q} (H_{m,k} \beta'_{j\,k} - \rho_{m,k} \alpha'_{j,k}) \nu_j^{z,c},$$

which is expressible as

$$2\pi i \sum_{j=1}^{q} (H\bar{\beta}' - \rho\bar{\alpha}')_{m,j} v_j^{z,c},$$

in which H denotes the matrix unity of p rows and columns, and $\bar{\mu}$ denotes the matrix obtained by interchange of rows and columns of a matrix μ, the so-called transposition of μ.

Equating this to the value found above, by integration round the infinities of the function, we have

$$u_m^{x_1, a_1} + \ldots + u_m^{x_s, a_s} \equiv N_{m,1} v_1^{z,c} + \ldots + N_{m,q} v_q^{z,c}, \quad m = 1, \ldots, p,$$

where N, a matrix of type (p, q), is given by

$$N = \bar{\beta}' - \rho\bar{\alpha}', \qquad (A').$$

By the transposition of an identity obtained above, however, we have $\bar{\rho} (\bar{\alpha} + \bar{\alpha}'\bar{\sigma}) - \bar{\beta} - \bar{\beta}'\sigma = 0$, or, since ρ and σ are symmetrical $(\bar{\rho} = \rho, \bar{\sigma} = \sigma)$, $\rho\bar{\alpha} - \bar{\beta} = (\bar{\beta}' - \rho\bar{\alpha}')\,\sigma$. Hence we also have

$$N\sigma = -\bar{\beta} + \rho\bar{\alpha}, \qquad (B').$$

Further, comparing

$$M = \alpha + \sigma\alpha', \quad M\rho = \beta + \sigma\beta',$$

with

$$N = \bar{\beta}' - \rho\bar{\alpha}', \quad N\sigma = -\bar{\beta} + \rho\bar{\alpha},$$

we see that the matrix associated with the correspondence from C to D, say

$$\Delta = \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix},$$

is replaced, in the reverse transformation from D to C, by

$$\nabla = \begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{\alpha}', & \bar{\alpha} \end{pmatrix},$$

which is of type $(2p, 2q)$, just as Δ is of type $(2q, 2p)$. We also have

$$N\,(1, \sigma) = (1, \rho)\,\nabla.$$

Further, using $(0)_p$, $(1)_p$ for zero and unit matrices of type (p, p), etc., and putting

$$\epsilon_{2p} = \begin{pmatrix} (0)_p, & -(1)_p \\ (1)_p, & (0)_p \end{pmatrix}, \quad \epsilon_{2q} = \begin{pmatrix} (0)_q, & -(1)_q \\ (1)_q, & (0)_q \end{pmatrix},$$

we have $\quad \epsilon_{2p}\nabla\epsilon_{2q} = \begin{pmatrix} \bar{\alpha}', & -\bar{\alpha} \\ \bar{\beta}', & -\bar{\beta} \end{pmatrix}\epsilon_{2q} = \begin{pmatrix} -\bar{\alpha}, & -\bar{\alpha}' \\ -\bar{\beta}, & -\bar{\beta}' \end{pmatrix} = -\bar{\Delta},$

and, similarly $\epsilon_{2q}\Delta\epsilon_{2p} = -\bar{\nabla}$; and we may also write

$$\epsilon_{2q}\Delta = -\overline{\epsilon_{2p}\nabla}, \quad \Delta\epsilon_{2p} = -\overline{\nabla\epsilon_{2q}}.$$

We may now express the relations between *circuits* on the surfaces C, D, arising by the direct and reverse correspondences; hitherto we have used only crossings. In the direct correspondence we had (p. 47)

a crossing of γ_k leads to
k-th column of α crossings $\delta + k$-th column of α' crossings δ',

a crossing of γ_k' leads to
k-th column of β crossings $\delta + k$-th column of β' crossings δ';

now, a crossing of γ_k' is obtainable by a positive circuit of (the positive rim of) γ_k, and a crossing of γ_k by a negative circuit of (the positive rim of) γ_k'; and similarly for δ and δ'. Hence, taking the facts written down in inverse order, and denoting positive circuits of γ_k, γ_k' by (γ_k), (γ_k'), etc., we may say that

(γ_k) leads to $\beta'_{1,k}(\delta_1) + \ldots + \beta'_{q,k}(\delta_q) - \beta_{1,k}(\delta_1') - \ldots - \beta_{q,k}(\delta_q')$,

(γ_k') leads to $-\alpha'_{1,k}(\delta_1) - \ldots - \alpha'_{q,k}(\delta_q) + \alpha_{1,k}(\delta_1') + \ldots + \alpha_{q,k}(\delta_q')$;

these may be represented by saying that the circuits

$$[(\gamma_1), \ldots, (\gamma_p), (\gamma_1'), \ldots, (\gamma_p')],$$

lead to $\quad \begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{\alpha}', & \bar{\alpha} \end{pmatrix} [(\delta_1), \ldots, (\delta_q), (\delta_1'), \ldots, (\delta_q')],$

or, still more briefly, by saying that $[(\gamma), (\gamma')]$ lead to $\nabla\,[(\delta), (\delta')]$. We have remarked that $\epsilon_{2p}\nabla = \bar{\Delta}\epsilon_{2q}$; thus we also have

$$\epsilon_{2p}\,[(\gamma), (\gamma')] \equiv \bar{\Delta}\epsilon_{2q}\,[(\delta), (\delta')],$$

which merely recovers the original definition of Δ in terms of crossings. By the reverse transformation we similarly have

$$[(\delta), (\delta')] \equiv \Delta\,[(\gamma), (\gamma')].$$

Suppose now that the second curve D coincides with the first. Then we have an (s, r) correspondence on the curve C, with relations

$$u_i^{z_1, c_1} + \ldots + u_i^{z_r, c_r} \equiv M_{i,1} u_1^{x, a} + \ldots + M_{i,p} u_p^{x, a}, \quad i = 1, \ldots, p,$$

in which (replacing ρ by the more usual notation τ for the period matrix) $M = \alpha + \tau\alpha'$, $M\tau = \beta + \tau\beta'$; and, also, relations

$$u_i^{x_1, a_1} + \ldots + u_i^{x_s, a_s} \equiv N_{i,1} u_1^{z, c} + \ldots + N_{i,p} u_p^{z, c}$$

with $N = \bar{\beta}' - \tau\bar{\alpha}'$, $N\tau = -\bar{\beta} + \tau\bar{\alpha}$.

The existence of the correspondence thus requires the equation (equivalent to p^2 equations) $\tau\alpha'\tau + \alpha\tau - \tau\beta' - \beta = 0$. This equation may express actually existing relations, with numerical coefficients, quadratic in the elements of the period matrix τ; or, it may be satisfied, if the numerical matrices α, α', β, β' be properly chosen, without any limitation for the matrix τ. We first enquire as to the possibility of the second alternative. The (m, n)th of the p^2 relations is

$$\sum_{i,j} \tau_{m.i}\alpha'_{i,j}\tau_{j,n} + \sum_{i} (\alpha_{m,i}\tau_{i,n} - \tau_{m,i}\,\beta'_{i,n}) - \beta_{m,n} = 0, \quad i, j = 1, \ldots, p;$$

for these p^2 relations to hold without restriction of the periods $\tau_{i,j}$, it is necessary and sufficient that

$$\beta_{m,n} = 0,\ \alpha'_{i,j} = 0;\ \alpha_{m,i} = 0,\ \beta'_{i,n} = 0, \quad i \neq m,\ i \neq n;\ \alpha_{m,m} = \beta'_{n,n},$$

for all values of m, n, i, j $(1, \ldots, p)$. Thus the matrices β, α' consist of zeros, and the matrices α, β' are both equal to the same diagonal matrix, of equal elements; namely using γ for a single integer, we have

$$\beta = 0,\ \ \alpha' = 0,\ \ \alpha = \beta' = -\gamma,\ \ \Delta = -\begin{pmatrix} \gamma, & 0 \\ 0, & \gamma \end{pmatrix} = -\gamma,\ \ \nabla = \Delta = -\gamma,$$

in accordance with $\epsilon^2{}_{2p} = -1$. And the two sets of p equivalences, of which we have deduced the second from the first, become, in this case,

$$u_i^{z_1, c_1} + \ldots + u_i^{z_r, c_r} + \gamma u_i^{x, a} \equiv 0, \quad u_i^{x_1, a_1} + \ldots + u_i^{x_s, a_s} + \gamma u_i^{z, c} \equiv 0,$$
$$i = 1, \ldots, p;$$

here γ may be any positive or negative integer, or zero. By the meaning of these equivalences (in virtue of the converse of Abel's Theorem), γ is what we have called the valency; and is the same for the reverse and direct correspondences.

Thus we have the result, first proved by Hurwitz (*Math. Ann.* XXVIII, 1887, p. 561), that, unless the periods of the matrix τ be connected by quadratic relations, of appropriate form, with numerical coefficients, every existing (s, r) correspondence, of a curve with itself, is a correspondence with valency. When such quadratic relations exist for the periods τ, the "valency coefficients" M, or N, are not arbitrary, but related to the conditions connecting the periods.

We can prove, however, that the number of coincidences for *any* (s, r) correspondence upon a single curve is given by

$$s+r-(\alpha_{11}+\ldots+\alpha_{pp}+\beta'_{11}+\ldots+\beta'_{pp}),$$

(of which, for a valency correspondence, the particular form is $s+r+2\gamma p$, as found before).

For this, we consider the function of (x) expressed by

$$\Phi \equiv \Pi_{x,c}^{z_1,c_1}+\ldots+\Pi_{x,c}^{z_r,c_r},$$

where Π denotes the elementary normal integral of the third kind, and $(z_1), \ldots, (z_r)$ are the places corresponding to the place (x). Here $(c_1), \ldots, (c_r)$ are the places corresponding to a place (a), but (c) is an arbitrary place.

With the notation employed above (p. 50), to distinguish two places of the Riemann surface which are opposite to one another on two rims of a period cut, we have

$$\Pi_{\bar{x},c}^{\bar{z}_i,c_i}=\Pi_{\bar{x},c}^{\bar{z}_i,z_i}+\Pi_{\bar{x},c}^{z_i,c_i}=\Pi_{\bar{x},c}^{\bar{z}_i,z_i}+\Pi_{\bar{x},x}^{z_i,c_i}+\Pi_{x,c}^{z_i,c_i},$$

so that

$$\Pi_{\bar{x},c}^{\bar{z}_i,c_i}-\Pi_{x,c}^{z_i,c_i}=\Pi_{\bar{x},c}^{\bar{z}_i,z_i}+\Pi_{z_i,c_i}^{\bar{x},x};$$

here, the second term on the right is the period, of the function $\Pi_{z_i,c_i}^{\zeta,c}$ (with ζ current), arising by the crossing of the period cut on which (x), $(\bar{x})$ lie; its value is therefore zero, or $2\pi i v_k^{z_i,c_i}$, for $k=1$, or $2, \ldots,$ or p. Thus we see that, for a crossing by (x) of a period cut (from the right rim to the left), the increment of the function Φ is $\sum_{i=1}^{r}\Pi_{\bar{x},c}^{\bar{z}_i,z_i}+\sum_{i=1}^{r}\Pi_{z_i,c_i}^{\bar{x},x}$; and further, by what was explained before, if, with the usual notation, this period cut be (a_k), this increment is $2\pi i\,(\alpha'_{1,k}v_1^{\bar{x},c}+\ldots+\alpha'_{p,k}v_p^{\bar{x},c})$, while, if this period cut be (b_k), the increment is

$$2\pi i\left(\beta'_{1,k}v_1^{\bar{x},c}+\ldots+\beta'_{p,k}v_p^{\bar{x},c}+\sum_{i=1}^{r}v_k^{z_i,c_i}\right),$$

namely, is

$$2\pi i\,(\beta'_{1,k}v_1^{\bar{x},c}+\ldots+\beta'_{p,k}v_p^{\bar{x},c}+M_{k,1}v_1^{x,c}+\ldots+M_{k,p}v_p^{x,c}).$$

We express now that the total increment of the function Φ, considered on the dissected Riemann surface, when taken round the aggregate of the contours of the p period cut pairs (a_k), (b_k), is equal to its increment taken round its infinities.

The two rims of the (a_k) cut, as contribution to $\int d(\Phi)$, will give the value of $\int d(\bar{\Phi}-\Phi)$ taken once along the positive rim of (a_k); this circuit is equivalent to a crossing of (b_k), and gives, therefore, by what we have seen, the corresponding period of the function

$$2\pi i\,(\alpha'_{1,k}v_1{}^{\bar{x},c}+\ldots+\alpha'_{p,k}v_p{}^{\bar{x},c}),$$

namely that at (b_k), which is

$$2\pi i\,(\alpha'_{1,k}\tau_{1,k}+\ldots+\alpha'_{p,k}\tau_{p,k}),$$

and this, as $\tau_{m,k}=\tau_{k,m}$, is the product of $2\pi i$ by the (k, k)-th element of the matrix $\tau\alpha'$, or, say, $2\pi i\,(\tau\alpha')_{k,k}$. Taking next the two rims of the (b_k) cut, these give $\int d(\bar{\Phi}-\Phi)$, taken once along the positive rim of (b_k); this circuit is equivalent to a recross of (a_k); thus, by what we have seen for $\bar{\Phi}-\Phi$, we obtain

$$-2\pi i\,(\beta'_{k,k}+M_{k,k}),\quad\text{or}\quad -2\pi i\,(\beta'+\alpha+\tau\alpha')_{k,k}.$$

Adding this to the former contribution, and taking all pairs (a_k), (b_k), the whole result is

$$-2\pi i\sum_{k=1}^{p}(\alpha+\beta')_{k,k}.$$

Now consider the infinities of the function

$$\Phi=\Pi_{x,c}^{z_1,c_1}+\ldots+\Pi_{x,c}^{z_r,c_r},$$

regarded as depending on (x). There is a logarithmic infinity, with multiplier unity, when (x) is at any one of the places $(z_1), \ldots, (z_r)$ which correspond to (x); say, at the C coincidences of the correspondence. There is a logarithmic infinity, with negative unity as multiplier, when (x) is at any of the places $(c_1), \ldots, (c_r)$, which correspond to (a); and there is such an infinity when (c) is at any of the places $(z_1), \ldots, (z_r)$ which correspond to (x), namely, when (x) is at any of the places $(a_1), \ldots, (a_s)$ which correspond reversely to the place (c). The total increment of Φ, round the infinities, is thus $2\pi i\,(C-r-s)$. Equating this to the former result, we have the anticipated formula

$$C=r+s-\sum_{k=1}^{p}(\alpha_{k,k}+\beta'_{k,k}),$$

where the last part is the sum of the diagonal elements of the matrix Δ, or, equally, of the matrix ∇.

It is easy to prove that if we have two correspondences T, T_1 upon the curve C, and form the product T_1T, whereby to (x) first correspond $(z_1), \ldots, (z_r)$ by T, and to (z_m) then correspond, say, $(z_{m,1}), \ldots, (z_{m,r_1})$, by T_1, then the matrices M, and Δ, for this product correspondence, are, respectively, $M_1 M$ and $\Delta_1\Delta$; and thus the number of coincidences, for this product correspondence, is given by $rr_1+ss_1-\Sigma(\Delta_1\Delta)_{k,k}$. Replacing Δ, by ∇_1, considering the product $T_1^{-1}T$, we have, for the number of pairs of places corresponding both for T and for T_1, $rs_1+r_1s-\Sigma(\nabla_1\Delta)_{k,k}$, which generalises the formula, due to Brill, for valency correspondences, given above (Ex. 3, p. 10).

We may also make an extension to the case of the correspondences previously considered between two curves C, D. Suppose we have two such correspondences, T and T', whereby, respectively to (x), of C, correspond (z_1), ..., (z_r), of D, and then (z_1'), ..., $(z'_{r'})$, of D. We may enquire, when does it happen that one of (z_1), ..., (z_r) coincides with one of (z_1'), ..., $(z'_{r'})$?

For this, take first the place (x_k), of C, which is one of the set (x_1), ..., (x_s) corresponding to the place (z_1) of D by the reverse correspondence T^{-1} from D to C; then the places $(z'_{k,1})$, ..., $(z'_{k,r'})$ of D, corresponding to the place (x_k), of C, by the correspondence T'; and then consider, on D, the correspondence by which, from (z_1), arises the aggregate of the sets $(z'_{1,1})$, ..., $(z'_{1,r'})$; ...; $(z'_{s,1})$, ..., $(z'_{s,r'})$. Namely, we consider, on D, the product correspondence $T'T^{-1}$, and consider the coincidences therein. The indices of this correspondence will be $(s'r, sr')$, and the associated matrix will be $\Delta'\nabla$. The number of times it happens that both T and T' lead from the point of C to the same point of D will thus be

$$rs'+r's-\Sigma(\Delta'\nabla)_{k,k}, \qquad k=1, \ldots, 2q.$$

It is easy to see that the latter term may be replaced by any of

$$\Sigma(\nabla'\Delta)_{h,h},\ \Sigma(\nabla\Delta')_{h,h},\ \Sigma(\Delta\nabla')_{k,k} \qquad h=1, \ldots, 2p.$$

Remark. It is clear from the fundamental equations, and the theory of the so-called inversion problem (Vol. v, p. 145, Ex. 4) that there exists on any curve a correspondence for which the valency coefficients M have arbitrary values, and one of the indices (r) of the correspondences is the genus, p, of the curve. By the use of the theta functions, Hurwitz (*Math. Ann.* XXVIII) obtains the value of the other index, and the number of coincidences, which, in the notation used here, is $\frac{1}{2}\Sigma[(1-\Delta)(1-\nabla)]_{k,k}$.

The theory of the correspondence between two curves has been further developed in a very interesting way, first by Severi, and then by Lefschetz, by means of the conception of a surface, of which every point represents the aggregate of a point of one curve and a point of the other curve. Such a surface may be realised geometrically by supposing the two curves to be in different independent spaces $[C]$, $[D]$, both lying in a space $[N]$, then joining, by a line, every point of one curve to every point of the other, and then taking the section of the resulting ∞^3 manifold by a prime of the space $[N]$. The resulting surface may be called the *product* of the two curves C, D. Upon this surface, say Σ, there will correspond, to every point (x) of the curve C, a curve K_x, the prime section of the cone of lines joining (x) to the curve D; and the aggregate of such curves on Σ will form a system of which one curve passes through an arbitrary point of Σ (we assume that the construction is so general that, from every point of Σ, there passes only one line to meet both C and D, determining thereby a single point on each curve); an algebraic system of curves, on a surface, of this character (that one curve of the system passes through any point of the surface) is called a *pencil* of curves. Each curve of the pencil will have the genus, q, of the curve D; and, as these curves correspond (1, 1) to the points of the curve C, the pencil itself is said to have the genus, p, of the curve C. The surface Σ possesses, also, another pencil of curves, K_z, each of genus p, this other pencil having genus q. Every curve of either pencil meets every curve of the other pencil in one point; and there is a single curve of each pencil through an arbitrary point of the surface Σ.

A correspondence between the curves C, D, in which, to each point (x) of C correspond r points of D, and to each point (z) of D correspond s points of C, say a (s, r) correspondence, gives rise to a ruled surface, formed by joining every (x) to the r corresponding points of D, and every (z) to the s corresponding points of C; and the section of this ruled surface by the prime, in which Σ lies, gives rise to a curve on this surface. This curve *represents* the correspondence, every point of the curve leading to a point of C and a point of D, associated with one another in the correspondence; conversely every such pair of associated points leads to a point of this curve. With every such point (x, z), of this curve, ϑ, we may take the $(r-1)$ other points $(x, z_2), \ldots, (x, z_r)$, where $(z), (z_2), \ldots, (z_r)$ are the points of D which correspond to the point (x) of C; and we may consider the aggregate of r curves, of the pencil K_z, which pass through the points $(x, z), (x, z_2), \ldots, (x, z_r)$, denoting this aggregate by ϑ_z. The aggregate ϑ_z, regarded as one composite curve, defines a linear system of curves on the surface Σ (see below, Chap. v). Similarly, with any point (x, z) of the curve ϑ, we may take $(s-1)$ other points $(x_2, z), \ldots, (x_s, z)$, wherein $(x), (x_2), \ldots, (x_s)$ are the points of C which correspond to the point (z) of D, and we may consider the aggregate, ϑ_x, of curves of the pencil K_x, which pass through the s points $(x, z), (x_2, z), \ldots, (x_s, z)$. This aggregate ϑ_x, regarded as a single composite curve, defines another linear system of curves on the surface Σ. The curve ϑ itself likewise defines a linear system on the surface.

A correspondence, T, between the curves C, D, may have the particular character that the sets of points $(z_1), \ldots, (z_r)$ of D, which arise from the points (x), all belong to the same linear series on D; in that case the valency coefficients M, of the foregoing theory, and the integers of the matrices α, β, α', β', are all zero; then the points $(x_1), \ldots, (x_s)$ of C, which correspond to a point (z) of D, are also a set of a linear series on C. Such a correspondence may be called a *correspondence of zero valency* between C and D. For such a correspondence it may be shewn that the linear system $|\vartheta|$, on Σ, defined by the curve which represents the correspondence, is the sum of the linear systems $|\vartheta_z|$, $|\vartheta_x|$ defined respectively by the curves ϑ_z and ϑ_x. Or we may say that there is the equivalence $\vartheta \equiv \vartheta_z + \vartheta_x$, or, also, that the linear system $|\vartheta - \vartheta_z - \vartheta_x|$ is *null*. This result we shall assume here; such a correspondence is represented by a single equation $\phi(x, z) = 0$, which is a polynomial in rational polynomials in the coordinates (x) of the curve C, and also in rational polynomials in the coordinates (z) of the curve D. Conversely, any curve ϑ, of the surface Σ, represents a correspondence between the curves C and D, in which, to (x), of C, correspond points (z) of D, represented by the curves K_z, forming ϑ_z, which pass through the points where the curve K_x meets the curve ϑ. If $\vartheta \equiv \vartheta_z + \vartheta_x$, as K_x does not meet any of the K_x curves of ϑ_x, the points $(x, z_1), \ldots, (x, z_r)$, where ϑ is met by K_x, are on the aggregate ϑ_z, and form a set of a linear series when (x) varies. A curve ϑ, for which $\vartheta \equiv \vartheta_z + \vartheta_x$, thus represents a correspondence of zero valency. This geometrical relation is thus equivalent to the analytical fact that, for the correspondence represented by the curve ϑ, in this case, the matrices M, N, Δ, ∇ are all zero; and conversely.

More generally, if $\lambda_1, \ldots, \lambda_t$ be integers, a correspondence represented by a curve on the surface Σ which is of the same linear system as the aggregate of curves given by $-\vartheta + \lambda_1 \vartheta^{(1)} + \ldots + \lambda_t \vartheta^{(t)}$, wherein $\vartheta^{(1)}, \ldots, \vartheta^{(t)}$ are, like ϑ, curves on Σ, is of zero valency if

$$\vartheta \equiv \vartheta_z + \vartheta_x + \lambda_1 [\vartheta^{(1)} - \vartheta_z{}^{(1)} - \vartheta_x{}^{(1)}] + \ldots + \lambda_t [\vartheta^{(t)} - \vartheta_z{}^{(t)} - \vartheta_x{}^{(t)}];$$

and this is consequently the same as the statement that, for the corresponding matrices Δ, we have $\Delta = \lambda_1 \Delta_1 + \ldots + \lambda_t \Delta_t$, there being similar equations, therefore, for the matrices M, N, ∇.

Now, the arithmetic theorem can be proved that, any matrix of integers Δ is expressible in a form $\lambda_1 \Delta_1 + \ldots + \lambda_t \Delta_t$, where $\Delta_1, \ldots, \Delta_t$ are appropriate particular matrices, and t is a limited number ($\leqslant 2pq$). Thus we can infer that any curve on the surface Σ can be expressed as equivalent to at most $2pq$ curves $\vartheta_1, \vartheta_2, \ldots,$ together with an aggregate of curves of the pencils K_x, K_z. This is a wide generalisation of a theorem already illustrated in the preceding volume (p. 216), in the case of a ruled surface (and a quadric, and cubic surface); and it is itself capable of much wider development (associating itself indeed with the existing so-called Picard integrals of a surface which have curves of logarithmic infinity). In the present case, the immediate application is to the expression of any possible correspondence between the curves C, D in terms of a finite number of such correspondences. Also, surfaces such as Σ, representing the pairs of points of two curves, have very interesting properties, considered further below, in Chapter VII (p. 282).

The following particular remark, dealing with the number of intersections of two curves ϑ, ϑ', on the surface Σ, may also be made. Suppose that the correspondences represented by the curves $-\vartheta+\lambda_1\vartheta^{(1)}+\ldots+\lambda_t\vartheta^{(t)}$, $-\vartheta'+\lambda_1'\vartheta^{(1)}+\ldots+\lambda_t'\vartheta^{(t)}$ are both of zero valency. Then we have seen that $\Delta'=\lambda_1'\Delta_1+\ldots+\lambda_t'\Delta_t$, and, also, from the same equation for ϑ, $\nabla=\lambda_1\nabla_1+\ldots+\lambda_t\nabla_t$; hence

$$\Delta'\nabla=\Sigma\lambda_i'\lambda_j(\Delta_i\nabla_j),$$

of which, if $d(\mu)$ be used for the sum of the diagonal elements of a square matrix μ, a particular consequence is $d(\Delta'\nabla)=\Sigma\lambda_i'\lambda_j d(\Delta_i\nabla_j)$; on the other hand, denoting by i the number of intersections (ϑ, ϑ'), of the curves ϑ, ϑ', the intersections of $\vartheta-\vartheta_z-\vartheta_x$ and $\vartheta'-\vartheta_z'-\vartheta_x'$, are given by

$$(\vartheta-\vartheta_z-\vartheta_x, \vartheta'-\vartheta_z'-\vartheta_x')=(\vartheta, \vartheta')-(\vartheta, \vartheta_z')-(\vartheta, \vartheta_x')-\text{ etc.},$$

and, if (s, r) be the indices for the correspondence ϑ, while (s', r') belong to ϑ', this is at once found to be $i-rs'-r's$. By a result given above (p. 56), however, this is the same as $-d(\Delta'\nabla)$. Thus we have $(\vartheta-\vartheta_z-\vartheta_x, \vartheta'-\vartheta_z'-\vartheta_x')=-d(\Delta'\nabla)$.

The preceding remarks, slight as they are, may serve to lead the reader to the remarkable paper of Severi, *Mem....Torino*, LIV, 1903, p. 32. In the present place only a still slighter indication is possible of a most striking paper by S. Lefschetz, *Annals of Maths.* XXVIII, 1927, wherein the expression, $rs'+r's-d(\Delta'\nabla)$, for the number of coincidences of two correspondences, is obtained by the methods of Analysis Situs. The comparison of this paper with the foregoing theory is full of interest and suggestion.

Part III. Correspondence and defective integrals. We have already seen (p. 22) that the existence of a $(1, \nu)$ correspondence between two curves involves the existence, on one of the curves, of a batch of integrals, of the first kind, in number less than the whole number of such integrals existing for that curve, whose periods depend likewise upon a defective number of independent periods. The equation we have obtained, for the general correspondence between two curves, $M(1, \rho)=(1, \sigma)\Delta$, and the similar equation for the correspondence between points of the same curve, give rise to further interesting results, as to the existence of systems of algebraic integrals, of the first kind, upon a curve, which have defective period systems. As such integrals are of great importance, we give now an introductory account of such results.

Some preliminary remarks should be made, which are of interest in themselves. Thinking first of the case of a single curve, of genus p, for which the normal integrals, $v_1, \ldots, v_p$, have a period system denoted as before by $(1, \tau)$, we consider a space $[2p-1]$, of dimension $2p-1$, which we denote by R_{2p}; and consider, in this space, a space of $p-1$ dimensions $[p-1]$, denoted by $[\tau]$, which is based on the p points of the space R_{2p} whose coordinates are, respectively,

$$(\tau_{11}, \ldots, \tau_{1p}; -1, 0, \ldots, 0), (\tau_{21}, \ldots, \tau_{2p}; 0, -1, 0, \ldots, 0), \ldots,$$
$$(\tau_{p1}, \ldots, \tau_{pp}; 0, \ldots, 0, -1).$$

Using $(x_1, \ldots, x_p; x_{p+1}, \ldots, x_{2p})$ for the homogeneous coordinates in R_{2p}, the space $[\tau]$ is thus the intersection of the p primes

$$x_s + \tau_{s1}x_{p+1} + \ldots + \tau_{sp}x_{2p} = 0, \qquad s = 1, \ldots, p.$$

We also consider, in R_{2p}, the *null system*, say ϵ, in which the polar prime of a point $(y_1, \ldots, y_{2p})$ is that given by the equation

$$x_1 y_{p+1} - x_{p+1} y_1 + \ldots + x_p y_{2p} - x_{2p} y_p = 0;$$

this equation may also be expressed, in matrix notation, by

$$\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} (x_1, \ldots, x_{2p})(y_1, \ldots, y_{2p}) = 0,$$

where, in the matrix of type $(2p, 2p)$, which in future we shall denote by ϵ, 0 and 1 denote respectively the zero and the unit matrices of p rows and columns. It is then at once clear that, the polar prime, in this Null system, of any point lying in the space $[\tau]$, wholly includes this space; and, conversely, that the pole, in the system ϵ, of any prime containing the space $[\tau]$, is a point of this space. This we may express by saying that the space $[\tau]$ is self-polar in regard to the null system ϵ. In fact,* if c denote $(c_1, \ldots, c_p)$, a general point of $[\tau]$ has, in R_{2p}, the coordinates

$$((\tau c)_1, \ldots, (\tau c)_p, -c_1, \ldots, -c_p),$$

and the polar prime of this, in the system ϵ, has the equation

$$c_1 x_1 + \ldots + c_p x_p + (\tau c)_1 x_{p+1} + \ldots + (\tau c)_p x_{2p} = 0,$$

which, if x' denote $(x_{p+1}, \ldots, x_{2p})$, is the same as

$$c_1 [x_1 + (\tau x')_1] + \ldots + c_p [x_p + (\tau x')_p] = 0,$$

so that this prime is one of the so-called star of primes containing $[\tau]$.

The space $[\tau]$, in virtue of Riemann's results, has the curious property of not containing any point whose coordinates $(x_1, \ldots, x_{2p})$

---

* If x, y denote rows $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_m)$, and a be a matrix of type (m, n), then $(ax)_s$ denotes $a_{s1}x_1 + \ldots + a_{sn}x_n$; and axy denotes $\Sigma(ax)_s y_s$.

are all real. For if $\tau_{sm}=\rho_{sm}+i\sigma_{sm}$, where ρ_{sm}, σ_{sm} are real, the equations of the primes defining τ shew that such a point would need to satisfy the p equations $\sigma_{s1}x_{p+1}+\ldots+\sigma_{sp}x_{2p}=0$, and would thus have $x_{p+1}=\ldots=x_{2p}=0$, because the determinant $|\sigma_{sm}|$ is known not to vanish (Vol. v, p. 145). From the equations of these primes, we should then also have $x_1=\ldots=x_p=0$.

By the polarity just remarked, or by simple direct proof, the space $[\tau]$ has also the property of not lying in any prime whose equation has wholly real coefficients. More generally, a space $[2p-1-n]$, or, say, R_{2p-n}, common to any n primes containing $[\tau]$, $(n\leqslant p)$, cannot contain more than $2p-2n$ *independent* points, for each of which all the coordinates are rational numbers—a point being dependent on others when its $2p$ coordinates are linearly expressible each by the like coordinates of these others, with coefficients the same for all the coordinates. The polar of this fact, with respect to the system ϵ, may also be stated. Analytically this property has the following enunciation: *let M be any matrix of p columns and (we may suppose) of p rows, whose rank is $n\,(n\leqslant p)$; let ξ be a matrix, whose elements are integers, with $2p$ rows and (we may suppose) of $2p$ columns. Let the identities expressed by the matrix equation $M\,(1,\tau)\,\xi=0$ hold. Then, the rank, r, of ξ, is equal to, or less than, $2p-2n$. It is thus sufficient to regard ξ as having only r independent columns.*

This result is important; we state the proof here in the simplest way, in a form often applied. In the first place, it is a consequence of the fact that the determinant of the imaginary parts of the periods τ_{sm} is not zero that, there exists no integral of the first kind of which all the $2p$ periods are real. Next, if $x_1, \ldots, x_{2p}$ be integers, and we consider the closed curve, on the Riemann surface representing the curve under consideration, which is equivalent to x_s crossings of the period oval (a_s), and x_{p+s} crossings of the conjugate oval (b_s), for $s=1, \ldots, p$, then the equation $M\,(1,\tau)\,(x_1, \ldots, x_{2p})=0$ expresses that, for each of the n independent integrals $V_1, \ldots, V_n$ contained in the formula

$$V_s=M_{s1}v_1+\ldots+M_{sp}v_p, \qquad s=1, \ldots, p,$$

the period, obtained by circuit of the closed curve spoken of, vanishes. Now consider an integral V given by

$$V=(\lambda_1+i\mu_1)\,V_1+\ldots+(\lambda_n+i\mu_n)\,V_n,$$

in which $\lambda_1, \ldots, \mu_n$ are all real; let $P_1+iQ_1, \ldots, P_n+iQ_n$ denote the periods, for $V_1, \ldots, V_n$ respectively, obtained by circuit of any specified closed curve. The period of V, for this circuit, will then be real if $\lambda_1Q_1+\mu_1P_1+\ldots+\lambda_nQ_n+\mu_nP_n=0$, so that $\lambda_1, \ldots, \mu_n$ can be chosen to make the periods of V real for any $2n-1$, or fewer, specified circuits. On the other hand, if r be the rank of the matrix

ξ, there being r independent columns in this matrix, the periods of all of $V_1, \ldots, V_n$ are zero (and hence real), for the r circuits corresponding to these columns. As there are $2p$ independent circuits on the Riemann surface, there will be $2p-r$ other circuits independent of these r. This number must then be greater than the number $2n-1$ of circuits mentioned above, since else the integral $\Sigma(\lambda_s+i\mu_s)V_s$ would have all its periods real. Wherefore $2n-1<2p-r$, or $r \leqslant 2p-2n$; which is the theorem to be proved.

Another statement of the result is that the period scheme, for a system of n independent integrals of the first kind, must have at least $2n$ independent columns, in which not every period is zero. When this number is exactly $2n$, the system of integrals is called regular.

It will now be proved that, when there exists one such regular defective system, of n independent integrals, with $2n$ independent columns of not all zero periods, then there also exists another regular defective system, consisting of $p-n$ integrals, independent of one another and of the integrals of the former system. This latter system, with a period scheme of $2n$ columns, may be called complementary to the former, the two systems together making up a set of p independent integrals.

If a general one of the former system of integrals, expressed in terms of the p normal integrals, be

$$\lambda_{s1}v_1+\ldots+\lambda_{sp}v_p, \qquad s=1, \ldots, n,$$

there will exist a matrix ξ, of $2p-2n$ columns, each of $2p$ integers, such that $\lambda(1, \tau)\,\xi=0$, these equations merely expressing that there are $2p-2n$ independent closed circuits for which all the integrals $(\lambda v)_s$ have zero periods. We prove then that there exists a matrix, μ, of $p-n$ rows and p columns (of rank $p-n$), and a matrix η, of $2n$ columns, each of $2p$ integers, such that $\mu(1, \tau)\,\eta=0$, there being no linear combination of the second system of integrals, that is, of the integrals

$$\mu_{t1}v_1+\ldots+\mu_{tp}v_p, \qquad t=1, \ldots, p-n,$$

which is the same as any linear combination of the integrals of the first system. In other words, the matrices λ, μ are such that there exists no set, c, of p quantities $c_1, \ldots, c_p$, other than all zero, for which the p equations $(\bar{\lambda}, \bar{\mu})\,c=0$ are satisfied.

We can, in fact, regard the $2p-2n$ columns of the matrix ξ as defining $2p-2n$ points of the space R_{2p}, with rational coordinates; the $2p-2n$ polar primes of these points, in the Null system ϵ, then intersect in a space $[2n-1]$, which can be regarded as based upon $2n$ rational points (that is, points of rational coordinates). In other words, we can determine a matrix, η, of integers, of $2p$ rows and $2n$ columns, such that $\bar{\xi}\epsilon\eta=0$, a typical one, of the $2p-2n$ equations to be satisfied by a column of η, being

$$-\xi_{1s}\eta_{p+1,t}-\ldots-\xi_{ps}\eta_{2p,t}+\xi_{p+1,s}\eta_{1t}+\ldots+\xi_{2p,s}\eta_{pt}=0,$$

for $s=1, \dots, 2p-2n$, t varying from column to column of η. The $2p$ elements $\eta_{1t}, \dots, \eta_{pt}, \eta_{p+1,t}, \dots, \eta_{2p,t}$, of a column of η, are thereby subject to $2p-2n$ equations, when, for the $\xi_{1s}, \dots, \xi_{2p,s}$, we put, in turn, all the columns of ξ. There are thus $2n$ independent columns for the matrix η; or the polar space is based on the $2n$ points given by these columns. It will be proved presently that the spaces $[\xi]$, $[\eta]$, thus explained, have no point in common.

Of the space $[\xi]$, based on the $2p-2n$ points given by the columns of ξ, there will be an intersection with the space $[\tau]$; this intersection is a space which, as lying in $[\xi]$, can be described as based upon a certain number of points, of each of which the coordinates are given by a column of a matrix of the form ξa, wherein a is a matrix of $2p-2n$ rows, and a certain number of columns; the number of such (independent) columns will be the number of independent points of the space which is the intersection in question. We prove that this number is $p-n$. For, first, it is an obvious remark that, if A be a matrix of type (p, q), and x a matrix of type (q, r), such that $Ax=0$, then the rank of x is q diminished by the rank of A; because, if α be the rank of A, the equations for the elements of a column of x which are expressed by $Ax=0$ are equivalent to α independent equations, and the q elements of a column of x are then expressible by $q-\alpha$ of them; so that x has $q-\alpha$ independent columns. This being noticed, in the equations $\lambda(1, \tau)\,\xi=0$, satisfied by ξ, as λ is of type (n, p), and of rank n, the rank of $(1, \tau)\,\xi$ is $p-n$; and, by the same remark, as $(1, \tau)\,\xi$ is of type $(p, 2p-2n)$, the rank of a matrix a, such that $(1, \tau)\,\xi a=0$, is $p-n$. The equations expressed by $(1, \tau)\,\xi a=0$ give, as the columns of ξa, the basic points of the space, say $[\tau\xi]$, of dimension $p-n-1$, in which the space $[\xi]$ meets the space $[\tau]$.

Now put $\xi=\begin{pmatrix} h \\ k \end{pmatrix}$, where h, k are matrices both of the type $(p, 2p-2n)$. Then, from $(1, \tau)\,\xi a=0$, we have the consequences

$$(h+\tau k)\,a=0, \quad ha+\tau ka=0, \quad \bar{a}\bar{h}+\bar{a}\bar{k}\bar{\tau}=0;$$

wherefore, from $\bar{\xi}\epsilon\eta=0$, using the last of these, we have

$$\bar{a}\bar{\xi}\epsilon\eta=0, \quad \bar{a}\,(\bar{h}, \bar{k})\,\epsilon\eta=0, \quad \bar{a}\bar{k}\,(-\tau, 1)\,\epsilon\eta=0,$$

and this is the same as $\bar{a}\bar{k}(1, \tau)\,\eta=0$; here $\bar{a}$ is of type $(p-n, 2p-2n)$, and $\bar{k}$ of type $(2p-2n, p)$, so that $\bar{a}\bar{k}$ is of type $(p-n, p)$. We put then $\mu=\bar{a}\bar{k}$, and have consequently the equations $\mu\,(1, \tau)\,\eta=0$; and these express that the $p-n$ integrals $(\mu v)_t$, or

$$\mu_{t1}v_1+\dots+\mu_{tp}v_p, \qquad t=1, \dots, p-n$$

form a defective system; all these in fact have zero columns of periods for circuits given by the matrix η. And further, as is expressed by $\bar{a}\,(\bar{h}, \bar{k})\,\epsilon\eta=0$, the space $[\eta]$, based on the points given by the columns of the matrix η, lies in the polar space, in regard to the Null system ϵ, of the space $\begin{pmatrix}h\\k\end{pmatrix}a$, or $[\xi a]$, which is the meet $[\tau\xi]$.

In the same way, for the matrix λ which determines the original system of defective integrals, we may substitute a matrix obtained by considering the meet $[\tau\eta]$. For let this meet be given by the columns of ηb, where b is, as above, of type $(2n, n)$, and is to be determined from $(1, \tau)\,\eta b=0$. Putting $\eta=\begin{pmatrix}l\\m\end{pmatrix}$, where l, m are matrices both of the type $(p, 2n)$, we have $\bar{b}\bar{l}+\bar{b}\bar{m}\tau=0$, and the equations $\bar{\eta}\epsilon\xi=0$ lead to

$$\bar{b}\bar{\eta}\epsilon\xi=0,\quad \bar{b}\,(\bar{l}, \bar{m})\,\epsilon\xi=0,\quad \bar{b}\bar{m}\,(-\tau, 1)\,\epsilon\xi=0,\quad \bar{b}\bar{m}\,(1, \tau)\,\xi=0;$$

if, however, we regard ξ as given, and the n equations expressed by $\lambda\,(1, \tau)\,\xi=0$ as equations for the p elements of a row of λ, then there will be n such rows, because $(1, \tau)\,\xi$ is of rank $p-n$. We may therefore take $\bar{b}\bar{m}v$ as the original system of defective integrals; and the space $[\xi]$ lies in the polar, with respect to ϵ, of the meet $[\tau\eta]$.

It is important, however, to shew that the $p-n$ integrals $\bar{a}\bar{k}v$, and the n integrals $\bar{b}\bar{m}v$, are linearly independent. Geometrically, as we first shew, this is the statement that the spaces $[\tau\xi]$, $[\tau\eta]$, both within $[\tau]$, have no common point. As $[\tau]$ is of dimension $p-1$, and these spaces are of respective dimensions $p-n-1$ and $n-1$, this would be exceptional. To prove this, remark that, if there be a set of $p-n$ quantities x, and a set of n quantities y, which, used as multipliers respectively for the $p-n$ integrals $\bar{a}\bar{k}v$, and the n integrals $\bar{b}\bar{m}v$, give a zero sum, then we shall have $kax+mby=0$; and hence, as $ha+\tau ka=0$, $lb+\tau mb=0$, we shall also have $hax+lby=0$; and these lead to $\xi ax+\eta by=0$; when this is so, the point $(ha, ka)x$, or ξax, of the space $[\tau\xi]$, is the same as the point $(lb, mb)\,y$, or ηby, of the space $[\tau\eta]$. To disprove the possibility contemplated, we invoke the theorem, found by Riemann (cf. Vol. v, Chap. vi, p. 145), that, if the matrix τ be written as $\rho+i\sigma$, where ρ, σ are matrices of real elements, and $n_1, \ldots, n_p$ be p real numbers, then the quadratic form σn^2 has a necessarily positive value (and is not zero). We have $(1, \tau)\,\xi a=0$; and hence $(h+\tau k)\,ax=0$, with an arbitrary set x. Thus the $2p$ quantities ξax, or (hax, kax), can be expressed in the form

$$(-\tau kax, kax),\quad \text{or}\quad \xi ax=\begin{pmatrix}-\tau\\1\end{pmatrix}z,$$

when z denotes the p quantities kax. Also we have

$$0=\bar{\eta}\epsilon\xi,\quad 0=\bar{\eta}\epsilon\xi\,(ax)\,(b_0y_0)=\epsilon\,(\xi ax)\,(\eta by)_0,$$

where b_0, y_0 are the conjugate complexes of b, y (and η is real). Thus, using the matrix rule $\mu x.\nu y = \bar{\nu}\mu xy$, we have from $\xi ax + \eta by = 0$

$$\begin{aligned} 0 &= \epsilon\,(\xi ax)(\xi ax)_0 \\ &= \epsilon\left[\begin{pmatrix} -\tau \\ 1 \end{pmatrix} z\right]\left[\begin{pmatrix} -\tau_0 \\ 1 \end{pmatrix} z_0\right] \\ &= (-\tau_0,\, 1)\,\epsilon\begin{pmatrix} -\tau \\ 1 \end{pmatrix} zz_0, \quad = (1,\, \tau_0)\begin{pmatrix} -\tau \\ 1 \end{pmatrix} zz_0, \quad = (\tau_0 - \tau)\, zz_0, \end{aligned}$$

which, if $\tau = \rho + i\sigma$, is $-2i\sigma zz_0$. By Riemann's theorem quoted this cannot vanish.

Thus, the integrals $\bar{a}\bar{k}v$, $\bar{b}\bar{m}v$ are linearly independent, and form, in all, the p independent integrals.

Having then proved that the spaces $[\tau\xi]$, $[\tau\eta]$ have no common point, their polar spaces, with regard to the Null system ϵ, will not lie in a prime. These polar spaces, we have shewn, include, respectively, the spaces $[\eta]$ and $[\xi]$; hence these do not lie in a prime. As they are of respective dimensions $2n-1$, $2p-2n-1$, they can, therefore, have no common point. Namely, the spaces $[\xi]$, $[\eta]$, defined by points whose (integer) coordinates give the period ovals for which the two systems of defective integrals have zero periods, are independent; and the $(2p-2n)+(2n)$ basic points of these give $2p$ independent period ovals.

In regard to the space $[\tau]$, we may add the further simple remark: the equation, $M(1, \tau) = (1, \tau)\,\Delta$, which arises for a transformation of a single curve into itself, shews that the linear transformation, in the space R_{2p}, with integer coefficients, which is expressed by $x' = \Delta x$, transforms any point of the space $[\tau]$ into another point of the same space. For it leads, from $(1, \tau)\,x = 0$, to $(1, \tau)\,x' = 0$. Conversely, for the existence of a transformation, $x' = ax$, in which the matrix a is of type $(2p, 2p)$, which changes the space $[\tau]$ into itself, the equations $(1, \tau)\,x' = 0$ must be a consequence of $(1, \tau)\,x = 0$; or $(1, \tau)\,ax = 0$ must be a consequence of $(1, \tau)\,x = 0$; thus we infer the existence of a matrix M such that $M(1, \tau) = (1, \tau)\,a$.

We pass now to consider the equations, which arose in considering the correspondence between two curves, $M(1, \rho) = (1, \sigma)\,\Delta$, $N(1, \sigma) = (1, \rho)\,\nabla$. It is clear, from consideration of the processes, of transposition, and interchange of rows and columns, by which ∇ may be obtained from Δ, or, more directly, from the identity $\epsilon_{2q}\Delta\epsilon_{2p} = -\bar{\nabla}$, that the rank of ∇ is the same as that of Δ. Denote this rank by δ, and the ranks of M and N respectively by μ and ν.

Then, if ξ be such a matrix of $2p$ rows that $\Delta\xi = 0$, this matrix will (by a remark made above) be of rank $2p - \delta$; and, by the former of the equations put down, we shall have $M(1, \rho)\,\xi = 0$. Hence, by a result proved, $2\mu + 2p - \delta \leqslant 2p$. Again, M being of type (q, p), if

L be such a matrix that $LM=0$, so that L is of rank $q-\mu$, the equation $M(1,\rho)=(1,\sigma)\Delta$ leads to $L(1,\sigma)\Delta=0$; and this, by the same result, shews that $2(q-\mu)+\delta\leqslant 2q$. By the comparison of these two consequences we infer that $\delta=2\mu$. Wherefore, in an equation $M(1,\rho)\xi=0$, the rank of ξ is $2p-2\mu$. By similar consideration of the equation $N(1,\sigma)=(1,\rho)\nabla$ we infer that $\delta=2\nu$, so that $\mu=\nu$; also, in an equation $N(1,\sigma)\eta=0$, the matrix η is of rank $2q-2\nu$. We shall denote μ, or ν, by n. The q integrals Mu (where $u_1, \dots, u_p$ are as above the normal integrals for the curve C) are then linear functions of n integrals, and these form, by what has been proved, a regular defective system on the curve C, with $2p-2n$ columns of zero periods; while the p integrals Nv are also linear functions of n integrals, likewise forming a regular defective system on the curve D, with $2q-2n$ columns of zero periods. Further, since a matrix L for which $LM=0$ is of a rank $q-n$, it follows, from the equations

$$v_j^{z_1,c_1}+\dots+v_j^{z_r,c_r}\equiv M_{j1}u_1^{x,a}+\dots+M_{jp}u_p^{x,a}, \quad j=1, \dots, q,$$

that there are $q-n$ independent integrals for the curve D, namely the integrals Lv, whose values, at the points $(z_1), \dots, (z_r)$ which correspond to the point (x) of C, have a sum which is independent of (x) (namely the same as when (x) is at (a)). By parity of reasoning, there are $p-n$ integrals on C whose sum values, at a set corresponding to (z) of D, are independent of (z). We have reached this conclusion in a particular case (p. 22 above).

The question, which we do not conclusively answer, naturally arises, whether the n defective integrals Nv, on D, and the $q-n$ integrals Lv, on D, where L is a matrix for which $LM=0$, are always independent. The former have been shewn to form a regular system; the latter also form a regular system, since, from $M(1,\rho)=(1,\sigma)\Delta$ and $LM=0$, we have $L(1,\sigma)\Delta=0$; and Δ is of rank $2n$. An equation $(1,\rho)\zeta=0$, in which ζ consists of integers, is impossible, since this would involve the existence of circuits for which all q integrals v have zero periods; hence an equation $N(1,\sigma)\eta=0$, with η integral, being the same as $(1,\rho)\nabla\eta=0$, involves $\nabla\eta=0$. Thus the columns of zero periods for the system Nv are given by a matrix of integers, η, such that $\nabla\eta=0$; and the columns of zero periods for the system Lv are given by Δ; here η has $2q-2n$ columns, and Δ has $2n$ columns. To find whether these systems form q independent integrals, we may take account of the equation $\bar{\Delta}\epsilon_{2q}\eta=0$, which follows from $\bar{\Delta}\epsilon_{2q}=\epsilon_{2p}\nabla$. When the answer to the question raised is affirmative, we shall have two complementary regular systems of defective integrals on both curves C and D.

Another remark, also only suggestive, dealing with the correspondence on a single curve, may be made. The equation $M(1,\tau)=(1,\tau)\Delta$

leads to $M^2(1, \tau) = M(1, \tau)\Delta = (1, \tau)\Delta^2$; and, in general, if $\phi(M)$ denote any integral polynomial in the matrix M, with coefficients which are single quantities, we have $\phi(M)(1, \tau) = (1, \tau)\phi(\Delta)$.

A very particular case, θ being a single number, is

$$(M-\theta)(1, \tau) = (1, \tau)(\Delta - \theta),$$

with the corresponding equation for the reverse transformation

$$(N-\theta)(1, \tau) = (1, \tau)(\nabla - \theta).$$

It can be easily shewn that the roots of the algebraic equation in θ expressed by $|N-\theta| = 0$ are the conjugate complexes of the roots of $|M-\theta| = 0$, and that $|\Delta - \theta| = |M-\theta||N-\theta|$. For we have, if τ_0 be the conjugate complex of τ, etc.,

$$\begin{pmatrix} 1, & \tau \\ 1, & \tau_0 \end{pmatrix} \Delta = \begin{pmatrix} M, & M\tau \\ M_0, & M_0\tau_0 \end{pmatrix} = \begin{pmatrix} M, & 0 \\ 0, & M_0 \end{pmatrix} \begin{pmatrix} 1, & \tau \\ 1, & \tau_0 \end{pmatrix},$$

and this shews that $|\Delta - \theta| = |M-\theta||M_0 - \theta|$; while, recalling

$$\Delta = \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix}, \quad M = \alpha + \tau\alpha', \quad M\tau = \beta + \tau\beta',$$

$$\nabla = \begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{\alpha}', & \bar{\alpha} \end{pmatrix}, \quad N = \bar{\beta}' - \tau\bar{\alpha}',$$

we have

$$\begin{pmatrix} 1, & \tau \\ 0, & 1 \end{pmatrix} \Delta \begin{pmatrix} 1, & -\tau \\ 0, & 1 \end{pmatrix} = \begin{pmatrix} M, & M\tau \\ \alpha', & \beta' \end{pmatrix} \begin{pmatrix} 1, & -\tau \\ 0, & 1 \end{pmatrix} = \begin{pmatrix} M, & 0 \\ \alpha', & \beta' - \alpha'\tau \end{pmatrix} = \begin{pmatrix} M, & 0 \\ \alpha', & \bar{N} \end{pmatrix};$$

and, for a single quantity θ, we have

$$\begin{pmatrix} 1, & \tau \\ 0, & 1 \end{pmatrix} \theta \begin{pmatrix} 1, & -\tau \\ 0, & 1 \end{pmatrix} = \theta \begin{pmatrix} 1, & \tau \\ 0, & 1 \end{pmatrix} \begin{pmatrix} 1, & -\tau \\ 0, & 1 \end{pmatrix} = \theta \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} = \theta,$$

thus $|\Delta - \theta| = |M-\theta||\bar{N} - \theta|$. The two equations

$$|\Delta - \theta| = |M-\theta||M_0 - \theta| = |M-\theta||\bar{N} - \theta|$$

establish the facts stated (cf. the author's *Abel's Theorem*, Camb. 1897, pp. 632, 638, and the references there given).

We may also consider derived correspondences associated with polynomials $\phi(\Delta)$; and the consequences of the fact that the matrix Δ satisfies a polynomial equation.

We have already (pp. 58, 59, Part II above) touched the question of the number of independent correspondences existing upon a curve, a correspondence being said to be dependent upon others when the matrix Δ, belonging thereto, is linearly expressible by the matrices Δ belonging to the others. It can be shewn, from the equation $\beta + \tau\beta' = (\alpha + \tau\alpha')\tau$, that certainly there are not more than $2p^2$ independent correspondences; for, if x, y both denote sets of p

integers, it is impossible to solve the equations $x+\tau y=0$; thus, in the equations quoted, the assumption of values for the $2p^2$ integers contained in α and α' leaves no ambiguity in the values of β and β'. But, in fact, it can be shewn that all existing so-called symmetrical correspondences on the curve, for which the matrices ∇, Δ are identical, are dependent on at most p^2 symmetrical correspondences; and all skew correspondences (for which $\nabla = -\Delta$) are likewise dependent on p^2 skew correspondences at most. We may prove these results in connexion with the linear transformation of the space $[\tau]$, considered above, remarking, from $\epsilon\Delta = \bar{\nabla}\epsilon, = -\overline{\epsilon\nabla}$, that, for a symmetrical correspondence, $\epsilon\Delta$ is a skew symmetric matrix (and a symmetric matrix for a skew correspondence).

For the theory of defective integrals many references may be given. The consideration of such integrals arises imperatively in the study of multiply periodic functions (cf. the writer's volume, Cambridge, 1907, Chap. viii); it arises also in the study of the Picard integrals associated with an algebraic surface, these integrals giving rise to a defective system of integrals upon a prime section of the surface (cf. Poincaré, *Ann. d. l'École norm.* xxvii, 1910, and *Sitzber. Berliner Math. Gesell.* x, 1911; also Severi, *Rend. Lincei*, xxx, 1921, p. 163, etc.). And the consideration arises also, we have seen, in the multiple correspondence of one, or of two, curves. The foregoing summary of this connexion has been suggested by the papers of Rosati referred to below. The theorem, that the existence of one regular defective system of integrals on a curve involves the existence of a complementary system, was first given in general by Poincaré, *Amer. J. of Math.* viii, 1886, p. 289; for particular cases of this, references are given in the writer's *Abel's Theorem*, p. 659; a proof of the theorem was given in the writer's *Multiply Periodic Functions*, p. 240. The proof here given, in terms of Rosati's geometrical point of view, seems very simple.

The papers of Rosati referred to are: *Ann. di Mat.* xxv, 1916, p. 1; *Atti...Torino*, l, 1915, p. 685 (see Scorza, *Rend. Lincei*, xxiii–xxv, 1914–16); *Atti...Torino*, li, 1916, p. 991; *ibid.* liii, 1918, p. 5; also *Ann. di Mat.* xxviii, 1919, p. 35. Many references are also given in a paper by Scorza, *Rend. Palermo*, xli, 1916, pp. 263–379.

CHAPTER II

SCHUBERT'S CALCULUS. MULTIPLE CORRESPONDENCE

Part I. Schubert's methods. Preliminary, as to notations. Characters of a manifold. The theory of correspondence so far developed deals with ∞^1 couples of corresponding points, and the number of coincidences of the points of a couple. Some extension is possible to cases when we have ∞^r couples of corresponding points, or, more generally, an infinite aggregate of corresponding linear *spaces*, of which we determine the number of couples of coinciding elements. This, and many other problems of enumeration, can be treated with the help of a symbolic calculus elaborated by Schubert; and some account of this calculus seems necessary, because, in intricate cases, its power has been shewn by its application to problems for which ordinary methods have failed to give a solution. The calculus is founded on the notion of representing a condition, to which a geometrical entity is to be subject, by an algebraic symbol. Then the alternative imposition of one *or* other of two independent conditions, is appropriately represented by the sum of the two chosen symbols, in either order; and the simultaneous imposition of both conditions is represented by the product of the two symbols, in either order. This multiplication then obeys the distributive law; while the associative law holds both for the sum and product of such symbols. The symbols, in the two operations, thus obey the ordinary laws of algebra. Subtraction of symbols may arise in equations between condition symbols, as a brief expression of additions; but division of the symbols does not arise. Schubert, in his remarkable book (*Kalkül der abzählenden Geometrie*, Teubner, 1879) ascribes the first traces of a calculus of conditions to Halphen, in his work for the characteristics of conics (*Compt. rend.* LXXVI, 1873, p. 1074; and *Bull. d. l. Soc. math.* I, 1873, p. 130), and to his own early work (*Gött. Nachr.* 1874, 1875). To Schubert is due also, beside the notation $[r]$ for a linear space of dimension r, a notation for a condition to which a linear space may be subject, which includes many familiar cases. For convenience we explain this notation at starting. It will be understood that the account of Schubert's methods given here is intended only to be introductory and illustrative; it is necessarily very incomplete.

It is implied generally that every geometrical entity for which conditions are postulated in the theory, is determined by the values of a certain number of parameters (and that the conditions are algebraic). A class of entities, of the same description, of which each depends on the values of m parameters, is often said to be of freedom m, or the aggregate of members of the class is said to be ∞^m; when i (independent) conditions are assigned for the parameters, the entities of the class which satisfy these conditions form a sub-class of freedom $m-i$. In particular, when $i=m$ this sub-class consists, in general, of a finite number of entities.

These conceptions are well illustrated by classes of linear spaces, say of dimension k, lying in space $[r]$, of dimension r; and the results, which are of frequent application, can be verified by detailed algebra.

(a) The spaces $[k]$, of dimension k, in space $[r]$, are of freedom $(r-k)(k+1)$, or of aggregate $\infty^{(r-k)(k+1)}$. If, in $[r]$, we denote by k' the dimension of a linear space $[r-k-1]$, *dual* to the space $[k]$, the freedom in question is $(k+1)(k'+1)$. For such a space necessarily meets (in one point) an arbitrary space $[r-k]$, and is determined by $(k+1)$ points of the space $[r]$. Hence, it is determined by its points of intersection with $(k+1)$ spaces $[r-k]$; and each of these spaces contains ∞^{r-k} points.

(b) In order that the space common to p primes of the space $[r]$ should be, not merely a space $[r-p]$, but a space $[m]$ of greater dimension, $m, \geqslant r-p$, it is necessary that these primes should satisfy $(m+1)(m-r+p)$ conditions. For $r-m$ general primes intersect in an $[m]$; the condition required here is that, choosing $r-m$ primes from among the p, the remaining $p-r+m$ primes should all contain the $m+1$ points which are necessary to define the space $[m]$.

(c) If two spaces $[h]$, $[k]$, in space $[r]$, do not intersect, the $h+1$ and $k+1$ points, on which they are respectively based, are independent; and the least-dimensional space which contains both $[h]$ and $[k]$ is an $[h+k+1]$. If however $[h]$ and $[k]$ have a common space $[m]$, which we may call their *meet*, then the least-dimensional space containing both $[h]$ and $[k]$, or, say, their *join*, is a $[j]$, wherein $m+j=h+k$. This is proved as in the case $m=0$, by considering the numbers of independent points on which the spaces are based. In order that the meet of an $[h]$ and $[k]$ should be an $[m]$, with $m \geqslant h+k-r$, the spaces must satisfy $(m+1)(r-j)$, or $(m+1)(m-h-k+r)$, conditions. For, in order that the $r-h$ and $r-k$ primes, by which the spaces $[h]$ and $[k]$ may be defined, should have, for their intersection, a space $[m]$, requires, by (b) preceding, a number of conditions equal to $(m+1)(m-r+r-h+r-k)$, which is $(m+1)(m-j)$.

This is also the number of conditions in order that a space $[j]$, which can be defined by $r-j$ independent primes, should contain a given space $[m]$. A particular case is when $j=r-1$; then the number of conditions is $m+1$.

(d) The spaces $[k]$ which contain a given space $[m]$, when $k \geqslant m$, are of freedom $(r-k)(k-m)$. For, any such space $[k]$ meets an arbitrary $[r-k]$ in a point, and the $[r-k]$ contains ∞^{r-k} points; while, a $[k]$ depends on $(k+1)$ points, of which $m+1$ may be taken in the $[m]$ through which the $[k]$ passes. Such a $[k]$ is then determined by its intersections with $k-m$ spaces $[r-k]$.

(e) If the space $[m]$ be not given, but lie anywhere in a given $[h]$, the result is different: The spaces $[k]$ which meet a given $[h]$ in spaces $[m]$ are of freedom $(r-k)(k-m)+(m+1)(h-m)$; for the freedom of spaces $[m]$ contained in a given $[h]$ is $(m+1)(h-m)$.

The same result is obtained by remarking that the spaces $[k]$ which meet a given $[h]$ in an $[m]$, may be found by taking the spaces $[j]$ which contain the $[h]$, of which the freedom is $(j-h)(r-j)$, by (d) preceding, with $j=h+k-m$; and then taking the spaces $[k]$ which lie in such a $[j]$, of which the freedom is $(j-k)(k+1)$. We easily verify that

$$(r-k)(k-m)+(m+1)(h-m)=(j-h)(r-j)+(j-k)(k+1),$$

the right side being $(k-m)(r-j)+(h-m)(k+1)$.

We now consider a certain composite condition, which may be required of a $[k]$ lying in the given space $[r]$. For this, we suppose a series of spaces, $[a_0]$, $[a_1]$, ..., $[a_k]$, to be given, of which each lies in the following space of this series. And the composite condition for $[k]$ is that it should meet $[a_0]$ in a point, $[a_1]$ in a line, ... $[a_h]$ in an $[h]$, ..., finally should lie in $[a_k]$. It is supposed that $a_i<a_{i+1}$, for $i<k$.

A $[k]$ meets an $[r-k]$ in a point, but meets an $[a_0]$ for which $a_0>r-k$ in a line at least. For the $[k]$ to meet the $[a_0]$ in a point, we must therefore have $a_0\leqslant r-k$; thus we put $a_0=r-k-\beta_0$, supposing $\beta_0\geqslant 0$. Similarly, a $[k]$ will meet an $[a_1]$ in more than a line if $a_1>r-k+1$; thus we put $a_1=r-k+1-\beta_1$, supposing $\beta_1\leqslant 0$. In general, we put $a_i=r-k+i-\beta_i$, with $\beta_i\leqslant 0$. Then $a_i<a_{i+1}$ involves $i-\beta_i<i+1-\beta_{i+1}$, or $\beta_i>\beta_{i+1}-1$; so that

$$\beta_0\geqslant\beta_1\geqslant\beta_2\geqslant\ \dots\ \geqslant\beta_k\geqslant 0,$$

with $\beta_0\leqslant r-k$. The composite condition described may be denoted by $(a_0, a_1, \dots, a_k)$, or also by $\{\beta_0, \beta_1, \dots, \beta_k\}$; and when necessary we may describe the former as the *dimensions symbol*, and the latter as the *condition symbol*. We proceed to prove that the composite condition is equivalent in all to $\beta_0+\beta_1+\dots+\beta_k$ conditions for the space $[k]$.

To state the proof of this clearly, we may write

$$[k]=[r-a_0][a_0+k]=[r-a_1+1][a_1+k-1]$$
$$=[r-a_2+2][a_2+k-2]=\dots,$$

meaning thereby that the $[k]$ may be regarded as the intersection of a space $[r-a_0]$ with a space $[a_0+k]$, or, in general, of a space $[r-a_i+i]$ with a space $[a_i+k-i]$. Then the space $[r-a_0]$ meets $[a_0]$ in a point, and, for $[k]$ to meet the $[a_0]$ in a point, we require the space $[a_0+k]$ to pass through this point. This requires $r-a_0-k$, or β_0, conditions. Suppose these to be satisfied, and $[k]$ to meet $[a_0]$ in a point. Then, similarly, the space $[r-a_1+1]$ meets $[a_1]$ in a line; and the $[r-a_1+1]$ was taken to contain the $[k]$, which now contains a point of $[a_0]$, and $[a_0]$ lies in $[a_1]$. Thus, the line in which the space $[r-a_1+1]$ meets $[a_1]$ contains already a point of $[a_1]$. We require that the $[a_1+k-1]$ should contain the line; and the $[a_1+k-1]$, like the $[r-a_1+1]$, contains already a point of $[a_1]$ lying on this line. Thus, for the $[a_1+k-1]$ to contain this line, it is sufficient that it contains some other point of this line. This requires $r-(a_1+k-1)$, or β_1, conditions, additional to the β_0 already found. Suppose these additional conditions also satisfied, and $[k]$ to meet $[a_1]$ in a line. Then, for the next step, the $[r-a_2+2]$ and the $[a_2+k-2]$ both contain $[k]$, and so contain this line; and this line, as lying in $[a_1]$, lies in $[a_2]$. The $[r-a_2+2]$ meets $[a_2]$ in a plane; we require that the $[a_2+k-2]$ should contain this plane. But this $[a_2+k-2]$, we have said, contains a line of $[a_2]$; it is thus only necessary that this $[a_2+k-2]$ should contain a further point of $[a_2]$. This requires $r-(a_2+k-2)$, or β_2, conditions. Thus the satisfaction of the partial condition which we may denote by (a_0, a_1, a_2) requires, in all, $\beta_0+\beta_1+\beta_2$ conditions. The argument may be continued, with the result we have enunciated for the whole composite condition.

In particular, if $\beta_{s+1}=0$, $\beta_{s+2}=\ldots=\beta_k=0$; namely, if

$$a_{s+1}=r-k+(s+1),\ a_{s+2}=r-k+(s+2),\ \ldots,\ a_k=r-k+k,\ =r,$$

then the number of conditions required is $\beta_0+\beta_1+\ldots+\beta_s$. In this case we may denote the composite condition simply by $(a_1, a_2, \ldots, a_s)$; though it is convenient to have an entry of $k+1$ indices, in order to indicate that we are expressing a condition for a $[k]$. We may also, in this case, denote the condition by $\{\beta_1, \ldots, \beta_s, 0^{k-s}\}$.

We may remark that the number of symbols $\{\beta_0, \beta_1, \ldots, \beta_k\}$, including the null symbol $\{0, 0, \ldots, 0\}$, is the binomial coefficient, $(r+1, k+1)$, which is the same as the number of homogeneous coordinates of a space $[k]$ in space $[r]$. For recalling

$$r-k \geqslant \beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_k \geqslant 0,$$

and putting

$$\beta_k=x_k,\ \beta_{k-1}=\beta_k+x_{k-1},\ \beta_{k-2}=\beta_{k-1}+x_{k-2},\ \ldots,\ \beta_0=\beta_1+x_0,$$

so that $x_0, x_1, \ldots, x_k$ are zero or positive integers, all values are possible for which $\beta_0 \leqslant r-k$, or $x_0+x_1+\ldots+x_k \leqslant r-k$; the number of sets of values, equal to the number of terms in a homogeneous polynomial of order $r-k$ in $1+(k+1)$ variables, is thus

$$(r-k+k+1,\ k+1), \text{ or } (r+1,\ k+1).$$

For instance, for a line in $[r]$, the number of possible conditions which may be imposed is $\frac{1}{2}r(r+1)$.

As examples of the notation, the condition that a line, in ordinary space [3], should lie in a given plane, is $(1, 2)$ or $\{1, 1\}$, requiring two conditions; or, the condition that a line should pass through a given point of a given plane, and lie in this plane, is $(0, 2)$ or $\{2, 1\}$, and is three-fold; or, again, the condition for a line to pass through a given point is $(0, 3)$ or $\{2, 0\}$, and is two-fold. Beside the ineffective condition $(2, 3)$, the other two possible conditions that can be required of a line are $(1, 3)$, or $\{1, 0\}$, the one-fold condition for the line to meet a given line, and the four-fold condition $(0, 1)$, or $\{2, 2\}$, for the line to pass through a given point of a given line, and coincide with this given line.

Ex. Among the $(r+1, k+1)$ conditions which a $[k]$, lying in an $[r]$, may be required to satisfy, find how many of these conditions are ρ-fold $(\rho = 0, 1, \ldots, (k+1)(r-k))$; for example, for a line in [3], there are two two-fold conditions, but the other four are all simple. For a plane in [4], the conditions $\{1, 1, 0\}$, $\{2, 0, 0\}$ are two-fold, the conditions $\{1, 1, 1\}$, $\{2, 1, 0\}$ are three-fold, and the conditions $\{2, 2, 0\}$, $\{2, 1, 1\}$ are four-fold; but the other four conditions are all simple.

The notation may be usefully combined with another notation which will be employed below. For instance we may denote the order of the curve of contact of the tangents drawn from a given point to a given surface in ordinary space [3], by the symbol $\epsilon\{2, 0\}(2)$, or $\epsilon(0, 3)(2)$, where the line symbol $\{2, 0\}$ or $(0, 3)$ refers to the condition that the line passes through a given point, the symbol ϵ denotes that two of the intersections of the line with the surface are coincident, and the point symbol (2) refers to the condition that this point of coincidence lies in an arbitrary plane. In a similar way, the number of tangents of the surface which pass through a given point, and lie in a given plane through this point, may be denoted by $\epsilon\{2, 1\}(3)$, or $\epsilon(0, 2)(3)$, where the line symbol $\{2, 1\}$ or $(0, 2)$ refers to the fact that the line passes through a given point and lies in a given plane containing this point, the symbol ϵ expresses that two of the intersections of the line with the surface are coincident, and the point symbol (3) merely expresses that the point of contact is subject to no further condition. The use of the symbol ϵ to refer to a coincidence of points is customary (after Schubert); and the symbol of a condition which is satisfied only in a finite number of cases is often used to express this number. More generally, for any algebraic manifold of k dimensions, say M_k, lying in space $[r]$, we may consider two conditions, both referring to the composite entity consisting of the tangent k-fold of M_k and its point of contact with M_k; these are the conditions ρ_i, ω_j respectively expressed by

$$\rho_i = \epsilon\{1^i, 0^{k+1-i}\}(r-k+i), \quad \omega_j = \epsilon\{j, 0^k\}(r-k+j),$$
$$(0 \leqslant i \leqslant k;\ 0 \leqslant j \leqslant k,\ 0 \leqslant j \leqslant r-k).$$

Here the symbol $\{\ \}$ refers, in both cases, to the $[k]$ which is the tangent space, and the symbol $(\ )$ to its point of contact; and ϵ indicates that it is a point of contact which is under consideration. In the first case, the symbol $\{1^i, 0^{k+1-i}\}$ is an i-fold condition for the ∞^k tangent spaces of the manifold M_k; so that the points of contact for which this is satisfied are an aggregate ∞^{k-i}. And, of such an aggregate, a finite number of points satisfy the point condition $(r-k+i)$, of lying in a space $[r-k+i]$. Similarly in the second case, with j for i. Both symbols therefore are satisfied by a finite number of points, and can be used to denote these numbers.

In the symbol ρ_i, the $\{1^i, 0^{k+1-i}\}$ can also be written as

$$(r-k-1, r-k, \ldots, r-k+i-2; r-k+i, r-k+i+1, \ldots, r)$$

where, to indicate the application to a space $[k]$, there are written $k+1$ indices, though those beginning with $r-k+i$ represent intersections which are satisfied without conditions. The index $r-k+i-2$ expresses that the tangent space, $[k]$, meets a given $[r-k+i-2]$ in a space $[i-1]$, namely lies in a prime, $[r-1]$, through this $[r-k+i-2]$. This condition, which is equivalent by itself to i conditions, if satisfied, ensures that the conditions denoted by the preceding indices of the symbol are all satisfied*. Some simple examples of the symbol ρ_i, for a surface $(k=2)$, in ordinary space $(r=3)$, will shew that it is familiar. For $i=0$, the tangent plane of the surface need satisfy no conditions, but its point of contact lies on a line; thus ρ_0 is the order of the surface. For $i=1$, the tangent plane must pass through a given point, and the point of contact lie on a given plane; thus ρ_1 is the order of the curve of contact of the tangent cone drawn to the surface from an arbitrary point, or the order of this cone; commonly called the rank of the surface. For $i=2$, the tangent plane of the surface must pass through a given line, the point of contact being anywhere in the space $[3]$; thus ρ_2 is the number of tangent planes of the surface which can be drawn through an arbitrary line, or the class of the surface.

In the symbol ω_j, the symbol for the tangent space $\{j, 0^k\}$ expresses that this meets a given $[r-k-j]$ in a point, and the factor $(r-k+j)$ that the point of contact lies on a given $(r-k+j)$. For the symbol to have meaning, j must be at most equal to the less of k and $r-k$. As a simple example of the meaning of ω_j, take the case of a surface $(k=2)$, in ordinary space $(r=3)$; then for $j=0$, the factor $\{j, 0^k\}$ implies no condition for the tangent plane, and the point factor $(r-k+j)$ only that the point of contact lies on a line. Thus ω_0 is the order of the surface, equal to ρ_0; but, for $j=1$, the tangent plane is to pass through a given point, and its point of contact is to lie on a given plane; so that ω_1 is the rank of the surface, equal to ρ_1.

---

* In general, the number of conditions for a $[k]$ to meet an $[a_s]$ in an $[s]$, is $(s+1)\beta_s$, and implies the whole condition $(a_0, a_1, \ldots, a_s)$ if every space of the series $[a_0], [a_1], \ldots, [a_s]$ is a prime of the following space, or $a_i = a_{i-1}+1$; in this case $\beta_0 = \beta_1 = \ldots = \beta_s$.

More generally, as an example of the notation, we compute ρ_i and ω_j for the manifold M_k, in $[r]$, which is the complete intersection of $r-k$ primals, given by equations $f_1=0, \ldots, f_{r-k}=0$. Let $x_0, \ldots, x_r$ denote the homogeneous variables, and denote $\partial f_m/\partial x_n$ by $f_{m,n}$. The condition that the tangent space at the point (x) of the manifold, which, with $y_0, \ldots, y_r$ as current coordinates, is given by the $r-k$ equations

$$y_0 f_{m,0}+\ldots+y_r f_{m,r}=0, \qquad m=1, \ldots, (r-k),$$

should meet the space $[r-k+i-2]$ which is given by the $k-i+2$ equations

$$y_0 c_{n,0}+\ldots+y_r c_{n,r}=0, \qquad n=1, \ldots, (k-i+2),$$

in a space $[i-1]$, is that this aggregate of $r-i+2$ equations for $y_0, \ldots, y_r$ should be equivalent only to $r-i+1$ equations (this being the number determining an $[i-1]$). Namely the condition is that, in the matrix of $r-i+2$ rows and $r+1$ columns

$$\left\| \begin{matrix} f_{m,0}, \ldots\ldots, f_{m,r} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n,0}, \ldots\ldots, c_{n,r} \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{matrix} \right\|,$$

all determinants of order $r-i+2$ should vanish. We are required to find the order of the manifold common to all the primals whose equations are given by the vanishing of these determinants, in which $x_0, \ldots, x_r$ are regarded as current variables.

For this we invoke a theorem, of frequent application, included in those noted at the end of this chapter (p. 108). Suppose we have a matrix of $p+1$ rows and $q+1$ columns $(q \geqslant p)$, in which the element in the (i, j)-th place, that is, in the i-th row and the j-th column, is of order r_i+c_j in the $r+1$ homogeneous coordinates $x_0, \ldots, x_r$; we can express the dimension, and the order, of the manifold upon which all the determinants, of $p+1$ rows and columns, in this matrix, vanish. The dimension is $r-(q+1-p)$, so that there is no common intersection when this number is negative, and the intersection is a set of points when this number is zero. The order can be expressed as the coefficient of t^{q+1-p} in the ascending expansion of a certain rational function of t, and, precisely, may be denoted by

$$\left[\frac{(1+c_1t)(1+c_2t)\ldots(1+c_{q+1}t)}{(1-r_1t)\ldots(1-r_{p+1}t)}\right]_{t^{q+1-p}};$$

for brevity, we shall generally refer to this as $[(c)/(-r)]_{q+1-p}$. In the case now under discussion, we have $q=r$, $p=r-i+1$; and we suppose $i \geqslant 1$, so that $q+1-p$, or i, is >0; further the orders of the elements of the matrix are the same in any row, being, for the first $r-k$ rows, respectively one less than the orders of $f_1, \ldots, f_{r-k}$, and zero for all succeeding rows; we may thus suppose all of $c_1, \ldots, c_{q+1}$ to be zero, and, of $r_1, \ldots, r_{p+1}$, all but $r_1, \ldots, r_{r-k}$ to be zero. The dimension of the manifold required is thus $r-i$, and, with the proper values of $r_1, \ldots, r_{r-k}$, the order* is

$$[(1-r_1t)^{-1} \ldots (1-r_{r-k}t)^{-1}]_{t^i}.$$

The points of the original M_k which we seek are the intersection of the M_k with the manifold thus determined. The dimension of this intersection is then $r-i-(r-k)$, or $k-i$; further, if n_m be the order of f_m, we have $r_m = n_m - 1$; so far as the term in t^k we can thus put

$$\rho_0 (1-r_1t)^{-1} \ldots (1-r_{r-k}t)^{-1} = \rho_0 + \rho_1 t + \ldots + \rho_k t^k,$$

where ρ_0, $=n_1 n_2 \ldots n_{r-k}$, is the order of M_k. And this determines $\rho_1, \ldots, \rho_k$.

The determination of ω_j for the complete intersection M_k is similar. In this case we are to express that the tangent space meets a given $[r-k-j]$ in a point; in other words, the $(r-k)+(k+j)$, or $r+j$, equations in $y_0, \ldots, y_r$,

$$y_0 f_{m,0} + \ldots + y_r f_{m,r} = 0, \qquad m=1, \ldots, (r-k),$$

$$y_0 c_{n,0} + \ldots + y_r c_{n,r} = 0, \qquad n=1, \ldots, (k+j),$$

are to be capable of solution. The determinant of the coefficients in every $r+1$ of these equations must therefore vanish. This we express by

$$\left\| \begin{matrix} f_{m,0}, & \ldots, & c_{n,0}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_{m,r}, & \ldots, & c_{n,r}, & \ldots \end{matrix} \right\| = 0,$$

in which there are $p+1$, $=r+1$ rows, and $q+1$, $=r+j$ columns; thus $q+1-p=j$, and we suppose $j \geqslant 1$; further, the order, in $x_0, \ldots, x_r$, of all the elements of any column of the matrix is the same; thus, in the notation used above, we may take $r_1, r_2, \ldots, r_{p+1}$ all zero, and all of $c_1, \ldots, c_{q+1}$ zero except $c_1, \ldots, c_{r-k}$; of these latter, however, $c_m = n_m - 1$. The rule then requires us to consider

$$[(1+c_1t) \ldots (1+c_{r-k}t)]_{t^j}.$$

The dimension of the intersection of M_k with the construct, of dimension $r-j$, which is given by the matrix, is $k-j$; thus the numbers $\omega_1, \omega_2, \ldots$, so far as ω_k (supposing $k \leqslant r-k$), are defined by

---

* We follow the usual notation of r for the dimension of the space considered; and we use here also r, the initial letter of the word *row*, for ease of memory. Confusion is not to be feared.

$$\omega_0 [1+(n_1-1)t] \ldots [1+(n_{r-k}-1)t] = \omega_0 + \omega_1 t + \omega_2 t^2 + \ldots,$$

where $\omega_0, = \rho_0, = n_1 \ldots n_{r-k}$.

In this case of a manifold M_k which is a complete intersection, the numbers ω_j are immediately expressible by the numbers ρ_i, or conversely. For, by what was found above,

$$\rho_0 [1+(n_1-1)t]^{-1} \ldots [1+(n_{r-k}-1)t]^{-1} = \rho_0 - \rho_1 t + \rho_2 t^2 - \ldots,$$

and hence, so far as the terms in t^k in the two factors,

$$\omega_0 \rho_0 = (\rho_0 - \rho_1 t + \rho_2 t^2 - \ldots)(\omega_0 + \omega_1 t + \omega_2 t^2 + \ldots),$$

leading, as $\omega_0 = \rho_0$, to

$$\begin{aligned} 0 &= \omega_1 - \rho_1, \\ 0 &= \omega_2 \rho_0 - \omega_1 \rho_1 + \omega_0 \rho_2, \\ 0 &= \omega_3 \rho_0 - \omega_2 \rho_1 + \omega_1 \rho_2 - \omega_0 \rho_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

where $\omega_j = 0$ for $j > r-k$.

Ex. 1. Consider the complete intersection of two quadrics in space [4], shewing that $\rho_0 = 4$, $\rho_1 = 8$, $\rho_2 = 12$, $\omega_1 = 8$, $\omega_2 = 4$, and obtain a direct geometrical verification.

Ex. 2. For the surface which is the complete intersection of three quadrics in [5], obtain $\rho_0 = 8$, $\rho_1 = \omega_1 = 24$, $\rho_2 = 48$, $\omega_2 = 24$; for the M_3 which is the intersection of two quadrics in [5], obtain $\rho_0 = 4$, $\rho_1 = \omega_1 = 8$, $\rho_2 = 12$, $\rho_3 = 16$, $\omega_2 = 4$, and verify these results geometrically.

The calculus of conditions applied to a line. A linear space $[k]$ in space $[r]$ may be required to satisfy any one of the Schubert conditions $(a_0, a_1, \ldots, a_k)$. It may however be required to satisfy, simultaneously, two such conditions, provided the total number of elementary conditions so involved is not greater than the number, $(k+1)(r-k)$, of parameters upon which a $[k]$ depends. When this total number of elementary conditions (say $\beta_0 + \beta_1 + \ldots + \beta_k$ for one factor of the product condition, and $\beta_0' + \beta_1' + \ldots + \beta_k'$ for the other factor) is equal to $(k+1)(r-k)$, there will be a finite number of spaces $[k]$ satisfying the two conditions. This total number is (it is believed), the sum of the numbers of spaces $[k]$ which satisfy certain alternative conditions which are all of the form $(a_0, a_1, \ldots, a_k)$. For instance, in ordinary space, the lines which satisfy the two conditions, of meeting two given intersecting lines, consist of those which pass through the point in which these intersect, together with those which lie in their common plane. In this way arises the problem of expressing the product of two (simultaneous) conditions, of the form $(a_0, a_1, \ldots, a_k)$, as a sum of (alternative) conditions, all of this form. We deal now with this problem in the case of lines, in space $[r]$.

Suppose that a line, in space $[r]$, is to satisfy both the conditions (a_0, a_1) and (b_0, b_1), or say, the composite condition $(a_0, a_1)(b_0, b_1)$. In condition symbols this is $\{r-1-a_0, r-a_1\}\{r-1-b_0, r-b_1\}$, and is equivalent to a number of elementary conditions equal to $2(2r-1)-a_0-a_1-b_0-b_1$. But a line in space $[r]$ depends on $2r-2$ parameters; the previous number must therefore not exceed $2r-2$. Hence it is necessary that $a_0+a_1+b_0+b_1 \geqslant 2r$.

We consider now the problem of expressing the product, or composite, condition, as a sum of (alternative) conditions all of the form (c_0, c_1). To obtain this expression we apply a notion which, in simple cases, is familiar: we assume that the number of solutions of an algebraic problem, when finite and definite, remains unaltered by such changes of the values of the constants, entering algebraically in the algebraic relations which express the problem, as leave the number of solutions, of the changed problem, still finite and determinate. In both problems it is to be understood that multiple solutions are to be counted an appropriate number of times. The underlying reason for the assumption is that the change in the values of the constants can be made in a continuous way, thereby effecting only a continuous change in the number of solutions; whereas one integer number cannot change continuously into another integer number.

As an example of this notion, consider the problem, in space of three dimensions, of finding the number of lines of an algebraic congruence which meet two given arbitrary lines. If p, q denote the order and class of a congruence (the numbers of its lines respectively passing through an arbitrary point and lying in an arbitrary plane), we already know (e.g. Vol. IV, p. 51) that the number of lines common to two congruences, (p, q) and (p', q'), is $pp'+qq'$. The lines meeting two arbitrary lines form a particular congruence with $p'=q'=1$. Thus the number of lines of (p, q), which meet two arbitrary lines, is $p+q$. This is the sum of the numbers of lines of the congruence which satisfy one or other of the two conditions, of passing through an arbitrary point, or of lying in an arbitrary plane. In order to obtain this result by an application of the notion under consideration, we seek such a special position of the two lines that either of these alternative conditions could be satisfied in the same figure. This is obviously so if the lines intersect; the respective numbers would then be p and q. The assumption made is that, the number of lines of the congruence (p, q) which intersect two *skew* lines, is also $p+q$.

The argument employed in this case leads to a particular conclusion for the product of two conditions, for lines in ordinary space.

As above, denote the condition for a line to pass through a given point, by $(0, 3)$; also the condition, for the line to lie in a given plane, by $(1, 2)$, where the first index in the bracket is inserted in order that the symbol may be understood to refer to a line; and, similarly, denote the condition for the line to meet a given line by $(1, 3)$. Then the argument and conclusion we have reached may be summarised by the equation $(1, 3)(1, 3)=(0, 3)+(1, 2)$; in this equation it is to be understood, unless the contrary is stated in any case, that the lines indicated by the first index, 1, in the two multiplied symbols on the left, are not assumed to be the same.

Any appearance of novelty in the argument is lessened by considering the representation of lines of the space [3] by the points of a quadric Ω in space [5] (cf. Vol. IV, as referred to). The given congruence is then represented by an algebraic surface lying in the quadric Ω; and we are to find the number of points of this surface which lie in the tangent spaces [4] at both of two arbitrary points of Ω; these lie in the solid, a space [3], in which these two tangent spaces intersect. The number is the order of the surface which represents the congruence, and is the same for any solid of general position in the space [5]. If, in particular, we take a solid which meets Ω in two planes, respectively of the first and second kind upon Ω, which meet in a line of Ω (of which the solid is the polar with respect to Ω), then, the intersections in question, since the surface lies wholly on Ω, must be in one or other of these two planes. The points of one of the planes represent lines through a point in the original space [3], the points of the other represent lines in a plane. The total number of points of the surface which lie in a solid is thus obviously $p+q$.

By an argument which is almost the same we prove now, in space $[r]$, under the limitations $a<r-1$, $b<r-1$, $a+b\geqslant r-1$, the symbolical equation

$$(a, r)(b, r)=(a+b-r+1, r)+(a, \overline{r-1})(b, \overline{r-1}),$$

where the upper bar in $\overline{r-1}$ is to indicate that the two spaces $[r-1]$ are the same space. In words this equation is expressed by saying that, the condition for a line to meet both of two arbitrary spaces $[a]$ and $[b]$, is of a kind which may be satisfied alternatively, either by the line meeting a space $[a+b-r+1]$; or by the line lying in a space $[r-1]$, and meeting both of two-spaces $[a]$ and $[b]$ which lie therein. The composite condition on the left, involves, we have seen, a number of elementary conditions equal to $r-1-a+r-1-b$, or $2r-2-a-b$. This is also the number of elementary conditions for a line to meet a space $[a+b-r+1]$; also, as the number of conditions for a line to lie in an $[r-1]$ is 2, $2r-2-a-b$ is also the number of elementary conditions for a line to satisfy the composite condition represented by the second term on the right. The equation is thus equivalent also to saying that, the number of lines satisfying the condition (a, r) (b, r), which also satisfy further specified $a+b$ elementary conditions (thus satisfying in all the number, $2r-2$, of elementary conditions which suffice to determine a finite number of lines), is equal to the sum of the numbers of lines which satisfy these $a+b$ conditions and, also, satisfy one of the two alternative conditions represented by the terms on the right of the equation.

To obtain the equation, we argue as follows: In general, the two spaces $[a]$, $[b]$ meet in a space $[a+b-r]$ (or do not meet, if $a+b=r-1$), and are not both contained in a space $[s]$ with $s<r$; suppose, however, in particular, that they intersect in a space $[a+b-r+1]$, and are therefore both contained in a prime space, $[r-1]$, of the given space $[r]$. Then, the lines meeting both $[a]$ and $[b]$ evidently consist of, (1), the lines meeting their common $[a+b-r+1]$, without, necessarily, lying in the $[r-1]$, together with, (2), the lines lying in this $[r-1]$, which meet both $[a]$ and $[b]$, without, necessarily, meeting the $[a+b-r+1]$ intersection of these. There may also, (3), be lines which satisfy both the conditions (1) and (2), in the particular figure considered. But the number of elementary conditions for a line to do this is greater than for either (1) or (2) alone, and the aggregate of lines of this kind is of less freedom than in either of these cases; such lines will not enter therefore in the solution for the general figure, wherein all the alternative conditions, in terms of which the composite condition is to be expressed, are to have the same *power* as this. The conclusion is illustrated by the example discussed, of the lines of a congruence in [3] which meet two lines.

We have thus justified the equation put down when $a<r-1$, $b<r-1$, $a+b\geqslant r-1$. The equation evidently remains true when one, or both, of a, b is as great as $r-1$, provided the second term on the right be omitted, that is if we put $(r-1, r-1)=0$, and replace the unrestrictive factor $(r-1, r)$ by 1. But, retaining the conditions $a<r-1$, $b<r-1$, we can repeat the argument, and obtain

$$(a, \overline{r-1})(b, \overline{r-1})=(a+b-r+2, r-1)+(a, \overline{r-2})(b, \overline{r-2}),$$

and the process can be continued, provided a and b are not too large. If, with $a+b\geqslant r-1$, we have $a<b+1<r$, we can proceed as far as the equation

$$(a, \overline{b+2})(b, \overline{b+2})=(a-1, b+2)+(a, \overline{b+1})(b, \overline{b+1}),$$

where $(b, \overline{b+1})=1$. By addition of the equations we thus have

$$(a, r)(b, r)=(a+b-r+1, r)+(a+b-r+2, r-1)+\dots$$
$$+(a-1, b+2)+(a, b+1),$$

with $a+b \geqslant r-1$, $a<b+1<r$; and this is conventionally true also if $a<b+1 \leqslant r$. Under these limitations this solves the problem of expressing the composite condition $(a, r)(b, r)$ as a sum of alternative conditions.

If, however, $a+b \leqslant r-1$, the spaces $[a]$, $[b]$, which do not intersect, lie both in the same space $[s]$, where $s=a+b+1$. In this case, a line which meets both $[a]$ and $[b]$ will lie in this space $[s]$, so that we have $(a, r)(b, r)=(a, s)(b, s)$. As we have $a+b \geqslant s-1$ (in fact $a+b=s-1$) and $b+1 \leqslant s$, it follows from the preceding case, provided $a<b+1$, that $(a, s)(b, s)=(0, s)+(1, s-1)+\ldots+(a, b+1)$. Reversing the order of the terms on the right, both results are included in the formula $(a, r)(b, r)=\sum_{t=0}^{\lambda}(a-t, b+1+t)$, where $a \leqslant b$, and λ is the less of the two numbers $r-b-1$, a.

Now consider $(a, p)(b, q)$, with $a<p$, $b<q$, the space $[a]$ lying in the space $[p]$, and $[b]$ in $[q]$. A line which satisfies this condition must lie in both the spaces $[p]$ and $[q]$, and this cannot be unless these intersect in a line at least, or $p+q-r \geqslant 1$. When this is not so, the symbol has the value zero. When it is so, a line satisfying the condition, as lying in the space $[p+q-r]$ common to $[p]$ and $[q]$ and meeting the space $[a]$, must meet the common space of the $[p+q-r]$ and the $[a]$; as both these two spaces lie in the space $[p]$, this common space is a space $[a+q-r]$. In the same way, for the line to meet the common space of the $[p+q-r]$, and to meet $[b]$, both these lying in the space $[q]$, the line must meet a certain space $[b+p-r]$. Whence the condition $(a, p)(b, q)$ may be modified, writing $(a, p)(b, q)=(a+q-r, \bar{k})(b+p-r, \bar{k})$, where $k=p+q-r$. Thus the previously obtained equation, replacing a, b, r respectively by $a+q-r$, $b+p-r$, k, leads to

$$(a, p)(b, q)=\sum_{t=0}^{\lambda}(a+q-r-t,\ b+p-r+1+t),$$

where, now, λ is the less of the two numbers $q-b-1$, $a+q-r$, and it is supposed, beside $p+q \geqslant r+1$, that $a+q \leqslant b+p$.

This result may be written in a form which is perhaps easier to grasp. Put $r-1-a=\alpha$, $r-p=\beta$, $r-1-b=\xi$, $r-q=\eta$, and use the condition symbols; then we have

$$\{\alpha, \beta\}\{\xi, \eta\}=\sum_{t=0}^{\lambda}\{\alpha+\eta+t,\ \beta+\xi-t\},$$

where the distinction between the two symbols on the left is by the inequality $\alpha-\beta \geqslant \xi-\eta$ (this being the relation previously written as $a+q \leqslant b+p$); and the series on the right continues until the first index has reached the value $\alpha+\xi$, and the second index has fallen to $\beta+\eta$, unless the first index reaches $r-1$ before this (this being the fact that λ is the less of the two numbers $\xi-\eta$, $r-1-(\alpha+\eta)$), in which case the series stops at this term. The second index in this term would be $\alpha+\xi-[r-1-\eta-\beta]$. The value of $\{\alpha, \beta\}\{\xi, \eta\}$ is zero unless $r-1 \geqslant \beta+\eta$ (this being the condition $p+q \geqslant r+1$); this is involved, however, in the condition $r-1 \geqslant \alpha+\eta$, which is obviously necessary in order that the term, for $t=0$, on the right, may have a meaning. Thus, beside $\alpha-\beta \geqslant \xi-\eta$, which we may look upon as indicating the order in which to write the factors on the left of the equation, the only condition in the application of the formula is that involved in the rule for the last term on the right.

One of the simplest examples of the formula is that considered above for the number of lines of a congruence in ordinary space which meet two given lines; for this we obtained a result which may be written $\{1, 0\}\{1, 0\}=\{2, 0\}+\{1, 1\}$. We may give other simple examples, also for ordinary space: (*a*), It is clear that a line through an arbitrary point cannot lie in an arbitrary plane. Thus the product $(0, 3)\,(1, 2)$ should have the value zero. In fact this is $\{2, 0\}\{1, 1\}$, which is zero because $2+1>2$ $(\alpha+\eta>r-1)$; (*b*), Again, the condition for a line to lie in two given planes is $(1, 2)^2$, which is $\{1, 1\}^2$; by the formula this is $\{2, 2\}$ or $(0, 1)$; namely, the line must coincide with a given line (the intersection of the planes); (*c*), Lastly, the condition for a line to meet four given lines is

$$(1, 3)^4=[(1, 3)^2]^2=[\{1, 0\}^2]^2=[\{2, 0\}+\{1, 1\}]^2$$
$$=\{2, 0\}^2+2\,\{2, 0\}\{1, 1\}+\{1, 1\}^2;$$

of the terms on the right, $\{2, 0\}^2=\{2, 2\}=(0, 1)$, while

$$\{1, 1\}^2=\{2, 2\}=(0, 1);$$

and $\{2, 0\}\{1, 1\}$, or say $\{\alpha, \beta\}\{\xi, \eta\}$, is zero because $\alpha+\eta>r-1$. Thus, as $(0, 1)=1$, we have 2 as the number of lines meeting four given lines in ordinary space.

We also give some examples for lines in space of four dimensions ($r=4$): (*a*) It is known that there is one line meeting three given lines; this follows from

$$(1, 4)^3=\{2, 0\}^3=\{2, 0\}[\{2, 2\}+\{3, 1\}]=\{2, 0\}\{3, 1\}=\{3, 3\}=(0, 1)=1,$$

the term $\{2, 0\}\{2, 2\}$, or, say, $\{\alpha, \beta\}\{\xi, \eta\}$, being zero because $\alpha+\eta>r-1$.

(*b*) There is one line through a given point to meet a given line and a given plane. And

$$(0, 4)(1, 4)(2, 4)=\{3, 0\}\{2, 0\}\{1, 0\}=\{3, 0\}[\{2, 1\}+\{3, 0\}]$$
$$=\{3, 0\}^2=\{3, 3\}=(0, 1)=1,$$

the term $\{3, 0\}\{2, 1\}$ being zero because $3+1>3$, as in the last example.

(*c*) There are five lines meeting six given planes. For, first for four planes,

$$(2, 4)^4=\{1, 0\}^4=[\{2, 0\}+\{1, 1\}]^2=\{2, 0\}^2+2\{2, 0\}\{1, 1\}+\{1, 1\}^2$$
$$=[\{2, 2\}+\{3, 1\}]+2\{3, 1\}+\{2, 2\}=3\{3, 1\}+2\{2, 2\},$$

and hence for six planes,

$$\begin{aligned}(2, 4)^6 &= [\{2, 0\}+\{1, 1\}][3\{3, 1\}+2\{2, 2\}]\\ &= 3\{2, 0\}\{3, 1\}+2\{2, 0\}\{2, 2\}+3\{3, 1\}\{1, 1\}+2\{2, 2\}\{1, 1\}\\ &= 3\{3, 3\}+2[0]+3[0]+2\{3, 3\}=5\{3, 3\}=5(0, 1)=5.\end{aligned}$$

Another example of the formula is to prove, for any space $[r]$, the following formula, which we require below (though a direct proof is possible), namely $(p, q)(r-2, r)=(p, q-1)+(p-1, q)$. This supposes $p<q-1$; but, for $p=q-1$, we have, to express that a line lying in a $[q]$, and meeting a $[r-2]$, meets their $[q-2]$ intersection, the particular case $(q-1, q)(r-2, r)=(q-2, q)$. Suppose, then, $p \leqslant q-2$; the formula required is

$$\{r-1-p, r-q\}\{1, 0\}=\{r-1-p, r-q+1\}+\{r-p, r-q\},$$

and this is obtained from the formula by putting

$$\alpha=r-1-p, \quad \beta=r-q, \quad \xi=1, \quad \eta=0,$$

for which $\alpha-\beta \geqslant \xi-\eta$.

More generally, in space $[r]$, we can use the formula, above, for the product of two line symbols, to obtain a formula for the number of lines which meet s given spaces of dimension $r-2$; that is, to find $(r-2, r)^s$, or $\{1, 0\}^s$. The result is

$$\{1, 0\}^s=\{s, 0\}+c_1\{s-1, 1\}+c_2\{s-2, 2\}+\ldots+c_p\{s-p, p\},$$

where p is the greatest integer in $\frac{1}{2}s$, and the coefficient c_i is, with $\binom{s}{i}$ for the binomial coefficient,

$$c_i=\binom{s}{i}\frac{s-2i+1}{s-i+1}.$$

This can be proved by induction; for the initial values, $s=1, 2, \ldots$, the formula can be proved directly; it is sufficient then to shew that, if the right side be multiplied by $\{1, 0\}$, the result can be arranged so as to give the same formula for $\{1, 0\}^{s+1}$. For this, denote the line symbol $\{m, s+1-m\}$ by λ_m, and notice that

$$\{s-k, k\}\{1, 0\}=\{s-k, k+1\}+\{s-k+1, k\}=\lambda_{s-k}+\lambda_{s-k+1};$$

thus, on multiplication by $\{1, 0\}$, the right side becomes

$$\lambda_s+\lambda_{s+1}+c_1(\lambda_{s-1}+\lambda_s)+c_2(\lambda_{s-2}+\lambda_{s-1})+\ldots+c_p(\lambda_{s-p}+\lambda_{s-p+1}),$$

or $\quad \lambda_{s+1}+(1+c_1)\lambda_s+(c_1+c_2)\lambda_{s-1}+\ldots+(c_{p-1}+c_p)\lambda_{s-p+1}+c_p\lambda_{s-p}$,

while, we see at once, $c_h+c_{h+1}=C_{h+1}$, where C_i is the value of c_i when s is changed to $s+1$. The new right side is thus

$$\lambda_{s+1}+C_1\lambda_s+C_2\lambda_{s-1}+\ldots+C_p\lambda_{s-p+1}+c_p\lambda_{s-p},$$

and the required result is obtained if we shew that the last effective term of this is

$$\binom{s+1}{q}\frac{s-2q+2}{s-q+2}\{s-q+1, q\},$$

where q is the greatest integer in $\frac{1}{2}(s+1)$. When s is even, we have $q=p$, and the last term $c_p\lambda_{s-p}$ which is written, is zero, since, in $\lambda_{s-p}=\{s-p, p+1\}$, we have $s-p=\frac{1}{2}s$, which is $<p+1$. When s is odd, say $2t+1$, we have $q=p+1$, $=t+1$, and it is to be verified that $c_p=C_{p+1}$; this is immediate. The formula for $\{1, 0\}^s$ is thus proved.

In particular, when $s=2r-2$, all the early terms $\{\alpha, \beta\}$, of the series, vanish because $\alpha>r-1$; the only surviving term is the last, for $p=r-1$. Thus, replacing $\{r-1, r-1\}$ by 1, the number of lines in space $[r]$ which meet $2r-2$ given spaces of dimension $r-2$, is $(2r-2)!/r!\,(r-1)!$, as we have seen above for $r=3$ and $r=4$.

Many other formulae have been found, for the solution of the general problem of expressing the product of a number of simultaneous conditions as a sum of alternative conditions. We can only give space for a few.

(1) For a line in space $[r]$, which depends on $2r-2$ parameters, to meet a space $[r-3]$, is two elementary conditions; there is thus a finite number of lines meeting $r-1$ given spaces $[r-3]$. This number is given by

$$(r-3, r)^{r-1}=\sum_{i=0}(-1)^i\binom{r-1}{i}\binom{2r-4-3i}{r-3},$$

where each term involves the product of two binomial coefficients.

(2) The number of spaces $[k]$, in space $[r]$, which satisfy two general conditions $(a_0, a_1, \ldots, a_k)$ and $(b_0, b_1, \ldots, b_k)$, and also satisfy the further e conditions of meeting e given spaces $[r-k-1]$, where e is such that the total number of elementary conditions imposed is equal to the number $(k+1)(r-k)$, of parameters on which a $[k]$ depends, is given by the formula

$$(a_0, a_1, \ldots, a_k)(b_0, b_1, \ldots, b_k)(r-k-1)^e=e!\,|\,c_{ij}\,|,$$

where $e!=1.2.3\ldots e$ and the determinant is of $k+1$ rows and columns $(i, j=0, 1, \ldots, k)$, and the element c_{ij} is $1/(a_i+b_{k-j}-r)!$, to be replaced by zero for every element in which $a_i+b_{k-j}<r$.

As an example, for the lines in [4] which pass through a given point, meet a given line, and meet a given plane, we have

$$(0, 4)(1, 4)(2)=\begin{vmatrix}[0!]^{-1}, & [(-3)!]^{-1}\\ [4!]^{-1}, & [1!]^{-1}\end{vmatrix}=1;$$

and for the lines in [4] which meet, in all, six planes, we have

$$(2, 4)(2, 4)(2)^4=4!\begin{vmatrix}[2!]^{-1}, & [0!]^{-1}\\ [4!]^{-1}, & [2!]^{-1}\end{vmatrix}=5.$$

(3) A particular case of the formula in (2) is when

$$(b_0, b_1, \ldots, b_k)=(r-k, r-k+1, \ldots, r)=1.$$

Then, with the appropriate value of e,

$$(a_0, a_1, \ldots, a_k)(r-k-1)^e = e!\,|[(a_i-j)!]^{-1}|.$$

In the determinant, the elements in the $(i+1)$-th row are

$$[a_i!]^{-1},\ [(a_i-1)!]^{-1},\ \ldots,\ [(a_i-k)!]^{-1},$$

and these, all multiplied by $a_i!$, are

$$1,\ a_i,\ a_i(a_i-1),\ \ldots,\ a_i(a_i-1)\ldots(a_i-k+1).$$

Thus the determinant, multiplied by $a_0!\,a_1!\ldots a_k!$, if the columns be suitably combined, becomes the determinant of which the general row consists of the elements $1, a_i, a_i^2, \ldots, a_i^k$. Wherefore

$$(a_0, a_1, \ldots, a_k)(r-k-1)^e = e!\prod_{r,s}^{0\ldots k}(a_r-a_s)/a_0!\,a_1!\ldots a_k!$$

(4) As a particular case of this last formula, we may take $(a_0, \ldots, a_k) = (r-k, r-k+1, \ldots, r) = 1$, with e equal to $(k+1)(r-k)$. Then we have the result that the number of spaces $[k]$ which meet $(k+1)(r-k)$ given spaces of dimension $r-k-1$ is

$$e!\,\frac{((k))\,((r-k-1))}{((r))},$$

or, say $e!((k))((k'))/((r))$, where $k' = r-k-1$ and $((m))$ denotes $1!\,2!\ldots m!$, and $e = (r-k)(k+1), = (k+1)(k'+1)$. This is also the number of spaces $[r-k-1]$ which meet $(k+1)(r-k)$ given spaces $[k]$.

This is a fundamental number. Writing the coordinates of $k+1$ points, of the original space $[r]$, which suffice to determine a space $[k]$, as the $k+1$ rows of a matrix of $r+1$ columns, the $(r+1, k+1)$ determinants of order $k+1$, which can be formed from this matrix, are the homogeneous coordinates of the $[k]$; these can be used to represent the $[k]$ by a point of space $[R]$, where $R = (r+1, k+1)-1$. The aggregate of all spaces $[k]$, in $[r]$, is then represented, in this space $[R]$, by a manifold M_e, of dimension e. The order of this manifold is the number we have found. For, by definition, the order of a manifold, of dimension e, is the number of its points which satisfy e equations linear in the coordinates of the space in which it lies; namely, the number of its points which lie on e primes of the space $[R]$. The condition for a space $[k]$, in the original space $[r]$, to meet a given $[r-k-1]$ is, however, a linear condition for the coordinates of the $[k]$; the coefficients in this linear equation, which are the coordinates of the $[r-k-1]$, are not general; but the relations which connect them are quadratic, not linear. Wherefore, the order of the manifold in $[R]$ which represents the $[k]$ of $[r]$ is given by the number of spaces $[k]$ which meet e given spaces $[r-k-1]$. All this is an easy generalisation of the familiar fact that the quadric, in space $[5]$, which represents the lines of ordinary space, has the order 2, this being the number of lines in ordinary space [3] which meet four given lines. More generally, the lines of space [4], as also the dually corresponding planes, are representable, in a space of 9 dimensions, by a manifold of dimension 6, whose order is 5, say $M_6{}^5$; and, for example, the planes of space [5], are representable, in a space of 19 dimensions, by a manifold of dimension 9, whose order is 42.

(5) Another noticeable formula, with the same notation, is

$$(a_0, a_1, \dots, a_k)(r-k-1) = (a_0-1, a_1, \dots, a_k) \\ + (a_0, a_1-1, a_2, \dots, a_k) + \dots + (a_0, a_1, \dots, a_{k-1}, a_k-1),$$

which, in condition symbols, is

$$\{\alpha_0, \alpha_1, \dots, \alpha_k\}\{1, 0^k\} = \sum_{i=0}^{k} \{\alpha_0, \dots, \alpha_{i-1}, \alpha_i+1, \alpha_{i+1}, \dots, \alpha_k\}.$$

From this formula we can build up, by steps, the formula for $(a_0, \dots, a_k)(r-k-1)^m$; it is necessary at each stage to omit the symbols on the right which are unmeaning because the indices are not in ascending order.

A generalisation in another direction is the formula

$$(a_0, \dots, a_k)(r-k-m) = \sum_j (a_0-j_0, a_1-j_1, \dots, a_k-j_k),$$

where, for brevity, $(r-k-m)$ means the condition that a $[k]$ meets a $[r-k-m]$, which is equivalent to m elementary conditions. Here the summation extends to all positive, or zero, integers $j_0, \dots, j_k$ whose sum is m, subject to $j_i < a_i - a_{i-1}, j_0 < a_0 + 1$.

Part II. The problem of multiple tangents of a manifold. Consider a primal, of order n, in space $[r]$, which we may denote by $M^n{}_{r-1}$, defined by a single perfectly general polynomial equation connecting the coordinates. In order that a line should touch the primal at one of its intersections with it, a single condition must be imposed upon the line; in order that, at this intersection, there should be i coincident points of the primal upon the line, $i-1$ conditions must be imposed. Lines with such a multiple intersection are thus ∞^k in aggregate, where

$$k = 2r-2-(i-1), \; = 2r-1-i;$$

and this is also the freedom of the points where such lines touch the manifold. If we impose, upon these multiple tangents, a further condition equal to $2r-2-i$ elementary conditions, or say of power $2r-2-i$, the lines satisfying all the conditions will be ∞^1, and will form a ruled surface; the locus of the points of multiple contact with the manifold will then be a curve. For a moment, let the order of this ruled surface be denoted by g, and the order of the curve of contact by b; each generator will have $(n-i)$ further intersections with the primal, and the curve which is the locus of these further points will have an order f given by $f=ng-ib$.

Now take a pencil of primes, that is, a series of primes all passing through a definite space $[r-2]$. Consider the correspondence between a prime, (α), of this pencil which contains a point of multiple contact, with the primal, of a generator of the ruled surface, and a prime, (β), of this pencil, which contains one of the simple intersections of the same generator. On each prime (β) there are f of the simple intersections spoken of, each associated with the multiple contact on the same generator; on each prime (α) there are b of the multiple contacts, each associated with the $n-i$ simple intersections on the same generator. Thus the correspondence between the primes (α), (β) is of indices f and $(n-i)\,b$. There are, therefore, $f+(n-i)\,b$ primes of the pencil which are coincidences of a pair of corresponding primes. Such coincidences of primes arise, (1), for lines of the ruled surface which have $(i+1)$ coincident intersections with the primal; denote the number of these by ϵ; they arise also, (2), for generators of the ruled surface which meet the base space $[r-2]$ of the pencil of primes; there are g such generators, and each of these gives $n-i$ coincidences of pairs of corresponding primes, the multiple intersection of the generator being taken with all the $n-i$ simple intersections in turn. On the whole, then, we have the equation $\epsilon+(n-i)\,g=f+(n-i)\,b$; and, in virtue of $f=ng-ib$, this leads to $\epsilon=(n-2i)\,b+ig$.

We modify this equation, in two stages, so that it may be a statement, not in regard to a point only, or a line only, but in regard both to a line and a point lying thereon; that the point lies in a space $[k]$ will be denoted by a symbol (k), that the line meets a space $[h]$ will be denoted by a symbol (h, r); further, that the point is to be thought of as the coincidence of j points on the line will be denoted by the symbol ϵ_j. Thus, as b is the number of i-pointic intersections of a line (the generator of the ruled surface) with the primal, which lie in a prime $[r-1]$, we replace b by $\epsilon_i\,(r-1)$; and as g is the number of the lines considered which meet a space $[r-2]$, we replace g by $\epsilon_i\,(r-2, r)$. The equation can then, first, be replaced by

$$\epsilon_{i+1}=(n-2i)\,\epsilon_i\,(r-1)+i\epsilon_i\,(r-2, r),$$

this being an abbreviation of the complete statement, for the composite entity consisting of a line and a point thereon, in which there would be supplied, in the three terms respectively, the factors $(r)(r-1, r)$, $(r-1, r)$, (r). This is an equation connecting numbers, obtained on the hypothesis that $2r-2-i$ elementary conditions have been imposed on the line, in addition to the condition of multiple intersection with the primal; without these additional conditions the line would not describe a ruled surface. We now suppose that these $2r-2-i$ elementary conditions for the line may be built up of a certain number s, of conditions for the line, with a number, t, of conditions for the multiple intersection with the primal (where $s+t=2r-2-i$); therein making the assumption that t elementary conditions, of the kind to be imposed, for this distinguishable point of the line, are equivalent to t conditions for the line. The s conditions for the line are that it lie in a space $[q]$ and meet a $[p]$ contained therein, or (p, q); this is equivalent, we have seen, to $(r-1-p)+(r-q)$, or to $2r-1-p-q$ elementary conditions; the t conditions for the multiple intersection are that it lie in a space $[r-p-q+i+1]$, that is, in $p+q-i-1$ primes, which is equivalent to $p+q-i-1$ elementary conditions. We have

$$s+t=2r-1-p-q+p+q-i-1=2r-2-i.$$

It is supposed that $p+q \geqslant i+1$. Denoting the condition for the point by $(r-p-q+i+1)$, we can now write the equation above in the form

$$\epsilon_{i+1}(p, q)(r-p-q+i+1)=(n-2i)\,\epsilon_i(r-1)(p, q)(r-p-q+i+1) \\ +i\epsilon_i(r-2, r)(p, q)(r-p-q+i+1),$$

of which each term, counting ϵ_j as of power $j-1$, is of the power $2r-2$, equal to the number of parameters on which a line depends. This equation, however, may be simplified: First, a point which lies both on a $[r-1]$ and on a $[r-p-q+i+1]$, lies on the $[r-p-q+i]$ in which they meet, so that, in the equation, $(r-1)(r-p-q+i+1)$ may be replaced by $(r-p-q+i)$. Second, we may use the formula, remarked above (p. 83), always true (for $p<q$) if impossible terms on the right are omitted, $(p, q)(r-2, r)=(p, q-1)+(p-1, q)$, to modify the last term on the right, so obtaining

$$\epsilon_{i+1}(p, q)(r-p-q+i+1)=(n-2i)\,\epsilon_i\,(p, q)\,(r-p-q+i) \\ +i\epsilon_i\,(p, q-1)(r-p-q+i+1)+i\epsilon_i(p-1, q)(r-p-q+i+1).$$

If then we put

$$H_i(p, q)=\epsilon_i(p, q)(r-p-q+i), \qquad p+q \geqslant i+1,$$

we have

$$H_{i+1}(p, q)=(n-2i)\,H_i(p, q)+iH_i(p, q-1)+iH_i(p-1, q).$$

The finite number of lines, $H_i(p, q)$, satisfying the condition $H_i(p, q)$, which is of power $2r-2$, can be obtained by induction from this equation if we obtain $H_1(p, q)$. This, being $\epsilon_1(p, q)(r-p-q+1)$, is the number of lines, lying in a $[q]$, which meet a $[p]$ lying therein, whose intersection with the primal $M^n{}_{r-1}$ lies on a space $[r-p-q+1]$. This last space meets $M^n{}_{r-1}$ in a manifold of dimension $r-p-q$, say $M^n{}_{r-p-q}$; unless $p=0$ this manifold will not in general intersect the space $[q]$, and there will be no lines in $[q]$ whose intersections with

M^n_{r-1} lie on the M^n_{r-p-q}; but, when $p=0$, the space $[q]$ will meet the M^n_{r-p-q} in n points, each of which can be joined to the point $[p]$, which lies in the $[q]$. Thus $H_1(p, q)$ is 0 or n, according as $p>0$ or $p=0$.

Now let P_i denote the polynomial in t, of order $i-1$, given by

$$P_i=n[n-1+t][n-2+2t]\ldots[n-(i-1)(1-t)],$$

and let $c_i(p, q)$ denote the coefficient of t^{-1} in the ascending expansion of $P_i(t^{-p-1}-t^{-q-1})$; in particular $c_1(p, q)$, the coefficient of t^{-1} in the ascending expansion of $n(t^{-p-1}-t^{-q-1})$, is 0 or n, according as $0<p<q$ or $0=p<q$. Thus $H_1(p, q)=c_1(p, q)$. While

$$(n-2i)\,c_i(p, q)+ic_i(p, q-1)+ic_i(p-1, q),$$

being the coefficient of t^{-1} in the ascending expansion of the product of P_i with

$$(n-2i)(t^{-p-1}-t^{-q-1})+i(t^{-p-1}-t^{-q})+i(t^{-p}-t^{-q-1}),$$

or

$$[n-i(1-t)][t^{-p-1}-t^{-q-1}],$$

is equal to $c_{i+1}(p, q)$. Thus in general we have $H_i(p, q)=c_i(p, q)$, or $\epsilon_i(p, q)(r-p-q+i)$ is equal to

$$[nn_1n_2\ldots n_{i-1}(t^{-p-1}-t^{-q-1})]_{t^{-1}},$$

where n_s means

$$n-s(1-t),\quad\text{and}\quad p+q\geqslant i>0,\quad 0\leqslant p<q;$$

this gives the number of lines, in space $[r]$, subject to the condition (p, q), which have i-pointic intersection with a primal M^n_{r-1}, with the point of contact in a space $[r-p-q+i]$. This number is expressible only by i, p, q, without r; as P_i is of order $i-1$ in t, the number is zero if $i-1<p$; or, we must suppose $i>p$. Further, we consider only those lines which lie in a $[q]$; thus, effectively, we consider only the M^n_{q-1} which is the section, by the $[q]$, of the original M^n_{r-1}. Thus, changing the notation by supposing $q=r$, the formula is, effectively,

$$\epsilon_i(p, r)(i-p)=[nn_1n_2\ldots n_{i-1}(t^{-p-1}-t^{-r-1})]_{t^{-1}}$$

with the obvious conditions $0\leqslant p<r$, $0\leqslant i-p\leqslant r$.

Some examples may be given:

(1) If $p=0$, $r=2$, and $i=1$ or 2, the formula gives, respectively, the order n, and the class $n(n-1)$, of a plane curve whose point equation is quite general. If $p=1$, $r=2$, $i=3$, the formula for $\epsilon_3(1, 2)(2)$ gives $3n(n-2)$ for the number of inflexional tangents of the curve.

(2) If $p=0$, $r=3$, $i=3$, the number, $\epsilon_3(0, 3)(3)$, of inflexional tangents of a surface, of order n, in ordinary space, which pass through a given point, is found as $n(n-1)(n-2)$. The points of the surface at which the inflexional tangents meet an arbitrary line, lie on a curve of order $\epsilon_3(1, 3)(2)$, which is $n(3n-4)$.

(3) If $p=2$, $r=3$, $i=4$, the order of the curve locus of 4-pointic tangents of a surface in ordinary space, is

$$\epsilon_4(2,3)(2), \;=\{n(n-1+t)(n-2+2t)(n-3+3t)(t^{-3}-t^{-4})\}_{t^{-1}},$$
$$=n[6(n-1)+3(n-2)+2(n-3)-6],$$

or $n(11n-24)$. In particular, for $n=3$, this gives the number, 27, of lines on a general cubic surface.

The fact that the points of 4-pointic contact, of tangents of a surface of order n in ordinary space, lie on another surface of order $11n-24$, was obtained by Salmon in 1849 (*Camb. and Dub. Math. J.* IV, 1849, p. 260). See also *Quart. J. of Math.* I, 1857, p. 336; *Phil. Trans.* A, CL, 1861, p. 229; and Clebsch, *Crelle*, LVIII, 1861, pp. 93, 106. Salmon, *Solid Geom.* 1882, p. 559, quotes from Cayley the remark that this surface is obtainable by eliminating m from the two equations

$$z_{11}+2mz_{12}+m^2z_{22}=0, \quad z_{111}+3mz_{112}+3m^2z_{122}+m^3z_{222}=0,$$

($z_{11}=\partial^2z/\partial x^2$, $z_{12}=\partial^2z/\partial x\partial y$, etc.); and proves (*loc. cit.* p. 567) that the surface touches the Hessian of the original surface. Salmon also gives (*ibid.*) the order, $2n(n-3)(3n-2)$, of the ruled surface generated by the 4-pointic tangents, which is obtainable as $\epsilon_4(1,3)(3)$.

(4) The number of 5-pointic tangents of a general surface of order n in ordinary space is $\epsilon_5(2,3)(3)$, $=5n(n-4)(7n-12)$, for $n\geqslant 4$.

(5) The simple case $\epsilon_i(i-1,r)(1)=n(i-1)!$ is in agreement with the fact that the i-pointic tangents, at a point of the primal M^n_{r-1}, form a cone of dimension $r-i+1$, and of order $(i-1)!$. For instance, the 2-pointic tangents lie in a prime. The 3-pointic tangents form a cone of dimension $r-2$ and order 2 (for $r=3$ they are 2 in number; for $r=4$, they form a conical sheet of order 2; for $r=5$ a quadric solid cone). The 4-pointic tangents form a cone of dimension $r-3$ and order 6 (for $r=4$ they are 6 in number).

(6) For a general value of r, $\epsilon_r(0,r)(r)$, $=n(n-1)\ldots(n-r+1)$ is the number of r-pointic tangents which can be drawn through an arbitrary point.

(7) The number of tangents of the primal with $(2r-1)$-pointic contact is $\epsilon_{2r-1}(r-1,r)(r)$. For example, in space [4], the number of tangents with 7-pointic contact is $35n(n-6)(7n-12)(3n-10)$.

(8) The order of the ruled surface generated by 5-pointic tangents of a primal in [4] is $\epsilon_5(3,4)(2)$, which is $10n(5n-12)$.

(9) The order of the curve of contact of 6-pointic tangents of a primal in [4] is $\epsilon_6(3,4)(3)$, which is $5n(45n^2-274n+360)$. In particular, a primal of order 5 in space [4], contains $5^3.23$ or 2875 lines (cf. Schubert, *Math. Ann.* XII, 1877, p. 192). In general a M^{2r-3}_{r-1} in $[r]$ contains $\epsilon_{2r-2}(r-1,r)(r-1)$ lines.

A more general problem than the one solved in the preceding investigation, is that of a line meeting a given primal with several points of coincident intersection, of assigned multiplicities, satisfying possibly other conditions also, for the line itself or the points of contact. This is an interesting problem, generalising the investigation of de Jonquières explained above (p. 39), of the variable curves in a plane, having coincidences of assigned multiplicities in their intersections with a given curve. There is here no space for a detailed account of Schubert's remarkable solution of this problem;

the reader who finds the introductory account of Schubert's notation which is given below (pp. 96 ff.) to be interesting, is referred to Schubert's chapter (*Kalkül d. abzähl. Geom.* 1879, pp. 228–44); and also to Zeuthen's *Lehrbuch d. abzähl. Methoden d. Geom.* 1914, pp. 372–76. Some particular examples, mostly given by Salmon, *Solid Geometry*, 1882 (or Ed. of 1915, pp. 277 ff.) may be quoted. See also Roth, *Proc. Camb. Phil. Soc.* XXVI, 1930, p. 43.

(1) If for a general surface of order n, in ordinary space, a line be inflexional at one point and ordinarily tangent at another point of the surface (thus satisfying $2+1$ elementary conditions in all), the locus of the points of inflexion is a curve of order $n(n-4)(3n^2+5n-24)$, while the locus of the points of ordinary contact is a curve of order $n(n-2)(n-4)(n^2+2n+12)$, and the lines generate a ruled surface of order $n(n-3)(n-4)(n^2+6n-4)$. From these it is easily found that the simple intersections generate a curve of order

$$n(n-4)(n-5)(n^3+6n^2-n-24).$$

For these results, see Salmon, *loc. cit.* pp. 558, 568, 570.

(2) In the same case, a line with three ordinary contacts generates a ruled surface of order $\frac{1}{3}n(n-3)(n-4)(n-5)(n^2+3n-2)$, the contacts generating a curve of order $\frac{1}{2}n(n-2)(n-4)(n-5)(n^2+5n+12)$, and the simple intersections a curve of order

$$\tfrac{1}{3}n(n-4)(n-5)(n-6)(n^3+3n^2-2n-12).$$

(Salmon, *loc. cit.* p. 569.)

(3) The lines having four ordinary contacts are in number

$$\tfrac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30).$$

(Salmon, *loc. cit.* p. 575, with reference to Schubert.)

(4) The number of lines with 4-pointic contact at one place, and an ordinary contact elsewhere, is $2n(n-4)(n-5)(3n-5)(n+6)$. (Salmon, *ibid.*)

(5) The number of lines which are inflexional at two places, is $\frac{1}{2}n(n-4)(n-5)(n^3+3n^2+29n-60)$. (Salmon, *ibid.*)

(6) The lines inflexional at one place and ordinarily tangent at two others are in number $\frac{1}{2}n(n-4)(n-5)(n-6)(n^3+9n^2+20n-60)$. (Salmon, *ibid.*)

(7) The number of points of the surface where both inflexional tangents have four coincident intersections is $5n(7n^2-28n+30)$. (Salmon, p. 572; Schubert, *Kalkül*, p. 246.) The surface has ∞^2 points, and the condition for each inflexional tangent is onefold. For a cubic surface the number is three times the number (45) of tritangent planes.

(8) The points where the two inflexional tangents of the surface coincide in a line having 4-pointic intersection are in number $2n(n-2)(11n-24)$. (Salmon, p. 573.)

(9) The inflexional tangents at the parabolic points of the surface, where the inflexional tangents coincide, generate a ruled surface of order $2n(n-2)(3n-4)$. (Schubert, p. 244.) The parabolic curve is the intersection of the surface with its Hessian, and the tangent planes of the surface at two coinciding points of the parabolic curve intersect in the inflexional line. Thus the number in question is the order of the developable surface enveloped by the tangent planes of the surface at the points of the parabolic curve; as such, it is given by Salmon (p. 546). Salmon also gives (p. 547) the order of the developable surface formed by the

tangent planes of the given surface, of order n, at the points of the curve of intersection of this surface with another surface of order k, namely $nk(3n+k-6)$. When $k=4(n-2)$, the second surface being the Hessian, this number is $4n(n-2)(7n-14)$; and this exceeds the number $2n(n-2)(3n-4)$ here given, by the number, $2n(n-2)(11n-24)$, given in Ex. 8 preceding. In general, the number of tangents of the curve (n, k) which are inflexional lines of the surface (n) is $nk(3n+2k-8)$; and the inflexional lines of the surface (n), at the points of the curve (n, k), form a ruled surface of order $nk(3n-4)$. (Salmon, p. 546.)

Part III. Correspondence of points of two manifolds. We consider now the generalisation of the principle of correspondence, as originally developed above for ∞^1 pairs of corresponding points, to cases of multiple correspondence. The results in the simpler cases were first enunciated by Salmon and Zeuthen (Salmon, *Solid Geom.* 1882, pp. 550, 598; Zeuthen, *Compt. rend.* LXXVIII, 1874, p. 1553). We may, for instance, in a plane, have ∞^2 pairs of points (P, P'), such that, to any position of P corresponds only a finite number of positions of P'; and similarly, to any position of P', a finite number of positions of P. This does not involve, however, necessarily, that points P, or P', occur everywhere in the plane. We may denote the number of positions of P' which correspond to an assigned position of P by $(0)\,(2)'$, the first symbol referring to the assigned point P, and the second symbol to the corresponding positions of P', anywhere in those parts of the plane where points P' occur; similarly, the number of positions of P when P' is assigned may be denoted by $(2)\,(0)'$. As P takes positions upon an arbitrary line, P' will take positions upon a certain curve, and there will be a certain number of points of this curve lying upon a second arbitrary line. We thus consider a number, $(1)\,(1)'$, of pairs of corresponding points, in which P and P' lie, respectively, on given arbitrary lines. The formula we arrive at expresses the sum $(0)\,(2)'+(1)\,(1)'+(2)\,(0)'$ in terms of the number of coincidences of P and P'. But such coincidences are of two kinds: It may happen that, as the pair of corresponding points, P, P', approaches a point C at which they coincide, the line PP' likewise approaches a definite line through C, which is independent of the manner in which P approaches to C; or, it may happen that the line PP', by suitable approach of these points to C, can be made to approach to any assigned line through C. For the number of points of coincidence of the second kind we may use the symbol $\epsilon\,(0, 2)\,(2)$, where the ϵ, after Schubert's use, indicates that we are considering a coincidence; the $(0, 2)$ refers to the line which is the ultimate position of PP' as these points approach C, and indicates that this line may be supposed to pass through an arbitrary point of the plane; and the (2) refers to the point of coincidence C, and indicates that we are considering all such coincidences as occur anywhere in the plane. Points of coincidence of the first kind, for which PP' ultimately takes up a definite position as P, P' approach C, are generally of infinite number, as we shall see immediately, and lie upon a curve, so that there is a definite number of such points lying upon an arbitrary line. The whole number of these points is therefore denoted by $\epsilon\,(1, 2)(1)$, where the symbol $(1, 2)$ has only a formal significance, as indicating that the ultimate line PP' cannot be supposed to pass through an arbitrary point of the plane, though it meets an arbitrary line, and the symbol (1) indicates that we consider all points of coincidence, C, of the kind in question, which lie upon an arbitrary line. Under the hypothesis that all the symbols have definite finite values, the equation we are to prove is

$$(0)(2)'+(1)(1)'+(2)(0)'=\epsilon(0, 2)(2)+\epsilon(1, 2)(1).$$

It must be understood that, when the contrary is not stated, the corresponding points P, P' are supposed to be so defined as to be distinguishable from one another. For instance, the theory applies when the P, P' are in a $(1, 1)$ correspondence defined by general linear equations. But, if this linear correspondence be involutory, two points, A, B, which are a pair of the involution, give two pairs of the correspondence; for P' is at B when P is at A, and also P' is at A when P is at B. The steps where the proofs of the fundamental formulae need such modification are briefly noticed below (p. 101).

Before proceeding to the proof of this equation, we consider the statements we have made as to the possible modes of coincidence. Denote the equations which express the correspondence, between the point (x, y) and the point (x', y'), for a limited region of the plane containing both these points, by

$$x'-f(x, y)=0, \quad y'-\phi(x, y)=0;$$

the functions f and ϕ, in the applications made, are generally algebraic; but we suppose them at least analytic and holomorphic (and single valued) in the regions in which they are used. The coincidences of the correspondences are then the points (x, y) which are the solutions of the equations $x-f(x, y)=0$, $y-\phi(x, y)=0$. If (a, b) be such a coincidence, and (x, y), (x', y') be both in the neighbourhood of this, the equations of the correspondence lead, in virtue of $a-f(a, b)=0$, $b-\phi(a, b)=0$, to

$$\begin{aligned} x'-x&=(x-a)[f_1(a, b)-1]+(y-b)f_2(a, b)+U_2, \\ y'-y&=(x-a)\,\phi_1(a, b) \qquad +(y-b)[\phi_2(a, b)-1]+V_2, \end{aligned}$$

where U_2, V_2 are series in $x-a$, $y-b$, involving squares and higher powers, and $f_1(x, y)=\partial f/\partial x$, etc. If then (a, b) do not lie on the curve $\Delta=0$, where

$$\Delta=\begin{vmatrix} f_1(x, y)-1, & f_2(x, y) \\ \phi_1(x, y), & \phi_2(x, y)-1 \end{vmatrix},$$

the equations, by an appropriate limiting value of the ratio $(y-b)/(x-a)$, lead to any assigned limiting value of $(y'-y)/(x'-x)$; that is, the joining line of the coinciding points may be regarded as capable of an arbitrary ultimate direction. Then we have a point included in the formula $\epsilon(0, 2)(2)$. But if (a, b) lie on the curve $\Delta=0$, the limiting value of $(y'-y)/(x'-x)$ is definite, expressible as $\phi_1(a, b)/[f_1(a, b)-1]$, or $[\phi_2(a, b)-1]/f_2(a, b)$; we have then a point of coincidence included in the formula $\epsilon(1, 2)(1)$. We omit the consideration of the cases in which neither of the alternative forms for $(y'-y)/(x'-x)$ is definite in value.

Similarly, we may have a correspondence in which there are ∞^3 pairs of corresponding points P, P', which then we suppose to lie in a space of three dimensions. Then we have the formula

$$(0)(3)'+(1)(2)'+(2)(1)'+(3)(0)'=\epsilon(0,3)(3)+\epsilon(1,3)(2)+\epsilon(2,3)(1);$$

here, on the left, the symbol $(i)\,(j)'$ means the number of pairs of corresponding points P, P', in which P lies in an arbitrary space $[i]$, while P' lies in another arbitrary space $[j]$; while, on the right, three possible kinds of coincidence are contemplated; these are classified by the freedom of the ultimate position of the joining line PP' as P and P' approach to a point of coincidence C, and it is assumed that this is correlated with the freedom of position of the associated points C. In particular $\epsilon(0, 3)(3)$ is the number of coincidences of P and P', of such character that the limiting position of the line PP' may be taken arbitrarily through the point of coincidence; $\epsilon(1, 3)(2)$ is the number of coincidences lying in an arbitrary plane, for which the ultimate PP' meets an arbitrary line, while, $\epsilon(2, 3)(1)$ is the number of coincidences lying on an arbitrary line, for which the ultimate PP' satisfies only the (nugatory) condition of meeting an arbitrary plane.

The general case is that of ∞^r pairs of corresponding points, in a space $[r]$. The appropriate equation, to be obtained, is immediately written down from the two preceding cases. On the right of this equation the general term, one of r terms, has the form $\epsilon(i, r)(r-i)$, for $i=0, \ldots, (r-1)$; this denotes the number of points of coincidence, lying in an arbitrary space $[r-i]$, of which the ultimate line PP' meets an arbitrary space $[i]$. For a point to lie in a $[r-i]$ requires i conditions, and for a line to meet an $[i]$ requires $r-1-i$ conditions. We must then explain the justification for the evident assumption, in the formula, that the conjoint figure, of a point of coincidence with the ultimate line PP' through this point, can satisfy, in all, $r-1$ conditions; the simple remark that there are ∞^r pairs of corresponding points, and the coincidence of a pair, upon the line which joins them, requires one condition, may not seem enough. On the left of the equation to be obtained, the general term, $(i)(r-i)'$, one of $r+1$ terms, requires $r-i+i$, that is r elementary conditions.

As in the case $r=2$, using non-homogeneous coordinates $x_1, \ldots, x_r$, we may represent the correspondence, when both P and P' are in a suitably limited space, by the r, regular analytic, equations $f_i(x_1, \ldots, x_r)-x_i'=0$, a coincidence $(a_1, \ldots, a_r)$ being a point for which the r equations $f_i(a_1, \ldots, a_r)-a_i=0$ hold. If then the corresponding points (x) and (x') are in the neighbourhood (a), we have, to the first approximation, r equations such as

$$x_i'-x_i=(x_1-a_1)f_{i1}+\ldots+(x_i-a_i)(f_{ii}-1)+\ldots+(x_r-a_r)f_{ir}, \quad \text{(I)},$$

where f_{ik} is the value at (a) of $\partial f_i/\partial x_k$. If then

(1), the determinant, Δ, of the coefficients of $x_1-a_1, \ldots, x_r-a_r$ on the right, be not zero, then the r equations $f_i(x_1, \ldots, x_r)-x_i=0$ are independent in the neighbourhood of (a); in this case, by suitable limiting values of the ratios of $x_1-a_1, \ldots, x_r-a_r$, that is, by a suitable direction of approach of (x) to (a), we can secure that the ultimate direction of PP', determined by the limits of the ratios of $x_1'-x_1, \ldots, x_r'-x_r$, is arbitrary. This corresponds to the hypothesis of discrete points of coincidence, and the term $\epsilon(0, r)(r)$. While

(2), if this determinant Δ be zero, but not every minor of $(r-1)$ rows and columns therein, then the equations $f_i(x_1, \ldots, x_r)-x_i=0$ determine a curve of points of coincidence lying in the neighbourhood of (a). In this case, the equations (I), by elimination of $x_1-a_1, \ldots, x_r-a_r$, imply a linear relation connecting $x_1'-x_1, \ldots, x_r'-x_r$; thus the ultimate line PP' may be, by suitable approach of (x) to (a), any line through (a) lying in a certain prime, and there is, in general, one such ultimate PP' meeting an arbitrary line of the space $[r]$. The equations (I) are only a first approximation; the more accurate statement is that the ultimate lines PP' lie on a cone, with vertex at (a), of dimension $r-1$, having a definite number of intersections with an arbitrary line of the space $[r]$. The symbol $\epsilon(1, r)(r-1)$ thus represents a definite number when the complete equations representing the correspondence are algebraic. Next

(3), if every first minor of the determinant Δ be zero, then the equations $f_i(x_1, \ldots, x_r)-x_i=0$ determine an ∞^2 of points of coincidence in the neighbourhood of (a), say a surface; and the equations (I) imply two necessary linear relations connecting $x_1'-x_1, \ldots, x_r'-x_r$; in this case, the ultimate lines PP' are any lines through (a) in a space $[r-2]$. Or, more generally, considering the complete equations to which the equations (I) are an approximation, these lines PP' are the generators of a conical locus, of dimension $r-2$, having a finite number of intersections with an arbitrary plane of the space $[r]$; and the symbol $\epsilon(2, r)(r-2)$ has a definite value, when the correspondence is algebraic.

And so in general, the term $\epsilon(i, r)(r-i)$ arises from points of coincidence lying on one or more manifolds, M_i, of dimension i, through any point of which the ultimate lines PP' are the generators of a cone of dimension $r-i$. For instance, when $r=3$, we have the cases, (1), of discrete coincidences, the lines PP' being all the lines through the point of coincidence; (2), of curves of points of coincidence, the ultimate lines PP' being the generators of a conical sheet, with vertex at the point of coincidence; and (3), of surfaces of points of coincidence, there being a finite number of ultimate positions PP' through a point of coincidence.

The reasonableness of the formula being thus made clear, we pass to a detailed proof. For the cases $r=2$ and $r=3$ this can be given without much difficulty so as to explain also Schubert's symbolism, as we first shew (Schubert, *Kalkül*, 1879, p. 45; see also *Math. Ann.* x, 1876, p. 54). But some readers may prefer to pass at once to the less symbolical general proof given later (pp. 99 ff.), which includes the cases $r=2$, 3.

Supposing the corresponding points P, P' to lie in a space [3], associate these points, respectively, with the letters p and q; the primary meaning of these letters is in equations of condition, in which they signify the condition of lying in an arbitrary plane, imposed on P or P' respectively; but here p denotes the number of corresponding pairs, P, P' whereof P lies in an arbitrary plane, with a corresponding meaning for q. Similarly let g denote the condition that a specified line, g, should meet an arbitrary line; in our case the specified line is the join, PP', of the corresponding points P and P'. Further let ϵ denote a coincidence, in our case of the corresponding points P, P'. We then postulate the symbolic equation $\epsilon=p+q-g$.

This is justified by first supposing that limitations equivalent to ∞^2 elementary conditions are imposed, so as to reduce the aggregate of pairs of points P, P' to ∞^1; and then considering the correspondence of a pair of planes, of a pencil of planes, with a line g as axis, of which one plane contains P and the other P'. In each plane of the pencil, in accordance with the definition of P, lie p points P, each of which gives a point P', and hence a corresponding plane; while, reversely, in such a plane, lie q points P', each of which gives a point P, and hence a corresponding plane. There is thus a correspondence, in the planes of the pencil, of indices (q, p), and there are $q+p$ coincidences of pairs of corresponding planes. These arise, however, from ϵ coincidences of the points P, P'; and from the g cases in which the line PP' intersects the axis of the pencil. Thus $\epsilon+g=q+p$, which is the symbolic equation put down. It will be seen that this equation, modified by such multiplications, indicating further conditions, as render the terms capable of concrete interpretation, is the basis of Schubert's method. We also employ the symbolic equations of condition $g^2=g_p+g_e$, $pg=p^2+g_e$, which introduce two fresh symbols, g_p, the condition that a line g should contain an arbitrary point, and g_e, the condition that a line g should lie in an arbitrary plane. The former of these equations is one we have remarked upon above (p. 78), expressing that the condition, for a line g, of meeting two arbitrary lines, is compounded of the alternative conditions, of passing through an arbitrary point and lying in an arbitrary plane; it may be looked upon as a particular case of a relation (p. 83) for lines in space $[r]$ expressible by

$$(r-2, r)(p, q)=(p-1, q)+(p, q-1).$$

The second of these equations is an equation of condition for the composite entity consisting of a line (g) with a distinguishable point (p) thereon. The left side is the condition that the point lies on a given arbitrary plane, while the line meets a given arbitrary line; in a notation already employed (p. 93) it may be denoted by $(2)(1, 3)$. On the right side, the term p^2 is the condition for the point to lie in two arbitrary planes, that is, to lie on an arbitrary line; supplying a unitary factor $(2, 3)$ as condition for the line g, the condition may be denoted also by $(1)(2, 3)$. The term g_e is the condition for the line g to lie in a given plane, and may be denoted by $(1, 2)$, to which, if desired, may be supplied the unitary factor (3) for the point p. The whole equation may thus be denoted by $(2)(1, 3)=(1)(2, 3)+(1, 2)$. As the equation is to be applied to cases in which, by imposing complementary conditions, it becomes an equation in the numbers of the composite entity (g, p) which satisfy the conditions, it may be justified, as explained before (p. 78), by taking the case when the arbitrary plane, in which p is to lie, contains the arbitrary line which g is to meet. The composite condition can then obviously be satisfied in two ways; either, by the point p, of the line g, lying on the arbitrary line (g not necessarily being in the plane); this corresponds to the term p^2 on the right: or, by the line g lying in the arbitrary plane, in which case the point p lies in this plane; this corresponds to the term g_e on the right. It may be similarly proved that, for a line in space $[r]$, associated with a distinguishable point thereon, there is an equation

$$(a, r)(b)=(b, r)(a)+(a, b),$$

where $a<b$; which may be proved by considering the particular case when the space $[a]$ lies in the space $[b]$. Assuming then

$$\epsilon=p+q-g,\ g^2=g_p+g_e,\ pg=p^2+g_e, \text{ and hence } qg=q^2+g_e,$$

the first gives, using the others,

$$\epsilon g=pg+qg-g^2=(p^2+g_e)+(q^2+g_e)-(g_p+g_e)=p^2+q^2+g_e-g_p,$$

and also gives

$$\epsilon p = p^2 + pq - pg = p^2 + pq - (p^2 + g_e) = pq - g_e;$$

hence

$$\epsilon g + \epsilon p = p^2 + pq + q^2 - g_p, \text{ or } p^2 + pq + q^2 = (g_p + \epsilon g) + \epsilon p.$$

This equation leads to the equation for the coincidences of a correspondence of ∞^2 pairs of points P, P' in a plane ϖ, enunciated above. For first, p^2, the condition that P lies in two arbitrary planes, and therefore in their line of intersection, is the condition for P to be at the point where this line meets the plane ϖ; thus p^2 is what was before denoted by $(0)(2)'$. And similar remarks hold for the other two terms, pq which is $(1)(1)'$, and q^2 which is $(2)(0)'$. Again, g_p, expressing that the line PP', in the plane ϖ, passes through an arbitrary point, not necessarily in the plane ϖ, is zero; while ϵg, expressing a coincidence in which (the limiting position of) the line PP' meets an arbitrary line, that is, passes through the point in which this line meets the plane ϖ, will be the condition we have denoted by $\epsilon(0, 2)(2)$; and similarly ϵp will be $\epsilon(1, 2)(1)$.

But we can proceed further in the same way, and obtain the equation for the coincidences of a correspondence of ∞^3 pairs of points P, P' in space. It is convenient to introduce another symbol, g_s, the condition for a line g both to pass through a given point and to lie in a given plane (to be a line of a given flat pencil); evidently we have $gg_e = g_s = gg_p$. Then, from $pg = p^2 + g_e$, $g^2 = g_e + g_p$, we have

$$p^2 g = p^3 + pg_e, \quad pg^2 = p^2 g + g_e g, = p^2 g + g_s, \quad pg^2 = pg_e + pg_p,$$

and, hence, $\quad p^3 + pg_e + g_s = p^2 g + g_s = pg^2 = pg_p + pg_e,$

so that $pg_p = p^3 + g_s$, and $qg_p = q^3 + g_s$; but, we found above $\epsilon p = pq - g_e$, and we have $\epsilon = p + q - g$; thus

$$\epsilon g_p = pg_p + qg_p - gg_p = (p^3 + g_s) + (q^3 + g_s) - g_s = p^3 + q^3 + g_s,$$

as also

$$\epsilon g_e = pg_e + qg_e - gg_e, \quad \epsilon p^2 = (pq - g_e)\,p = p^2 q - pg_e, \quad \epsilon q^2 = pq^2 - qg_e;$$

so that

$$\epsilon g_p + \epsilon g_e + \epsilon p^2 + \epsilon q^2 = (p^3 + q^3 + g_s) + (pg_e + qg_e - g_s) \\ + (p^2 q - pg_e) + (pq^2 - qg_e),$$

and hence

$$p^3 + p^2 q + pq^2 + q^3 = \epsilon g_p + \epsilon p^2 + \epsilon(q^2 + g_e) = \epsilon g_p + \epsilon qg + \epsilon p^2,$$

if we recall that $qg = q^2 + g_e$. Here ϵg_p is $\epsilon(0, 3)(3)$, while ϵqg is $\epsilon(1, 3)(2)$, and ϵp^2 is $\epsilon(2, 3)(1)$, the factor $(2, 3)$ being supplied *pro forma*. The equation is therefore exactly that enunciated,

$$\sum_{i=0}^{3} (i)(3-i)' = \sum_{i=0}^{2} \epsilon(i, 3)(3-i).$$

We now give a less symbolic proof, by induction, of the equation for the coincidences in the general case of a correspondence of ∞^r pairs of corresponding points P, P' (Pieri, *Rend. Palermo*, v, 1891, p. 252; see also Pieri, *Rend. Ist. Lombardo*, XXVI, 1893; XXVII, 1894; XXVIII, 1895). The proof is to be built up from the two formulae

$$\sum_{i=0}^{r-1}(i)(r-i)'-(0, r)(r-1)=\sum_{i=1}^{r-1}\epsilon(i, r)(r-i), \qquad (a),$$

$$(r)(0)'+(0, r)(r-1)=\epsilon(0, r)(r), \qquad (b),$$

which, by addition, give the desired formula

$$\sum_{i=0}^{r}(i)(r-i)'=\sum_{i=0}^{r-1}\epsilon(i, r)(r-i), \qquad (c).$$

The equation (c) holds for $r=1$, becoming then Chasles' fundamental formula, $(0)+(0)'=\epsilon$, in the form $(0)(1)'+(1)(0)'=\epsilon(0,1)(1)$. We thus assume ($c$) to hold when $r-1$ is put for r; from this the equation (a) is deduced. The equation (b) is capable of direct proof, without induction, provided $r>1$; for $r=1$ it becomes the Chasles' formula $(0)'+(0)=\epsilon$.

We first give the direct proof of (b), with $r>1$. By considering a pencil of primes in the space $[r]$, all through a base space $[r-2]$, and considering two primes which join this base to corresponding points P, P' as corresponding primes, we deduce the fundamental symbolic equation of condition $(r-1)+(r-1)'-(r-2, r)=\epsilon$, where $(r-1)$ is the condition for P to lie in a prime, $(r-1)'$ the same condition for P', $(r-2, r)$ is the condition for the line PP' to meet a space $[r-2]$, and ϵ is the condition for coincidence of P and P'. This is in fact Schubert's fundamental equation, discussed above (p. 96). From this equation we have

$$(0, r)(r-1)+(0, r)(r-1)'-(r-2, r)(0, r)=\epsilon(0, r)(r), \quad (d).$$

We combine this with the equation

$$(0, r)(r-1)'-(0, r-1)=(r)(0)', \qquad (e),$$

which we now prove. Consider, of the ∞^r couples of corresponding points P, P', those for which the line PP' passes through an arbitrary point; this is an $(r-1)$-fold condition, and there will be ∞^1 couples P, P' subject to this condition, namely P' (as well as P) will describe a curve. The correspondence being, as is assumed here throughout, of algebraic character, this curve will be algebraic. Its order will be the number of points P' lying in an arbitrary prime when the line PP' passes through an arbitrary point, namely $(0, r)(r-1)'$. This curve locus of P' will pass through the arbitrary fixed point as many times as there are pairs, of the ∞^r original pairs P, P', for which P' is at this arbitrary point; namely the curve has a multiple point of order $(r)(0)'$ at this arbitrary point. The order of the curve is equal to this multiplicity together with the number of intersections of the curve with an arbitrary prime through this point, which is $(0, r-1)$. Thus we have

$$(0, r)(r-1)' = (r)(0)' + (0, r-1);$$

which is the equation (e). From this and (d), since

$$(0, r-1) = (r-2, r)(0, r),$$

equation (b) follows.

Assuming this, we can prove that the equation (a), as written, follows from the hypothesis that this equation holds when $r-1$ is put for r. Take a fixed point O, and a fixed prime Ω (not containing O); and, from the ∞^r original pairs of corresponding points P, P', consider the ∞^{r-1} pairs in which P lies in Ω; projecting each of the points P', which correspond to such points P, from O, into points P'' which lie in Ω, we have a correspondence of ∞^{r-1} pairs P, P'' in Ω. For this correspondence, from the hypothesis made for the equation (a), we have

$$\sum_{j=0}^{r-1} (j)(r-1-j)'' = \sum_{j=0}^{r-2} \epsilon'(j, r-1)(r-1-j),$$

where $(r-1-j)''$ refers to a position of P'' in a space $[r-1-j]$ lying in Ω, and ϵ' refers to a coincidence of P and P'', which lies in a $[r-1-j]$ of Ω, the ultimate line PP'' meeting a $[j]$ of Ω. But, the point P'' will lie in a $[r-1-j]$ of Ω, if and only if P' lie in the $[r-j]$ which joins the $[r-1-j]$ to O. The left side of this equation is therefore the same as the terms $\sum_{i=0}^{r-1} (i)(r-i)'$ occurring on the left side of the equation (a). The right side of this equation has, for first term, $\epsilon'(0, r-1)(r-1)$, referring to coincidences of P and P'' in which the ultimate PP'' has an arbitrary direction in Ω. Such a coincidence arises when the point P', corresponding to P, is itself in Ω and coincides with P, in such a way that the ultimate line PP' can be regarded as meeting an arbitrary line (through O); such a coincidence also arises when P and P', not coinciding, are in a line through O. We may thus replace the term $\epsilon'(0, r-1)(r-1)$ by $\epsilon(1, r)(r-1) + (0, r)(r-1)$, referring to the original ∞^r correspondence. For $j > 0$, however, the term $\epsilon'(j, r-1)(r-1-j)$ of the right side, or say the term $\epsilon'(i-1, r-1)(r-i)$, where $i = j+1$, so that $i > 1$, is the same as the term $\epsilon(i, r)(r-i)$ applied to the original correspondence in $[r]$. The right side of this equation is thus $(0, r)(r-1) + \sum_{i=1}^{r-1} \epsilon(i, r)(r-i)$, applied to the original correspondence. Thus the equation (a) is established.

Thus, finally, the desired formula (c) is obtained.

Ex. 1. For the composite entity consisting of a line with two distinguishable points thereon, we have used the unsymmetric equation (*b*). Prove also the symmetric equation

$$(0)(r)'+(r)(0)'+(0, r-1)=\epsilon(0, r)(r),$$

and the unsymmetric generalisation of (*b*)

$$(r-i)(i)'+(i, r)(r-i-1)-(i-1, r)(r-i)=\epsilon(i, r)(r-i).$$

Ex. 2. For a correspondence of ∞^2 pairs of corresponding points in a plane, prove that $(1)(1)'$ is equal to the number of pairs of corresponding, but not coinciding, points upon an arbitrary line, together with the number, $\epsilon(1, 2)(1)$, of pairs of coinciding corresponding points upon the line.

Ex. 3. We have remarked that, in the preceding theory, it is supposed that the corresponding points P, P' are distinguishable from one another, by their geometrical construction or otherwise. Consider now a correspondence of points P, P', in a plane, which are harmonic conjugates of one another, in regard to a fixed point O, and a fixed line l, to examine the application of the formulae given, to this involutory case. The points P, P' are in line with O, and l is a range of self-corresponding points at each of which the ultimate line PP' has a definite direction, passing through O. In this case, in the formula $\epsilon g=p^2+q^2+g_e-g_p$, of p. 97, we have $p^2=q^2=1$, $g_e=g_p=0$, and hence ϵg, which we denote by $\epsilon(0, 2)(2)$, becomes equal to 2. Similarly, the formula (*d*), of p. 99, for $r=2$, namely $(0, 2)(1)+(0, 2)(1)'-(0, 1)=\epsilon(0, 2)(2)$, putting $(0, 1)=0$, gives again $\epsilon(0, 2)(2)=2$. The general formula of coincidence

$$(0)(2)'+(1)(1)'+(2)(0)'=\epsilon(0, 2)(2)+\epsilon(0, 1)(1)$$

is then valid, in the form $1+1+1=2+1$. For a general homographic transformation in a plane, it is valid, because $\epsilon(0, 2)(2)=3$ and $\epsilon(0, 1)(1)=0$.

Ex. 4. Another involutory correspondence in a plane is that of the pairs of points in which the plane is met by the two inflexional tangents at a point of a surface, say of order n. For the indices of the correspondence we have $(0)(2)'=(2)(0)'=n(n-1)(n-2)$, a point P of the plane, not on the surface, being on an inflexional tangent at a point X, of the surface, if X lie on the first and second polars of P. It has been remarked (p. 91) that the ruled surface formed by the inflexional tangents at the parabolic points of the surface is of order $2n(n-2)(3n-4)$. Hence, taking account of the involutory character of the correspondence, we infer $\epsilon(1, 2)(1)=2n(n-2)(3n-4)+2n$; also, the number of pairs of inflexional tangents which meet a line is $n(n-1)^2$, and each gives two pairs of corresponding points on this line. Hence, by Ex. 2 above, we have, for the correspondence in the plane,

$$(1)(1)'=2n(n-1)^2+2n(n-2)(3n-4)+2n, \quad =4n(2n^2-6n+5).$$

Ex. 5. As an application of Ex. 4, the number of points of the surface, for which the inflexional tangents meet two given arbitrary lines in a plane—either by one inflexional tangent of the pair meeting each of the two lines, or by both meeting the same—is

$$4n(2n^2-6n+5)+n(n-1)^2+n(n-1)^2, \quad =2n(5n^2-14n+11).$$

Again, the number of points of the surface for which the inflexional tangents both meet the same repeated line of the plane, is

$$2n(n-2)(3n-4)+2n+4n(n-1)^2, \quad =2n(5n^2-14n+11).$$

If we regard the pair of lines in the plane, or the repeated line, as alternative degenerations of the absolute conic of the plane, either of these formulae gives the number of umbilici of the general surface of order n. Cf. Schubert, *Kalkül*, p. 244; and Voss, *Math. Ann.* IX, 1876, p. 241.

Ex. 6. As an example of Schubert's symbolism, the following determination of the number of torsal lines of a ruled surface may be given (Schubert, *Kalkül*, p. 60, No. 22; see also p. 196, F. 1. And Chap. I above, p. 26). Of the ∞^1 generators consider a correspondence between two generators g and h; let g also denote the number of corresponding pairs in which g meets an arbitrary line, with a similar meaning for h; β denote the condition that g and h meet an arbitrary plane in points of a ray of an arbitrary flat pencil of rays in that plane, and ϵ the condition that g and h coincide, being "consecutive". Then, by considering a pencil of rays in a plane, we have the equation of condition $g+h=\epsilon+\beta$. If also σ be the condition that two generators g, h intersect without coinciding, we again have, considering an arbitrary plane, $\sigma+\epsilon=\beta$. Thus $\sigma=2\beta-g-h$. We have however $\epsilon g, =\epsilon h, =$order of the ruled surface; also $\epsilon\beta$ is the class of a plane section of the surface. Thus $\epsilon\sigma$, which is the number of torsal generators, being $2\epsilon\beta-\epsilon g-\epsilon h$, or $2\epsilon\beta-2\epsilon g$, is equal to $2(2n+2p-2-n)$, if n be the order of the surface, and p the genus of a plane section.

Ex. 7. The following example of the general formula gives what we may describe as the point-equivalence of an unusual intersection of two manifolds in space $[r]$, whose usual intersection consists of points. Let $M_s{}^m$ and $N_t{}^n$ denote two algebraic manifolds in space $[r]$, of orders m and n, whose dimensions s and t are complementary, namely $s+t=r$; suppose that these have, as part intersection, a manifold F_k, of dimension k, $(k>0)$, at every point of which $M_s{}^m$ and $N_t{}^n$ have definite distinct tangent spaces, $[s]$ and $[t]$; but otherwise meet only in points. We prove that the number, ξ, of these points is

$$\xi=mn-\sum_{i=0}^{k}\epsilon(i,\,k+1,\,k+2,\,\ldots,\,r)(r-i),\ \text{or}\ mn-\sum_{i=0}^{k}\epsilon\{k-i,\,0^{r-k}\}(r-i),$$

where $\epsilon\{k-i, 0^{r-k}\}(r-i)$ denotes the number of points of the section of F_k by a space $[r-i]$, at which the space $[r-k]$, or $[s+t-k]$, containing the tangent spaces of M and N (which both pass through the tangent space $[k]$ of F), meets an arbitrary space $[i]$. In space $[r]$, this is equivalent to $r-(r-k)-i$, or $k-i$, elementary conditions, while the condition for the point to lie in a space $[r-i]$ is equivalent to i elementary conditions; there will then presumably be a finite number of such points existing on the F_k. The symbol $\epsilon\{k-i, 0^{r-k}\}(r-i)$, relating to the join, $[r-k]$, of the tangent spaces $[r]$, $[s]$, is different from the grade $\omega_{k-i}=\epsilon\{k-i, 0^k\}(r-i)$, defined above (p. 73), relating to the meet $[k]$, tangent to F_k, of these tangent spaces $[r]$, $[s]$.

The formula is a direct application of the equation

$$\Sigma(i)(r-i)'=\Sigma\epsilon(i,\,r)(r-i),$$

applied to the correspondence of points P, P', in which P is any point of $M_s{}^m$, and P' is any point of $N_t{}^n$; as, to any point P, of M, there corresponds any one of the ∞^s points P' of N, and to any point P' of N correspond ∞^r positions of P, anywhere on M, there are in all ∞^{s+t} or ∞^r couples of corresponding points. By the same condition, however, as there is no position of P at an arbitrary point of the space $[r]$, the term $(0)(r)'$ is zero; and so, likewise, for the same reason, are all the terms $(1)(r-1)'$, $(2)(r-2)'$, ... which precede the term $(r-s)(r-t)'$ on the left side of the equation; this term is however equal to mn, there being m points P of the manifold M_s^m lying on an arbitrary space $[r-s]$, and, corresponding to each, n points P' of the N_t^n lying on an arbitrary space $[r-t]$; as spaces $[r-i]$, with $i>t$, do not intersect N_t^n, the succeeding terms, of the series on the left of the equation, all vanish. Now consider the right side of the equation. Let ξ be the number of points common to M and N which do not lie on F; such a point is a coincidence of the correspondence; and, we assume, the points P, P', of M and N, can be made to approach this intersection so that the ultimate line PP' is any line through this; there is thus such a line through an arbitrary point of the space $[r]$. Wherefore, these intersections furnish a contribution ξ to the first term, $\epsilon(0, r)(r)$, on the right of the equation. The other coincidences of the correspondence are the common points of M and N which lie on F; at such a point, we assume, the ultimate line PP' is any line, through the point, which lies in the space $[r-k]$ which is the join of the tangent spaces of M and N, at this point. Just as, for example, if P, P' be points of two curves which intersect at O, the line PP', when P and P' approach to O, is any line in the plane containing the tangent lines of the curves at O. The number of such ultimate lines PP' which intersect an arbitrary space $[i]$ is therefore the number of such joining spaces $[r-k]$ which intersect this space $[i]$. When $i>k$, the space $[i]$ being arbitrary, this number is zero; but for less values of i, the terms $\Sigma\epsilon(i, r)(r-i)$, of the correspondence equation, for coincidences lying on F, furnish the terms $\sum_{i=0}^{k} \epsilon\{k-i, 0^{r-k}\}(r-i)$. These terms together then give the point-equivalence of the intersection F, of M and N, whose remaining intersections have the number ξ given by the formula.

Ex. 8. As an illustration of the formula $(a, r)(b)-(b, r)(a)=(a, b)$, for a line on which is a distinguishable point, which has been quoted (p. 97), the following may be taken. Suppose that we have an algebraic system of manifolds M_k, of dimension k, in space $[r]$, the system being of aggregate ∞^ρ. Further let a, b be such that $0 \leqslant a < b \leqslant r$, and $a+b=2(r-k)-\rho$. Consider the aggregate of all possible tangent lines of all these manifolds; on each M_k are ∞^k points, at each of which (we assume) are ∞^{k-1} tangent lines; the aggregate of all these tangent lines is thus $\infty^{2k-1+\rho}$. The condition (a, b), or $\{r-1-a, r-b\}$, for a line is of power $2r-1-a-b$, and this, by the assumed relation, is $2k-1+\rho$. Of all the tangent lines, of the system of manifolds, there is thus a finite number, which satisfy the condition (a, b), of lying in an arbitrary $[b]$ and meeting an arbitrary $[a]$ therein. The number $(a, r)(b)$ is that of points, on the manifold M_k, where these are intersected by a space $[b]$, at which the tangent lines meet a space $[a]$; with a similar meaning for $(b, r)(a)$.

For instance, for $r=3$, $k=1$, $a=1$, $b=2$, taking an algebraic system of ∞^1 curves in ordinary space, the number (a, b) is that of these curves which touch an arbitrary plane; the number $(1, 3)(2)$ is the order of the ruled surface generated by the tangents of these curves at points where the curves meet an arbitrary plane; and the number $(2, 3)(1)$, in which the symbol $(2, 3)$ is unitary, is the number of curves of the system which meet an arbitrary line.

This particular case, for ordinary space, is remarked by Schubert (*Kalkül*, p. 27). The general formulation was made by Pieri.

Ex. 9. A simple application of the principles here developed is to the normals of a surface in ordinary space. Suppose this is of order n, of rank r, and of class n'. Let ϖ denote the plane of the Absolute conic; let P' be the pole, in regard to the Absolute conic, of the line in which the plane ϖ is met by the tangent plane at a point P of the surface. We define the normal at P as the line PP'. It may easily be seen that, as P describes a plane section of the surface, P' describes a curve in ϖ of order r; and that the points P, P' are then in (1, 1) correspondence. Thus the normals of the surface at the points of a plane section generate a ruled surface of order $r+n$; and there are r normals of the surface in a general plane. But, in particular, there are ∞ normals in the plane ϖ, enveloping a curve of class $r+n$. To find how many normals of the surface pass through an arbitrary point, O, of the space, join O to any point P of the surface, and let the joining line meet the plane ϖ in Q; as before let P' be the point of the plane ϖ determined from P. Consider the correspondence of ∞^2 pairs Q, P' in the plane ϖ; the normals to the surface from O correspond to the coincidences of this correspondence. In the general case, these are finite in number, and given by the formula $(0)(2)'+(1)(1)'+(2)(0)'$, where (0), (1), (2) refer to Q and $(2)'$, $(1)'$, $(0)'$ refer to P'. The terms herein are easily seen to be n, r and n', respectively; so that the number of normals of the surface from O is $n+r+n'$.

We may also find the number of lines normal to two given general surfaces. For the order and class of the congruences of normals of these two surfaces, respectively, are given by $(n_1+r_1+n_1', r_1)$ and $(n_2+r_2+n_2', r_2)$; while there is a curve of class n_1+r_1 in the plane ϖ, whose tangents satisfy the definition of a normal of the first surface, and a similar curve of class n_2+r_2 for the second surface. The number of normals common to the two surfaces is, therefore,

$$(n_1+r_1+n_1')(n_2+r_2+n_2')+r_1r_2-(n_1+r_1)(n_2+r_2),$$

or

$$(n_1+r_1)n_2'+(n_2+r_2)n_1'+r_1r_2+n_1'n_2'.$$

Schubert obtains the number, r, of normals of a surface which lie in an arbitrary plane, by supposing the Absolute conic, defined tangentially, to degenerate into two pencils of lines, with centres I, J, together with the joining line IJ, say g, counted doubly. Then the normals of the surface, lying in an arbitrary plane, are assumed to be the same in number as those in a plane which passes through the point I; and these consist of the tangent lines of the surface, in this plane, which pass through I, which are r in number. Again, the normals of the surface from an arbitrary point are assumed to be the same in number as those passing through the point I; and these consist, first, of n lines, all coinciding with g, one at each intersection of the surface with this line g; then, second, of r tangents to the surface from I, in an arbitrary plane through I; and, then, lastly, of n', arising from points of contact of tangent planes of the surface drawn through the line g. See *Kalkül*, p. 16, where the reference for the results is to Sturm, *Math. Ann.* VII, 1873, p. 567. Schubert also considers the Absolute conic, defined punctually, as degenerating into two lines (*loc. cit.* p. 244).

Ex. 10. For two general primals, of orders n, n', in Euclidean space of r dimensions, the number of common normals (defined, as above, by an Absolute quadric in a particular prime space $[r-1]$) is given by Pieri (*Rend. Palermo*, V, 1891, p. 323), with reference to Fouret (*Bull. d. l. Soc. Math. d. France*, VI, 1878, p. 43) in the form

$$\sum_{k=1}^{r}\sum_{j=0}^{r-k}\rho_{k+j}\rho'_{r-j},$$

where ρ_i is the rank, defined on p. 73 above, given by

$$\epsilon\{1^i, 0^{r-i}\}(i+1), \; = n(n-1)^{i-1}$$

in general, for $i=1, \ldots, r$. For instance, when $r=4$, the number is

$$(\rho_1\rho_4'+\rho_2\rho_3'+\rho_3\rho_2'+\rho_4\rho_1')+(\rho_2\rho_4'+\rho_3\rho_3'+\rho_4\rho_2')+(\rho_3\rho_4'+\rho_4\rho_3')+\rho_4\rho_4'.$$

Putting $\rho_1'=\rho_2'=\ldots=\rho'_{r-1}=0$, $\rho_r'=1$, the number of normals of a single primal which pass through an arbitrary point, is $\rho_1+\rho_2+\ldots+\rho_r$.

Part IV. Pairs of corresponding spaces $[k]$, in space $[r]$. If, in space $[r]$, we have ∞^e pairs of corresponding spaces $[k]$, where $e. =(r-k)(k+1)$, is the freedom of such spaces, then, in case the coincidences of two corresponding spaces are only finite in number, this number is $\Sigma(a_0, \ldots, a_k)(r-a_k, \ldots, r-a_0)'$, where the two Schubert symbols refer to the two spaces of a pair, respectively, and the summation extends over all sets of integers $a_0, \ldots, a_k$, subject to $0 \leqslant a_0 < a_1 < \ldots < a_k \leqslant r$.

This summation, as we have remarked (p. 72), contains, therefore, in general $(r+1, k+1)$ terms. For $k=0$, the sum is that, $\Sigma(i)(r-i)$, obtained for ∞^r pairs of points. For ∞^2 pairs of lines in a plane, the number of coincidences, when finite, is

$$(0, 1)(1, 2)'+(0, 2)(0, 2)'+(1, 2)(0, 1)', \text{ or } l+cc'+l',$$

where l is the number of the first elements (lines) of a pair, which coincide with an arbitrary line, c is the number which pass through an arbitrary point, and l', c' have the same meanings for the second elements of a pair; a simple application is to the case when the ∞^2 lines are the tangents of two curves in the plane, and any tangent of one curve corresponds to every tangent of the other. For an ∞^4 of pairs of lines in ordinary space [3], the number of coincidences is

$$(0, 1)(2, 3)'+(0, 2)(1, 3)'+(0, 3)(0, 3)'+(1, 2)(1, 2)'+(1, 3)(0, 2)' +(2, 3)(0, 1)';$$

for instance, if we consider two congruences of lines, and any line of one of these congruences is regarded as corresponding to every line of the other, the formula reduces to Halphen's well-known result $(0, 3)(0, 3)'+(1, 2)(1, 2)'$, for the number of lines common to the congruences; or, as another instance, for the number of lines common to a ruled surface and a complex of lines, we have $(1, 3)(0, 2)'$, the product of the orders of the ruled surface and the complex. For an ∞^3 of pairs of corresponding planes in ordinary space, the formula gives

$$(0, 1, 2)(1, 2, 3)'+(0, 1, 3)(0, 2, 3)'+(0, 2, 3)(0, 1, 3)' +(1, 2, 3)(0, 1, 2)';$$

for instance, there are ∞^1 planes tangent to a cone, and ∞^2 planes passing through the tangent lines of a curve; the formula then gives the number of tangent lines of the curve which touch the cone; and, it reduces to $(0, 2, 3)(0, 1, 3)'$, the product of the class of the cone and the rank of the curve.

The general formula above was enunciated and proved by Severi, *Rend. Lincei*, IX, 1900, p. 321. In what follows we shall illustrate this proof by carrying it out when $r=3$, for the case $k=1$. For the case of lines see also Pieri, *Atti...Torino*, XXV, 1890, p. 365. For the cases when the pairs of corresponding elements are those of a manifold ∞^s, each taken with those of a manifold ∞^t, with $s+t=r$ (as in all the examples given above), see Schubert, *Mittheil. d. math. Ges. Hamburg*, I, 1886, p. 134.

In general the proof of the formula is by induction; being supposed to be proved for space $[r-1]$, it is deduced for space $[r]$. As we have proved it to hold for points in space [2], and can hence deduce it, by duality, for lines in space [2], this is sufficient. And in proceeding now to prove the theorem for lines in space [3], we can assume it true in space [2].

Consider, in ordinary space [3], an aggregate, Γ, of ∞^4 pairs of corresponding lines. From this aggregate we select an aggregate, γ, of ∞^1 pairs of corresponding lines, so chosen that all the coinciding pairs of the aggregate Γ are equally coincidences in γ; to find the number of coincidences in Γ, it is thus sufficient to find the coincidences in γ. The deduction of γ from Γ is as follows: Take an arbitrary point P, and an arbitrary plane ϖ; for a pair of lines of Γ to intersect is a one-fold condition; for the point of intersection to lie on the plane ϖ is a one-fold condition; for the plane of the lines to pass through P is a further one-fold condition. The aggregate γ consists of the ∞^1 pairs from Γ which satisfy these three conditions. Evidently any pair of coinciding lines of Γ is a pair of coinciding lines of γ. In the aggregate γ, the point of intersection of a pair of the lines describes a curve in the plane ϖ, whose order we denote by m; and the common plane of a pair envelops a cone, of vertex P, whose class we denote by μ. The two lines of a pair, in the aggregate γ, are (as in all the cases discussed in this chapter) supposed to be distinguishable; we denote them, respectively, by l and l'. Now we take a general plane α, and therein a point A, and determine a correspondence of the rays of the flat pencil (α, A); namely, by the condition that two such rays correspond when the first ray meets a line l of the aggregate γ, and the second ray meets a corresponding line l'. As it is a one-fold condition for a line to meet a given line, there will, of the aggregate γ, be a finite number of pairs for which the line l meets a given ray of the pencil (α, A), say y pairs; and as it is a two-fold condition for a line to lie in a given plane, the lines l' of these y pairs will meet the plane α in points, and so determine y rays of the pencil (α, A) corresponding to the first given ray. Similarly, a second given ray of the pencil is met by a definite number, say y', of lines l' of the aggregate γ, and the corresponding lines l determine y' rays of the pencil, of the first kind. There is thus a (y', y) correspondence of rays of the pencil;

and therefore there are $y+y'$ rays of coincidence in the pencil, each met by both the corresponding lines l, l' of a pair of the aggregate γ. These $y+y'$ coincidences in the pencil (α, A) may arise in three ways: (1), through a coincidence of the lines l, l' of γ; (2), because the point of intersection of l and l' is on the plane α—that is, at one of the m points where the plane α meets the curve, of the plane ϖ, which is the locus of such intersections; (3), because the pair l, l' meet the plane α in points lying in line with A, that is, because the plane containing l and l' passes through A—namely, when this plane is one of the μ tangent planes from A to the cone of vertex P enveloped by the planes (l, l'). The number of coincidences of lines of a pair l, l', of the aggregate γ, is thus $y+y'-m-\mu$.

Now the number y was that of the lines l, of a pair (l, l') of γ, which meet an arbitrary line. By considering the case when this line lies in the plane ϖ, whereby in general y is unaltered, we can find an expression for $y-m$. First, consider the original aggregate Γ of ∞^4 pairs of lines l, l'; and then, of these, the ∞^2 pairs which satisfy the two-fold condition that the line l lies in the plane ϖ; project the lines l', of these ∞^2 pairs, from P, obtaining thence lines l'' in the plane ϖ. We thus have ∞^2 pairs of corresponding lines (l, l''), in the plane ϖ, of which every l'' arises from a line l', of the aggregate Γ, of which the corresponding line l lies in the plane ϖ. If, in the plane ϖ, the line l'' satisfy the Schubert condition represented by (b_0, b_1), the line l', from which it arises, satisfies, in the space [3], a condition (b_0+1, b_1+1), wherein, however, $0 \leqslant b_0 < b_1 \leqslant 2$. Assuming the formula under discussion to be proved for a plane, the number of coincidences of lines l, l'' in the plane ϖ, is given by $\Sigma(a_0, a_1)(2-a_1, 2-a_0)'$, with $0 \leqslant a_0 < a_1 \leqslant 2$. Wherefore, the number of pairs of lines l, l', of the aggregate Γ, which satisfy the conditions that l lies in the plane ϖ and its corresponding line l' lies with l in a plane through P, is given by $\Sigma(a_0, a_1)(3-a_1, 3-a_0)'$, with $0 \leqslant a_0 < a_1 \leqslant 2$. All these pairs (l, l') evidently belong to the aggregate γ, and are all the pairs of the aggregate γ of which l lies in ϖ. This being so, consider, as suggested, the number y, of lines l, of a pair (l, l') of the aggregate γ, which meet an arbitrary line, p, of the plane ϖ. These consist of, (1), those lying in the plane ϖ; their number has just been expressed; (2), those, not in this plane, whose intersection with the corresponding line l' is at one of the m points in which the line p meets the curve in the plane ϖ, which is the locus of intersections of lines l, l' of the aggregate γ. Thus we have $y = m + \Sigma(a_0, a_1)(3-a_1, 3-a_0)'$. A like argument applied to the lines l' of the aggregate γ leads to the equation

$$y' = m + \Sigma(a_0, a_1)'(3-a_1, 3-a_0),$$

with the same limitations for the values of a_0 and a_1.

We can now prove further that the order m of the curve in the plane ϖ, and the class μ of the cone with vertex at P, which arise from the definition of the aggregate γ, are such that

$$m-\mu=(0,3)(0,3)'-(1,2)(1,2)',$$

where the line symbols refer to the lines l, l' of the aggregate Γ. For first, m, by definition, is the number of points of intersection, of two corresponding lines l, l' of Γ lying in a plane through P, which lie in the line p of the plane ϖ. Consider now how this number is made up when P lies on this line p: the point, H, where l, l' meet, may lie on p at a point which does not coincide with P; then the plane of l and l', which contains P, will contain the line p. Let u be the number of such cases. But, the intersection H may fall at P; then the plane of l, l' need not contain the line p. As P is, essentially, an arbitrary point, the number of such cases is that denoted, for the lines l, l' of the aggregate Γ, by the symbol $(\bar{0},3)(\bar{0},3)'$, the notation $\bar{0}$ denoting that the same arbitrary point is used in both symbols. Thus we have $m=u+(\bar{0},3)(\bar{0},3)'$.

By a dual argument, in which the use of the plane ϖ and the point P is interchanged, we may similarly prove $\mu=u+(1,\bar{2})(1,\bar{2})'$. The cases for which, in considering m, we obtained the number u, are evidently self-dual in the sense indicated; and the dual of the possibility of both l and l' passing through P is that in which they both lie in ϖ.

Finally, then, the number $y+y'-m-\mu$, of coincidences of a corresponding pair (l, l') in the aggregate Γ, is expressed by

$$\Sigma(a_0,a_1)(3-a_1,3-a_0)'+\Sigma(3-a_1,3-a_0)(a_0,a_1)' \\ +(\bar{0},3)(\bar{0},3)'-(1,\bar{2})(1,\bar{2})',$$

subject to $0\leqslant a_0<a_1\leqslant 2$. If we suppose that when the line l, of a corresponding pair (l, l'), passes through an arbitrary point, the condition for l' to pass through the same point, remains two-fold, as for an arbitrary point, we may replace $(\bar{0},3)(\bar{0},3)'$ by $(0,3)(0,3)'$. With a similar hypothesis we may replace $(1,\bar{2})(1,\bar{2})'$ by $(1,2)(1,2)'$. It is then easy to see that the whole formula reduces to

$$\Sigma(a_0,a_1)(3-a_1,3-a_0)', \qquad 0\leqslant a_0<a_1\leqslant 3,$$

which we desired to obtain.

For the proof of the general case, for the ∞^e aggregate Γ of corresponding spaces (l, l'), of dimension k, where $e=(k+1)(r-k)$, in space $[r]$, we can similarly select, from Γ, an aggregate ∞^1, γ, constituted by the pairs (l, l') which meet in a space $[k-1]$, lying in a given prime ϖ, whose containing space $[k+1]$ passes through a given point P. And then the proof proceeds as here.

Appendix. Some enumerative formulae. Consider a matrix of $(p+1)$ rows and $(q+1)$ columns $(p\leqslant q)$, of which the element in the (i, j)th place is homogeneous in the $(r+1)$ coordinates of a space $[r]$, of order $r_i + c_j$. We consider the manifold given by the simultaneous vanishing of all determinants of the matrix which have $(c+1)$ rows and columns. For suitable c such a manifold exists, and is of dimension $r-(p-c+1)(q-c+1)$. To express the order of this manifold, let

$$(1+c_1t)\ldots(1+c_{q+1}t)=\sum_{i=0}^{q+1}\gamma_i t^i,\quad (1-r_1t)\ldots(1-r_{p+1}t)=\sum_{i=0}^{p+1}\rho_i t^i,$$

so that $\rho_1, \rho_2, \ldots$ are alternately negative and positive; form the determinant of $p+q-2c+2$ rows and columns

$$\left|\begin{array}{lllll}
1, & \rho_1, & \rho_2, & \ldots, & \rho_{p+q-2c+1} \\
0, & 1, & \rho_1, & \ldots, & \rho_{p+q-2c} \\
\multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\multicolumn{5}{l}{0,\ \ldots,\ 0,\ \ldots,\ 1,\ \ldots,\ \rho_{p-c+1}} \\
1, & \gamma_1, & \gamma_2, & \ldots, & \gamma_{p+q-2c+1} \\
0, & 1, & \gamma_1, & \ldots, & \gamma_{p+q-2c} \\
\multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\multicolumn{5}{l}{0,\ \ldots,\ 0,\ \ldots,\ 1,\ \ldots,\ \gamma_{q-c+1}}
\end{array}\right|,$$

in which there are $q+1-c$ rows containing symbols ρ, and $p+1-c$ rows containing symbols γ. The determinant is of a form familiar in considering the resultant of two polynomials (Vol. v, Chap. VIII, p. 202). The order of the manifold in question is given by the value of this determinant.

In particular, if $c=p$, this is a determinant of $q-p+2$ rows and columns, with only one row containing elements γ, the last of the rows containing elements ρ consisting of the two elements $1, \rho_1$. In this case the determinant is equal to the coefficient of t^{q-p+1} in the ascending expansion of

$$(1+\gamma_1 t+\ldots+\gamma_{q+1}t^{q+1})/(1+\rho_1 t+\ldots+\rho_{p+1}t^{p+1}).$$

In terms of the numbers r and c, associated with the columns and rows of the original matrix to specify the orders of its elements, this order of the manifold spoken of may be remembered in the abbreviated form $[(c)/(-r)]_{t^{q-p+1}}$.

If, as another particular case, we have $r_1=r_2=\ldots=r_{p+1}=0$, the order of an element of the original matrix depending then only on the column in which the element occurs, then the determinant giving the order of the manifold in question reduces to

$$\left|\begin{array}{lll}
\gamma_{q+1-c}, & \ldots, & \gamma_{q+1-c+p-c} \\
\gamma_{q-c}, & \ldots, & \gamma_{q+1-c+p-c-1} \\
\multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
\gamma_{q-p+1}, & \ldots, & \gamma_{q-c+1}
\end{array}\right|,$$

with $p-c+1$ rows and columns. If, still more particularly, we have also $c_1=c_2=\ldots=c_{q+1}=\mu$, so that every element of the original matrix is of order μ, then $\gamma_i=\mu^i(q+1,\,i)$, where $(q+1,\,i)$ is a binomial coefficient. In this case, putting m for $q+1$, and transposing rows and columns, the determinant which gives the order may be written

$$\mu^{(p-c+1)(q-c+1)}\begin{vmatrix} (m,c) & (m,c+1) & \ldots, & (m,p) \\ (m,c-1) & (m,c) & \ldots, & (m,p-1) \\ \cdot & \cdot & \ldots, & \cdot \\ (m,2c-p), & (m,2c-p+1), & \ldots, & (m,c) \end{vmatrix},$$

where each element is a binomial coefficient, to be replaced by zero when its second index is negative. It can however be proved that

$$\begin{vmatrix} (m,c) & ,(m,c+1) & ,\ldots, & (m,c+d) \\ (m,c-1), & (m,c) & ,\ldots, & (m,c+d-1) \\ \cdot & \cdot & \ldots, & \cdot \\ (m,c-d), & (m,c-d+1), & \ldots, & (m,c) \end{vmatrix} = (m,c)\frac{\{m+d,\,d\}}{\{c+d,d\}\{m-c+d,d\}},$$

in, which, if $((k))$ mean $1!2!\ldots k!$, the notation $\{n,\,d\}$ stands for $((n))/((d))\;((n-d))$; it is understood that $d\geqslant 0$ and $c+d<m$. For the case under consideration we have $d=p-c$; thus the required order of the manifold, given by the vanishing of all determinants of order $c+1$, in a matrix of type $p+1$, $q+1$, wherein each element is of order μ in the coordinates, is

$$\mu^{(p-c+1)(q-c+1)}(q+1,\,c)\,\frac{\{q+p+1-c,\,p-c\}}{\{p,\,p-c\}\{q+p+1-2c,\,p-c\}}.$$

This agrees with a result found by Segre, *Rend. Lincei*, IX, 1900, p. 253.

The preceding results assume the original matrix to be of general form. If this be square $(p=q)$, and be either symmetrical, or skew symmetrical, general formulae can also be given. In the symmetrical case, if the order of the (i,j)th element of the matrix, supposed to be of $p+1$ rows and columns, be r_i+r_j, wherein either $r_1,\,\ldots,\,r_{p+1}$ are all integers, or else $r_1-\frac{1}{2},\,\ldots,\,r_{p+1}-\frac{1}{2}$ are all integers, the dimension of the manifold upon which all determinants, in the matrix, of $c+1$ rows and columns, vanish, is $r-\frac{1}{2}(p-c+1)(p-c+2)$; and if we define $h_1,\,h_2,\,\ldots$ by

$$(1-r_1t)\ldots(1-r_{p+1}t)=\sum_{i=0} h_i t^i,$$

then the order of this manifold is

$$2^{p-c+1}\begin{vmatrix} h_1 & ,\,h_0, & 0, & 0, & 0, & \ldots\ldots \\ h_3 & ,\,h_2, & h_1, & h_0, & 0, & \ldots\ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{2p-2c+1}, & \cdots & \cdots & \cdots & , & h_{p-c+1} \end{vmatrix},$$

where the determinant has $p-c+1$ rows and columns. In particular, if every element of the original matrix be of order μ, and $r_1=r_2=\ldots=\frac{1}{2}\mu$, we have $h_s=2^{-s}\mu^s(p+s, p)$, where $(p+s, p)$ is the binomial coefficient. In this case the order of the manifold is

$$\mu^e \prod_{i=0}^{p-c} (2p-c+1, 2i+1)/(2p-c+1, i),$$

with $e=(p-c+2, 2)$. When the original matrix is skew symmetric, but p is odd and c is odd, precisely the same formulae hold with the one change, that the two given values for the order of the manifold must be divided by 2^{p-c+1}.

These results are due to Giambelli (Segre, *Enzyk. Math. Wiss.* III, C 7, pp. 828, 856, 930). The following references may be given: Salmon, *Higher Algebra*, 1866, Lesson XVIII, p. 212; S. Roberts, *Crelle*, LXVII, 1867 (with reference to de Jonquières, *Crelle*, LXVI); Schubert, *Jahresber. Deut. math. Ver.* IV, 1894–5; Vahlen, *Crelle*, CXIII, 1894; Pieri, *Rend. Palermo*, XI, 1896; Segre, *Rend. Lincei*, IX, 1900; Palatini, *Rend. Lincei*, XI, 1902; Nanson, *Mess. Math.* XXXIII, 1903; Kohn, *Archiv d. Math. u. Phys.* IV, 1903; Giambelli, *Rend. Lincei*, XII, 1903; *ibid. Atti...Torino*, XXXVIII, 1903; *ibid. Mem. Ist. Lombardo*, X, 1903 and XI, 1904; *ibid. Atti...Torino*, XLI, 1905.

CHAPTER III

TRANSFORMATIONS AND INVOLUTIONS FOR THE MOST PART IN A PLANE

THE present chapter is designed to indicate some preliminary theorems, for the most part dealing with plane geometry, which it is desirable for the reader to have in mind in dealing with the subsequent theory of surfaces. The treatment is only introductory, but can be supplemented by the wide literature, extending over a long time, which treats of the questions referred to.

Cremona transformations in a plane. Take a set of distinct points in a plane; and consider the plane curves, of given order, say n, which have multiple points of assigned order at these; suppose that the assigned orders are such that the general curve satisfying the conditions has an equation of the form $\lambda u + \mu v + \nu w = 0$, wherein $u = 0$, $v = 0$, $w = 0$ are particular curves of the system, and λ, μ, ν are arbitrary parameters; suppose further that any two curves of the system have only one intersection, outside the assigned points common to all. Then the equations $x'/u = y'/v = z'/w$, corresponding to any point, (x, y, z), whose coordinates enter in u, v, w, define another point, (x', y', z'), of the plane. Conversely, if (x', y', z') be any given point of general position, these equations define a point (x, y, z), which is the common point, outside the given base points, of the two curves $x'v - y'u = 0$, $x'w - z'u = 0$. Thus the equations are capable of solution in the form $x/u' = y/v' = z/w'$, where u', v', w' are homogeneous polynomials of the same order in x', y', z'. There is thus established a $(1, 1)$ birational correspondence of points (x, y, z), (x', y', z') of the plane.

We may denote the prescribed multiple points by a scheme $(i_1^{p_1}, i_2^{p_2}, \ldots)$, meaning that the curves are to have p_1 distinct i_1-ple points (with tangents all unassigned), p_2 i_2-ple points (with unassigned tangents), and so on. Suppose that these numbers are such that

$$p_1 i_1 + p_2 i_2 + \ldots = 3n - 3, \quad p_1 i_1^2 + p_2 i_2^2 + \ldots = n^2 - 1,$$

which we may denote by $\Sigma i = 3n - 3$, $\Sigma i^2 = n^2 - 1$. The second equation secures that two curves of the system which are such as have a finite total number (n^2) of common points, have one intersection not at the base points; and, if the prescribed conditions at the base points lead to linearly independent conditions for the coefficients in the general curve which is subject thereto, the number of such curves which are linearly independent will be

$$\tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}\Sigma i(i+1), \ = \tfrac{1}{2}(n^2 + 3n + 2) - \tfrac{1}{2}(n^2 - 1 + 3n - 3), \ = 3,$$

as we have supposed above. But the numerical conditions, even when independent, do not ensure that the resulting curves are without a common part, which is a necessary condition for the proof that any two curves of the system have only one free intersection. For instance, quintic curves with the prescribed base system $(3^2, 1^6)$ break up into the line joining the two triple points, and quartic curves with the base $(2^2, 1^6)$, of which any two have 2 intersections outside this base.

Suppose now that the conditions at the base are independent, and that the resulting curves have no common part, so that, conversely, the numerical conditions are satisfied; and the reverse expressions for x, y, z in terms of x', y', z' follow as above. Then the curve $\lambda u+\mu v+\nu w=0$ has a genus given by

$$\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma i(i-1),$$

that is $\frac{1}{2}(n^2-3n+2)-\frac{1}{2}(n^2-1-3n+3)$, or zero, the same as that of the line $\lambda x'+\mu y'+\nu z'=0$, to which it corresponds. The curves of the plane (x', y', z') which correspond to the lines $\lambda x+\mu y+\nu z=0$, given by $\lambda u'+\mu v'+\nu w'=0$, will have an order equal to the number of common solutions, for general values of λ, μ, ν and p, q, r, of $\lambda u'+\mu v'+\nu w'=0$ and $px'+qy'+rz'=0$; this is the same as for $\lambda x+\mu y+\nu z=0$ and $pu+qv+rw=0$; thus the curves

$$\lambda u'+\mu v'+\nu w'=0$$

have also the order n. From the (1, 1) correspondence between the two planes, they will then be of genus zero, and their base system will satisfy equations which we can represent by

$$\Sigma j=3n-3,\ \Sigma j^2=n^2-1,$$

provided their base points be distinct. We consider only cases when this is so.

To any point, (x, y, z), in the neighbourhood of one of the base points, O, of the curves $u=0$, $v=0$, $w=0$, will correspond a point (x', y', z') which, in general, will assume a definite position as (x, y, z) approaches O along a definite line. The points (x', y', z') so obtained, which we may speak of as corresponding to the neighbourhood of O, will describe a curve, of which the points correspond severally to the rays of the pencil of lines through O. This curve will then be rational. We can obtain the equation of this curve by supposing u, v, w expressed in non-homogeneous coordinates ξ, η, both vanishing at O; and, then, in $x'/u=y'/v=z'/w$, retaining only terms of the lowest order in ξ, η, we have the ratios of x', y', z' expressed by polynomials, in ξ/η, whose order, i, is the common multiplicity of $u=0$, $v=0$, $w=0$ at the point O. In general terms, it appears that the base points in the plane (x, y, z) correspond severally to curves in the plane (x', y', z'), which we may call *exceptional* (or fundamental) curves. Similarly, there are exceptional curves in the plane (x, y, z), which correspond to the base points in the plane (x', y', z'). Such a curve has the property that the substitution, in u, v, w, of the coordinates of a point of this curve, gives, ultimately, for (x', y', z'), the same values whatever be the point, namely the coordinates of the corresponding base point in the plane (x', y', z'). Using u_1 for $\partial u/\partial x$, etc., it follows that, if (x, y, z) be a point of an exceptional curve of the plane (x, y, z), there are ratios of dx, dy, dz for which the three equations

$$u_1dx+u_2dy+u_3dz=0,\quad v_1dx+v_2dy+v_3dz=0,$$
$$w_1dx+w_2dy+w_3dz=0,$$

are all satisfied. The exceptional curves are thus part of the curve expressed by the vanishing of the Jacobian of u, v, w, or, say, $\partial(u, v, w)/\partial(x, y, z)$. The order of this Jacobian is $3n-3$; we have seen that, in general, an exceptional curve, in the plane (x, y, z), has the order, j, of the base point, in the plane (x', y', z'), to which it corresponds, and that $\Sigma j=3n-3$. Thus we infer that the Jacobian curve referred to is made up of the exceptional curves; and likewise, in the general case under consideration, the Jacobian curve $\partial(u', v', w')/\partial(x', y', z')=0$ is made up of the exceptional curves in the plane (x', y', z').

After the case of linear transformations, the simplest application of the preceding theory is for $n=2$. The curves $\lambda u+\mu v+\nu w=0$ are then conics, with three distinct simple base points. We may thus take $x'=yz$, $y'=zx$, $z'=xy$, with the reverse formulae $x=y'z'$, $y=z'x'$, $z=x'y'$. The exceptional curves, in either plane, are now the joining lines of the base points; and in fact the Jacobian of yz, zx, xy is $2xyz$. In this simple case, however, we can see directly that the restriction we have imposed, of distinctness in the base points of the curves $u=0, v=0, w=0$, is not necessary for obtaining a (1, 1) relation of the two planes. We can use a system of conics subject to the three conditions of passing through two points but with a given tangent at one of these; that is, with three base points of which the third is "infinitely close" to the second. In fact the equations $x'=x^2$, $y'=xy$, $z'=yz$ have the unique reverse $x=x'y'$, $y=y'^2$, $z=x'z'$, and the base points in the plane (x', y', z') are of similar character to those in the plane (x, y, z). The exceptional curves, in either plane, consist then of the line joining the two distinct base points, taken twice over, together with the line joining the coincident base points. As the reader is aware, the quadratic transformations have been used to establish an analysis of multiple points of a plane curve, when these consist of limits of simple distinct multiplicities lying infinitely close. With this analysis it is possible to attach a definite meaning to a prescription for a curve, that it should have multiple points with other multiplicities infinitely close thereto, all of assigned orders. The results obtained are simple; but the justification, to be complete, is long, and we do not give space to it here. The reader may consult the references given below.

A very general (1, 1) transformation of the plane, which includes the quadratic transformation just referred to, is that which is effected by curves, of order n, having one general $(n-1)$-fold multiple point, and $(2n-2)$ simple base points, supposed distinct from one another, and from the $(n-1)$-fold point. The numbers of the appropriate specification, $[(n-1)^1, 1^{2n-2}]$, evidently satisfy the equations $\Sigma i = 3n-3$, $\Sigma i^2 = n^2-1$, namely $(n-1)+(2n-2)=3n-3$, and $(n-1)^2+2n-2=n^2-1$. If the coordinates (x, y, z) be taken with the $(n-1)$-fold point at $x=0=y$, the equations for the transformation are of the forms

$$x' = zu_{n-1}+u_n, \quad y' = zv_{n-1}+v_n, \quad z' = zw_{n-1}+w_n,$$

where, for $i=n-1$, $i=n$, the symbols u_i, v_i, w_i denote homogeneous polynomials in x, y, of order i. But we can conveniently modify the coordinates x', y', z'. For there exists, in general, a definite curve of order $(n-1)$, with a $(n-2)$-fold point at $x=0=y$, passing through the $(2n-2)$ assigned base points, these conditions being equivalent to $\frac{1}{2}(n-2)(n-1)+2n-2$, or $\frac{1}{2}(n-1)(n+2)$ conditions. If this particular curve have the equation $za_{n-2}+a_{n-1}=0$, where a_{n-2}, a_{n-1} are definite homogeneous polynomials in x, y, then we can take the coordinates x', y', z', so that

$$x' = x(za_{n-2}+a_{n-1}), \quad y' = y(za_{n-2}+a_{n-1}), \quad z' = zb_{n-1}+b_n,$$

where again b_{n-1} and b_n are homogeneous in x and y. Writing, momentarily, $Y=y/z$, $Z=z/x$, the equations of transformation are then $Y'=Y$, $Z'=(Zxb_{n-1}+b_n)/(Zx^2a_{n-2}+xa_{n-1})$; if then a_i', b_i' be the same polynomials as a_i, b_i, with x', y' put for x, y, these equations lead to $Z=(-Z'x'a'_{n-1}+b_n')/(Z'x'^2a'_{n-2}-x'b'_{n-1})$. Thus the reverse transformations are

$$x = x'(z'a'_{n-2}-b'_{n-1}), \quad y = y'(z'a'_{n-2}-b'_{n-1}), \quad z = (-z'a'_{n-1}+b'_n);$$

these are obtained from the direct transformations by replacing a_{n-1}, b_{n-1} respectively by $-b'_{n-1}$, $-a'_{n-1}$, and a_{n-2}, b_n respectively by a'_{n-2}, b'_n. This reverse transformation is effected by certain curves of order n in (x', y', z'), likewise with an $(n-1)$-fold point at $x'=0=y'$, and $(2n-2)$ other simple base points.

Consider now the exceptional curves in the plane (x, y, z). The rational curve $za_{n-2}+a_{n-1}=0$ is the exceptional curve which corresponds to the neighbourhood of the multiple base point at $x'=0=y'$ in the second plane. For, in the equations which express x, y, z in terms of x', y', z', if we retain only the lowest powers of x', y', we find that x, y, z are in the ratios of $x'a'_{n-2}$, $y'a'_{n-2}$, $-a'_{n-1}$; so that we have $za_{n-2}+a_{n-1}=0$. Similarly, a line joining the multiple base point $(0, 0, 1)$, in the plane (x, y, z), to one of the $2n-2$ points, other than $(0, 0, 1)$, which are common to the two curves

$$za_{n-2}+a_{n-1}=0, \quad zb_{n-1}+b_n=0,$$

is an exceptional curve corresponding to one of the $2n-2$ simple intersections of the two curves $z'a'_{n-2}-b'_{n-1}=0$, $z'a'_{n-1}-b'_n=0$. The aggregate of these lines is given by $a_{n-1}b_{n-1}-a_{n-2}b_n=0$; and if the transformation be denoted by $x'/u=y'/v=z'/w$, the Jacobian of u, v, w is $n(za_{n-2}+a_{n-1})(a_{n-1}b_{n-1}-a_{n-2}b_n)$. The exceptional curves of the plane (x', y', z') are similar.

It is clear that the quadratic transformation considered above is a particular case of the present transformation $(n=2)$. More generally, the present transformation may be regarded as the result of a succession of a finite number of quadratic and linear transformations. For if, momentarily, we put $\xi=x/y$, $\eta=z/y$, with $\xi'=x'/y'$, $\eta'=z'/y'$, the transformation is $\xi'=\xi$, $\eta'=(\eta\alpha+\beta)/(\eta\gamma+\delta)$, where α, β, γ, δ are polynomials in ξ, of respective orders $n-1$, n, $n-2$, $n-1$, for which it is supposed that $\alpha\delta-\beta\gamma$ is not identically zero. Provided neither α, nor γ, nor δ, is identically zero, this transformation is made up by the succession of the following four, in which ξ remains unaltered, but η is changed by

$$\eta_1=\frac{\delta}{\gamma}\left(\eta\frac{\gamma}{\delta}+1\right), \quad \eta_2=\frac{1}{\eta_1}, \quad \eta_3=\frac{\beta\gamma-\alpha\delta}{\gamma^2}\eta_2, \quad \eta'=\frac{\alpha}{\gamma}\left(\eta_3\frac{\gamma}{\alpha}+1\right);$$

to prove the statement made, in the general case contemplated, it is thus sufficient to shew that each of the three transformations following, in which m denotes a rational function of ξ, namely

$$(\xi'=\xi, \eta'=\eta+1), \quad (\xi'=\xi, \eta'=1/\eta), \quad (\xi'=\xi, \eta'=m\eta),$$

is obtainable by a finite number of linear and quadratic transformations. Of these three, however, the first is the linear transformation $x'=x$, $y'=y$, $z'=z+y$; and the second is the quadratic transformation $x'=xz$, $y'=yz$, $z'=y^2$; the third, considering in detail the possible forms for the rational function m, is obtainable by combining a transformation of one of the two forms

$$\xi'=\xi, \ \eta'=(a\xi+b)\,\eta, \quad \text{or} \quad \xi'=\xi, \ \eta'=(a\xi+b)^{-1}\eta,$$

in which a, b are constants, with a transformation of the form $\xi'=\xi$, $\eta'=\mu\eta$, in which μ is rational in ξ, but with either its denominator or its numerator of one less order than those of m. The two particular transformations are the quadratic transformations in which x', y', z' are respectively proportional to xy, y^2, $z(ax+by)$ and $x(ax+by)$, $y(ax+by)$, yz. By continued reduction of the rational functions m, μ, ... thus arising, the general theorem is therefore proved by induction. If one of α, γ, δ be zero the corresponding theorem is similarly obtained. With $\alpha\delta-\beta\gamma\neq 0$, the theorem is therefore true.

We have supposed the $(n-1)$-fold point, and the $(2n-2)$ points, of the transforming curves, to be distinct. This is not essential. An extreme case is the transformation $x'=xy^{n-1}$, $y'=y^n$, $z'=zy^{n-1}+b_n$, for which $a_{n-2}=0$ and $a_{n-1}=b_{n-1}=y^{n-1}$. In this case the exceptional curves consist of $y^{n-1}=0$ (replacing $za_{n-2}+a_{n-1}=0$), and $y^{2n-2}=0$ (replacing the $2n-2$ exceptional lines). The composition of this transformation, by linear and quadratic transformations, is easily obtained.

In such a case, when the transforming curves (of order n) have coincident base points, the necessary conditions for a (1, 1) transformation are that the base points imposed are equivalent to $\frac{1}{2}(n+1)(n+2)-3$ linear conditions, that they absorb n^2-1 intersections of any two of the curves, and are such that the curves are irreducible; though, when the analysis of complex multiple points into a succession of ordinary multiple points following one another in infinitely near neighbourhood, has been carried out, the conditions can be stated as in the case of distinct base points. Utilising the description of the multiple points obtained by such an analysis, it can be shewn that any (1, 1) birational transformation of the plane can be obtained by compounding transformations of the character just considered (with base points $(n-1)^1$, 1^{2n-2}). This was proved by Castelnuovo, *Atti...Torino*, XXXVI, 1900–1, p. 861 (cf. Alexander, *Trans. Amer. Math. Soc.* XVII, 1916, p. 295). As we have shewn that such a transformation is compounded of quadratic transformations, it follows that the general (1, 1) transformation may be compounded of quadratic transformations (for a recent direct proof of this see Enriques-Chisini, *Teoria Geom.* III, 1924, p. 170).

The theory of the resolution of a complex singularity by a succession of quadratic transformations was enunciated by Noether (*Math. Ann.* IX, 1875; *Math. Ann.* XXIII, 1883). The theorem of the decomposition of a (1, 1) transformation into quadratic transformations was given, almost at the same time, by Noether, *Math. Ann.* III, 1870, p. 167; by Rosanes, *Crelle*, LXXIII, 1870; and by Clifford (see Cayley, *Papers*, VII, p. 223, and Clifford, *Papers*, p. 542); but the completion of the proof (for cases of complex base points) is due to a remark made by Segre (*Atti...Torino*, XXXVI, 1900–1901, p. 645). A recent exposition of Noether's analysis of a singular point, is given in Enriques-Chisini, *Teoria Geom.* II, pp. 327–535. A detailed account of (1, 1) birational transformations, in the plane, and in ordinary space, with an exhaustive bibliography, is found in Hudson, *Cremona Transformations* (Cambridge, 1927). See also Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie*, II, 1880, pp. 188–219; Brill-Noether, *Die Entwicklung der Theorie der algebraischen Functionen*, 1894, VI. Abschnitt; and Coble, *Bulletin of Nat. Research Council, Washington, D.C.*, No. 63, 1928, Chaps. IV and VIII.

The transformation of specification $(n-2)^1, 1^{2n-2}$, which we have discussed, is called after its discoverer, de Jonquières (*Battaglini's Gior.* XXIII, Naples, 1885, dating from 1859). If in the formulae

$$x'=x(za_{n-2}+a_{n-1}),\quad y'=y(za_{n-2}+a_{n-1}),\quad z'=zb_{n-1}+b_n,$$

the polynomial b_{n-1} is the negative of the polynomial a_{n-1}, then the reverse formulae are of precisely the same form. In this case, if we suppose the planes, and coordinates, (x, y, z) and (x', y', z'), to be identified, the formulae determine, in the plane (x, y, z), an *involution* of pairs of points, in which any point of the plane belongs to one such pair. This *de Jonquières involution* may be simply defined as consisting of pairs of points, of which the points of a pair have coordinates (x, y, z) and (x, y, z') and lie upon a line through the point $(0, 0, 1)$, the values of z, z' being connected by the relation $zz'a_{n-2}+(z+z')a_{n-1}-b_n=0$; the points of a pair are thus harmonic conjugates of one another in regard to the two points (other than $(0, 0, 1)$ in which their joining line meets the curve whose equation is $z^2a_{n-2}+2za_{n-1}-b_n=0$. In general terms, the curve is any curve of order n with a $(n-2)$-ple point; it is of genus $n-2$, and of hyperelliptic character; and, it can in fact be conversely shewn that any hyperelliptic plane curve can be birationally reduced to this equation. The curve in this case is the locus of self-corresponding points of the involution, there being two such points upon any line through $(0, 0, 1)$. For this involution, the exceptional curve, corresponding to the multiple point, given as in the general case by the equation $za_{n-2}+a_{n-1}=0$, is the first polar of $(0, 0, 1)$ in regard to the curve locus, Ω, of self-corresponding points; the curves $(\lambda x+\mu y)(za_{n-2}+a_{n-1})+\nu(zb_{n-1}+b_n)=0$, which determine the involution $(b_{n-1}=-a_{n-1})$, have in common the points of contact of the $2n-2$ tangents from $(0, 0, 1)$ to this curve Ω, and these tangents, given by $a_{n-2}b_n+a^2{}_{n-1}=0$, are the exceptional curves corresponding thereto.

It is a general problem, to represent the ∞^2 pairs of a plane involution, each by a point, say (ξ, η, ζ), of a plane, so that conversely to every point (ξ, η, ζ) corresponds one such pair; it is a theorem of importance that this can always be done, as we shall see. In the case of the de Jonquières involution, a simple solution of this problem is to take, corresponding to the pair (x, y, z), (x, y, z') of the involution, the point $\xi:\eta:\zeta=x:y:\frac{1}{2}(z+z')$; then, when (ξ, η, ζ) is given, the pair are determined by taking $x:y=\xi:\eta$, the values of z/x and z'/x being given by the equation

$$(z^2\xi-2zx\zeta)\,\xi\alpha_{n-2}-x^2(2\zeta\alpha_{n-1}-\beta_n)=0,$$

where $\alpha_{n-2}, \alpha_{n-1}, \beta_n$ are the same polynomials in ξ, η as are a_{n-2}, a_{n-1}, b_n in x, y. This is easy to verify. In a less simple, but more representative manner, following Bertini (*Rend. Ist. Lombardo*, XXII, 1889, p. 773), we may solve this problem as follows: It can be shewn, as below, that a system of curves exists in the plane with the property that all curves of the system, which pass through an arbitrary point, pass also through the point corresponding thereto in the involution (having in general no other common points, outside the base points of the system). Let such a system of curves have the equation $\lambda_0\phi_0 + \ldots + \lambda_r\phi_r = 0$, with $r \geqslant 2$; then consider the point $(\xi_0, \ldots, \xi_r)$, in a space $[r]$, for which $\xi_0, \ldots, \xi_r$ are in the ratios of $\phi_0, \ldots, \phi_r$. As the point with coordinates x, y, z, which occur in $\phi_0, \ldots, \phi_r$, varies in the plane, the point $(\xi_0, \ldots, \xi_r)$ describes a surface; conversely, to any point (ξ) of this surface, correspond the pair of points in the plane common to all the curves $\xi_i\phi_j - \xi_j\phi_i = 0$; and these, by hypothesis, are a pair of the involution. Thus, if the surface obtained be rational, its points being in (1, 1) birational correspondence with the points of a plane, then, it is clear, the points of this plane form a representation of the couples of the involution. To find, for the de Jonquières involution, a system of curves, of freedom at least 2, with the property that all curves of the system described through an arbitrary point of the plane pass also through the point which corresponds thereto in the involution, denote the curve, Ω, considered above, which is the locus of coincident pairs of the involution, by the equation $z^2u + 2zv + w = 0$, where u, v, w are homogeneous polynomials in x and y, respectively of orders $n-2$, $n-1$ and n. On any line through the point (0, 0, 1), a pair of points which are harmonic in regard to the points in which this line meets the curve Ω, are a pair of the involution; these lines form a system of curves of freedom 1; but a system of greater freedom is obtainable by curves Ω', given by an equation of the form $z^2u' + 2zv' + w' = 0$, in which u', v', w' are homogeneous polynomials in x, y of respective orders $n'-2$, $n'-1$, n'; we require only that the curve Ω' should meet any line through (0, 0, 1) in points harmonic in regard to the intersections of this line with the curve Ω. The condition for this relation is $u'w - 2v'v + w'u = 0$, which leads to $n + n' - 1$ homogeneous conditions for the $3n'$ unspecified homogeneous coefficients in the polynomials u', v', w'; thus the curves Ω' are of freedom at least $2n' - n$. At a point of intersection of a curve Ω' with the curve Ω, other than the $(n-2)(n'-2)$ intersections which fall at (0, 0, 1), the tangent line of one of the two curves passes through (0, 0, 1); this is in accordance with

$$nn' = (n-2)(n'-2) + 2(n-1) + 2(n'-1).$$

If the general curve Ω' have an equation $\lambda_0\phi_0 + \ldots + \lambda_{2n'-n}\phi_{2n'-n} = 0$, the equations $\xi_0/\phi_0 = \ldots = \xi_{2n'-n}/\phi_{2n'-n}$ determine, in space $[2n'-n]$, a surface whose points represent the pairs of the plane involution; and, as two curves Ω', beside the fixed intersections at (0, 0, 1) and at the points of contact of tangents drawn from this point to Ω, have $n'^2-(n'-2)^2-2(n-1)$, or $2(2n'-n-1)$ intersections—which will consist of pairs of points corresponding in the involution—the surface will be of order $2n'-n-1$. Here n' is arbitrary (subject to $2n'-n \geqslant 2$); when n is odd, we may take $n'=\frac{1}{2}(n+3)$, and obtain a quadric surface in ordinary space. It may be shewn that the points of the generators of one system, of this surface, represent the pairs of the plane involution lying on lines through the point $(0, 0, 1)$; while the generators of the other system correspond to the ∞^1 curves Ω' for which $n'=\frac{1}{2}(n+1)$. When n is even, if we take $n'=\frac{1}{2}n+2$, we obtain a cubic surface in space of four dimensions. This is known to be rational, the coordinates of a point being expressible, in terms of two parameters θ, ϕ, by θ^2, θ, 1, $\phi\theta$, ϕ; such a surface is obtainable, if we take a rational quartic curve in the space $[4]$, and, thereon, any involution of pairs of points, as the locus of the chords of the curve which join the pairs of this involution. In general, however, it is known that (irreducible) surfaces of order $r-1$ in space $[r]$, as in this case, with $r=2n'-n$, are always rational, being, unless $r=5$, ruled surfaces, normal in this space $[r]$, that is, unobtainable as projections of surfaces of the same order lying in higher space (Del Pezzo, *Rend....Napoli*, 1885, 1886; Bertini, *Geom. d. iperspazi*, 1907, p. 319). That, in our case, the representative surfaces are rational, follows also, because, as the curves Ω' are hyperelliptic, the prime sections are hyperelliptic curves (Castelnuovo, *Rend....Palermo*, iv, 1890, p. 73).

Ex. 1. A particular property of the related curves Ω, Ω' considered here, suggested by a note of Bertini's (*Atti...Lincei*, Transunti (3^a), i, 1877, p. 92; cf. Caporali, *Rend....Napoli*, xxi, 1882, p. 227, or *Mem. d. Geom.* 1888, p. 164), may be noted. Let the curves Ω, Ω' be given, as above, by equations $z^2u+2zv+w=0$ and $z^2u'+2zv'+w'=0$, the relation $u'w-2v'v+w'u=0$ being assumed. Denote the point $(0, 0, 1)$ by O; the $2(n-1)$ points of contact of tangents from O to Ω, that is, the points, other than O, common to Ω and the curve $zu+v=0$, be denoted by A; and the $(n-2)$ points of intersection, other than O, of the tangent lines at O, with the curve (one on each of the tangents), be denoted by T. Use a similar notation for the curve Ω'. Then, neglecting intersections at O, it may be proved:

(1), That the intersections of Ω, Ω' are $A+A'$, as has been remarked above; (2), that the first polar of O, with respect to Ω, meets Ω', partly in the set A, and partly in the set T', the "tangentials" of O on Ω'; or, say, $(\Omega', zu+v)=A+T'$. This is in accordance with

$$n'(n-1)-(n-2)(n'-2)=(2n-2)+(n'-2).$$

Similarly we have $(\Omega, zu'+v')=A'+T$. For the relation connecting Ω and Ω' involves, we easily see, the two identities

$$w\Omega'+w'\Omega=2(vz+w)(v'z+w'),\quad (zu'+v')(zu+2v)-u'\Omega=u(zv'+w'),$$

from which the results can be read off.

For instance, let Ω be a quintic curve, with triple point at O. Let the tangents at O meet the curve again in P, Q, R; let any conic through

O, P, Q, R meet the quintic curve again in the four points A, B, C, D. There are eight points of contact of tangents drawn from O to the quintic curve; these lie, with the five points O, A, B, C, D, upon a cubic curve, touching OA, OB, OC, OD at A, B, C, D. The conic $OABCD$ is the first polar of O in regard to this cubic curve. Moreover, the tangential point of O, upon this cubic curve, lies on the first polar of O with respect to the quintic curve, being the other intersection of this first polar with the cubic.

Ex. 2. Suppose that the three polynomials, $uv'-u'v$, $uw'-u'w$, $vw'-v'w$, arising for the curves Ω, Ω' have no common factor; denote them, respectively, by u'', $2v''$, w'', so that, beside the harmonic relation $uw'-2vv'+wu'=0$, for u, v, w and u', v', w', we have also the corresponding relations for both the couples $(u, v, w; u'', v'', w'')$, $(u', v', w'; u'', v'', w'')$. Consider a curve $z^2u''+2zv''+w''=0$, or, say, Ω''. By any one of the three curves Ω, Ω', Ω'' a plane de Jonquières involution may be determined. Shew that any two of these involutions carried out in succession, in either order, leads to the third.

Ex. 3. Shew that the plane quadratic transformation expressed by $x'=y(z-x)$, $y'=z(z-x-y)$, $z'=z(z-x)$, or, in non-homogeneous variables, by $x'=y$, $y'=(1-x-y)/(1-x)$, or, say, $(x', y')=T(x, y)$, is such that $T^5=1$. This gives rise therefore to sets of five points in the plane, of which any set is determined indifferently by any one of its points, say to an involution of sets of five points. Consider also the quadratic involution, U, expressed by $x'=-x/(1-x)$, $y'=y/(1-x)$, which is a harmonic inversion, with $(2, 0)$ as centre, and $x=0$ as axis. It can be shewn that the products of U with the powers of T, in all possible ways, give rise to a finite group of plane transformations having precisely the structure of the symmetric group of permutations of five symbols (Burnside, *Groups*, 1911, p. 469; Moore and Slaught, *Amer. J. Math.* XXII, XXIII, 1900 and 1901). This example is indicative of a wide range of possible investigations. (Cf. Wiman, *Math. Ann.* XLVIII, 1897, p. 195; also p. 278, and p. 299 below.)

Bertini's four types of involution in a plane. If a transformation in a plane expressed by the rational equations $x'=\phi(x, y)$, $y'=\psi(x, y)$, where (x, y), (x', y') are the non-homogeneous coordinates of two points, be such that $\phi(x', y')=x$ and $\psi(x', y')=y$, then the two points are the pairs of an involution, of sets of two points. By any $(1, 1)$ Cremona transformation of the plane, whereby (x, y) is placed in $(1, 1)$ relation with a point (ξ, η) and (x', y') is placed in the same relation with a point (ξ', η'), we can hence deduce an involution of pairs of points (ξ, η) and (ξ', η'). The question then arises whether there is a finite number of involutions, sets of two points, from which all others are so derivable, by application of Cremona transformations. This question was answered, in the affirmative, by Bertini (*Ann. d. Mat.* VIII, 1877; see the further references below); and his results have been confirmed from other points of view. By Cremona transformation, all plane involutions, of sets of two points, can be obtained from *four* types: namely (i), the familiar harmonic inversion, expressible by the linear equations $x'=x$, $y'=-y$, $z'=z$; (ii), the de Jonquières involution, considered above, expressible by equations $x'=x$, $y'=y$ with

$zz'u+(z+z')v+w=0$, where u, v, w are homogeneous polynomials in x, y, of respective orders $n-2, n-1$ and n; (iii), the so-called Geiser involution, of which the pair of a set is formed by the residual intersections of two cubic curves described through seven given points; and (iv), the appropriately called Bertini involution. This may be defined by sextic curves prescribed to have eight given double points; two such curves have four further intersections; but, all sextic curves with the eight given double points which pass through another arbitrary point P, have a single further point P' in common; and the pair (P, P') are a pair of the involution in question.

We do not prove this theorem of the reduction to four types of involution; but it is desirable to have a familiarity with these four involutions. Of the de Jonquières involution we have already treated, explaining a method by which the pairs can each be represented by a single point of a plane. We can pass over the case of the harmonic inversion with only the notice of such representations. An obvious one is to represent the pair $(x, y, z), (x, -y, z)$, of the involution, by the point (ξ, η, ζ), where $\xi=x^2, \eta=y^2, \zeta=xz$. A second more geometrical representation may be given by a construction in space; the points P, P', of a pair of the involution, may be regarded as arising by projection from two points Q, Q' of a quadric surface, whose joining line passes through a fixed point O (the centre of the harmonic inversion in the plane); the involution is represented by the points of intersection of the lines QQ' with the plane which is the polar plane of O in regard to the quadric surface. The point from which P, P' are projected to Q, Q' lies on the section of the quadric surface by this polar plane.

Pass now to the Geiser involution (Geiser, *Crelle*, LXVII, 1867, p. 78; Milinowsky, *Crelle*, LXXVII, 1874, p. 263). We have defined it as consisting of the pairs of variable intersections of two plane cubic curves of a system having seven given fixed common points. We obtain an interesting representation of the involution in space; we shew that the points of the plane, at which a pair of the involution are coincident points, describe a sextic curve with double points at the seven base points; we shew that the order of the functions by which a point P', of a pair, is expressed in terms of P, is eight, or, more precisely, that, as P describes an arbitrary line, P' describes an octavic curve with triple points at the seven base points, determining the exceptional curves of the transformation from P to P'; and we shew how to represent the pairs of the involution by the points of a plane. From the seven base points, choose out six; and consider the cubic surface, which, as is well known, is obtained, as a representation of the plane, from the cubic curves which pass through these six points. These curves are then represented by the plane sections of the surface. And the lines of the plane are represented by rational cubic curves on the surface, which have no intersections with the six lines of the surface which represent the base points in the plane. The cubic curves of the plane, through the original seven base points, are thus represented by plane sections of the cubic surface which all have a single common fixed point on the surface, say O. Hence the pairs of the involution in the plane are represented by points Q, Q' of the cubic surface, whose joining line passes through the fixed point O. Incidentally, the *pairs* of the involution are represented by the points in which lines through O meet an arbitrary fixed plane. The coincidence points, of pairs on the cubic surface, are then the points of contact of tangent lines to the surface drawn from O; the locus of these points is known to be a sextic curve, of genus 3, having a double point at O. This curve, meeting an arbitrary plane section in six points, and meeting, as we know, every line on the cubic surface in two points, represents a curve in the plane with a double point at each of the *seven* base points; as such a curve has twelve fixed intersections, with each of the fundamental cubic curves of the plane, at the base points of these, it is of order six, having six other intersections with each of the plane cubic curves. The order of the functions which express the plane involution is the number of pairs P, P' in which each of P and P' lies on an arbitrary line in the plane. This is then the number of pairs Q, Q', on the cubic surface, collinear with O, in which each of Q and Q' lies on an arbitrary rational cubic curve of the surface, both belonging to the system of cubic curves which do not intersect the lines of the surface representing the six base points in the plane. Two such cubic curves have a single intersection, corresponding to the intersection of the lines in the plane which the curves represent; the line from O to this intersection is one of the nine common generators of the cones, with vertex at O, which stand on these curves. The other eight common generators meet the curves in such pairs Q, Q' as we seek. Thus 8 is the order of the functions representing the plane involution. Consider one of the six lines of the cubic surface representing the six base points of the cubic curves in the plane; the plane joining O to this line is met by a rational cubic curve of the surface which does not meet this line, in three points; and the line joining O to one of these intersects the line; there are therefore three pairs Q, Q', of the involution, in which Q lies on the cubic curve and Q' lies on the line. Thus the octavic curve in the plane which is the locus of P' when the other point P, of a pair of the involution, moves on an arbitrary line, has a triple point O at each of the six base points of the system of plane cubic curves. The curve has also a triple point at the seventh point of the plane, represented by the point O of the surface; for, the tangent plane of the surface at O meets a cubic curve on the surface in three points, and as a point Q describes this curve, the corresponding point Q' will pass three times through O. A plane octavic curve with 7 triple points has genus $\frac{1}{2}7.6-21$, or zero; this is the genus of the curve described by P' when P describes a line. The exceptional curves, in the plane transformation by which we pass from a point P to the other point P' of the involutory pair, are the curves described by P' when P is in the neighbourhood of one of the seven triple base points of the octavic curves. It may be proved that these curves are cubics, passing through these seven points, each with a double point at one of them; in fact, on the cubic surface, the tangent plane at O meets the surface in a curve with a double point at O, and a plane drawn through a line of the surface meets the surface otherwise in a conic having two intersections with the line. Recurring to the introductory remarks in regard to Cremona transformations made above, we notice that the equations $\Sigma i=3n-3$, $\Sigma i^2=n^2-1$ are satisfied here, with $n=8$ and $i=3$ (seven cases); the Jacobian of the three octavic curves which can be constructed with seven triple points is the product of the seven exceptional curves, the multiplicity of the Jacobian, at each of the seven s-ple points of the octavics ($s=3$), being $3s-1$.

Ex. There is similarly an involution in ordinary space of three dimensions, determined by quadric surfaces having six given fixed points, of which the pairs are the residual intersections of three quadrics through the base points. It is well known (e.g. *Principles of Geometry*, IV, p. 156) that the three-fold space can be represented on a rational three-fold locus, of order 3, say $M_3{}^3[4]$, in space of four dimensions, by means of quadric surfaces having five base points in the original space. The involution is then represented by pairs of points Q, Q' on this manifold, whose joining line passes through a fixed point O thereof. The locus of coincident points Q, Q' in this involution is then a surface which projects from O (on to space of three dimensions) into a quartic surface with sixteen nodes (the Kummer surface). The reader may determine the surface in the original space which is the locus of coincident points; and also the order of the functions representing the involution. As the prime section of the $M_3{}^3[4]$ is a cubic surface, the present involution leads to the Geiser involution. The present involution is considered in Hudson's *Cremona Transformations*, 1927, p. 323.

We now consider the Bertini involution. Defining the *characteristic series*, for a linear system of curves with assigned points, as the series determined on any curve of the system, outside the base points, by the other curves of the system, we may assume the theorem that the redundancy of the assigned conditions at the base points, is equal to the index of specialness of the characteristic series on any of the curves—and, in particular, is zero when the number of points in a set of the characteristic series exceeds the number of points in a canonical set on any curve of the system. Consider now sextic curves in the plane with 8 assigned double points (O). Such curves are hyperelliptic, of genus 2, and have a characteristic series of sets of 4 points; thus the base points are independent, and the curves, linearly independent of one another, are 4 in number. Let P be an arbitrary point of the plane, and let $\phi=0$ be the cubic curve through the eight points (O), and through P, while $\psi=0$ is a particular independent curve through the points (O) only. The curves $\phi=0$, $\psi=0$ will have common a further point, say K, which is common to all cubic curves through the base points (O), and independent of P. The general sextic curve, with double points at (O), which passes through P, is thus given by an equation $\phi(\lambda\phi+\mu\psi)+f=0$, wherein λ, μ are arbitrary parameters, and $f=0$ is a particular sextic curve of this character. The curve $\phi=0$ is met by $f=0$ in another point P', beside P; and all the sextic curves $\phi(\lambda\phi+\mu\psi)+f=0$, having the points (O) as double points, which pass through P, pass through P' also. Conversely, when P' is assigned, an unique position for P is determined by the same construction. Thus the involution (P, P') is defined. It will appear that the curves which enter into the expression of P' in terms of P, are of order 17, with each of the points (O) as six-fold points (so that the involution may be said to be of order 17); and also that the curve of united points, where P and P' coincide, is of order 9, with the points (O) as triple points. Further, as also follows from this, the transformation from P to P' has, as exceptional curves, sextic curves, each with a double point at seven of the points (O), and a triple point at the remaining point.

We consider three representations of the involution. For the first, let $\phi=0$, $\psi=0$ be any two independent cubic curves through the base points (O). The general sextic curve with these as double points is representable linearly by means of the four curves $\phi^2=0$, $\phi\psi=0$, $\psi^2=0$, $f=0$, where $f=0$ is a particular sextic curve having the points (O) for double points. Now take a point (ξ,η,ζ,τ) in three-fold space, such that ξ, η, ζ, τ have the ratios of ϕ^2, $\phi\psi$, ψ^2 and f. The points of the original plane are then represented by the points of a quadric cone, with equation $\zeta\xi=\eta^2$, whose vertex arises from the point, K, common to all cubic curves through the eight points (O); any generator of this cone represents a particular cubic curve $\phi+\nu\psi=0$. If a point, P, of the plane, determine the point (ξ, η, ζ, τ) of the cone, then, as all the sextic curves of the plane, with the base (O), which pass through P, also pass through another point, P', it follows that the ratios of ϕ^2, $\phi\psi$, ψ^2, f at P' are the same as at P; and thus the point (ξ,η,ζ,τ) represents the two points P, P', forming a pair of the plane involution. The four intersections of two sextic curves with base (O), outside this base, are then represented by the two points in which the cone is met by a line. We have seen that a pair P, P' of the involution lie on a cubic curve with simple points at (O); let this be

$\phi=0$; then the united points of the involution, which lie on $\phi=0$, are the double points of the ∞^1 series of sets of two points, determined on $\phi=0$ by the sextics with double points at (O), that is, by the curves with equation of the form $C\psi^2+Df=0$, where C, D are parameters. Such double points are 4 in number, but one of these is the point K, where $\phi=0$, $\psi=0$ intersect; and there are three others. As we may take, in place of $\phi=0$, any cubic curve $\phi+\lambda\psi=0$, we see that the curve of united points in the plane is represented, on the cone, by a curve meeting each generator in three points, other than the vertex. This curve is then the sextic intersection of the cone with a cubic surface not passing through the vertex; for clearness denote this sextic curve on the cone by γ. As in general, the neighbourhood of any one of the double base points (O), is represented on the cone by a curve of order 2, a conic; the plane of this, determined by the form of the general sextic curve, $A\phi^2+B\phi\psi+C\psi^2+Df=0$, near this base point, has six intersections with the sextic curve γ. Thus the curve of united points in the plane has a triple point at each of the points (O); and the order, say ν, of this curve, is such that $6\nu-8.3.2$ is the order, 6, of the curve γ, or $\nu=9$. This curve of united points is then of genus 4, and the original sextic curves, with base (O), are the adjoint curves of order 6 of this nonic curve. To determine the order of the curves expressing the plane involution, we are to find the number of pairs of which one point lies on one arbitrary line, while its corresponding point lies on another such line. Either line is represented on the quadric cone by a sextic curve, given, we suppose, by the intersection of the cone with a cubic surface. The number of pairs required is then only the number of intersections of two such sextic curves, since each point of such a curve represents a pair of corresponding points in the plane. The number of intersections of two such sextics is 18 (being the number of common points of the cone and two cubic surfaces); but from these must be omitted the point arising from the intersection of the two lines in the plane. There remain then 17 pairs; so that the curves representing the involution in the plane are of order 17. If these curves pass x times through each of the points (O), there will remain $17.6-8.2.x$ intersections, of any one of these curves, with the general sextic curve of base (O), in the plane; as the curve on the cone which represents a line of the plane meets an arbitrary plane section of the cone in 6 points, we must thus have $17.6-16x=6$, or $x=6$. This deduction is confirmed by the equations $\Sigma i=3n-3$, $\Sigma i^2=n^2-1$, with $n=17$, $i=6$. The exceptional curves, corresponding to the sextuple points of these representative curves of order 17, can be similarly found; as above stated, they are rational sextic curves, each with 7 double points and one triple point, at points of the set (O).

The Bertini involution may be obtained also by considering a certain involution in space of three dimensions. In this space consider seven arbitrary fixed points, which we denote by (H); all quadric surfaces through (H) have another common point, which we denote by K. An elliptic quartic curve, of the kind which is determined by the intersection of two quadric surfaces, can be put through 8 independent points. There passes then such a curve through the points (H) and any arbitrary point, P, of the space, other than K. Upon this curve, another point, P', may be determined, by describing a *quartic* surface, say $F=0$, to have a double point at each of the seven points (H), and to pass through P; the seven double points with P constitute 15 of the 16 intersections of the surface with the quartic curve. This point P' is, in fact, the same, whatever be the quartic surface subject to the conditions. For, let $\phi_1=0$, $\phi_2=0$, $\phi_3=0$ be three independent quadric surfaces passing simply through the points (H), of which $\phi_1=0$ and $\phi_2=0$ are two which also contain the point P; thus $\phi_1=0$, $\phi_2=0$ contain the quartic curve on which P was taken; then, if a, b, c, d, e be arbitrary parameters, a quartic surface having double points at the points (H), and passing through P, is given by the equation $a\phi_1{}^2+b\phi_2{}^2+c\phi_1\phi_2+\phi_3(d\phi_1+e\phi_2)+F=0$; this is then the general quartic surface of the description, because the equation of a quartic surface contains 35 terms, and to have a double point at a given point involves 4 conditions, and $35-7.4-1=6$. All these surfaces meet the quartic curve $\phi_1=0$, $\phi_2=0$ in the same point P'. Thus we reach the conclusion that all quartic surfaces described to have double points at the seven points (H), which pass through an arbitrary general point P, pass through another point P'; and this point P' similarly determines P. The pairs P, P' form the involution in question. From this involution, we may determine the Bertini involution in a plane, by first deducing an involution of pairs of points on a quadric surface, and then projecting this on to a plane. For consider, of the space involution just described, those pairs, (P, P'), of which P lies on a definite quadric surface, say $\phi_1=0$, put through the seven points (H); by what we have seen, the corresponding point P' equally lies on this quadric surface; it is given in fact as the point common to the octavic curves on $\phi_1=0$, having double points at the points (H), and passing through P, which are defined by quartic surfaces of the form $b\phi_2{}^2+e\phi_2\phi_3+F=0$. Each of these quartic surfaces meets the two generators, of the quadric surface $\phi_1=0$, which pass through any particular point H_1, of the set (H), in two points not at H_1; the octavic curves thus also meet these two generators each in two points not at H_1. If we now project the quadric surface $\phi_1=0$, from the point H_1, on to a plane, the octavic curves project into sextic curves, with double points at the two points of the plane lying on the generators at H_1, and with six other double points, projections of the points (H) other than H_1. The sextic curves of this system through an arbitrary point Q of the plane (the projection of P) all pass through another point Q' (the projection of P')—which is Bertini's result. The involution in space is dealt with, in this way, by Hudson, *Cremona Transformations*, Cambridge, 1927, pp. 97, 326, 127.

Ex. The pairs of points of the involution in three dimensions, which we have considered, can be represented by the points of a three-fold locus, in a manner analogous to that followed for the pairs of points of the Bertini involution in the plane. For, if $\phi_1=0$, $\phi_2=0$, $\phi_3=0$ be three independent quadric surfaces passing through the seven fixed points (H), the general quartic surface having these points as double points has an equation with seven parameters entering linearly, containing the six terms ϕ_i^2, $\phi_i\phi_j$ and another term F, where $F=0$ is a particular quartic surface of the kind in question. By taking homogeneous coordinates $\xi_0, \ldots, \xi_6$, of a space [6], respectively in the ratios of ϕ_1^2, ϕ_2^2, ϕ_3^2, $\phi_2\phi_3$, $\phi_3\phi_1$, $\phi_1\phi_2$, F, the three-fold space is represented upon a point cone in this space [6], with vertex at $(0, 0, 0, 0, 0, 0, 1)$, of which a generator, characterised by definite ratios of ϕ_1, ϕ_2, ϕ_3, represents an elliptic quartic curve in the space [3], passing through the points (H). The cone meets the prime $x_6=0$ in the so-called Veronese surface, of order 4, upon which $\xi_0, \ldots, \xi_5$ have the ratios of ϕ_i^2, $\phi_i\phi_j$; and consists of the lines joining its vertex to the points of this surface. Any point of this cone represents a pair of points in the involution in space [3]. Like the Veronese surface, the cone is rational; but—it may be remarked—it is not to be assumed, in general, that the three-fold locus which represents the pairs of an involution in three-fold space, is necessarily rational (see below, in this chapter). The characteristics of the involution may be determined in connexion with the cone. In particular, it may be shewn that the united points are represented by a surface of order 12 lying thereon, the intersection of the cone with a cubic primal of the space [6].

A third way of considering the Bertini involution is by representing the plane upon a cubic surface so that six of the eight double points of the defining sextic curves become lines of the surface. The sextic curves have then six intersections, not at the base points, with the cubic curves, through the base points, which are represented by the plane sections of the surface. Thus the sextic curves are represented by sextic curves on the cubic surface, having two intersections with each of the lines of the surface which represent the six base points in the plane; the curves on the surface have, moreover, two double points, at the points, say A and B, which represent the remaining two of the original eight double points of the defining sextic curves in the plane. The sextic curves on the cubic surface, by which the corresponding involution thereon is to be determined, are thus given by quadric surfaces subject only to the condition of touching the surface at the points A, B. Let C be the third fixed point in which the line AB meets the cubic surface. If P be an arbitrary point of the surface, let T be the tangential point of C on the cubic curve in which the plane PAB meets the surface; thus T is on the section of the surface by the tangent plane at the fixed point C. The plane, TAB, say γ, meets a quadric surface drawn to touch the cubic surface at A and B, in a conic touching the section of the surface, by the plane γ, in A and B; and this conic contains P if the quadric surface contains P; the conic has then another intersection with the section γ. By the ordinary theory of residuation on the cubic curve γ, this remaining intersection is the third point, say P, in which the line TP meets the surface. Thus P' is uniquely determined when P is given, and the relation of these points is mutual. Moreover P' is common to all the sextic curves on the cubic surface representing the original defining sextic curves in the plane, which pass through P. This constitutes then another proof of the existence of the Bertini involution in the plane. Denoting the point where the line PP' meets the fixed line AB by U, a pair of the involution is given by taking any point T on the section of the cubic surface by the tangent plane at C, and joining this to any point U of the line AB; then P, P' are the two points, other than T, in which this line meets the surface. As every plane through the line AB meets the section of the surface by the tangent plane at C, in only one point T beside C, there is one line such as TU through an arbitrary point of space; thus an incidental consequence is that the pairs of the involution may be represented by the points in which an arbitrary fixed plane is met by the lines TU. That, as we have otherwise seen, the order of the curves representing the plane involution is 17, follows if we shew that there are 17 lines which meet the four curves: (1), the line AB; (2) and (3), two cubic curves on the cubic surface representing lines in the plane; (4) the section by the tangent plane at C. This is the number given by the formula of Ex. 10*d*, p. 30, of Chapter I, namely

$$2n_1n_2n_3n_4-\Sigma n_pn_qi_{rs}+\Sigma i_{pq}i_{rs},$$

if we put $n_1=1$, $n_2=3$, $n_3=3$, $n_4=3$, with $i_{12}=0$, $i_{13}=0$, $i_{14}=2$, $i_{23}=1$, $i_{24}=3$, $i_{34}=3$. To find the curve of united points of the involution on the cubic surface, we are to consider the lines TU which touch the surface. These it is easy to see form a ruled surface having the line AB as six-fold line, and the rational tangent section, by the tangent plane at C, as a triple line; this ruled surface is then of order 9. The curve of united points is then represented on any plane by the curve in which this plane meets the ruled surface. This is a curve of order 9, with one six-fold point, on the line AB, and three triple points, these being on the line in which the tangent plane at C is met by the plane of the nonic curve. A curve of this character, we can easily see, has an equation which is capable of the form

$$x^3z^3(x+mz)^3+yx^2z^2(x+mz)^2u_2+y^2xz(x+mz)u_4+y^3u_6=0,$$

where u_i is homogeneous in x and z, of order i. By putting x^{-1}, y^{-1}, z^{-1} for x, y, z; then $z+mx=z_1$; and then $z_1=1$, this equation reduces to $y^3+y^2v_2+yv_4+v_6=0$, where v_i is a polynomial in x, of order i. This is a sextic curve having the singularity which is described, in Noether's phraseology, as two consecutive triple points. That the curve of united points is equivalent to such a curve, follows from the preceding representation of the pairs of a Bertini involution by the points of a quadric cone; we saw that the curve of united points is representable by the intersection of this cone with a cubic surface. The projection of this intersection on to a plane, from an arbitrary point of the cone, gives such a sextic curve. We see also, from this point of view, that the curve is equivalent to a quintic curve having one point of self-contact. (For this curve, the reader may compare Noether, *Math. Ann.* XXXIII, 1889, p. 543; and also Brill u. Noether, *Deut. math. Ver.* III, p. 512; Schottky, *Crelle*, CIII, 1888, p. 185.) But we may consider the curve of united points on the cubic surface on which the Bertini involution is represented; it is the locus of points of contact of quadric surfaces having beside two given contacts with the cubic surface, at A and B. This curve has triple points at A and B, and can be shewn (as was remarked by Cremona, *Ann. d. Mat.* VIII, 1877, p. 273 f.n.) to be the intersection of the given cubic surface U with another cubic surface V. The surface V touches U at A and B; the tangent planes of U at these points each meet V in three lines, one of which is the intersection of these planes; and the two lines, lying on V, which intersect in A, are the inflexional tangents of U at A (with a similar relation at B) (Hilton, *Journ. Lond. Math. Soc.* I, 1926, p. 2; and W. L. Edge).

The four involutions and rational double planes. The fact, due to Bertini, that all involutions of pairs of points, in a plane, are reducible by Cremona transformations of the plane, to one of the four types we have considered: (i), Harmonic inversion; (ii), a de Jonquières involution; (iii), a Geiser involution; (iv), a Bertini involution, is intimately related with a theorem due to Noether (*Sitzungsber. Phys.-med. Soc.* Erlangen, X, 1878, p. 81); this theorem states that, of the so-called double planes, namely the surfaces with an equation of the form $t^2z^m=\phi(x, y, z)$, where ϕ is a homogeneous polynomial in x, y, z, those which are rational are reducible, by a Cremona transformation of the plane (x, y, z), to one of the three in which the curve $\phi(x, y, z)=0$ is (i), an ordinary quartic curve; (ii), a sextic curve with two consecutive triple points, considered above; (iii), a curve of order $2n$ with a $(2n-2)$-fold point. That the surfaces should be rational means that there exist two functions, ξ and η, both rational, and homogeneously of zero order, in the four coordinates x, y, z, t, which are such that, in virtue of the equation of the surface, the ratios of x, y, z, t are expressible rationally in ξ and η. For a summary proof that the surfaces are rational in the three cases named, the reader may consult *Proc. Lond. Math. Soc.* XII, 1913, p. 35. It is clear that the surface $t^2z^m = \phi(x, y, z)$, contains the pairs (x, y, z, t), $(x, y, z, -t)$, of an involution existing thereon; the curve of united points $\phi = 0$, in the three cases named, is such as arises, respectively, for a Geiser involution, a Bertini involution, and a de Jonquières involution. In this connexion, the reader may consult Bertini, *Ann. d. Mat.* VIII, 1877; Bertini, *Rend. Ist. Lombardo*, XIII, 1880; Bertini, *ibid.* XXII, 1889; also Lüroth, *Rationale Flächen und involutorische Transformationen*, Freiburg, 1889; beside the fundamental paper of Noether (Erlangen, 1878), referred to above. Also Vol. V, p. 110, Ex. 3.

Involutions of sets of more than two points. The involutions of sets of two points which we have considered may be defined by rational transformations of the coordinates which have the property that, on repetition, they lead to the identical transformation. We may similarly consider transformations which lead to the identical transformation after $k(>2)$ repetitions. Such is the transformation $(x', y') = T(x, y)$, remarked above (Ex. 3, p. 121), given, in non-homogeneous coordinates, by $x' = y$, $y' = (1 - x - y)/(1 - x)$, for which $T^5 = 1$. In this case there are sets of 5 points in the plane, of which any set is determined, by the repetition of the same rational formula, from any point of the set. More generally we may consider the possibility of sets of points, in any space, of which a set, and only one set, is determinable algebraically to contain an arbitrary point of the space; thus the set is equally determinable from every point of the set. Such a system of sets is called an involution; and is of *index* 1. We may in fact have involutions with the property that there are i sets containing an arbitrary point of the space; such an involution is said to be of index i. Involutions arising by repetitions of a single transformation were considered by Kantor, *Compt. rend.* fév. 1885; see *Atti...Napoli*, IV, 1892.

An interesting question in regard to an involution is whether its *sets* can be represented by the points of a (linear) space, so that every set of the involution gives one point of this space (whose coordinates will then be rational symmetric functions of the coordinates of the points of a set of the involution), and conversely any point of this space gives one set of the involution (the rational symmetric functions of the coordinates of the points forming a set of the involution being expressible rationally by the coordinates of the representative point). When this is so, the involution is said to be rational. We have illustrated the question, by shewing that the involutions of sets of two points in a plane are rational; it is true that all involutions in a plane are rational, however many be the points in a set; we give some account of the proof of this, due to Castelnuovo, in what follows. It is believed, however, that in higher space there exist involutions which are not rational.

The question of the rationality of an involution arises naturally in connexion with the theory of rational curves, or surfaces, or manifolds of higher dimension. Consider, first, a curve. Suppose that it is found that the coordinates of a point of the curve are expressible rationally in terms of a parameter, the equations which determine the curve being satisfied identically when the appropriate expressions are substituted for the coordinates. If we interpret this parameter as a coordinate on a line, to any point, P, of the line, corresponds, by rational equations, a single point, Q, of the curve. But it may happen that the same point Q of the curve arises, by the same formula, from several different points, P, of the line; this will be so if the parameter is not expressible as a single-valued (rational) function of the coordinates of the point of the curve. Then, to any point P, of the line, will correspond, first, a single point, Q, of the curve, and then, the set P, P', ..., of all the points of the line from which this point Q may equally be determined. The sets (P, P', ...) evidently form an involution on the line. In this case it was proved by Lüroth (*Math. Ann.* IX, 1876, p. 163; see *Principles of Geometry*, II, p. 136) that a new parameter can be chosen, a rational function of the original one, of which the coordinates of the point of the curve are rational functions, which is itself conversely expressible as a rational function of the coordinates of the point of the curve. This new parameter represents then the sets of the involution on the line, which is therefore said to be rational. Similarly, in the case of a surface, it may be that the coordinates of a point, Q, of the surface are known to be rational functions of two parameters, which we may interpret as the coordinates of a point, P, of a plane; and the functions may be such that to the point Q correspond reversely, not only P, but also other points, P', ..., of the plane; the sets (P, P', ...) then constitute an involution in the plane. In this case also, as has been stated, the sets of the involution can be represented uniquely, both directly and reversely, by the points R of another plane; thus the points Q of the surface are in unique correspondence with the points R of this other plane; and the surface is rational in this sense. But, contrary to what may seem natural, the same procedure is not possible for manifolds of more than two dimensions. For instance, consider, in space of four dimensions, the general cubic primal, that is the locus, of three dimensions, which is characterised by a single homogeneous equation of order 3 in the coordinates; it is easy, as we shall see, to express the coordinates of a point of the primal by rational functions of three parameters. But, interpreting these as coordinates of a point in space of three dimensions, two points of this space, at least, give rise to the same point of the primal.

The process which has been followed, in order to shew that any involution in a plane is rational, consists of two steps, which take the ideas we have explained in reverse order: (1), It can be shewn that any involution in a plane can be represented by an algebraic surface. This means (under the hypothesis, adopted throughout, that the points $(P, P', \dots)$ of a set of the involution, are determined uniquely from any one of these, P, by algebraic equations), that we can find a surface, of points Q, of which the coordinates are rationally determinable from those of the points $P, P', \dots$, of any set of the involution; and are, therefore, rational in the coordinates of any one point, P, of the set; while, conversely, any rational symmetric function of the coordinates of $P, P', \dots$ is expressible rationally by the coordinates of Q. This theorem, which holds equally, with obvious changes, for an involution in space of any number of dimensions, is often assumed as obvious. But we shall formulate a proof. We may speak of a surface of this character as *representing* the involution; it will have properties, as regards the curves which lie upon it, which arise from the existence of the involution. The next step, (2), consists in deducing from these properties, with the help of general properties of surfaces, that this representative surface is rational. Here we must be content merely to indicate what is necessary for the step (2) to be made.

By what has been said, an involution in a plane consists of sets of, say n, points, with, say, non-homogeneous coordinates $(x, y), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})$, such that any rational algebraic symmetrical function, of the coordinates of these n points, is expressible rationally in terms of the coordinates of any one of them, say of (x, y). We consider then, first, the whole aggregate of the functions which are rational algebraic symmetric functions of $(x, y), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})$; every function of this aggregate is a rational algebraic function of (x, y); and any rational function of two, or more, of the functions of this aggregate, is itself a function of the aggregate. Denote the aggregate of such functions by K. It may happen that there are two functions, belonging to the aggregate, in terms of which all other functions of the aggregate are expressible uniquely, and therefore rationally; this is the case when the involution is rational. In detail, however, let $u(x, y), v(x, y)$ be two functions of the aggregate K; and consider, for general assigned values of ξ and η, the solutions, for x and y, of the two equations $u(x, y) = \xi$, $v(x, y) = \eta$, excluding: (1), the solutions which do not vary with ξ, η; (2), the solutions which give indeterminate values for $u(x, y)$ or $v(x, y)$. By the nature of the aggregate K, if (x, y) be a solution of these two equations, so also are the other pairs

$(x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$ defining, with (x, y), a set of the involution; and if these are all, for a given pair ξ, η, then the variable point (ξ, η) represents a set of the involution, which is therefore rational. But, generally, the solutions of these two equations may be taken to consist of the points of several sets of the involution. Now, let $w(x, y)$ be any other function of the aggregate K; thus $w(x, y)$ has the same value at all the points of any single set of the involution. If these values of $w(x, y)$ are different for all the different sets of the involution which are given by the equations $u(x, y)=\xi$, $v(x, y)=\eta$, then the assignment of the value of the function $w(x, y)$, say ζ, together with the values ξ, η of $u(x, y)$ and $v(x, y)$, will suffice to identify just one set of the involution. Suppose the equations $u(x, y)=\xi$, $v(x, y)=\eta$ are satisfied by μ sets of the involution, and consider the μ values ζ of $w(x, y)$ which correspond thereto; any rational symmetric function, of these values of ζ, will be uniquely determinate when ξ, η are given, and will therefore be expressible rationally in terms of ξ and η. Thus there exists an equation $F(\zeta, \xi, \eta)=0$, rational in all of ζ, ξ, η, which becomes an identity when ξ, η, ζ are replaced by $u(x, y)$, $v(x, y)$, $w(x, y)$; and the order of this equation, in ζ, is μ when the μ sets in question lead to different values of ζ; but may be less than μ if $w(x, y)$ takes the same value for at least two of the sets of the involution which satisfy $u(x, y)=\xi$, $v(x, y)=\eta$. For simplicity, we prove that there certainly exist functions $w(x, y)$, in the aggregate K, for which the μ values in question are different. Let $\rho(x, y)$ be any rational function of (x, y); denote the μ sets (x, y) which satisfy the equations $u(x, y)=\xi$, $v(x, y)=\eta$, for a definite pair of values of ξ and η, by $(x, y), \ldots, (x_{n-1}, y_{n-1})$ and $(x^{(i)}, y^{(i)}), \ldots, (x_{n-1}^{(i)}, y_{n-1}^{(i)})$, with $i=1, 2, \ldots, (\mu-1)$. The function $w(x, y)$ which is the product of $\rho(x, y)$, $\rho(x_1, y_1), \ldots, \rho(x_{n-1}, y_{n-1})$ is a function of the aggregate K. Consider the μ sets, of each n equations,

$$\rho(x, y)=k, \ldots, \rho(x_{n-1}, y_{n-1})=k_{n-1};$$
$$\rho(x^{(i)}, y^{(i)})=k^{(i)}, \ldots, \rho(x_{n-1}^{(i)}, y_{n-1}^{(i)})=k_{n-1}^{(i)},$$

for $i=1, \ldots, (\mu-1)$; we assume that the arbitrary rational function $\rho(x, y)$ can be chosen so that these $n\mu$ equations are all satisfied for such values of the numbers k on the right that the μ products $kk_1 \ldots k_{n-1}$, $k^{(i)}k_1^{(i)} \ldots k_{n-1}^{(i)}$ are all different. The function $w(x, y)$, defined in terms of $\rho(x, y)$ as we have stated, has then different values for the chosen values of ξ and η; its values will therefore not be repeated for general values of ξ and η. We may thus assume that the equation $F(\zeta, \xi, \eta)=0$, is of order μ in ζ. Hence we can deduce that any function, say $t(x, y)$, of the aggregate K, is rationally expressible by $u(x, y)$, $v(x, y)$ and $w(x, y)$, when $w(x, y)$ is chosen as we have proved to be possible. The function $t(x, y)$ is in fact determinate when a set of the involution is given, and such a set is identified, we have remarked, when the values of $u(x, y)$, $v(x, y)$, $w(x, y)$ are given; the necessarily algebraical expression of $t(x, y)$, in terms of these three functions, is therefore rational. But we can give a rule for the computation of $t(x, y)$, as a rational function. Denote by $b_1, \ldots, b_\mu$ the values of $w(x, y)$ for the μ sets of the involution which satisfy $u(x, y)=\xi$, $v(x, y)=\eta$; the values of $t(x, y)$ for these sets are not necessarily all different; denote by $c_1, c_2, \ldots$ the various *different* values of $t(x, y)$. Let θ be a number which is different from every one of the finite number of fractions $(b_i-b_j)/(c_p-c_q)$. Then the values of $w(x, y)-\theta t(x, y)$ cannot be the same for any two of the μ sets in question, since $b_i-\theta c_p=b_j-\theta c_q$ is impossible (even when $c_p=c_q$). It follows then, by the argument above, that the function $w(x, y)-\theta t(x, y)$, or say $\zeta-\theta\tau$, satisfies an equation $F_1(\zeta-\theta\tau, \xi, \eta, \theta)=0$, rational in ξ, η, θ and $\zeta-\theta\tau$, and of order μ in the last. We can then suppose that $F(\zeta, \xi, \eta)=F_1(\zeta, \xi, \eta, 0)$. As the equation $F_1=0$ is unaffected when θ varies, it can be differentiated with regard to θ; this gives $-\tau\partial F_1/\partial\zeta+\partial F_1/\partial\theta=0$, which, putting $\theta=0$, gives the rational expression of τ in terms of ζ, ξ, η. In its application to the involution we may sum up this result by saying: *It is possible to find a surface of which the coordinates of any point are rational symmetric functions of the coordinates of the n points of a set of the plane involution; and any such function is expressible rationally by the coordinates of the representative point of this surface. The surface is thus an adequate reversible representation of the sets of the involution.*

The functions of the aggregate K, defined as rational symmetric functions of the coordinates of the n points of a set $(x, y), \ldots, (x_{n-1}, y_{n-1})$ of the involution, are all rational functions of x and y only, with the property of having the same value at all the points of such a set. Conversely, any rational function of x, y, say $u(x, y)$, which has this property, can be explicitly expressed as symmetric in the n points, in the form $[u(x, y)+\ldots+u(x_{n-1}, y_{n-1})]/n$. We may thus speak of the aggregate K as consisting of the (rational) *invariants* of the involution. As the aggregate consists of an infinite number of functions, it may be of interest to exhibit a practical method of computing a finite number of such invariants, from which the functions $u(x, y)$, $v(x, y)$, $w(x, y)$ of the preceding theory may be selected. Let θ be an undetermined number; consider the product of the n functions $t-(x+\theta y), \ldots, t-(x_{n-1}+\theta y_{n-1})$. This is of the form

$$\sum_{k=0}^{n} t^k[p+p_1\theta+\ldots+p_{n-k}\theta^{n-k}],$$

wherein, by the definition of the involution, $p, p_1, \ldots, p_{n-k}$ are expressible as rational functions of x and y; the product contains $2+3+\ldots+(n+1)$, or $\frac{1}{2}n(n+3)$, of these rational functions of x, y, of which, however, some may identically be constants. By the theory of symmetric functions, every rational symmetric function of $(x, y), \ldots, (x_{n-1}, y_{n-1})$ is expressible rationally by these $\frac{1}{2}n(n+3)$ functions (cf. Netto, *Algebra*, II, 1900, who refers to Poisson, *J. d. l'école polyt.* XI. Cah. (An X), p. 199). Thus every invariant of the involution is expressible rationally by these $\frac{1}{2}n(n+3)$ functions. The functions $u(x, y)$, $v(x, y)$, of our theory, were any two invariants of the involution; for these then we may take any two of the $\frac{1}{2}n(n+3)$ functions p_i which are not constants; the function $w(x, y)$ was an invariant, of sufficient generality, and is a rational function of the $\frac{1}{2}n(n+3)$ functions. All invariants of the involution are thus expressible rationally by any two, not constant, functions chosen from the $\frac{1}{2}n(n+3)$ functions p_i, and a further function, itself rationally expressible by these $\frac{1}{2}n(n+3)$ functions. The three chosen functions are connected by a rational equation. The necessary and sufficient condition for the involution to be rational is that all the invariants of the involution should be rationally expressible in terms of two properly chosen invariants; or, that all rational functions of the functions p_i should be rationally expressible in terms of two properly chosen rational functions of these; or, the condition is that the surface expressed by $F(\zeta, \xi, \eta)=0$ should be rational. The condition is satisfied when all the functions p_i are rational functions of two of these functions.

Ex. 1. Consider the simple involution of which the pairs of points, in non-homogeneous coordinates, are (x, y) and (x^{-1}, y^{-1}). In this case, two invariants of the involution are $u(x, y)$, $v(x, y)$ given by

$$u(x, y)=(x-1)^2(x+1)^{-2}, \text{ and } v(x, y)=(y-1)^2(y+1)^{-2}.$$

But the equations $u(x, y)=\xi$, $v(x, y)=\eta$ are satisfied by the two pairs of the involution $[(x, y), (x^{-1}, y^{-1})]$, $[(x^{-1}, y), (x, y^{-1})]$, or the number μ, of the theory above, is 2. If we take, however,

$$w(x, y)=(x-1)(y-1)(x+1)^{-1}(y+1)^{-1},$$

this is an invariant with different values for the two sets. Calling this last ζ, we have $\zeta^2=\xi\eta$, and every invariant is a rational function of ξ, η, ζ; in fact, therefore, is a rational function of ξ and ζ, the involution being rational. In this case, the product of $t-x-\theta y$ and $t-x^{-1}-\theta y^{-1}$ is $t^2-2tP+Q$, where P, Q, expressed in terms of ξ, η, ζ, are given, respectively, by

$$P=(1+\xi)(1-\xi)^{-1}+\theta(1+\eta)(1-\eta)^{-1},$$

$$Q=1+\theta^2+2\theta[(1+\xi)(1+\eta)-4\zeta](1-\xi)^{-1}(1-\eta)^{-1},$$

so that an inspection of P and Q might suggest the choice of ξ, η and ζ. Another set of three characteristic functions, for the involution, is given by ξ_1, η_1, ζ_1, where $\xi_1=\frac{1}{2}(x+x^{-1})$, $\eta_1=\frac{1}{2}(y+y^{-1})$, $\zeta_1=\frac{1}{2}(xy^{-1}+yx^{-1})$, which are connected by $\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2-2\xi_1\eta_1\zeta_1=1$, likewise representing a rational surface.

Ex. 2. Consider the involution, in space of three dimensions, with non-homogeneous coordinates x, y, z, of which a pair is (x, y, z), (y, x, mz^{-1}), in which m is a given rational and symmetric function of x, y, only. Four invariants of this involution are ξ, η, ζ, τ, given by

$$\xi = \tfrac{1}{2}(mz^{-1}+z), \qquad \eta = \tfrac{1}{2}(mz^{-1}-z)(x-y)(x+y)^{-1},$$
$$\zeta = -(x+y)^2(x-y)^{-2}, \quad \tau = x+y,$$

and these are a characteristic set, only one pair of the involution arising when the values of these four are given. For we have $xy = \tfrac{1}{4}\tau^2(1+\zeta^{-1})$, and m is thus a given rational function of ζ and τ. Whence, when ζ and τ are given, the couple (x, y) is given, save for ambiguity between this and (y, x); and m is given. Thus if ξ and η be also given, $mz^{-1}+z$ is given, and $mz^{-1}-z$ is given save for sign, the signs being interchanged by interchange of x and y; whence, the only set satisfying the equations when ξ, η, ζ, τ are given, beside (x, y, z), is (y, x, mz^{-1}). Thus, by what we have proved, all invariants of the involution are rationally expressible by ξ, η, ζ, τ, which are connected by the equation $\xi^2+\zeta\eta^2 = m$, where m is a given rational function of ζ and τ. For the involution to be rational, all invariants must be rationally expressible in terms of three properly chosen invariants; thus, in particular, ξ, η, ζ, τ must be rationally expressible in terms of three properly chosen rational functions of these; in other words, the manifold expressed, in the four-dimensional space in which ξ, η, ζ, τ are coordinates, by the equation $\xi^2+\zeta\eta^2 = m$, must be rational. Conversely, if this manifold be rational, three rational functions of ξ, η, ζ, τ exist—which therefore are three invariants of the involution—in terms of which ξ, η, ζ, τ, and therefore all other invariants, are rationally expressible, and the involution is rational. In this case, with p, q, r undetermined, the product of $t+px+qy+rz$ and $t+py+qx+rmz^{-1}$ is t^2+tP+Q, where $P = (p+q)\tau+2r\xi$, and

$$Q = \tfrac{1}{4}(p^2+q^2)\tau^2(1+\zeta^{-1})+\tfrac{1}{2}pq\tau^2(1-\zeta^{-1})+r^2m+pr(\xi+\eta)\tau+qr(\xi-\eta)\tau.$$

There are many cases for which the manifold $\xi^2+\zeta\eta^2 = R(\zeta, \tau)$, where R is a given rational function of the arguments ζ, τ, is rational. For instance, if $R = \tau^2+\zeta Z$, where Z is a given rational function of ζ only, the equation is $\xi^2-\tau^2 = \zeta(Z-\eta^2)$, and if we take $\xi+\tau = \theta\zeta$, $\xi-\tau = \theta^{-1}(Z-\eta^2)$, then ξ and τ are rational in η, ζ and θ. But it seems impossible that the manifold should be rational for all forms of the rational function R. There is thus an *a priori* presumption that not all involutions, except on a line or in a plane, are rational.

The consideration of the preceding involution is suggested by the case of the cubic primal, represented by the vanishing of a single polynomial of the third order in the coordinates, in space of four dimensions. This primal contains lines (in fact ∞^2); take one of these, and take two points on this line. The tangent primes of the primal at these two points intersect in a plane; this plane contains the line, and meets the primal, beside, in a conic. Conversely, an arbitrary point of the primal may be joined to the line by a plane. The coordinates of a point of the primal are thus expressible rationally by three parameters: one to determine a point of the conic, the other two to determine the points of intersection of the conic with the line. But these parameters are not rational functions of the coordinates of the point of the primal, there being no rational discrimination between the parameters of the two points of the line (cf. Richmond, *Proc. Camb. Phil. Soc.* xv, 1910, p. 116; Enriques, *Rend.... Lincei*, xxi, 1912, p. 81, who ascribes the preceding remark to Noether, and quotes, as another example, the three-fold intersection of a quadric and a cubic primal, in space of five dimensions, with reference to Fano, *Atti... Torino*, 1908).

The surface representing a plane involution cannot contain an irrational pencil of curves. Having thus dealt in detail with the representation of a plane involution by a surface (or, indeed, of any involution, existing in a linear space, by an appropriate manifold) we now approach the question whether such a surface is necessarily rational. We can prove without difficulty that the surface cannot be an irrational ruled surface, or indeed any surface upon which there exists an irrational pencil of curves. By a *pencil* of curves on a surface is meant a system of which there is just one curve of the system passing through an arbitrary point of the surface, so that the aggregate of the curves of the system is ∞^1. Such a system may be *linear*, being given by the intersection of the surface with a, singly infinite, linear family of surfaces, with equation of the form $\phi+\lambda\psi=0$; when the surface under consideration is in higher space, the linear pencil is given by a family of *primals*, of equation $\phi+\lambda\psi=0$. But the pencil may not be linear: it may be that there exist two definite rational functions of the coordinates, ϕ and ψ, not necessarily polynomials, whose values on the surface are connected by a definite irreducible rational equation $F(\phi, \psi)=0$, satisfied identically in virtue of the equations which determine the surface; then, if ξ, η be numbers connected by the equation $F(\xi, \eta)=0$, part (at least) of the intersection of the given surface with the primal whose equation is $\phi=\xi$, is a curve lying also on the primal $\psi=\eta$. In other words, if $F(\xi, \eta)=0$ be regarded as the equation of a plane curve, to every point (ξ, η), of this curve, corresponds a curve on the surface, upon which both $\phi=\xi$ and $\psi=\eta$; conversely, in virtue of the identity $F(\phi, \psi)=0$ on the surface, the values which $\xi=\phi$ and $\eta=\psi$ take, at an arbitrary point of the surface, correspond to a point of the curve $F(\xi, \eta)=0$, and lead to a curve of the system passing through this point. The system of curves is thus ∞^1, and one curve of the system, lying on the surface, passes through every point of the surface. When the curve $F(\xi, \eta)=0$ is rational, there is a rational function of ξ, η, say $\theta=r(\xi, \eta)$, in terms of which ξ, η have such rational expressions that the condition $F(\xi, \eta)=0$ is identical, irrespective of the equations which determine the surface; in this case, to any point of the curve $F(\xi, \eta)=0$, characterised by the value of θ, there is a curve on the surface upon which $\theta=r(\phi, \psi)$; and the equations $\phi=\xi$, $\psi=\eta$ are a consequence of this. The system of curves then becomes the linear ∞^1 system given, for variable θ, by $\theta=r(\phi, \psi)$; this is the first case considered. If, however, $F(\xi, \eta)=0$ be not a rational curve, we have an *irrational pencil* of curves on the surface. The most familiar example of the theory is the case of a ruled surface in ordinary space, for which $F(\xi, \eta)=0$ determines the plane section, and $\phi=\xi$, $\psi=\eta$ determine a generator. Conversely it will subsequently (see Note, at the end of the Chapter) be noticed that a surface upon which there exists a pencil of curves of which every one is rational, is birationally reducible to a ruled surface (or is a rational surface, when the pencil of curves is rational, or linear).

After this explanation of an irrational pencil of curves on a surface, we prove now that a surface on which such a pencil exists cannot be the representative surface for a plane involution. For clearness, let the coordinates in the plane be X, Y. The coordinates of a point of the surface will be rational in X, Y, and unaltered when (X, Y) is replaced by any other point of the set of the involution determined by (X, Y). A curve of an irrational pencil, supposed to exist on the surface, given by $\xi=\phi$, $\eta=\psi$, will thus give rise, on the plane, to a curve which satisfies equations of the forms

$$\xi=\Phi(X, Y), \quad \eta=\Psi(X, Y),$$

wherein Φ, Ψ are rational in their arguments; and there will exist an identical equation $F[\Phi(X, Y), \Psi(X, Y)]=0$, in which X, Y are arbitrary. This is impossible unless the curve $F(\xi, \eta)=0$ is a rational curve; it can conceivably hold for a rational curve for a particular numerical value of one of X, Y. The representative surface of the involution cannot, therefore, contain an irrational pencil of curves.

The rationality of the representative surface of a plane involution. The original very remarkable proof given by Castelnuovo in 1893 (*Math. Ann.* XLIV, pp. 145–55) that the representative surface of a plane involution is rational, has in effect been modified by subsequent work of the same writer, and of Enriques; first, by an investigation, given by Castelnuovo, of the conditions in general for a surface to be rational (*Mem. d. Soc. Ital. d. Scienze*, X, 1896, pp. 103–23); and then, by the results given in a paper by Castelnuovo and Enriques (*Ann. d. Mat.* VI, 1901). For any linear system of curves in a plane, without common base points, the number of points in a canonical set on any one of the curves is less than the number of intersections of this curve with any other curve of the system; and, for a linear system with assigned base points a "completed" canonical set likewise contains less points than the "completed" number of intersections of the curve with another curve of the system; here the completed canonical set consists of the whole number of intersections of the curve with the adjoint curve defining the set, including those at the base points, and the completed number of intersections of two curves includes their intersections at the base points. In the last paper referred to it is proved, on the basis of several particular theorems already known, that, if a surface, which has no multiple points, contain a linear system of curves for which the completed canonical sets contain a number of points less than the completed number of intersections of two curves of the system, then such a surface either contains an irrational pencil of curves, or is a rational surface. It had been proved in Castelnuovo's original paper of 1893, that the representative surface of a plane involution necessarily contains a linear system of curves for which the inequality above explained is fulfilled. We have proved that the surface does not contain an irrational pencil of curves. We assume that a representative surface can be found which has no multiple points. The conclusion, then, is that the surface is rational; and, therefore, as explained above, the plane involution is also rational. The general condition given above may be enunciated also thus: consider a surface, in space of any dimensions, not having multiple points; let the surface contain a linear system of curves with assigned base points, of typical multiplicity t; let these curves be of genus p, and n be the grade of the system, or the number of intersections of two curves of the system outside the base points; define the completed genus, π, of a curve, and the completed grade, ν, by $\pi = p + \frac{1}{2}\Sigma t(t-1)$, $\nu = n + \Sigma t^2$, and call $2\pi - 2 - \nu$ the *canonical number* of a curve of the system (see p. 222, below). We find $2\pi - 2 - \nu \leqslant 2p - 2 - n$. Then, *if there be such a linear system for which the canonical number is negative, the surface may be birationally transformed to a ruled surface.* The theorem is true when the linear system is of zero freedom; that is, consists of a single isolated curve, provided $\pi > 0$. The less detailed form of the result, which we have stated above, namely that a surface with the specified linear system of curves must contain an irrational pencil of curves, or be rational, is sufficient for our present purpose. We cannot here give the proof of this theorem. But we may give, in barest outline, the proof that the representative surface of a plane involution contains a linear system of curves of negative canonical number. For brevity, let two points of the plane which belong to the same set of the involution be called *conjugate*. To any curve on the representative surface will correspond a curve of the plane, with the property that all the conjugate points, of any point of the curve, equally lie on this curve; conversely any curve of the plane with this property corresponds to a curve of the surface; we may describe such a curve of the plane as "belonging to" the involution. A particular curve of this character is obtainable by taking an arbitrary line u, and then the curve ϕ which is the locus of the sets of points conjugate to all the points of the line; and then considering the composite curve $u + \phi$. Let k be the order of the curve of united points of the involution in the plane, the locus of points of which each coincides with one of its conjugate points; let j be the number of pairs of conjugate points which lie on an arbitrary line of the plane (called by Caporali the *class* of the involution; *Rend....Napoli*, XVIII, 1879, p. 212). The curve ϕ meets the line u in its k intersections with the locus of united points, and also in the j pairs of conjugate points lying on u. Thus ϕ is of order $k+2j$ and $u+\phi$ is of order $k+2j+1$. The irreducible curve $u+\phi$ defines a linear system of curves in the plane, all belonging to the involution (having base points at the exceptional points of the plane of which the conjugate points are not a finite set but all the points of an exceptional curve). From this system there arises a linear system of curves on the representative surface. *This is the system of which we can prove the canonical number to be negative.* The number of intersections of two curves of this system, on the surface, is equal to the number of sets of the involution, in the plane, which are common to the degenerate curve $u+\phi$, and to a general curve, of order $k+2j+1$, of the linear system defined thereby; from the nature of $u+\phi$, any set lying thereon has one point on u; the number of sets required is thus the number of intersections, with u, of a curve whose order is $k+2j+1$. The genus of the curves on the surface which correspond to the plane curves defined by $u+\phi$ is more difficult to find; it is in fact equal to j. As $2j-2-(k+2j+1)$, equal to $-(k+3)$, is negative, it follows that the canonical number in question is certainly negative.

Ex. 1. A very simple example is the involution before taken (p. 136) of which, in homogeneous coordinates, the sets are the pairs (x, y, z), (x^{-1}, y^{-1}, z^{-1}). The points conjugate to the points of a line u, of equation $lx+my+nz=0$, then describe the conic ϕ given by $lx^{-1}+my^{-1}+nz^{-1}=0$. The system defined by the curve $u+\phi$ then consists of the cubic curves given by $A(xz^{-1}+x^{-1}z)+B(yz^{-1}+y^{-1}z)+C(xy^{-1}+x^{-1}y)+D=0$; these have 3 base points, and any two of them meet beside in three pairs of the involution. With the invariants before used, $\xi=(x-z)^2(x+z)^{-2}$, $\eta=(y-z)^2(y+z)^{-2}$, $\zeta=(x-z)(y-z)/(x+z)(y+z)$, we have

$$xz^{-1}+x^{-1}z=2(1+\xi)(1-\xi)^{-1},$$
$$xy^{-1}+x^{-1}y=2[(1+\xi)(1+\eta)-4\zeta](1-\xi)^{-1}(1-\eta)^{-1},$$

and the system of cubic curves corresponds to the intersection of the cone $\xi\eta-\zeta^2=0$ with the quadric surfaces

$$P(\xi-\zeta)+Q(\eta-\zeta)+R(\xi\eta-\zeta)+H(1-\zeta)=0,$$

that is to a system of elliptic quartic curves. These curves contain the point $(1, 1, 1)$ and have, beside, two fixed points of contact on the cone. For this involution $k=0$, the united points being the four points $(\pm 1, \pm 1, 1)$; further, on an arbitrary line there lies one pair of conjugate points, or $j=1$. Thus j is the genus of the curves in which the cone is met by the system of quadric surfaces, and $2j+1$ is the number of variable intersections of two of these curves.

Ex. 2. For the involution of sets of five points obtainable from one point (x, y) of the set by repetitions of the transformation

$$x'=y,\quad y'=(1-x-y)/(1-x),$$

(Ex. 3, p. 121), obtain a representative surface, shewing that this is rational, and contains a linear system of curves of negative canonical number.

Number of cyclical sets in a correspondence. We have noticed that particular involutions are obtainable by repetitions of a (single valued, algebraic) transformation which is periodic (as in Ex. 3, p. 121). More generally, now, consider a correspondence between pairs of points, of such character that the number of coincidences is finite; we can find a formula for the number of cyclical sets, to be defined, contained therein.

In the first place, for definiteness, consider such a correspondence as we have already considered (Chap. II, p. 92), of ∞^r pairs of points in space $[r]$, in which, to any point P, corresponds a finite number of points Q, and, reversely, to Q, corresponds a finite number of points P. Denote the correspondence by T; assume the correspondence to be such that P can be at any point of the space; then, when P takes the position of any one of the points Q which correspond to the original P, there will correspond points R, which we may denote by $T(Q)$; as already explained (p. 6), the aggregate of all such points R, for all the points Q, or $T(P)$, may be denoted by $T^2(P)$, and regarded as arising from P by the forward correspondence T^2. Assume then, further, that the correspondence is such that the numbers of coincidences in all the successive correspondences $T, T^2, T^3, \ldots$ are finite. We have proved, then (p. 99), that the number of coincidences for the correspondence T is $g(T)$, where $g(T)=(0)(r)'+(1)(r-1)'+\ldots+(r-1)(1)'+(r)(0)'$, in which $(0)(r)'$ denotes the number of points Q which correspond to an arbitrary point P, similarly $(r)(0)'$ denotes the number of points P which correspond to an arbitrary point Q, in the reverse correspondence; and, more generally, $(i)(j)'$ means the number of pairs of corresponding points P, Q, of which P lies in an arbitrary space of i dimensions, $[i]$, while Q lies in an independent arbitrary space $[j]$, with $i+j=r$. The correspondence being supposed to be expressed by algebraic equations, the number $(i)(j)'$ is the order of the algebraic manifold described by all the points Q when P describes the linear space $[i]$. As a simple example, consider the correspondence in a plane, between points (x, y) and (ξ, η), which is determined by equations $x^p=\xi^r$, $y^q=\eta^s$, wherein p, q, r, s are positive integers; then, as (x, y), or P, describes an arbitrary line $lx+my+n=0$, the rs points (ξ, η), or Q, which correspond thereto, describe a curve whose equation is obtained by rationalising the equation $l\xi^f+m\eta^g+n=0$, where $f=rp^{-1}$ and $g=sq^{-1}$. Suppose now, with a slight change of notation, that the points Q arising from P, are denoted by $Q^{(1)}$; we may write $(Q^{(1)})=T(P)$; next that all the points $T(Q^{(1)})$, arising from the aggregate of points $Q^{(1)}$, are denoted by $Q^{(2)}$, so that we write $(Q^{(2)})=T^2(P)$, and so on. Finally, that among the points $(Q^{(\nu)})$ given by $T^\nu(P)$, there is one particular point $Q^{(\nu)}$ which coincides with P; this is then a coincidence of the correspondence T^ν, and there is a succession of $\nu+1$ particular points $P, Q^{(1)}, \ldots, Q^{(\nu)}$ such that $Q^{(1)}=T(P)$, $Q^{(2)}=T(Q^{(1)}), \ldots$, $Q^{(\nu)}=T(Q^{(\nu-1)})$; moreover, the ν first points of this set will be such that, starting from *any one* of them, we can come back to it by ν successive applications of the correspondence T. In saying this, it is supposed that no one of the points $Q^{(1)}, Q^{(2)}, \ldots, Q^{(\nu-1)}$ coincides with P. Such a set as $P, Q^{(1)}, \ldots, Q^{(\nu-1)}$ is then called here a cyclical set. The result to which we proceed gives the number of *cyclical* sets which exist for a given value of ν, supposing the numbers of coincidences $g(T), g(T^2), \ldots$, for the various powers of T, to be known. For example, if T be a Cremona transformation in a plane, of order n, that is, a $(1, 1)$ correspondence in which the point Q describes a curve of order n when P describes a line, then, in the most general case, the correspondence T^2 is of order n^2; and, by the formula $(0)(2)'+(1)(1)'+(2)(0)'$, the number of coincidences for this correspondence T^2 is $1+n^2+1$. This, however, includes the $1+n+1$ coincidences for T itself. Therefore the number of coincidences *proper* to T^2, is n^2-n; and hence the number of cyclical sets of 2 points belonging to T is $\frac{1}{2}(n^2-n)$.

In general, among the $g(T^\nu)$ coincidences of T^ν will be found, not only the coincidences of T itself, but also the points of any existing cyclical set of d points, of which d is a divisor of ν; in particular the points of a cyclical set of ν points. Let γ_i denote the number of cyclical sets of i points; any such set gives i coincidences. Hence, if we put $p(T^i)$ for $i\gamma_i$, we have the equation

$$g(T^\nu)=p(T)+\Sigma p(T^d)+p(T^\nu),$$

wherein d denotes in turn every divisor of ν other than unity and ν itself.

The equations of this form, for successive values of ν, enable us to express $p(T^\nu)$ in terms of functions $g(T^i)$. We have in fact the theorem (*Proc. Lond. Math. Soc.* XXI, 1891, p. 30): If $f(a_1, \ldots, a_p)$ be any number uniquely determinate from the numbers $a_1, \ldots, a_p$, and symmetrically dependent thereon, and we put

$$F(a_1, \ldots, a_m)=f(a_1, \ldots, a_m)+\underset{\mu}{\Sigma\Sigma}\, f(a_i, a_j, \ldots, a_k)+\Sigma f(a_s)+f(0),$$

where $a_i, a_j, \ldots, a_k$ are in turn every $m-\mu$ of $a_1, \ldots, a_m$, and $f(0)$ is arbitrary, then, conversely,

$$F(a_1, \ldots, a_n)=F(a_1, \ldots, a_n)+\underset{\mu}{\Sigma\Sigma}\,(-1)^\mu F(a_p, a_q, \ldots, a_r)$$
$$+(-1)^{n-1}\Sigma F(a_s)+(-1)^n F(0),$$

where $F(0)=f(0)$, and $p, q, \ldots, r$ are in turn every selection of $n-\mu$ of $a_1, \ldots, a_n$. As a consequence of this: let $f(n)$, $F(n)$ be two determinate functions of the positive integer n, such that

$$F(n)=f(1)+\Sigma f(d)+f(n),$$

where d denotes in turn all the divisors of n, other than unity and n itself; also, let the expression of a positive integer $\nu(>1)$, in terms of its prime factors (>1), be $\nu = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$; then we have

$$f(\nu) = F(\nu) - \sum_{i=1}^{s} F(\nu/p_i) + \sum_{i \neq j} F(\nu/p_i p_j) + \ldots + (-1)^s F(\nu/p_1 \ldots p_s);$$

cf. Dedekind, *Crelle*, LIV, 1857, p. 25; Moebius, *Werke*, IV, 1887, p. 591. Hence, we have the result

$$p(T^\nu) = g(T^\nu) - \Sigma g(T^{\nu_i}) + \Sigma g(T^{\nu_{i,j}}) - \ldots + (-1)^s g(T^{\nu_{1,2,\ldots,s}}),$$

where ν_i has all the values ν/p_i, $\nu_{i,j}$ has all the values $\nu/p_i p_j$, $\ldots$, $\nu_{1,2,\ldots,s}$ has all the values $\nu/p_1 \ldots p_s$.

We make application of this to the case of the correspondence of ∞^r pairs of points in space $[r]$, referred to, under the hypotheses we have stated. On the assumption, already made, that the number of coincidences in the correspondence T^ν is finite, this number, $g(T^\nu)$ is, in fact, given by

$$g(T^\nu) = [(0)(r)']^\nu + [(1)(r-1)']^\nu + \ldots + [(r-1)(1)']^\nu + [(r)(0)']^\nu.$$

For, if the locus of points Q, corresponding to P in the correspondence T, when P describes a linear space $[i]$, be of order k, namely $(i)(r-1)'$, then, when P describes a manifold, M_i^m, also of dimension i, but of order m, the points Q will describe a locus of order mk. Whence, considering the sequence of corresponding points, P, $Q^{(1)}$, $Q^{(2)}$, $\ldots$, before described, when P describes a space $[i]$, the point $Q^{(1)}$ will describe a locus of order k, thence $Q^{(2)}$ will describe a locus of order k^2, and so on, and finally $Q^{(\nu)}$ will describe a locus of order k^ν. We can then use this form for $g(T^\nu)$ in the formula for $p(T^\nu)$.

A particular case is that of a general Cremona transformation in a plane, for which $(0)(2)' = (2)(0)' = 1$, and $(1)(1)'$ is the order of the transformation, say n. Then $g(T^\nu) = n^\nu + 2$. But, when $\nu > 1$, the addition to $g(T^\nu)$ of a number independent of ν does not affect the value of $p(T^\nu)$, since, for $s > 0$, the sum, for $i = 0, \ldots, s$, of the terms $(-1)^i (s, i)$, in which (s, i) is the binomial coefficient, is equal only to $(1-1)^s$ or zero. Hence, in general, for $\nu > 1$,

$$p(T^\nu) = n^\nu - \Sigma n^{\nu_i} + \Sigma n^{\nu_{i,j}} - \ldots + (-1)^s n^{\nu_{1,\ldots,s}},$$

where $\nu_i = \nu/p_i$, $\nu_{i,j} = \nu/p_i p_j$, $\ldots$, $\nu_{1,\ldots,s} = \nu/p_1 \ldots p_s$. And the number of cyclical sets is $p(T^\nu)/\nu$. Thus the number of cyclical sets of two points is $\frac{1}{2}(n^2 - n)$; and, for example, is 1 for a quadratic Cremona transformation. If this be expressed by $ux' = vy' = wz'$, where u, v, w are linear in x, y, z, such a cyclical pair must also satisfy $u'x = v'y = w'z$; the elimination of x', y', z', from these, leads to two cubic curves having in common the three exceptional points $v = 0 = w$, $w = 0 = u$, $u = 0 = v$, and the four coincidences of the original transformation given by $ux=vy=wz$; the two remaining intersections constitute the cyclical pair. For cyclical sets of three points the number is $\frac{1}{3}(n^3-n)$, which is 2 for a general quadratic transformation. More generally, in space $[r]$, with coordinates $(x_0, \ldots, x_r)$, let $y_0, \ldots, y_r$ denote general (homogeneous) linear functions of $x_0, \ldots, x_r$, and $y_0', \ldots, y_r'$ the same linear functions of $x_0', \ldots, x_r'$; we may consider the correspondence between $(x_0, \ldots, x_r)$ and $(x_0', \ldots, x_r')$ given by the equations

$$y_0'=x_0^{-1},\ y_1'=x_1^{-1},\ \ldots,\ y_r'=x_r^{-1}.$$

When $(x_0', \ldots, x_r')$, and therefore also a point $(y_0', \ldots, y_r')$, describes a linear space of dimension $r-s$, the point $(x_0, \ldots, x_r)$ will describe a manifold of $r-s$ dimensions given as the intersection of s primals each given by an equation of the form

$$a_{i0}x_0^{-1}+a_{i1}x_1^{-1}+\ldots+a_{ir}x_r^{-1}=0, \qquad (i=1, \ldots, s).$$

It can be shewn that this manifold is of order (r, s), that is $r!/s!(r-s)!$, having the vertices of the fundamental simplex as multiple points of order $(r-1, s)$, the edges of this simplex as multiple lines of order $(r-2, s)$, and so on, in general the bounding spaces $[k]$ of the simplex multiple of order $(r-1-k, s)$, the bounding spaces $[r-s-1]$ being simple thereon. Thus, in this case we have $g(T^\nu)=1+(r, 1)^\nu+(r, 2)^\nu+\ldots+(r, 1)^\nu+1$. In particular, in ordinary space, we have $g(T^\nu)=2+2.3^\nu$, and the number of cyclic sets, of ν points, is $2\nu^{-1}[3^\nu-\Sigma.3^{\nu_i}+\Sigma.3^{\nu_{i,j}}-\ldots+(-1)^s 3^{\nu_{1,\ldots,s}}]$. This result was given by S. Kantor, *Compt. rend.* xc, 1880, pp. 1156–8, with proof. The result he states without proof for $r>3$ is, however, not obtained by the formula given above.

Another application of the ideas developed here is to a *general* correspondence of two points on a curve of genus p, for which the indices are α, β and the valency is γ. In this case

$$g(T^\nu)=\alpha^\nu+\beta^\nu-2p(-\gamma)^\nu$$

(Severi, *Mem.... Torino*, LI, 1902, p. 88), under the same hypothesis as before, that the number of coincidences of T^ν is finite; an exception arises, for example, if, among the points corresponding to Q, where $Q=T(P)$, the point P occurs for every P, in which case the number of cyclical sets of two points is infinite.

Note in regard to surfaces containing a pencil of rational curves. In connexion with the proof that the surface representing a plane involution is rational, it is interesting to bear in mind the theorem that a surface having a pencil of rational curves is either rational, when the pencil is rational, or is birationally equivalent to a ruled surface whose genus is the same as that of the pencil of curves. We remark upon the proof of this theorem.

Lemma. Upon a rational curve, in space $[r]$, of order n, there exists a linear series of sets of $n-2$ points, of freedom $n-2$, a g_{n-2}^{n-2}, simple and without fixed point, which is determinate without any irrationality, when the curve is given. For a rational plane curve of order n, with multiple points, typically i-fold, supposed resolved in Noether's manner, we may assume, (*a*), that

$$\Sigma i(i-1)=(n-1)(n-2);$$

(*b*), that the number of independent conditions to be satisfied at the multiple points by an adjoint curve is not greater than $\frac{1}{2}\Sigma i(i-1)$; (*c*), that an adjoint curve, as involving symmetrically the whole set of multiple points, is determinable rationally. Thence, considering adjoint curves of order $n-2$, these determine a series of sets of each $n(n-2)-\Sigma i(i-1)$ or $n-2$ points, with freedom at least $\frac{1}{2}(n-2)(n+1)-\frac{1}{2}(n-1)(n-2)$, or $n-2$; and, therefore, with exactly this freedom (so that the multiple points furnish independent conditions for the adjoint curve). This is the series g_{n-2}^{n-2} in question.

For a rational curve of order n, in space $[r]$, we may assign rationally a space $[r-3]$; there are then the equivalent of $\frac{1}{2}(n-1)(n-2)$ chords of the curve which meet this space, because the curve can be projected from this space $[r-3]$ into a plane curve of order n with the equivalent of $\frac{1}{2}(n-1)(n-2)$ double points. We may then consider the most general cone of order $(n-2)$, having this space $[r-3]$ as vertex, described to pass through these $\frac{1}{2}(n-1)(n-2)$ chords. Thereby we obtain a g_{n-2}^{n-2} upon the curve. This series is evidently simple and without fixed points, because its freedom is equal to its grade.

Application of the lemma. Provided $n-2 \geqslant 2$, the series can be used (as in Vol. v, Chap. I, p. 7) to make a (1, 1) birational transformation of the curve to one of order $n-2$; and this process can be continued until the given curve is transformed either into a rational cubic curve, or into a conic. Thus we have the inference: The expression of a rational curve in terms of a variable parameter involves no irrationality in the coefficients of the equations which determine the curve if we assume the coordinates of a single point upon the curve to be given, when necessary (as in the case of a conic).

Surfaces with a rational pencil of rational curves. That such surfaces are rational was proved by Noether (*Math. Ann.* III, 1870, p. 161; wherein the preceding lemma for plane curves is given). His proof consists in establishing that there is upon the surface a rational curve, determinable rationally, which does not belong to the pencil of curves, but meets every curve of the pencil in one point. The points of the surface can then be expressed rationally (and reversibly) by two parameters, one determining the position of the point upon a curve of the pencil, the other determining the position of the intersection of this curve with the rational director curve, upon this director curve.

To illustrate the argument consider the case of a surface of order n, in ordinary space, which has an $(n-2)$-fold line; this surface contains the linear pencil of conics obtained by planes through the axis. Its equation may be supposed of the form

$$z^2a+2zth+t^2b+zA+tB+C=0,$$

wherein a, h, b, A, B, C are homogeneous polynomials in x, y, the first three of order $n-2$, while A, B are of order $n-1$, and C is of order n. A provisional proof of the existence of a suitable rational director curve upon this surface, which can be confirmed by more detailed examination, is easily given. Let u, v be homogeneous polynomials in x, y of order $n-2$, and w a similar polynomial of order $n-3$. The equations $z=u/w$, $t=v/w$ then represent a rational curve of order $n-2$, having the axis $x=0=y$ as an $(n-3)$-fold chord. That this curve lies on the surface requires the identical vanishing of a homogeneous polynomial, in x, y, of order $3n-6$; and there are in u, v, w just $3n-5$ (non-homogeneous) disposable coefficients with which to effect this, by the solution of linear equations. With u, v, w so determined, put $y=1$ for simplicity of statement, denote u/w by ζ and v/w by τ; and substitute, in the equation of the surface, $t=\tau+\theta(z-\zeta)$. Then, by use of

$$C=-[a\zeta^2+2h\zeta\tau+b\tau^2+A\zeta+B\tau],$$

after removing the factor $z-\zeta$, it is possible to express z as a rational function of x and θ; hence t is also such a function, and the surface is proved to be rational.

Surfaces with an irrational pencil of rational curves. In accordance with a paper of Enriques (*Math. Ann.* LII, 1899) there exists also a rational director curve, cutting each curve of the pencil in one point, when this pencil of curves is not rational. The point of the surface is then expressible once more in terms of two parameters, one expressing the position upon a rational curve of the pencil, the other expressing (irrationally) the curve of the pencil on which the point is taken. This is the general character of a ruled surface. Reference may be made (as Enriques remarks) to Painlevé's Stockholm lectures on Differential Equations (1895, lithographed, Paris, 1897, pp. 255–341), where a surface is under consideration which is capable of birational transformation into itself by a finite continuous group of transformations. Upon this surface there are rational curves, the locus of a point under a one-parameter subgroup; and upon each such curve are two united points. The locus of one of these gives the unisecant curve required for Enriques' demonstration, whose existence then, in Painlevé's case, does not require the proof which is necessary in general.

CHAPTER IV

PRELIMINARY PROPERTIES OF SURFACES IN THREE AND FOUR DIMENSIONS

WE consider an algebraic surface of order n, in ordinary space, in which the coordinates are x, y, z, t; and certain properties of the tangent cone drawn to this surface from an arbitrary point. We suppose, unless the contrary be stated, that the multiple points of the surface consist at most of the points of a double curve, whereon are triple points of the surface, through which pass three sheets of the surface; so that such points are also triple points of the double curve, with tangents not generally lying in a plane. Reasons will be given later for the conclusion that any surface can be birationally transformed to a surface of this character, so that the supposition involves no essential limitation of generality in the surfaces dealt with. At the same time, it is often convenient to suppose that, beside the double curve, the surface considered has isolated multiple points; and this possibility will also be taken into account in some of the results given.

Some familiar facts may be recalled first, particularly for the case when there is no such double curve, the surface being expressed by a polynomial equation of complete generality in the coordinates. At a general point of the surface there is a plane, called the *tangent plane*, which is the locus of lines through the point which have two intersections with the surface coinciding at this point; such a line has then $n-2$ further intersections with the surface. Thus the tangent plane meets the surface in a curve having a double point at the point of contact, whereof the tangents, each meeting the surface in three points coinciding at the point, are in general distinct. These two lines are the generating lines, at this point, of the quadric surface—called the *polar quadric* of the point of contact (x_0, y_0, z_0, t_0)—whose equation is $(x\,\partial/\partial x_0)^2 f(x_0, y_0, z_0, t_0)=0$, where $f(x, y, z, t)=0$ is the equation of the surface, and $x\,\partial/\partial x_0$ stands for $x\,\partial/\partial x_0+y\,\partial/\partial y_0+z\,\partial/\partial z_0+t\,\partial/\partial t_0$; this quadric surface touches the tangent plane of the given surface at the point (x_0, y_0, z_0, t_0). The two particular lines are called the *asymptotic* or the *inflexional* lines of the surface at this point. The statement is easily proved by considering the equation in λ, $f(\lambda x_0+x, \ldots, \lambda t_0+t)=0$, which defines the intersections of the surface with the line joining the points (x_0, y_0, z_0, t_0), (x, y, z, t). Further, if (x', y', z', t') be another point of the surface, the line in which the tangent plane at (x_0, y_0, z_0, t_0) is met by the tangent plane at (x', y', z', t'), has for limit, as the latter point approaches (x_0, y_0, z_0, t_0), a line through this, whose direction is the harmonic conjugate, in regard to the inflexional lines thereat, of the direction in which (x', y', z', t') approaches (x_0, y_0, z_0, t_0). Two such lines, in the tangent plane at (x_0, y_0, z_0, t_0), which are harmonic conjugates in regard to the inflexional lines, are often called simply *conjugate* lines at the point. At a less general point of the surface it may happen that the inflexional lines coincide, the section by the tangent plane, at this point, having here a cusp; then the polar quadric of this point, having two intersecting generators coincident, must be a cone (and conversely). If the polar quadric be written $a_{11}x^2 + \ldots + 2a_{12}xy + \ldots = 0$, the condition for this is the vanishing of the determinant $|a_{ij}|$, of which each element is a partial differential coefficient, of the second order, such as $\partial^2 f/\partial x_0{}^2$ or $\partial^2 f/\partial x_0 \partial y_0$, if the point in question be (x_0, y_0, z_0, t_0). Thus this point lies on a surface of order $4(n-2)$; this surface is called the *Hessian* of the original; its intersection with the original surface is called the *parabolic curve* on the surface; at a point of this curve the tangent plane is said to have *stationary contact* with the surface. We may speak of the line, in which the two inflexional lines at a parabolic point coincide, as the *parabolic* line, to be distinguished from the tangent line of the parabolic curve at this point. At a parabolic point one of two conjugate directions, defined as above by a harmonic pencil, must be along the parabolic line; so that the tangent planes, at *all* points (x', y', z', t') of the surface, ultimately pass through this parabolic line as (x', y', z', t') approaches the parabolic point.

Ex. Of the inflexional lines of the general surface $f=0$, at points of the curve in which this surface is met by another surface of order n', there are $nn'(3n-4)$ which meet an arbitrary line, and $nn'(3n+2n'-8)$ which touch the common curve, in general.

We enquire now: (i), what is the number of the inflexional lines of the surface which pass through an arbitrary point of space, remarking that, as there are ∞^2 inflexional lines and two conditions are required for a line to contain a given point, we may expect a definite number of such lines; and (ii), what is the number of tangent planes of the surface, having stationary contact with this, which pass through an arbitrary point of space, remarking that there are ∞^1 such tangent planes and one condition is required for a plane to contain a given point?

As to the enquiry (i), in order that an inflexional line, at a point (x, y, z, t) of the surface, may pass through a point (ξ, η, ζ, τ), this last must lie both on the tangent plane and on the polar quadric of (x, y, z, t). This point must then satisfy the two equations denoted by $(\xi\,\partial/\partial x)f=0$ and $(\xi\,\partial/\partial x)^2 f=0$; these equations, when (x, y, z, t) are regarded as current coordinates, which represent what are called respectively the *first* and *second polars* of (ξ, η, ζ, τ) in regard to the surface, are of orders $n-1$ and $n-2$; the common curve of these polars thus meets $f=0$ in $n(n-1)(n-2)$ points. Conversely, when the surface has no multiple points, the conditions ensure that an inflexional line, at the point (x) of the surface, passes through the point (ξ), and the number of such lines is $n(n-1)(n-2)$. But, if the surface has a double curve, there is a difference; in this case, the equation $(\xi\,\partial/\partial x)f=0$ is satisfied, wherever (ξ) may be, for all points (x) of the double curve; and the enveloping cone, of tangent planes of the surface which pass through (ξ), breaks up into the cone joining (ξ) to the double curve, and another *proper* enveloping cone; the first polar of (ξ) contains the double curve, and its remaining intersection with the surface constitutes the *proper curve of contact* of the enveloping cone from (ξ); thus if the order of this proper curve of contact be denoted by μ_1, and the order of the double curve by ϵ_0, we have $n(n-1)=\mu_1+2\epsilon_0$. There is likewise change as to the intersection of the surface with the second polar of (ξ): Evidently, if the second polar of (ξ) pass through a point (x), then the polar quadric of (x) passes through (ξ). Now, at any point of the double curve, the polar quadric of this point breaks up into the two tangent planes of the surface at this point, as may be seen by taking particular coordinates with origin at this point, it being known that the polar surfaces are unaffected by linear transformation of the coordinates; and, of the ∞^1 tangent planes of the surface at points of the double curve, there will be a number passing through an arbitrary point (ξ). Thus, the second polar of (ξ) passes through as many points of the double curve. If the number of such points be denoted by ρ, it appears on examination that the number of intersections of the second polar of (ξ), with the proper curve of contact, at ordinary points of this, is lessened precisely by ρ; and hence that the number of points of the surface at which an inflexional tangent passes through (ξ) is $\mu_1(n-2)-\rho$. In general we denote the number of points of the surface at which an inflexional tangent line passes through an arbitrary point, (ξ), by κ. Thus, when there is a double curve, we have $\kappa=\mu_1(n-2)-\rho$. This will be proved below. For the case when there is no such curve we have shewn that $\kappa=n(n-1)(n-2)$.

Now, (ii), consider the points of the surface at which the tangent plane, from the arbitrary point (ξ), has stationary contact with the surface. At such a point consider the two lines, both in the tangent plane, one joining the point to (ξ), the other the tangent line of the curve of contact of tangent planes from (ξ) to the surface. By the harmonic property spoken of, one of these must coincide with the parabolic line at the point. It can be shewn (see the analysis below) that, in general, it is the tangent of the curve of contact which coincides with the parabolic line, the other possibility involving that the curve of contact has a multiple point at the point considered. We shall denote by i the number of tangent planes of the surface, drawn from an arbitrary point (ξ), which have stationary contact with the surface.

The two kinds of points of the surface we have considered, of numbers respectively κ and i, correspond to characters of the enveloping cone drawn from the point (ξ). Take an arbitrary plane section of this cone; we desire to make it clear that, in general, on a generator of this enveloping cone which is an inflexional line of the surface, there is a cusp of this plane section; and, on a generator containing a point of stationary contact, there is an inflexion of this plane section; which thus has κ cusps and i points of inflexion. In fact, at a point (x) of the surface for which the joining line (x, ξ) is an inflexional line, the curve of contact of the enveloping cone with the surface touches the line (x, ξ) at (x), and we have the same generating line of the cone for two coinciding tangent planes; while, at a point (x) of stationary contact, we have the same tangent plane for two coinciding points of the curve of contact.

Consider the two possibilities analytically. For the case where an inflexional line passes through the point (ξ), take non-homogeneous coordinates x, y, z, with origin at the point of the surface, and $z=0$ for the tangent plane. The equation of the surface may be represented by

$$z+xy+(xz, yz, z^2)+(x^3, x^2y, \ldots, z^3)+\ldots=0,$$

the axes x, y being along the inflexional lines. The first polar of a point $x=a, y=0, z=0$, on the inflexional line $z=0, y=0$, has an equation of the form

$$a\{y+(z, 0, 0)+(x^2, xy, \ldots)+\ldots\}+(n-1)z+(n-2)xy+\ldots=0;$$

this meets the tangent plane $z=0$ in a curve given by

$$a\{y+(x^2, xy, \ldots)+\ldots\}+(n-2)xy+\ldots=0,$$

of which the first approximation $(a \neq 0)$ is of the form $y-kx^2=0$. This verifies what was said above, that, when an inflexional line, at a point (x) of the surface, passes through an external point (ξ), the curve of contact, of the tangent planes from (ξ) to the surface, touches the line (x, ξ) at the point (x). Next, for the other case, when a tangent plane to the surface from (ξ) has stationary contact with the surface at a point (x), the equation of the surface, expressed by non-homogeneous coordinates, with origin at the point of contact, is of the form

$$z+\tfrac{1}{2}y^2+(xz, yz, z^2)+(x^3, x^2y, \ldots)+\ldots=0;$$

thus the first polar of a point $x=a$, $y=b$, $z=0$ is given by an equation

$$a\{\ldots\ (z,\ 0,\ 0)+(x^2,\ xy,\ \ldots)+\ldots\}+b\{y+(0,\ z,\ 0)+(x^2,\ \ldots)+\ldots\}$$
$$+(n-1)\,z+\tfrac{1}{2}(n-2)\,y^2+\ldots=0,$$

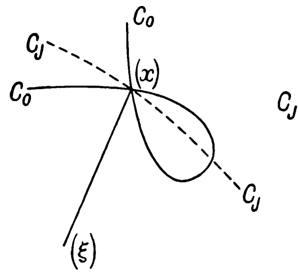
and the intersection of this with $z=0$ is given by

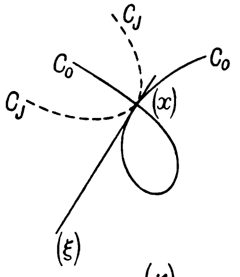
$$a\{\ldots(x^2,\ xy,\ \ldots)+\ldots\}+b\{y+(x^2,\ \ldots)+\ldots\}+\tfrac{1}{2}(n-2)\,y^2+\ldots=0;$$

provided $b \neq 0$, that is, in the original description, provided the point (ξ) is not on the parabolic line at the point (x), this is also, to the first approximation, of the form $y-kx^2=0$; so that the curve of contact has the parabolic line for tangent. When $b=0$, the curve of contact has a double (or higher) point, at the point of stationary contact.

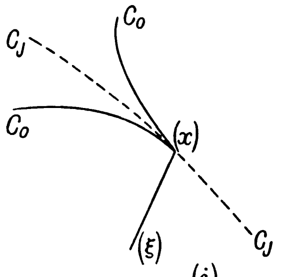
We may consider the matter from another point of view. The sections of the surface by all possible planes through the external point (ξ), form a linear system of curves on the surface. This system is such that one curve goes through any two arbitrary points of the surface; it is given by the intersection of the surface with another system of surfaces, whose equation is of the form $\lambda\vartheta+\mu\phi+\nu\psi=0$, in which $\vartheta=0$, $\phi=0$, $\psi=0$ represent definite surfaces (in our case planes through (ξ)), and λ, μ, ν are variable parameters. We may consider the more general case when $\vartheta=0$, $\phi=0$, $\psi=0$ are any surfaces of the same order; a system of curves on the surface so determined is called a *net of curves*. Of such a net of curves we may consider the *Jacobian curve*; this is defined as the locus of points of the surface at which a curve of the net has a double point. For an ordinary point of the surface $f=0$, the condition for this is that the surface should have the same tangent plane as the surface $\lambda\vartheta+\mu\phi+\nu\psi=0$; the Jacobian is then part (or the whole) of the intersection of $f=0$ with the surface expressed by the vanishing of the four-rowed determinant $\partial(\vartheta,\ \phi,\ \psi,f)/\partial(x,\ y,\ z,\ t)$. The curves of the net which pass through a point of the Jacobian curve, this being supposed to be an ordinary point of $f=0$, form a pencil of curves, obtainable by the intersection of $f=0$ with surfaces whose equation is of the form $pU+qV=0$, in which p, q are variable, but U and V are definite; we may suppose that $U=0$ gives the curve of the net which has a double point at the point considered. Thus the point is such that all curves of the net which pass through it have the same tangent line thereat, that of the curve given by $V=0$. For the sake of comparison with the simple case considered above, we may describe this common tangent, at the point (x) of the Jacobian curve, as the $(x,\ \xi)$ line; in that particular case, the locus of double points of plane sections of $f=0$ made by planes through the external point (ξ), is the curve of contact of tangent planes of the surface drawn from (ξ). As in that case, we may prove in general that the

(x, ξ) line is the harmonic conjugate of the tangent line of the Jacobian curve, C_J, with respect to the tangent lines of the curve, C_0, of the net which has a double point at (x). We return to this general point of view later.





(κ)



(i)

Consider now in further detail the case when the net of curves is formed by the plane sections of the surface whose planes contain an external point (ξ); and suppose the surface has no double curve, or other multiple points. The enveloping cone has then an order, N, equal to $n(n-1)$, the same as the order of the curve of contact; this curve is now the complete intersection of the surface with the first polar of (ξ), which is a surface of order $n-1$. The *class* of the enveloping cone, defined by the number of its tangent planes which contain an arbitrary line through its vertex, and thus equal to the number of tangent planes of the *surface* through this line, is also equal to the number of intersections, on the surface, of the two curves of contact of tangent planes to the surface drawn from any two points of this line. It is thus given by $N' = n(n-1)^2$. These numbers N, N' are respectively the order and class of any plane section of the enveloping cone. This plane section has also, we have proved, a number K of cusps given by $K = n(n-1)(n-2)$. Thus we have $N' - N = K$, a relation not generally true for a plane curve. If, further, this plane section have D double points, T double tangents, and I inflexions, and no other singularities, we can employ the formulae of Plücker

$$N' = N^2 - N - 2D - 3K, \quad I - K = 3(N' - N),$$
$$2(T - D) = (N' - N)(N' + N - 9),$$

and thence deduce

$$I = 4K = 4n(n-1)(n-2), \quad D = \tfrac{1}{2}n(n-1)(n-2)(n-3),$$

together with

$$T = \tfrac{1}{2}n(n-1)(n-2)(n^3 - n^2 + n - 12).$$

Of these, the number I, by what we have said, will be the number of tangent planes with stationary contact which pass through an arbitrary external point (ξ); but an independent proof of this arises from the form of I itself, which is the number of intersections of three surfaces, namely the given surface, of order n; the Hessian of this, of order $4(n-2)$; and the first polar of (ξ), of order $n-1$. The number D will be the number of tangent lines from (ξ) which touch the surface in two distinct points. The number T will be the number of tangent planes from (ξ) which have two distinct points of contact with the surface; for instance, in the case of a cubic surface, $T=27$, a plane from (ξ) to any one of the 27 lines of the surface having, we know, two distinct points of contact with the surface, lying on this line. The numbers D, K are such that

$$D+K=\tfrac{1}{2}n(n-1)^2(n-2);$$

we also obtain this equation directly, in an independent way, thus: If we denote $(x\partial/\partial\xi)^r f(\xi)/r!$ by f_r, this being of order r in (x, y, z, t), the result of the substitution of $(\xi+\lambda x, \eta+\lambda y, \ldots)$ for $(x, y, \ldots)$, in $f(x, y, z, t)=0$, is $f_0+\lambda f_1+\lambda^2 f_2+\ldots+\lambda^n f_n=0$; this determines the intersections with $f=0$ of the line joining the points (ξ), (x). In order that this line may touch the surface, the two first derivatives of this equation in λ must have a common root; for the line to have two distinct contacts with the surface these derivatives must have two different common roots, and for the line to have three coincident intersections with the surface, these derivatives must have two coincident common roots. To express this, we form (as in Vol. v, Chap. VIII) a determinant of $2(n-1)$ rows and columns, of which the elements of the first $n-1$ rows are, beside zeros, the coefficients, nf_0, $(n-1)f_1$, $(n-2)f_2$, $\ldots$, f_{n-1}, of one derivative of the λ equation, beginning respectively in the first, second, $\ldots$, $(n-1)$th place; and the elements of the last $n-1$ rows are, beside zeros, the coefficients, f_1, $2f_2$, $3f_3$, $\ldots$, nf_n, of the other derivative, beginning likewise in the first, second, $\ldots$, $(n-1)$th place of these rows, respectively. The vanishing of this determinant expresses that the λ equation has a repeated root; the terms in the expansion of this determinant are however isobaric in the suffixes, and thus of equal order in (x, y, z, t); this order is then equal to that of the single term $(nf_0 . nf_n)^{n-1}$ in this expansion, and so is $n(n-1)$. This gives the equation of the enveloping cone to the surface drawn from (ξ). To form the condition that the first derivatives should have two common roots, we form from this determinant another, of $2(n-2)$ rows and columns, which can be obtained from this one by leaving out the $(n-1)$th and the $(2n-2)$th rows of it, as well as the last two columns. The new determinant has, in its expansion, the term $(nf_0 . (n-1)f_{n-1})^{n-2}$, and is of order $(n-1)(n-2)$ in x, y, z, t. The vanishing of this gives the cone containing both the double and the cuspidal edges of the previous enveloping cone drawn from (ξ).

Hence we have $2(D+K)=n(n-1).(n-1)(n-2)$, as was deduced. We may likewise obtain a direct proof of the value of D, by supposing (ξ) to be on the surface, and considering the lines drawn tangent to the surface at this point. Then we consider the λ equation, obtained from the former by replacing f_0 and f_1 by zero, $f_2+\lambda f_3+\ldots+\lambda^{n-2}f_n=0$. For the tangent line (x, ξ) to have further contact with the surface, the two first derivatives of this must have a common root; the determinant whose vanishing expresses this fact contains the term $(f_2.f_n)^{n-3}$, and is of order $(n+2)(n-3)$ in (x), and of order $(n-2)(n-3)$ in (ξ). By this latter fact, the point of contact (ξ), when (x) is a given point on the double tangent of the surface, lies on a surface of order $(n-2)(n-3)$, beside lying on $f_0=0, f_1=0$, which are of respective orders $n, n-1$ in (ξ); as the double tangent contains two such points, the number of such tangents is $\frac{1}{2}n(n-1)(n-2)(n-3)$; this proves independently the value of D deduced above. That the determinant is of order $(n+2)(n-3)$ in (x) expresses that this is the number of tangents which can be drawn to a general plane curve of order n from a double point of the curve.

We may also consider the genus, π, of the plane section of the enveloping cone. By the Plücker formulae this is given by $2\pi-2=n(n-1)(2n-5)$; and, by the correspondence between the generators of the cone, and the points of the curve of contact, π is also the genus of this curve. We may find this genus, also, from the general formula (Vol. v, Chap. VIII) for the genus of the complete curve of intersection of two surfaces, namely, here, the original surface and the first polar of the external point (ξ). Incidentally, we thus have an independent determination of $D+K$, and thence of D when K is found. By the same theory, any canonical set of points on the curve of contact lies on a surface of order $2n-5$; and a canonical set of points is obtained by taking a canonical set of generators of the enveloping cone, which can be found by the intersection of this cone with a cone of order $n(n-1)-3$ drawn through its double and cuspidal generators.

Ex. 1. A very simple example is the case of an ordinary cubic surface, to which no double tangent lines can be drawn from an arbitrary external point (ξ). The curve of contact is of order 6 and genus 4, and has 6 tangent lines which pass through the vertex (ξ) of the enveloping cone, these being cuspidal generators of the cone. The points of contact of these tangents lie on a plane, the second polar of the vertex (ξ), and on a conic. A cubic cone, put through these 6 tangents, meets the curve of contact in 6 other points, and these lie in a plane; in particular there is a cubic cone meeting the enveloping cone only along its 6 cuspidal generators.

Ex. 2. By the formulae given above (Vol. v, Chap. VIII), for the curve which is the complete intersection of a general surface of order n, and the first polar of an arbitrary external point, of order $n-1$, the rank, r; the number, h, of chords of the curve from an arbitrary point; the class,

n'; and the number, ν', of planes containing two tangent lines of the curve which pass through an arbitrary point, are given, respectively, by

$$r = n(n-1)(2n-3),\quad h = \tfrac{1}{2}n(n-1)^2(n-2),\quad n' = 6n(n-1)(n-2),$$

$$\nu' = \tfrac{1}{2}n(n-1)(2n-3)\{n(n-1)(2n-3)-10\} + 4n(n-1).$$

If a curve in space be projected from a point at which χ tangents of the curve intersect, this curve having no double points, or cusps, or inflexions, or double tangents, prove that, *for the resulting plane curve*,

$$N' = r-\chi,\quad D = h-\chi,\quad K = \chi,\quad T = \nu' - (r-4)\chi + \tfrac{1}{2}\chi(\chi-1),\quad I = n' - 2\chi.$$

Shew that, when $\chi = n(n-1)(n-2)$, the values for N', D, K, T, I which are found in the text for the curve of contact, result from these two sets of formulae.

Ex. 3. For a cubic surface, $f_x^3 = 0$, with the notation $u_0 = f_\xi^3$, $u_1 = f_\xi^2 f_x$, $u_2 = f_\xi f_x^2$, $u_3 = f_x^3$, prove that the enveloping cone from the point (ξ) has the equation $(u_0^2 u_3 - 3u_0 u_1 u_2 + 2u_1^3)^2 + 4(u_0 u_2 - u_1^2)^3 = 0$.

Ex. 4. If u_r be a homogeneous polynomial in x, y, z, of order r, the discriminant in regard to θ of the form $u_0\theta^n + u_1\theta^{n-1} + \ldots + u_n = 0$, represents, when equated to zero, with x, y, z as coordinates, a plane curve of order $n(n-1)$, with $\tfrac{1}{2}n(n-1)(n-2)(n-3)$ double points, and $n(n-1)(n-2)$ cusps lying on a curve of order $(n-1)(n-2)$.

Ex. 5. If a plane sextic curve have 6 nodes which lie on a conic, then the sextic is touched by another conic in 6 points which lie on a cubic curve through the nodes (Rohn, *Math. Ann.* xxv, 1885, p. 598). Any plane sextic curve with 6 nodes can be birationally transformed to a sextic curve with 6 nodes which lie on a conic.

Ex. 6. In general, the Hessian of a surface contains the points of the surface for which the polar quadric is a cone, and the points at which the inflexional lines coincide. A ruled surface (see below, Ex. 7, p. 165) has generators, called *torsal* generators, each with the property that the tangent plane of the surface is the same at every point of the generator; and the ruled surface has a double curve. Salmon has proved that the Hessian of a ruled surface meets the surface only in its torsal generators and its double curve, having the former as double lines, and the latter as a four-fold curve. For example, the Hessian of the ruled cubic surface $x^2z + y^2t = 0$ is $x^2y^2 = 0$.

Ex. 7. A related theorem, for a surface of order n, containing a line, is that all the intersections of the Hessian with this line are contacts, so that their number is $2(n-2)$. If the equation of the surface be $x\phi + y\psi = 0$, the tangent plane of the surface at any point $(0, 0, z_1, t_1)$ of the line is given by $xu_1 + yv_1 = 0$, where u_1, v_1 are the polynomials, homogeneous of order $n-1$ in z_1, t_1, obtained by substituting $(0, 0, z_1, t_1)$ in ϕ, ψ; thus any plane containing the line $x = 0 = y$ touches the surface in $n-1$ points, where the section of the surface by this plane meets the line. And the points of the line whereat the second inflexional line of the surface coincides with this (inflexional) line, are the points where this plane section, for a suitably chosen plane, touches the line. These points arise for values of z_1/t_1 satisfying the two equations $x\,\partial u_1/\partial z_1 + y\,\partial v_1/\partial z_1 = 0$, $x\,\partial u_1/\partial t_1 + y\,\partial v_1/\partial t_1 = 0$, and are $2(n-2)$ in number. Only in these points can the Hessian meet the line; so that all its intersections are contacts. The argument assumes that the tangent plane of the surface is not the same at all points of the line. When this is so, the equation of the surface may be taken to be $x\phi + y^2\vartheta = 0$; and it can be verified that the line lies wholly on the Hessian, which has, further, the tangent plane of the surface for its tangent plane at every point of the line (Salmon).

The surface with a double curve. In what now follows we deal in some detail with surfaces having a double curve, but with no other multiple points beside triple points, which are triple points for this double curve, at which three sheets of the surface meet. This apparent limitation is suggested by the following considerations: It has been proved (Vol. v, Chap. II) that any curve may be changed, by birational correspondence, to one with no singular points, lying possibly in space of higher dimension than that of the original curve. It is believed, similarly (see the references given below) that any surface may be obtained, by birational transformation, from one without multiple points; and that such a representative non-singular surface may be supposed to be in space of five (or less) dimensions. If such a surface, in space [5], be projected into space [4], the resulting surface will also have no multiple points (for general projection) other than double points of a particular kind, at which two sheets of the surface cross with only point-intersection, like the intersection of two planes in this space. There are in fact ∞^4 chords of a surface, and, for a line in space [5] to pass through a given point, 4 conditions must be satisfied. Thus, in the general projection of a non-singular surface in [5], there will be a finite number of chords of the surface passing through the centre of projection, and these will give rise to the double points spoken of, on the surface in [4]. Such double points, of a surface in [4], with, in general, only point-intersection, at which there are two distinct tangent planes of the surface, will be spoken of as *accidental* double points. They have also been called *improper* double points*, for a reason which will appear. If, now, a surface in [4], with no other singularity than such accidental double points, be projected, from a general point of its space, into space [3], we shall obtain therein a surface with a double curve. For, now, though the surface still has ∞^4 chords, only 3 conditions are necessary for a line to pass through a given point. Thus, through an arbitrary point of the space [4], there pass ∞^1 chords of the surface, forming a conical sheet, whose intersection with the space [3] will give the double curve of the projected surface in that space. As two conditions are necessary for a plane, in space [4], to pass through a given point, a certain number of the ∞^2 tangent planes, of the surface in [4], will pass through an arbitrary point; the line from this point to the point of contact of such a plane is the limit of chords of the surface;

---

* Severi, "Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni...", *Rend. Palermo*, xv, 1901. When the surface in space [5] is the Veronese surface, of order 4 (whose general point has coordinates ξ^2, η^2, ζ^2, $\eta\zeta$, $\zeta\xi$, $\xi\eta$, where ξ, η, ζ are parameters), no chord of the surface can be drawn from an *arbitrary* point, and the projected surface in [4] has no double point. But this surface, and cones, are the only surfaces in space [5] for which this exception arises.

thus, on the double curve on the surface in [3], there will be a certain number of points at which the two tangent planes of the surface are not different; these are called (after Cayley) the *pinch points* of the double curve (Cayley, *Papers*, VI, p. 123, of date 1868). There will also in general be a finite number of triple chords of the surface in [4], passing through an arbitrary point; these give rise to triple points of the surface in [3], triple both for the surface and the double curve. The accidental double points, of the surface in [4], will project into points of the double curve, of the surface in [3], which will be ordinary points of this curve. Thus plane sections of the surface in [3] will all have the same genus; and prime sections of the surface in [4], made by linear spaces of dimension 3 (primes), will be curves of the same genus; in particular the genus of a prime section through an accidental double point is the same as of a general prime section. This was the reason for the name *improper*, applied to these double points*.

The formulae which we proceed to give for surfaces in ordinary space [3], with only a double curve and triple points thereon, were originally given by Salmon (cf. Cayley, *Papers*, VI, p. 329), in connexion with investigations in regard to reciprocal surfaces, and on the hypothesis of the existence of a cuspidal curve as well as a double curve. The proof given by Salmon, in part tentative, will be found in his *Solid Geometry* (1882, p. 581), and is indicated below. The formulae were revised, and extended in great detail, by Zeuthen, *Math. Ann.* X, 1876, pp. 446–546. The order of the surface we denote most often by μ_0 (instead of n). The enveloping cone to the surface from an arbitrary point will break up into the cone standing on the double curve, and a *proper cone of contact*, as has already been said; the order of the proper cone of contact, which is also the order of the proper curve of contact, and the class of a general plane section of the surface, we denote by μ_1. The class of the proper cone of contact, which is also the class of an arbitrary plane section of this cone, and is also the number of tangent planes of the surface passing through an arbitrary line (the *class* of the surface), we denote by μ_2. As in the earlier part of this chapter, we denote by κ the number of inflexional lines of the surface which pass through an arbitrary point, this being in general the number of cusps of an arbitrary plane section of the proper enveloping cone. And we denote by i the number of tangent planes of the surface having stationary contact, which pass through an arbitrary point; this is in general, as we have seen, the number of inflexions of an arbitrary plane section of the proper enveloping cone. The genus of a general plane section of the surface will often be denoted by p, and the genus of the proper curve of contact of the enveloping cone, from an arbitrary point, by π. The order of the double curve we denote by ϵ_0, the rank of this curve by ϵ_1, the number of its triple points by t, and the number of its pinch points by ν_2. As an auxiliary number, we use ρ for the class of the developable surface formed by the tangent planes of the surface at the points of the double curve, namely the number of such tangent planes passing through an arbitrary point. The surface is supposed to be irreducible; but evidently the corresponding characters can be formed for a composite surface by addition of the characters for the component surfaces, if a proper correction be introduced for the intersections of the components.

---

* For the reduction of a surface to one without multiple points, the following references may be of use: (1), Segre, *Ann. d. Mat.* XXV, 1897; also, *Rend. Palermo*, XXX, 1910, and *Atti...Torino*, XXXVI, 1901, p. 635; (2), Beppo Levi, *Atti...Torino*, XXXIII, 1897, p. 66; also, *Ann. d. Mat.* XXVI, 1897, and *Torino Mem.* XLVIII, 1898, and *Rend. Lincei*, VII, 1898; (3), O. Chisini, *Bologna Mem.* VIII, 1921; (4), Albanese, *Rend. Palermo*, XLVIII, 1924; also *Rend. Lincei*, XXXIII, 1924, p. 13, and Severi, *Atti...Veneto*, LXXIX, 1920, p. 929.

With these notations we have the following five equations:

(I), $\mu_0(\mu_0-1)=\mu_1+2\epsilon_0$; (II), $\mu_1(\mu_0-2)=\kappa+\rho$;

(III), $\epsilon_0(\mu_0-2)=\rho+3t$; (IV), $2\rho-2\epsilon_1=\nu_2$;

(V), $\mu_2+2\nu_2=\mu_1+\kappa$,

of which we proceed to give proofs. The equation (I) is obvious by considering the curve of intersection of the surface with the first polar of an arbitrary point; this is of order μ_0-1, and contains the double curve. It is obtainable also by the formula for the class of a plane section of the surface, namely $\mu_1=2\mu_0+2p-2$, compared with the definition of p, given by $2p-2=\mu_0(\mu_0-3)-2\epsilon_0$. The equation (II) expresses that the number of intersections of the proper curve of contact, of tangent planes of the surface from an arbitrary point O, with the second polar of this point O, which is of order μ_0-2, is equal to the sum of: (a), the number, κ, of points of the surface at which an inflexional tangent line passes through O—and this is justified just as in the case discussed earlier when the surface has no double curve—and, (b), the number of points of the double curve for which one of the two tangent planes of the surface passes through O. In regard to this, the point in question is a point of contact of a tangent plane from O and thus lies on the proper curve of contact; but it is also on the second polar of O, because O is on the polar quadric of the point, which breaks up into the two tangent planes of the surface at this point. The matter is made clearer by considering the surface as arising by projection from a surface in space [4]. Suppose that this surface is projected from a point L, and that l is a line through L meeting the space [3], on to which the projection is made, in a point O: of the ∞^2 tangent planes of the surface in [4], there will be ∞^1 which meet the line l, and their points of contact will describe a curve, say σ, on this surface. The projection of the curve σ is the proper curve of contact, of tangent planes of the projected surface in [3], which pass through O. If a point P, of the curve σ, be one of the two intersections, with the surface in [4], of a chord of this surface from L, and the tangent plane of this surface at P meet the line l in a point other than L, we obtain, on projection, a point of the double curve of the surface in [3], at which one of the two tangent planes passes through O, this point being on the proper curve of contact of tangent planes from O; but, if the tangent plane of the surface in [4], at the point P, passes through L, then P projects into a pinch point of the double curve in space [3], and the tangent plane of the surface in [3] does not, as a rule, pass through O. Thus the pinch points of the surface in [3] lie on the proper curve of contact of tangent planes from every point O (a fact we shall need below), but are in general different from the points of the double curve contributing to ρ. The formula (II) states that the only ordinary points of the surface in [3], for which the tangent plane and the polar quadric both pass through O, are the (κ) points; and the only multiple points are the (ρ) points; we assume that this is so. More detailed consideration of the behaviour of a first polar at a pinch point is found in Note I, at the end of this chapter. Coming now to equation (III), this expresses that the intersections of the second polar, of the point O, with the double curve, consist of the (ρ) points just considered, and the triple points, each counted triply. That these latter points are on the second polar follows because the polar quadric, of a triple point of the surface, vanishes identically. It appears clear that in general there are no other than these two kinds of intersection. Further justification of the equation arises below, Chap. VI, p. 254. The equation (IV) is in fact only the dual of a theorem which has been obtained previously (Chap. I, p. 25), that an algebraic curve of order ν, lying on a ruled surface of order n (in the present application a developable surface), which meets every generator in 2 points, touches $2\nu-2n$ generators. The dual statement is that, for a developable surface of class ν (i.e. having ν tangent planes through an arbitrary point), which is such that two of its tangent planes pass through every tangent line of a certain curve of rank n, it arises $2\nu-2n$ times that the two tangent planes of the developable surface, through a tangent line of the curve, coincide with one another. In the application to be made, the developable surface is that formed by the tangent planes of the given surface at the points of its double curve (so that ν is to be replaced by ρ), and the curve is the double curve (so that n is to be replaced by ϵ_1). Thus $2\rho-2\epsilon_1$ is the number, ν_2, of the pinch points of the given surface. Equation (V) may be obtained by an easy generalisation of a result previously found

(Vol. v, p. 218); this result was that, if two surfaces of orders m, m', both passing through a curve C, of genus π, have for their remaining intersection two curves C_1, C_2, simple on both surfaces, then a surface of order $m+m'-4$ drawn through C_1 and C_2, and the double points of C, if such exist, cuts on C, other than at the intersections of C with C_1 and C_2 and the double points of C, a canonical set of $2\pi-2$ points. The generalisation we require is that this remains true when the curves C_1 and C_2 coincide in a double curve of the surface of order m, the surface of order $m+m'-4$ having then this curve as a double curve. This is in fact a particular case of a theorem, due to Noether, given below. In the application we make, the curve C is the proper curve of contact of the tangent planes to the fundamental surface under consideration, drawn from a point O; the two surfaces passing through this are the fundamental surface of order $(m=)\,\mu_0$, and the first polar of O, of order $(m'=)\,\mu_0-1$; and the curves C_1, C_2 coincide in the double curve of the fundamental surface. The proper curve of contact is of order μ_1 and of genus π; we have seen that this curve passes through the pinch points of the double curve, and also through the (ρ) points of this, for which one tangent plane of the surface passes through O. These points together will then absorb $2\nu_2+2\rho$ of the intersections of the surface of order $m+m'-4$ (here equal to $2\mu_0-5$) with the proper curve of contact; hence we have

$$\mu_1(2\mu_0-5)=2\nu_2+2\rho+2\pi-2.$$

The genus π of the proper curve of contact is obtainable by remarking that the rank of this curve, the number of its tangents which meet an arbitrary line, which we may take to pass through O, is $\kappa+\mu_2$; for, an inflexional line of the surface which passes through O is a tangent line of the curve of contact; and a tangent line of the curve of contact which meets a line through O lies in a tangent plane of the surface passing through this line; therefore, by the ordinary formula for the rank of a curve, we have

$$\kappa+\mu_2=2\mu_1+2\pi-2.$$

Using this, and the equation (II), namely $\mu_1(\mu_0-2)=\kappa+\rho$, which enables us to eliminate ρ, we find the result stated, or

$$\mu_2+2\nu_2=\mu_1+\kappa.$$

This is a remarkable result, expressing the number of pinch points of the double curve in terms of the order, the class and the number of cuspidal edges of the proper enveloping cone. Other investigations of equations (I)–(V) arise below.

The general theorem of Noether ("Theorie des eindeutigen Entsprechens, u.s.w.," *Math. Ann.* VIII, 1875, § 7), of which we have used a particular case in proving equation (V), has already been referred to

(Vol. v, p. 218); it is as follows: If through a curve C there pass two surfaces F_1, F_2, of respective orders n_1, n_2, whose further intersection consists of curves, such that a j_1-fold curve of F_1 is a j_2-fold curve of F_2, while a k_1-fold node of F_1 is a k_2-fold node of F_2, then the complete canonical series on C is obtained by surfaces of order n_1+n_2-4, which have every such common (j_1, j_2) curve as a (j_1+j_2-1)-fold curve, and every such common (k_1, k_2) node as a node of such character that the surfaces are there met in (k_1+k_2-2) points by every branch of C passing through such node; which moreover pass through every point of contact of F_1 and F_2 occurring at simple points of these. Beside the simple application just made, for equation (V), an application is made in the next chapter (p. 227) to the case when the curve C is defined, on a surface F of order n, as the residual intersection of this surface with a surface, ϕ, of order $n-4$, which passes $(j-1)$ times through every j-fold curve of F, and $(k-2)$ times through every k-fold node of F. The canonical series on C is then determined by surfaces of order $n+n-4-4$, or $2n-8$, passing $j+j-1-1$, or $2(j-1)$ times, through every j-fold curve of F, and $k+k-2-2$, or $2(k-2)$ times, through every isolated k-fold node of F. A particular surface of this description is then given by the aggregate of two of the surfaces ϕ, of order $n-4$, which we have described. Wherefore, if π be the genus of the curve C, or (F, ϕ), and ν be the number of intersections of two such curves, (F, ϕ) and (F, ϕ'), we have $2\pi-2=2\nu$. This will be found to be an important result. (Cf. p. 223.)

The equations (I)–(V) contain nine characters, μ_0, μ_1, μ_2, ν_2, ϵ_0, ϵ_1, t, κ, ρ, any five of which can be expressed thereby in terms of the other four. For instance, in terms of the properties of the double curve, and the order of the surface, that is, in terms of ϵ_0, ϵ_1, t, μ_0, we have $\mu_1=\mu_0(\mu_0-1)-2\epsilon_0$, and

$$\mu_2=\mu_0(\mu_0-1)^2-(7\mu_0-12)\,\epsilon_0+4\epsilon_1+15t.$$

And we may add the expression for the number of inflexions of an arbitrary plane section of the proper enveloping cone, in terms of the double curve. For, by Plücker's formula, we have $i-\kappa=3(\mu_2-\mu_1)$, which is equivalent here to either of $i=4\kappa-6\nu_2$, $i=4(\mu_2-\mu_1)+2\nu_2$; thus we obtain

$$i=4\mu_0(\mu_0-1)(\mu_0-2)-24(\mu_0-2)\,\epsilon_0+12\epsilon_1+48t;$$

this is the number of tangent planes of the surface from an arbitrary point with stationary contact. We also have

$$\nu_2=2(\mu_0-2)\,\epsilon_0-2\epsilon_1-6t, \quad \kappa=\mu_0(\mu_0-1)(\mu_0-2)-3(\mu_0-2)\epsilon_0+3t.$$

The equation (V) will be modified if the surface have isolated multiple points. Thus, if there be δ isolated ordinary double points, though the formulae for π and ρ are as before, the equation we have employed is changed to

$$\mu_1(2\mu_0-5)-2\delta=2\nu_2+2\rho+2\pi-2,$$

and equation (V) becomes $\mu_2+2\nu_2=\mu_1+\kappa-2\delta$. As the Plücker formula $i-\kappa=3(\mu_2-\mu_1)$ remains true, we hence deduce

$$i=4(\mu_2-\mu_1)+2\nu_2+2\delta.$$

More generally, if there be isolated ordinary multiple points, of which the typical order is denoted by k, we have

$$\kappa + \mu_2 = 2\mu_1 + 2\pi - 2, \quad \mu_1(\mu_0 - 2) = \kappa + \rho + \Sigma k(k-1)(k-2),$$

of which the latter is a modified form of equation (II); and we also have $\mu_1(2\mu_0 - 5) = 2\nu_2 + 2\rho + 2\pi - 2 + \Sigma k(k-1)(2k-3)$, so that equation (V) becomes $\mu_2 + 2\nu_2 = \mu_1 + \kappa - \Sigma k(k-1)$. Hence, again with $i - \kappa = 3(\mu_2 - \mu_1)$, we obtain $i = 4(\mu_2 - \mu_1) + 2\nu_2 + \Sigma k(k-1)$. From this, taking, for a plane section of the proper enveloping cone, the dual of a well-known result, namely $i + \mu_1 = 2\mu_2 + 2\pi - 2$, we deduce $2\pi - 2 = 2\mu_2 + 2\nu_2 - 3\mu_1 + \Sigma k(k-1)$.

For completeness and comparison it seems desirable to quote here still more general formulae given by Noether ("Sulle curve multiple di superficie algebriche," *Ann. d. Mat.* v, 1871–3; quoted in his paper in *Math. Ann.* VIII, 1875, already referred to). For a surface with multiple curves which have no mutual intersections and no multiple points—typically of multiplicity j on the surface, of order ϵ_0 and rank ϵ_1—the surface having also isolated general multiple points (typically of multiplicity k), the formulae of Noether lead, as a particular case, to

$$\mu_1 = \mu_0(\mu_0 - 1) - \Sigma j(j-1)\epsilon_0, \qquad \nu_2 = \Sigma(j-1)\{2(\mu_0 - j)\epsilon_0 - j\epsilon_1\},$$

$$\mu_2 = \mu_0(\mu_0 - 1)^2 - \Sigma k(k-1)^2 - \Sigma(j-1)\{(3j+1)\mu_0 - 2j(j+1)\}\epsilon_0 + \Sigma j^2(j-1)\epsilon_1,$$

$$i = 4\mu_0(\mu_0 - 1)(\mu_0 - 2) - 4\Sigma j(j-1)(3\mu_0 - 2j - 2)\epsilon_0 + 2\Sigma j(j-1)(2j-1)\epsilon_1.$$

For the general formulae see the Note (II, p. 180) at the end of this chapter.

For the case when the surface has no multiple points beside a double curve with triple points, if we use Plücker's formula for the double points of a plane section of the proper enveloping cone, namely $2D = \mu_1^2 - \mu_1 - \mu_2 - 3\kappa$, we find that the number of tangent lines to the surface, from an arbitrary point, which have two points of contact, is given by

$$D = \tfrac{1}{2}\mu_0(\mu_0 - 1)(\mu_0 - 2)(\mu_0 - 3) + 2\epsilon_0^2 - (\mu_0^2 - 5\mu_0 + 7)\epsilon_0 - 2\epsilon_1 - 12t;$$

and we can compute the number, T, of twice touching tangent planes, from an arbitrary point, from the well-known formula $2(T - D) = (\mu_2 - \mu_1)(\mu_2 + \mu_1 - 9)$. Again, the genus, π, of the proper curve of contact, whose rank is $\mu_2 + \kappa$, as we have remarked, computed with help of $2\pi - 2 = \mu_2 - 2\mu_1 + \kappa$, $= 2\mu_2 + 2\nu_2 - 3\mu_1$, is given, for this case of a surface with a double curve, by

$$2\pi - 2 = \mu_0(\mu_0 - 1)(2\mu_0 - 5) - 2(5\mu_0 - 11)\epsilon_0 + 4\epsilon_1 + 18t.$$

Reversely, in terms of μ_0, μ_1, μ_2, ν_2, the characters of the double curve are given by

$$2\epsilon_0 = \mu_0(\mu_0 - 1) - \mu_1, \quad 2\epsilon_1 = 2\mu_0\mu_1 - 2\mu_1 - 2\mu_2 - 5\nu_2,$$

$$6t = \mu_0(\mu_0 - 1)(\mu_0 - 2) - 3\mu_0\mu_1 + 4\mu_1 + 2\mu_2 + 4\nu_2,$$

and therefore, the genus, P, of the double curve, computed from $\epsilon_1 = 2\epsilon_0 + 2P - 2$, is given by

$$2(2P-2) = 2\mu_0\mu_1 - 2\mu_0(\mu_0 - 1) - 2\mu_2 - 5\nu_2.$$

Ex. 1. For the surface with only a double curve, and triple points thereon, triple for the surface, the formulae shew that the numbers μ_1, ν_2, ϵ_1, $\mu_2 + \frac{1}{2}\nu_2$ are all even; that the two numbers κ, $\mu_2 - \mu_1 - \nu_2$ divide by 3; and the number i divides by 24 (cf. Roth, *Camb. Phil. Soc. Proc.* xxv, 1929, p. 395).

Ex. 2. It has been shewn (p. 155) that, if a line from an arbitrary point meet the surface in $\mu_0 - 4$ simple points, and also in two points each a coincidence of two intersections, then these two points lie on a certain surface of order $(\mu_0 - 2)(\mu_0 - 3)$. Let then D, E denote, respectively for the proper curve of contact of tangent planes from an arbitrary point, and for the double curve of the surface, the number of chords of these curves passing through this point; so that

$$\mu_2 = \mu_1^2 - \mu_1 - 2D - 3\kappa, \quad \epsilon_1 = \epsilon_0^2 - \epsilon_0 - 2E - 6t;$$

also, let $[\mu_1\epsilon_0]$ denote the number of common generators of the two cones with vertex at this arbitrary point, (1), the proper cone of contact, (2), the cone through the double curve, excluding common generators through actual intersections of the proper curve of contact and the double curve; then prove that

$$[\mu_1\epsilon_0] = \mu_1\epsilon_0 - 2\rho - \nu_2, \quad \mu_1(\mu_0 - 2)(\mu_0 - 3) = 2D + 2[\mu_1\epsilon_0],$$
$$\epsilon_0(\mu_0 - 2)(\mu_0 - 3) = 4E + [\mu_1\epsilon_0];$$

subtracting $\mu_1\mu_0(\mu_0 - 1)$ and $\epsilon_0\mu_0(\mu_0 - 1)$, respectively, from the last two equations, using the equation $\mu_0(\mu_0 - 1) = \mu_1 + 2\epsilon_0$, and the values above given for D, E, and also the two equations

$$\mu_1(\mu_0 - 2) = \kappa + \rho, \quad \epsilon_0(\mu_0 - 2) = \rho + 3t,$$

infer the results $\mu_2 + 2\nu_2 = \mu_1 + \kappa$, $2\rho - 2\epsilon_1 = \nu_2$. This, essentially, is Salmon's deduction of equations (IV), (V) from (I), (II) and (III).

Ex. 3. For a ruled cubic surface with a double line, whose equation may be supposed to be $x^2z - y^2t = 0$, prove that $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 3$, $\nu_2 = 2$, $\epsilon_0 = 1$, $\epsilon_1 = 0$, $\kappa = 3$, $\rho = 1$, $i = 0$, obtaining the positions of the pinch points, and of the (κ) and (ρ) points in this case, for a given external point. Shew that the proper curve of contact of tangent planes, from a point (a, b, c, d), is the rational quartic curve given, in terms of a parameter θ, by $x = \theta(a\theta - b)$, $y = \theta^2(a\theta - b)$, $z = \frac{1}{2}\theta^2(d\theta^2 - c)$, $t = \frac{1}{2}(d\theta^2 - c)$, and find the intersections of this curve with the double line $x = 0 = y$. Prove that the (κ) points lie on the quadric surface expressed by

$$adyz - 3bdxz + 3acyt - bcxt = 0,$$

which meets the curve also on $x = 0 = y$ and on $z = 0 = t$; and that these (κ) points lie also on the plane given by $2acx - 2bdy + a^2z - b^2t = 0$, which also contains the (ρ) point.

Ex. 4. For a quartic surface with a double line, so that $\mu_0 = 4$, $\epsilon_0 = 1$, $\epsilon_1 = 0$, $t = 0$, prove that $\mu_1 = 10$, $\mu_2 = 20$, $\nu_2 = 4$, $\kappa = 18$, $\rho = 2$, $i = 48$.

Ex. 5. For a quartic surface with a double conic (a cyclide), from $\mu_0 = 4$, $\epsilon_0 = 2$, $\epsilon_1 = 2$, $t = 0$, deduce $\mu_1 = 8$, $\mu_2 = 12$, $\nu_2 = 4$, $\kappa = 12$, $\rho = 4$, $i = 24$.

Ex. 6. For the Steiner quartic surface, with three concurrent double lines, from $\mu_0 = 4$, $\epsilon_0 = 3$, $\epsilon_1 = 0$, $t = 1$, deduce $\mu_1 = 6$, $\mu_2 = 3$, $\nu_2 = 6$, $\kappa = 9$, $\rho = 3$, $i = 0$. These can be verified by considering the dual surface, a four-nodal cubic surface. The intersections of the proper curve of contact with the double curve may be computed directly, taking for the equation of the surface $y^2z^2+z^2x^2+x^2y^2=2xyzt$.

Ex. 7. For a general ruled surface, the characters can be expressed in terms of the order of the surface, μ_0, and the genus of the general plane section, p. A fundamental fact is that $\mu_2=\mu_0$; for, every tangent plane is a plane through a generator, and every plane through a generator is a tangent plane; so that tangent planes through an arbitrary line, each containing a generator, must be those containing the generators which meet the line, and, conversely, all the planes through the arbitrary line and generators which meet it, are tangent planes. From this we have μ_1, the class of a plane section of the surface, equal to $2\mu_0+2p-2$, or to $2\mu_2+2p-2$. The order, ϵ_0, of the double curve is given by

$$\epsilon_0=\tfrac{1}{2}\mu_0(\mu_0-1)-\tfrac{1}{2}\mu_1, \text{ or } \tfrac{1}{2}(\mu_0-1)(\mu_0-2)-p.$$

We consider now the direct deduction of the number of pinch points of the double curve, ν_2. On any generator, there is associated with any point a certain plane, the tangent plane of the surface at this point, which is the ultimate position of the plane joining this point to a consecutive generator; and there is associated with every plane through the given generator a certain point, the point of contact of this plane, which is the ultimate position of the intersection of this plane with a consecutive generator. But there are generators for which the associated plane is the same for every point of the generator, and the associated point is the same for every plane through the generator. Such may be described as *torsal* generators, or, more particularly, if not very clearly, as generators each of which intersects its consecutive. In general, the points of intersection of generators describe a curve on the surface, the double curve, but the pair of generators intersecting at any point of this are not consecutive; the two tangent planes of the surface at such a point of the double curve are the planes joining the tangent line of the curve to the two generators which intersect at this point. If the two tangent planes coincide (and the point is a pinch point of the double curve), it must be because the two generators are consecutive and ultimately coincident; and, conversely, a torsal generator meets the double curve in a pinch point. The number ν_2 is thus the number of torsal generators. Of the value of ν_2, which is $2\mu_0+4p-4$, various proofs may be given; the simplest is, perhaps, as follows: Take an arbitrary line, and, through this, two arbitrary planes; establish a correspondence between points P, P' of the line, by drawing a tangent line PT to the section of the surface by the first plane, touching this in T; let the generator of the surface which contains T meet the section of the surface by the second plane in T'; and let the tangent line of the second section at T' meet the given line in P'. The correspondence on the line so established is evidently symmetrical, and the indices of this correspondence are both equal to the number, $2\mu_0+2p-2$, of tangents PT which can be drawn from P to the first section. There are, therefore, $2(2\mu_0+2p-2)$ cases of coincidence of P and P'. Among these are the coincidences which arise at the points where the given line meets the surface; through each of these μ_0 points both the first and second sections of the surface pass; we count each of these as two coincidences, so obtaining $2\mu_0$ in all. The other coincidences of P and P' arise when the tangent plane of the surface at T, defined by PT and the generator at T, coincides with the tangent plane of the surface at T', which is on the generator through T; in other words, TT' must be a torsal generator. Thus, if the number of such be ν_2, we have $\nu_2+2\mu_2=2(2\mu_0+2p-2)$; which proves the formula cited. Another proof is obtained by representing the generators of the surface by the points of a curve lying on a quadric Ω in space of five dimensions (cf. Vol. IV, pp. 40 ff.). Then two non-consecutive intersecting generators determine a flat pencil of lines; this is represented in the space [5] by a line, lying on the quadric Ω, which is a chord of the curve thereon representing the generators of the ruled surface. Thus two consecutive intersecting generators give rise to a tangent line of this curve, itself lying wholly on Ω. To find the number of torsal generators of the ruled surface is thus to find the number of tangent lines of this curve on Ω which lie wholly thereon. The order of this curve, it is easily seen, is the number of generators of the ruled surface which meet an arbitrary line, namely is μ_0. Now, let P be any point of the representative curve on Ω; the lines through P which lie on Ω, lie, we know, on the tangent prime of Ω at P, which meets the curve in two coincident points at P, and hence in μ_0-2 other points. There are thus μ_0-2 chords of the curve, say PP', lying on Ω, through any point P of the curve. The ends P, P', of such a chord, are then in correspondence on the curve; and this correspondence, evidently symmetrical with both indices equal to μ_0-2, has a valency 2, because the points P' which correspond to P are on a prime, with double intersection at P (see Chap. I, p. 3, preceding). The number of coincidences of P and P', namely the number of tangents of the curve which lie wholly on Ω, is thus $2(\mu_0-2)+2p.2$; for the genus of the curve on Ω, whose points are in (1, 1) correspondence with the generators of the ruled surface, and hence with the points of a plane section of this, is p. Thus $\nu_2=2\mu_0+4p-4$, as before. Two other proofs of this result, also by the theory of correspondence, are given by Schubert; one of these will be found in Chap. II preceding (p. 102).

Having found ν_2, we deduce, for the rank of the double curve of the ruled surface,

$$\epsilon_1=\mu_0\mu_1-\mu_1-\mu_2-\tfrac{5}{2}\nu_2, \ =2(\mu_0-2)(\mu_0-3)+2(\mu_0-6)p.$$

The genus P, of this curve, is thence given by

$$2P-2, \ =\mu_0\mu_1-\mu_0(\mu_0-1)-\mu_2-\tfrac{5}{2}\nu_2, \ =(\mu_0-5)(\mu_0+2p-2), \ =\tfrac{1}{2}(\mu_0-5)\nu_2,$$

and the number of triple points, given by

$$t=\tfrac{1}{6}\mu_0(\mu_0-1)(\mu_0-2)-\tfrac{1}{2}\mu_0\mu_1+\tfrac{2}{3}\mu_1+\tfrac{1}{3}\mu_2+\tfrac{2}{3}\nu_2,$$

is
$$t=(\mu_0-4)\{\tfrac{1}{6}(\mu_0-2)(\mu_0-3)-p\}.$$

These values lead to

$$i, \ =4(\mu_2-\mu_1)+2\nu_2, \ =0; \quad \kappa, \ =\mu_2+2\nu_2-\mu_1, \ =3(\mu_0+2p-2), \ =\tfrac{3}{2}\nu_2,$$
$$\rho, \ =\epsilon_1+\tfrac{1}{2}\nu_2, \ =(\mu_0-2)(2\mu_0-5)+2p(\mu_0-5),$$

while the genus of the proper curve of contact of tangent planes from an arbitrary point, being given by $2\pi-2=\mu_2+\kappa-2\mu_1$, $=2p-2$, is that of the plane sections of the surface, as should evidently be the case. If we assume that for a ruled surface there are no tangent planes through an arbitrary point which have stationary contact, or $i=0$ (cf. Salmon's result above, p. 156, Ex. 6), we have, from $\nu_2=2(\mu_1-\mu_2)$, a further determination of ν_2. These general formulae are under the hypothesis that the double curve is irreducible; for cases of reducible double curve the reader should consult Edge, *Ruled Surfaces* (Cambridge, 1931).

Ex. 8. A torsal generator of a ruled surface is such that the tangent plane, at any point P of this generator, is the same; and the point of contact of any plane ϖ, through this generator, is the same. If the generator be given by the equations $x=t\xi+lz$, $y=t\eta+mz$, where x, y, z, t

are the homogeneous coordinates, ξ, η are connected by a certain algebraic equation, and l, m are algebraic functions of ξ and η, prove that the differential condition for the generator to be torsal is $d\eta . dl - d\xi . dm = 0$; further, that the unique tangent plane is then given by $(x - t\xi - lz)\,d\eta - (y - t\eta - mz)\,d\xi = 0$; and the unique point of contact has the homogeneous coordinates $\xi\,dl + l\,d\xi$, $\eta\,dl + m\,d\xi$, $d\xi$, dl.

Ex. 9. It will be of importance later to consider the intersection of a general surface with surfaces passing through the double curve. We may consider here the case of a ruled surface, though this is, in some respects, exceptional. Denote the order of the surface temporarily by n (instead of μ_0), and recall that the order, genus, and number of triple points of the double curve, respectively, are given by $\epsilon_0 = \frac{1}{2}(n-1)(n-2) - p$, $P - 1 = \frac{1}{2}(n-5)(n+2p-2)$ and $t = (n-2, 3) - (n-4)p$, where $(n-2, 3)$ is the binomial coefficient. Then, first, as an arbitrary plane through a generator meets the surface further in a curve of order $n-1$, of which one intersection with the generator is the point where the plane touches the surface, we infer that a generator of the surface meets the double curve in $n-2$ points. Hence there can be no surface of order $n-3$ passing through the double curve, since such a surface would contain every generator. But, second, we can prove that there are surfaces of order $n-2$ passing through the double curve, $n+p-1$ of which are linearly independent; and that the remaining intersection of one of these with the ruled surface consists of $n+2p-2$ generators; the aggregate of these surfaces meets an arbitrary plane in the complete system of curves of order $n-2$ adjoint to the section of the surface by this plane. For surfaces of order $n-2$ through the triple points of the double curve have $\epsilon_0(n-2) - 3t$ remaining intersections with the double curve; and this number, being $2P - 2 + n(n-2)$, is greater than $2P - 2$; such surfaces therefore cut on the double curve a series of freedom $\epsilon_0(n-2) - 3t - P - \xi$, where ξ is zero if this series is complete. Surfaces of order $n-2$ through the double curve thus have a freedom

$$\tfrac{1}{6}(n-1)n(n+1) - 1 - t - \{\epsilon_0(n-2) - 3t - P - \xi + 1\},$$

which is $n + p - 2 + \xi$. Such surfaces, however, determine, on an arbitrary plane, adjoint curves of order $n-2$, of this plane section; and there can be no surfaces of this kind containing the plane, since the residual surfaces would be surfaces of order $n-3$ passing through the double curve. The surfaces of order $n-2$ thus determine on the plane a system of curves of freedom $n+p-2+\xi$, which cannot, however, be greater than the freedom $n+2p-2-p$, of the adjoint system of this order. Thus $\xi = 0$, the surfaces of order $n-2$ determine the complete adjoint system of curves of order $n-2$, and the system of surfaces is of freedom $n+p-2$. Moreover, a surface of order $n-2$, through the double curve, will meet the generator passing through any general point, of a plane section, which lies on this surface of order $n-2$, in $n-1$ points in all; and will thus contain this generator. Such a surface thus contains the $n+2p-2$ generators passing through the points in which this surface meets a plane section of the ruled surface; and, since $n(n-2) - 2\epsilon_0 = n + 2p - 2$, these generators together form the complete residual intersection with the ruled surface beyond the double curve. If we now consider a surface of order $n-1$ passing through the double curve, this will meet the surface in a residual curve of order $2n+2p-2$, having one intersection with every generator, other than on the double curve. This residual curve can then be expressed, in the symbolical way previously explained (Vol. v, p. 216), in the form $\varpi + (n+2p-2)g$, where ϖ denotes a plane section and g denotes a generator; two such residual curves then meet in $(\varpi, \varpi)+2(n+2p-2)(\varpi, g)$ points, that is $n+2(n+2p-2)$ points. In the notation we have used above, this is $\mu_2+\nu_2$. A particular surface of order $n-1$ through the double curve is the first polar of an arbitrary external point, in regard to the ruled surface; the residual intersection of this is the proper curve of contact for this point; this agrees then with the number $\mu_2+\nu_2$, since the proper curve of contact contains the pinch points. We may compute the freedom of surfaces of order $n-3+k$ passing through the double curve ($k \geqslant 1$), as in the case above when $k=1$; and find $\frac{1}{2}nk(k+1)+k(p-1)+\frac{1}{6}k(k-1)(k-2)-1$. This number reduces to -1 for $k=0$; and to $-p-1$ for $k=-1$, of which an interpretation will occur in the next chapter. It can be shewn that these surfaces of order $n-3+k$ intersect a curve on the ruled surface which is given by the intersection of this with a general surface of order $k+1$, in the canonical series of this curve (Enriques, *Mem. Soc. Ital. d. Scienze*, x, 1896, p. 56). The pinch points are further considered in Note I, at the end of this chapter (p. 176).

Ex. 10. The following three results (quoted from Severi, "Intersezioni...", *Mem. Torino*, LII, 1903, No. 19) are of importance in the sequel. A surface, of order μ_0, has a double curve of order ϵ_0 and rank ϵ_1, with ν_2 pinch points, and certain triple points triple for the surface. A surface of order μ_0' is drawn through the double curve, having as residual intersection with the given surface, a curve, of order ϵ_0' and rank ϵ_1', meeting the double curve in i points. Then

$$\epsilon_0'=\mu_0'\mu_0-2\epsilon_0, \quad i=2(\mu_0'-\mu_0+1)\epsilon_0+\epsilon_1+\tfrac{3}{2}\nu_2,$$
$$\epsilon_1'=\mu_0'\mu_0(\mu_0'+\mu_0-2)-2(3\mu_0'-\mu_0)\epsilon_0-2\epsilon_1-3\nu_2.$$

For example, for a quartic surface with a double line, a cubic surface through the line has, as residual intersection, a curve of order 10 and genus 10, meeting the line in 6 points.

Shew, in particular, that with $\mu_0'=\mu-4$, $\nu_2=2(\mu_0-2)\epsilon_0-2\epsilon_1-6t$ (cf. p. 162, above) we obtain $i=3(\mu_0-4)\epsilon_0-2\epsilon_1-9t$, and

$$\epsilon_1'=2\mu_0(\mu_0-3)(\mu_0-4)-2(5\mu_0-18)\epsilon_0+4\epsilon_1+18t.$$

Further, let two surfaces of orders m and n be drawn through the double curve; let their residual curves of intersection with the given surface, supposed irreducible, have $p_1^{(1)}$, $p_2^{(1)}$ for genera, respectively; let the residual curve of intersection of the two surfaces of orders m, n, with one another, assumed to have in all $2\epsilon_0(m+n-2)-2\epsilon_1-9t$ intersections with the double curve, have $p_{12}^{(2)}$ further intersections with the given surface. Then prove that

$$p_1^{(1)}-1+p_2^{(1)}-1-2p_{12}^{(2)}=\tfrac{3}{4}\mu_0(m-n)^2+(2\mu_0-8-m-n)\{\tfrac{1}{4}\mu_0(m+n)-\epsilon_0\}.$$

Thus, when $m=n, p^{(1)}-1-p^{(2)}=(\mu_0-4-m)(\mu_0 m-2\epsilon_0)$, which is a formula of subsequent interest (cf. p. 223).

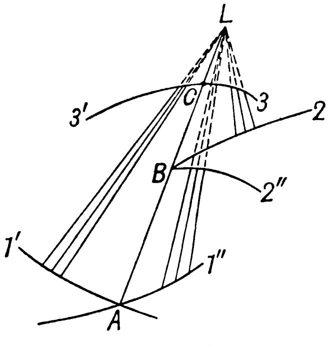
Ex. 11. By means of the formulae given earlier in this chapter for a general surface with a double curve, and triple points thereon, we can prove that $\epsilon_0(\mu_0-4)-3t=2P-2+\frac{1}{2}\nu_2$. When $\nu_2>0$, this shews that surfaces of order μ_0-4, or of higher order, passing through the triple points of the double curve, cut thereon a series which is not special. If, now, we define a number p_n by means of $12p_n=2\mu_2-8\mu_1+12\mu_0-12+\nu_2$, it can be proved, as in Ex. 9, that, if surfaces of order μ_0-3+k, through the triple points of the double curve, cut thereon, for $k=-1, 0, 1, \ldots$, a non-special series, of incompleteness ξ_k, then the tale (= freedom + 1) of such surfaces through the double curve is

$$\tfrac{1}{2}\mu_0 k(k+1)+k(p-1)+\tfrac{1}{6}k(k-1)(k-2)+p+p_n+\xi_k.$$

Of such surfaces, a certain number will wholly contain an arbitrary plane, the number, namely, of surfaces of order $n-3+k-1$ which exist passing through the double curve. Let this number be denoted by ϖ_{k-1}; then the tale of the system of curves, of order $n-3+k$, adjoint to the plane section of the surface, determined on this plane by surfaces of order $n-3+k$ passing through the double curve, is obtained by subtracting ϖ_{k-1} from the number just set down. In particular, for $k=0$, if the number, ϖ_{-1}, of surfaces of order $n-4$ which exist passing through the double curve, be denoted by p_g, then the surfaces of order $n-3$, through the double curve, determine, on an arbitrary plane, a system of curves adjoint to the section of the surface by this plane, of order $n-3$, with freedom $p-1-(p_g-p_n)+\xi_0$. Picard (Picard-Simart, *Fonctions algébriques*, II, 1906, p. 438) has proved, by transcendental methods, using a formula of Castelnuovo, a theorem which is equivalent to saying that $\xi_0=0$. Picard's theorem is thus equivalent to the result that *the non-special series, on the double curve of the surface, which is determined by surfaces of order $n-3$ put through the triple points of the curve, is complete.* It would be of great importance for the general theory of surfaces to have an elementary direct proof of this. For a ruled surface there are no surfaces of order $n-4$, or $n-3$, through the double curve, as we have shewn; but the results found for ruled surfaces, as to the completeness of series for surfaces of order $n-2$, or more, hold for surfaces in general, as Picard's theory shews. (Cf. Ex. 8, p. 269, below.)

Surfaces in space of four dimensions. We now consider a surface in space [4] which is the projection of a non-singular surface in space [5], as has been explained. As there are ∞^4 chords of a surface, and for a line to pass through a given point in space [5] requires 4 conditions, we expect that a finite number of chords of a surface in [5] will pass through a general point of this space. Further enquiry shews in fact that this is so save in two cases only, which are, when the surface is a cone, and when it is the surface, called the Veronese surface; this last is the surface of which the coordinates $(x_0, \ldots, x_5)$ of a general point may be taken, in terms of three arbitrary parameters, as $x_0=\xi^2$, $x_1=\eta\zeta$, $x_2=\eta^2$, $x_3=\zeta\xi$, $x_4=\zeta^2$, $x_5=\xi\eta$; the surface lies on the six quadrics whose equations are given by the vanishing of the quadratic minors in the symmetrical determinant whose rows consist in turn of the elements x_0, x_5, x_3; x_5, x_2, x_1; x_3, x_1, x_4; the surface itself, and all its chords, lie on the cubic primal expressed by the vanishing of this determinant. Thus no chords of the surface can be drawn from any point of the space [5] which does not lie on this primal; the surface contains (∞^2) conics, whose planes lie on this primal, and a chord of the surface is a chord of such a conic. We consider then a surface in [5] of which a finite number of chords passes through an arbitrary point of this space. If this surface be projected, from such an arbitrary external point, into space [4], the surface obtained, which we shall denote by ψ, will have a finite number of isolated double points, each lying on one of the chords from the point of projection; at such a point of the surface ψ there will be two tangent planes, the projections of the tangent planes of the surface in [5] at the points where this is met by the chord. These two tangent planes will in general meet only at this point, and every line through the point may be regarded as a limit of a chord of the surface. Such double points we shall call *accidental*. In particular it may happen that the two tangent planes at the double point meet in a line, if the point in [5], from which the projection is made, be suitably taken. Again, the chords of the surface ψ, in [4], are ∞^4 in aggregate; thus, there pass ∞^1 such chords through an arbitrary point L of this space; these chords form a conical sheet, a surface, with vertex at L. For the moment, let the order of this surface, the number of points in which it meets a general plane, be called S. Every generator of the conical sheet meets the surface ψ in two points, forming a curve on this surface; denote this curve by γ, and, for a moment, denote the order of this curve, the number of its intersections with a general prime space [3], by N. The accidental double points of the surface ψ, where two sheets of the surface have point intersection, are double points of the curve γ, where it crosses itself. The surface ψ is met by a prime in a curve; let h be the number of chords of this curve which can be drawn through an arbitrary point of this prime. In particular, when this prime is taken through L, these chords are generators of the conical sheet, of chords of ψ from L, and each meets the curve γ in two points; thus $2h$ is the number of intersections of the curve γ with the prime through L, this prime meeting the conical sheet only in chords of ψ. Wherefore, $N=2h$. Again, S, the order of this conical sheet, being the number of its intersections with a plane, is the number of chords of the surface ψ which lie in the prime joining L to this plane; and, conversely, in a prime through L, a plane meets every chord of ψ which lies in this prime. Wherefore, $S=h$, and $N=2S$. Now take an arbitrary general plane ϖ, and a pencil of primes through this plane. Any one such prime meets the curve γ in N points; and each of these points, say P, by projection from L, gives another point, P', of the curve γ; while this point P' determines another prime of the pencil through ϖ. We may thus regard any prime of this pencil as corresponding to N other primes of this pencil, namely there is a, clearly symmetrical, correspondence, of indices N, N, between the primes passing through the plane ϖ. There are then $2N$ cases of coincidence of corresponding primes. Such coincidences, we assume, are of only two kinds: namely, such a coincidence arises when the points P, P', on a pair of corresponding primes, and on a chord of the surface ψ passing through L, coincide with one another; and, a coincidence arises when the chord PP' meets the base plane ϖ. A coincidence of P with P' arises for every tangent line of the surface ψ which passes through L; let ν_2 denote the number of such tangent lines, which is the same as the number of tangent planes of ψ which pass through L; a coincidence of P with P' also arises at a double point of the surface ψ; let d denote the number of such double points. At a coincidence of P and P' where the tangent line of ψ passes through L, the chord from L touches the curve γ; at a coincidence at a double point, the curve γ crosses itself, and not only does P coincide with P', but also P' coincides with P. Thus we say that the number of coincidences of corresponding primes which arise by coincidence of P with P' is ν_2+2d. Coincidences of corresponding primes which arise because the chord PP' meets the plane ϖ, occur for each of the S points in which the plane ϖ meets the conical sheet of chords from L to the surface ψ; these, however, likewise involve, not only that P' coincides with P, but also that P coincides with P', and count for $2S$. Thus we have $\nu_2+2d+2S=2N$, or as $N=2S$, we have $\nu_2+2d=N$. In what follows we constantly denote the order of the curve γ by ζ_0, so that $N=\zeta_0$; and we denote the order of the double curve, on the surface in [3], into which ψ projects from L, by ϵ_0; so that $\epsilon_0=S=\frac{1}{2}\zeta_0$. Hence, the number of the accidental double points of the surface ψ is $d=\frac{1}{2}(\zeta_0-\nu_2)$. Further, as the condition for a plane, in space [4], to meet a line, is one-fold, there will be a curve of points on ψ, at each of which the tangent plane of the surface meets a given arbitrary line; the order of such curve we denote by μ_1. Thus, μ_1 is also the number of tangent lines, of the curve section of ψ by a prime drawn through a given arbitrary line, which meet this line, namely the rank of such prime section. Hence, if the prime section have no cusps, by the properties of a curve in space [3], we have $\mu_1=\mu_0(\mu_0-1)-2h$, where h, as above, is the number of so-called apparent double points of such prime section, and we have seen that $h=S=\epsilon_0$. Thus we have $\mu_0(\mu_0-1)=\mu_1+2\epsilon_0$. In fact, if a line l, from L, meet the space [3] on which we project, in the point O, and μ_1 be defined by the tangent planes of the surface ψ which meet the line l, it is clear that μ_1 is the order of the proper enveloping cone from O to the surface in [3] into which ψ projects; thus the equation just obtained is the equation (I), which was proved above in space [3] by use of the first polar of the point O. With this equation the number of accidental double points of the surface ψ, which was found as $d=\epsilon_0-\frac{1}{2}\nu_2$, can be expressed also by $d=\frac{1}{2}\mu_0(\mu_0-1)-\frac{1}{2}\mu_1-\frac{1}{2}\nu_2$. The proof of the value of d we have given is due to Severi ("Intorno ai punti doppi...," already referred to, ***Rend. Palermo***, xv, 1901). Another proof arises below (Chap. VI, p. 259, footnote), in connexion with the formulae for the intersection of manifolds in space [4].

We have now defined the characters μ_0, μ_1, μ_2 for a surface in space [4]; and these give rise, by projection to a surface in space [3], to the characters already denoted by the same symbols. We may also define μ_2, for the surface ψ in [4], so that, on projection, it gives the class of the surface in space [3]: That a plane, of the ∞^2 tangent planes of the surface ψ, should meet a given arbitrary plane in the space [4], not in a point, but in a line, requires two conditions; there is thus a finite number of the tangent planes of ψ which satisfy this condition. This number we denote by μ_2. When we take the arbitrary plane through the point L, from which we project to [3], the tangent planes in question become the tangent planes, of the resulting surface in [3], which pass through a line, namely the line in which the arbitrary plane meets the space [3]; or μ_2 is the class of the projected surface. Consider now in more detail the origin of the triple points on the double curve of the surface in [3], obtained by projection. Among the ∞^1 chords of the surface ψ which pass through the point L, there will be a finite number satisfying the one-fold condition of meeting ψ in a further point; thus, there is a finite number, say t, of trisecant chords of ψ through L. On projection from L to space [3], these give rise to triple points of the double curve of the surface obtained, which are evidently also triple points of this surface. Denote now by A, B, C the points of the surface ψ lying on a trisecant line from L; for the moment, denote the surface obtained from ψ, by projection from L, by ϕ. We may regard the single point of ϕ which arises by projection of both B and C as a point of the double curve of ϕ, the point, namely, where this curve is met by the sheet of ϕ which arises by projection of the neighbourhood of the point A on ψ; and a similar remark may be made in regard to each of the other two pairs C, A and A, B. The tangents at the triple point of the double curve on ϕ thus arise as the intersections of the three pairs of tangent planes of the sheets of ϕ which meet at this point. The conical surface formed by the chords of ψ from L will, correspondingly, have three sheets meeting in the trisecant line $LABC$; the chord LBC will be one of a continuum of chords, lying in one sheet of the cone, meeting ψ in two curves, respectively through B and C; say $B2$ and $C3$; likewise the chord LCA will be one of a continuum of chords lying in another sheet of the cone, meeting ψ in two curves, respectively through C and A; say $C3'$ and $A1'$; and the chord LAB will be one of a continuum of chords, meeting ψ in two curves; say $A1''$ and $B2''$. The conical sheet of chords to ψ from L thus meets ψ in a curve having a double point at each of A, B, C, the two branches of the curve which cross at A being $A1'$, $A1''$; those which cross at B



being $B2''$, $B2$; and those which cross at C being $C3$, $C3'$. The two sheets $LA1'$, $LA1''$ have curve intersections with ψ at A, but the sheet $LBC23$ has only point intersection with ψ at A. And similarly at B and C.

As a concrete example of these relations which can be examined in detail, we may take the surface which is the locus of intersection of pairs of osculating planes of a rational quartic curve in [4]; this is in fact the same as the projection from space [5] of the Veronese surface referred to above. In terms of three independent parameters ξ, η, ζ, the coordinates of a point of this surface may be represented by ξ^2, $\xi\eta$, $\frac{1}{3}(2\eta^2+\xi\zeta)$, $\eta\zeta$, ζ^2. It can be shewn without difficulty that a single trisecant of this surface passes through an arbitrary general point, that through this trisecant there pass three planes each meeting the surface in a conic, and that each conic contains two of the three points of the surface which lie on the trisecant.

Without much difficulty we can state five equations, for the surface ψ in space [4], which contain the equations (I)–(V) which were found for the surface ϕ in [3] into which this projects from the point L. The order of the curve γ, in which the surface ψ is met by the conical sheet of chords from L, we denote by ζ_0; it is given by $\zeta_0=2\epsilon_0$, where ϵ_0 is the order of the double curve on the surface ϕ. By the *rank* of a curve, in space [4], we mean the number of its tangent lines which meet an arbitrary plane; for the curve γ we denote the rank by ζ_1. Taking a plane, which we denote by λ for the moment, which passes through L, a tangent line of the curve γ, which meets the plane λ, will project from L, into a tangent line, of the double curve of the surface ϕ, which meets the line of intersection of λ with the space [3] in which ϕ exists—unless the tangent line in question passes through L. The tangent lines of the surface ψ which pass through L, we have remarked, are tangent lines of the curve γ, and we have used ν_2 for their number. Using ϵ_1, as before, for the number of tangent lines of the double curve of the surface ϕ which meet an arbitrary line of its space, we thus have the relation $\zeta_1=2\epsilon_1+\nu_2$, every pertinent tangent line of the double curve arising from two tangent lines of the curve γ. Next, from the tangent planes of the surface ψ at points of the curve γ, which are ∞^1 in aggregate, denote the number which meet an arbitrary general line of the space [4], by R. In particular, take this line, say l, to pass through L, and let it meet the space [3], on which we project, in the point O. There will be ν_2 such tangent planes of ψ meeting l in L; any other will project into a tangent plane of the surface ϕ, at a point of the double curve of this, which has the property of passing through the point O. We have denoted the number of such tangent planes of ϕ by ρ. Thus we have the equation $R=\rho+\nu_2$. Evidently this equation is in accord with the fact we have previously remarked that, for the surface ϕ, in space [3], the proper curve of contact of tangent planes, from an arbitrary point, passes through the (ρ) points, and through the pinch points, of the double curve of ϕ. We may also define, for the surface ψ, the number called κ for the surface ϕ, which was the number of stationary generators of the proper enveloping cone, or the number of inflexional lines of ϕ passing through an arbitrary point of the space [3]. In the space [4], the prime joining L to the tangent plane at a point P of ψ, meets ψ in a curve having a double point at P; the number κ is the number of points P for which one of the tangent lines of this curve, at P, meets an arbitrary line l drawn through L. With the slight changes thus described, the equations found (p. 159) for a surface in [3] lead to the following equations for the surface ψ:

(I), $\mu_0(\mu_0-1)=\mu_1+\zeta_0$; (II), $\mu_1(\mu_0-2)=\kappa+R-\nu_2$;
(III), $\epsilon_0(\mu_0-2)=R-\nu_2+3t$; (IV), $2\rho=\zeta_1$, or $2R=\zeta_1+2\nu_2$;
(V), $\mu_2+2\nu_2=\mu_1+\kappa$.

We can also obtain an interesting result in regard to the genus of the curve γ; this genus we denote by Q. As in the case of curves in space [3], we have, of course, $\zeta_1=2\zeta_0+2Q-2$, which is obtained by considering the coincidences of pairs of points of the sets of ζ_0 points, obtained on γ by primes through an arbitrary plane. Another form of this equation is $2Q-2=2\epsilon_1+\nu_2-4\epsilon_0$, and hence

$$2Q-2=2\mu_0(\mu_1-\mu_0+1)-2\mu_2-4\nu_2.$$

This result is obtainable also by considering the (2, 1) correspondence between the curve γ, and the double curve of the surface ϕ; Zeuthen's formula (p. 19 above) then gives $2Q-2-\nu_2=2(2P-2)$, which, with the value of P given above (p. 164) leads to the same form for Q. Another formula for Q, in terms of κ, is

$$Q-1+\kappa=(\mu_1-\mu_0)(\mu_0-1).$$

The genus Q satisfies the equation $2Q-2+6t=\zeta_0(\mu_0-4)$, as the formulae shew. Geometrically, this expresses that a primal of order μ_0-4, drawn through the double points of the curve γ, which lie on the trisecants of ψ drawn from L, meets γ, beside, in a set of $2Q-2$ points. It will appear (see p. 243, below) that such a set is a canonical set on γ. This is an interesting result.

Ex. 1. For the surface in [4], referred to above (p. 173), which is the projection of a Veronese surface in [5], obtain the values $\mu_0=4$, $\mu_1=6$, $\mu_2=3$, $\nu_2=6$, $\zeta_0=6$, $\zeta_1=6$, $t=1$, $d=0$, $Q=-2$, $\kappa=9$, $\rho=3$, and obtain these geometrically. This surface projects from a point into the Steiner quartic surface in space [3].

Ex. 2. For a ruled surface in [4], the number of accidental double points, in general $\frac{1}{2}(\zeta_0-\nu_2)$, is $\frac{1}{2}(\mu_0-2)(\mu_0-3)-3p$. Also $2Q-2=(\mu_0-4)\nu_2$, and $3t=(\mu_0-4)d$, while ζ_1, or $2\epsilon_1+\nu_2$, is $2(\mu_0-2)(2\mu_0-5)+4(\mu_0-5)p$. Segre (*Enzyk. d. Math. Wiss.* III, C 7, p. 913) states that the lines having 3 coincident intersections with the ruled surface form a ruled surface of order N, and their points of contact form a curve of order E, for which $N=6\{\mu_0+3(p-1)\}$, and $E=3\mu_0+8(p-1)$. Formulae giving these numbers for any surface in [4] are found by Roth, *Proc. Camb. Phil. Soc.* xxv, p. 395, with references to Severi.

Ex. 3. We may compute directly the characters of a surface in [4] which is the complete intersection of two general primals of orders m, m'. For this surface, $\mu_0=mm'$; and μ_1, which is the rank of the curve in which the surface is met by an arbitrary prime, is the rank of the complete curve of intersection of two surfaces of orders m, m' in space [3]; so that $\mu_1=mm'(m+m'-2)$. It is convenient to use symbols for $m-1$, and $m'-1$, say k and k'; thus $\mu_1=mm'(k+k')$. To find the number, μ_2, of tangent planes of the surface which meet a given arbitrary plane in a line, let $f=0, f'=0$ be the equations of the primals which define the surface, and $a_0x_0+\ldots+a_4x_4=0, b_0x_0+\ldots+b_4x_4=0$, be two primes containing the given plane; then, if the tangent plane of the surface, at the point (x), meet the plane in a line of which (ξ) is any point, the four equations

$$\xi_0\partial f/\partial x_0+\ldots+\xi_4\partial f/\partial x_4=0,\quad \xi_0\partial f'/\partial x_0+\ldots=0,\quad \xi_0 a_0+\ldots=0,\quad \xi_0 b_0+\ldots=0,$$

are satisfied, and, as equations for the determination of (ξ), are equivalent only to three equations. Thus, in the matrix of four rows and five columns formed with the coefficients of $\xi_0, \ldots, \xi_4$ in these equations, all the determinants of four rows and columns must vanish. Conversely, this is a sufficient condition for the geometrical relation. Thus (x) lies on a locus, a surface, whose order can be found by a rule, previously explained (Chap. II, pp. 75, 109), which we have associated with a symbol $[(c)/(-r)]_{t^{q+1-p}}$. In this case, the orders of the elements in any row of the matrix are the same, these orders being respectively $k, k', 0, 0$; thus the order of the surface is equal to the coefficient of t^2 in the ascending expansion of $(1-kt)^{-1}(1-k't)^{-1}$, namely is $k^2+kk'+k'^2$. This surface intersects the given surface in $mm'(k^2+kk'+k'^2)$ points, and this is the value of μ_2. Similarly, to find ν_2, the tangent plane at any point of the surface is the intersection of the tangent primes of $f=0, f'=0$, and the points of $f=0$ whereat the tangent prime passes through an arbitrary point are on the first polar of this point in regard to $f=0$. Thus, the number ν_2, of tangent planes of the surface (f, f') which pass through an arbitrary point is $\nu_2, =mm'kk'$. This result also furnishes an example of the rule quoted; for, if the arbitrary point be given as the intersection of the four primes $(a_0, \ldots, a_4), \ldots, (d_0, \ldots, d_4)$, the matrix, of five rows and six columns, of which the general row consists of the elements $\partial f/\partial x_i$, $\partial f'/\partial x_i$, a_i, b_i, c_i, d_i, for $i=0, \ldots, 4$, has its columns respectively of orders k, k', $0, 0, 0$; and the coefficient of t^2 in the expansion of $(1+kt)(1+k't)$ is kk'. From the values of $\mu_0, \mu_1, \mu_2, \nu_2$, by the preceding formulae, we find

$$\zeta_0=mm'kk',\quad d,\ =\tfrac{1}{2}(\zeta_0-\nu_2),\ =0,\quad \zeta_1=2mm'kk'(k+k'-1),$$

$$2Q-2=2mm'kk'(k+k'-2),\qquad t=\tfrac{1}{6}mm'kk'(k-1)(k'-1),$$

as also $\kappa-\nu_2=mm'(k+k')(k+k'-1),\quad \rho+\nu_2=mm'kk'(k+k'),$

$$i+2\nu_2=4mm'(k+k')(k+k'-1).$$

Of these, for instance, t is also easily obtained directly by the known determinantal method (Vol. v, Chap. VIII) of finding the condition that the two equations in λ, $f_0+\lambda f_1+\lambda^2 f_2+\ldots+\lambda^m f_m=0$, $f_0'+\lambda f_1'+\ldots=0$, should have three common roots—where $f_i=(x\partial/\partial\xi)^i f(\xi)/i!$, etc.

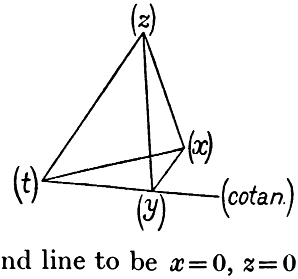
It may also be remarked (see p. 244 below) that the number of chords, from an arbitrary point lying on the surface (f, f'), which have three other intersections with the surface, is $6[m_3m_3'-\tfrac{1}{3}m_2m_2'+\tfrac{1}{3}m_1m_1'-1]$, where $m_i=m!/i!(m-i)!$.

Note I; in regard to pinch points. It has appeared in this chapter (p. 162) that the presence, on a surface of order n, in space of three dimensions, of a double curve of order ϵ_0, and class ϵ_1, with t triple points, involves a diminution of the class $n(n-1)^2$ of the surface of amount $(7n-12)\,\epsilon_0-4\epsilon_1-15t$. We now remark a relation of this formula with that for the pinch points of the double curve, whose number is ν_2, $=2(n-2)\,\epsilon_0-2\epsilon_1-6t$, by means of which, in terms of the double curve, either μ_2 or ν_2 can be found when the other is known. Take an arbitrary general line, m, and consider the curve of intersection of the first polars, in regard to the surface, of two points of the line. For both polars the double curve of the surface is a simple curve; the curve, θ, which we study, is the remaining curve of intersection, of order $(n-1)^2-\epsilon_0$. We assume, the proof being considered below, that this curve θ passes, (i), simply through each point of the double curve at which the tangent line of this curve meets the given line m; (ii), simply through each triple point of the double curve; (iii), through each pinch point in such a way as to have three coincident intersections with the surface at this point. At the points under (i) and (ii), the curve θ will have, respectively, 2, and 3, intersections with the surface. The number of intersections of θ with the surface, at points not on the double curve, is thus $n\{(n-1)^2-\epsilon_0\}-2\epsilon_1-3t-3\nu_2$. Each of such intersections is, however, a point of contact with the surface of one of the μ_2 tangent planes which can be drawn through the line m. Thus we have $\mu_2+3\nu_2=n\{(n-1)^2-\epsilon_0\}-2\epsilon_1-3t$; this is the relation between μ_2 and ν_2 which we had in view. The present direct proof of this may replace the proof of one of the equations (I)–(V) previously considered (p. 159). With the double curve itself, the curve θ has a number of intersections given by $\epsilon_1+3t+\nu_2$, which, by the value of ν_2 found above, is the same as $2(n-2)\,\epsilon_0-\epsilon_1-3t$. Consider now the assumptions we have made. It can be shewn that the first polar of a point (ξ, η, ζ, τ) has, for tangent plane, at a point (x, y, z, t) of the double curve, the plane, through the tangent line of the double curve at this point, which is the harmonic conjugate of the point (ξ) in regard to the tangent planes of the surface at (x); this is then the same for any two points (ξ) which lie on a plane through the tangent line of the double curve; wherefore, the first polars of two points (ξ), (ξ_1), whose join meets this tangent line, meet in a curve having a double point at (x); one branch of this curve is the double curve; thus, the residual curve of intersection, θ, of the two polars passes through (x). This proves the assumption (i). The assumption (ii) is also very easy to justify. For, both first polars have a conical double point at the triple point of the double curve; and, of the four common generators of the asymptotic cones of these surfaces at this point, three are the tangent lines of the double curve. The residual intersection, θ, of the two polars, thus passes simply through the triple point. Assumption (iii) is less obvious: at a pinch point, the common tangent plane of the two sheets of the surface, which we may call here the *pinch plane*, meets the surface in a curve having a triple point; of the three tangent lines of this curve, at this point, two coincide, namely along the tangent line of the double curve; to the third tangent we apply the name used by Cayley (*Papers*, VI, p. 335), and call it the *cotangent*. Every general line, in the pinch plane, through the pinch point, has three coincident intersections with the surface at this point; but the tangent line of the double curve, and the cotangent, have each four intersections. The *proper curve of contact*, of tangent planes to the surface from an arbitrary point, O, *passes through the pinch point, and touches the cotangent*; the first polar of O has the pinch plane for tangent plane (as follows from the harmonic relation remarked above for an ordinary point of the double curve). One of the inflexional lines of the first polar surface, at the pinch point, is the tangent line of the double curve; in general the cotangent is *not* the other inflexional line. The curve θ, the residual curve of intersection of two first polars, may have, in general, any direction in the pinch plane, and has then just three coincident intersections with the surface at the pinch point. We first prove these statements directly in space of three dimensions, and then consider the proof which arises from regarding the surface as obtained by projection from [4]. For the first, we assume that it is sufficient to regard the double curve as a line. Let this line be taken for $x=0$, $y=0$, the pinch plane being $x=0$. In non-homogeneous coordinates the equation of the surface, $f=0$, can then be supposed to be of the form

$$x^2(1+ax+cz+u)+2xy(Ax+Cz+v)$$
$$+y^2(Px+Rz+Qy+w)=0,$$

where u, v, w are of the second and higher orders in x, y, z. The pinch plane $x=0$ then meets the surface doubly in $y=0$, and also in the line $x=0$, $Rz+Qy=0$; by modifying the coordinate z we can suppose the second line to be $x=0$, $z=0$, so that $Q=0$; this line is then the cotangent. The point (x), or $y=z=t=0$, of the coordinate tetrahedron, may be supposed to be an arbitrary point of space; it will then be general to consider the first polar of this point, $(1, 0, 0, 0)$. This is given by $f_1=0$ where, with u_1 for $\partial u/\partial x$, etc.,

$$f_1=2x(1+ax+cz+u)+2y\,(Ax+Cz+v)+x^2(a+u_1)$$
$$+2xy(A+v_1)+y^2(P+w_1);$$



this surface has an ordinary point at the origin (t), the tangent plane being the pinch plane $x=0$. Remarking the identity

$$2f-xf_1=-ax^2+xy(2Cz+Py)+2Ry^2z$$
$$+\text{terms of fourth order at least,}$$

we see that the proper curve of contact, of tangent planes of the surface drawn from the point (x), touches the cotangent $x=0$, $z=0$. Further, not only is the point (x) arbitrary, but the points (z), (y) are also arbitrary upon the tangent line of the double curve, and upon the cotangent, respectively. We can therefore suppose a second arbitrary point of space to be $(1, \eta, \zeta, 0)$. To terms of the second order in x, y, z, the first polar of this point is given by

$$f_1+\eta\{2x(Ax+Cz)+2y(Px+Rz)\}+\zeta\{cx^2+2Cxy+Ry^2\}=0,$$

and has also $x=0$ for tangent plane at the origin. But the curve of intersection of this with $f_1=0$ has tangent lines, lying in $x=0$, given by $Ry(2\eta z+\zeta y)=0$, so that the curve θ, the residual of the double line in the intersection of the two first polars we have considered, has the line $x=0$, $2\eta z+\zeta y=0$ for tangent at the origin, and has three coincident intersections with the surface at the pinch point. Only when the line, which joins the two points of which the first polars have been taken, meets the cotangent (or $\zeta=0$), does the curve $\theta=0$ touch the cotangent. With this analysis the reader should also consult Zeuthen, *Math. Ann.* x, 1876, where *close* points are considered as well as pinch points. It is instructive further, to consider the pinch point of the surface ϕ, in [3], as arising when we project a surface ψ, in space [4], from a point L, as above (p. 170). Any chord LPP', of the surface ψ, through L, gives rise to a single point on the double curve of ϕ, at which, for a general position of P, the tangent planes are the projections of the tangent planes of the surface ψ at P and P'; the tangent line of the double curve of ϕ, at the corresponding point, is the intersection of the space [3] with the plane through L which contains the tangent lines, at P and P', of the curve γ which projects into the double curve. If T be a point of the curve γ at which P and P' coincide, the tangent line of γ at T passing through L, then, as P and P' approach to T, the tangent planes of ψ at P and P' tend to coincidence with the tangent plane of ψ at T; this plane, however, passes through L, and does not project into the pinch plane, at the point S, of the double curve of ϕ, into which T projects. This pinch plane is the limit of the intersection, with the space [3], of the prime which may be defined as containing the tangent plane of ψ at T and a chord LPP' which is approaching the line LT; the tangent line, of the double curve of ϕ at S, is the limit of the intersection of the space [3] with the plane containing LT and LPP'; the cotangent at the pinch point S is the intersection of this space [3] with the tangent plane of ψ at T. If l be an arbitrary line through L, meeting the space [3] in O, and σ be the curve on ψ, defined by the property that the tangent plane of ψ at a point of this curve meets the line l, then σ projects into the proper curve of contact, of tangent planes to ϕ from O. The curve σ passes through T, because the tangent planes of ψ, at points near to T, ultimately meet the line l in L; moreover, the curve σ touches the tangent plane of ψ at T, so that, on ϕ, *the proper curve of contact touches the cotangent.* It does not follow, however, that two curves, σ and σ', on ψ, arising from two lines, l and l', through L, touch one another at T. For, consider a (1, 1) correspondence, between the points of ψ in the neighbourhood of T, and points of the plane (l, l') in the neighbourhood of L, determined by the point intersection of the plane (l, l') with a tangent plane of ψ at a point near T; then, to two lines l, l' through L, in the plane (l, l'), will correspond different directions from T on ψ, in general. This remark will be of importance below.

Ex. 1. We have already considered (Ex. 3, p. 164) the cubic ruled surface $x^2z - y^2t = 0$, of which the points are expressible by two parameters by $x = \phi$, $y = \theta\phi$, $z = \theta^2$, $t = 1$. For this surface the point (t) is a pinch point, with $y = 0$ as pinch plane, and the line (tx) as cotangent. Similarly (z) is a pinch point, with $x = 0$ as pinch plane and the line (zy) as cotangent. The proper curve of contact, of tangent planes from the point (a, b, c, d), whose points are there expressed in terms of θ, meets the line (xy) in two points, touches the line (tx) at (t), touches the line (zy) at (z), and meets the double line (tz) in a further point. This surface is the projection, from the point (u), in the four-fold space in which the coordinates are x, y, z, t, u, of the surface whose points are given, in terms of two parameters, by $x = \phi$, $y = \theta\phi$, $z = \theta^2$, $t = 1$, $u = \theta$. The tangent plane of this surface in [4], at the point (θ, ϕ), is given by $2\theta(\theta x - y) + \phi(z - \theta^2 t) = 0$, $z - 2u\theta + t\theta^2 = 0$; in particular, at the point (t), for which $\theta = \phi = 0$, the tangent plane is the plane (tux). The various statements made above can be easily tested in this case.

Ex. 2. Shew that if X, Y, Z respectively denote $yt - z^2$, $yz - xt$, $zx - y^2$, the elimination of θ from the two equations

$$X + hY + \theta Z = 0, \quad aX + bY + (\theta^2 + c)Z = 0,$$

leads to the equation of a rational ruled quartic surface; and, if these equations lead to $X/\lambda = Y/\mu = Z/\nu$, so that λ, μ, ν are definite polynomials in θ, shew that the generator (θ) of the quartic surface is the intersection of the planes $\lambda x + \mu y + \nu z = 0$, $\lambda y + \mu z + \nu t = 0$. Shew further that, on the double curve of this surface, which is given by $X = Y = Z = 0$, the point $(0, 0, 0, 1)$ is a pinch point, at which the (torsal) generator is $y - hx = 0$, $z - hy = 0$. Also, that the proper curve of contact, for tangent planes to this surface from the point (ξ, η, ζ, τ), is the (sextic) locus of the point where the generator (θ) is met by the plane

$$(2\theta - a)A + (2h\theta - b)B + (\theta^2 - c)C = 0,$$

wherein, for brevity, A, B, C denote, respectively $y\tau - 2z\zeta + t\eta$, $-x\tau + y\zeta + z\eta - t\xi$, $x\zeta - 2y\eta + z\xi$; and shew that this curve touches the generator at $(0, 0, 0, 1)$, and that this generator is the cotangent at this pinch point (cf. Cayley, *Papers*, VI, p. 127).

Ex. 3. And so in general, on a ruled surface, to use a crude description, let O be a pinch point on the double curve, and Q, Q' be the points, on the proper curve of contact, which lie on two generators PQ, PQ' through a point P of the double curve; then, as P approaches O, the generators PQ, PQ' "open out" until, when P is at O, they are in the same line—the (torsal) generator at O, and the points Q, Q' both approach to P, and coincide, at O, in the point of contact of the proper curve of contact with the torsal generator.

Ex. 4. Verify the results, for the neighbourhood of a pinch point, given above in this Note, by forming the equation of the enveloping cone drawn from $(1, 0, 0, 0)$, namely, by forming the discriminant, in regard to θ, of the equation. It is useful to notice that the θ-discriminant of an equation of the form $a+b\theta+c\theta^2+\ldots=0$ has a form $aU+b^2V$, where U, V are polynomials in $a, b, c, \ldots$. This is easily seen from the Sylvester determinantal form of this discriminant.

Note II, to p. 163 preceding. Noether's formulae, quoted by himself ("Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde, Zw. Auf.," *Math. Ann.* VIII, 1875), from his own paper, "Sulle curve multiple di superficie algebriche," *Ann. d. Mat.* v, 1871–3, are as follows:

The surface, of order n, has j-fold curves C_j, of order m_j, and rank r_j; a curve C_j has s_{jj} actual double points, and meets a curve $C_{j'}$ in $s_{jj'}$ further points $(j \leqslant j')$. The surface also has k-fold nodal points, P_k, through which a curve C_j passes with h_{jk} branches, at which the osculating cone (with no further multiple generators) has class ν_k. Then the class of the surface, say μ_2, and the class of the envelope of its stationary tangent planes, say i, are given by

$$\begin{aligned}\mu_2=n(n-1)^2-\sum_j (j-1)\{(3j+1)n-2j(j+1)\}\, m_j+\sum_j j^2(j-1)r_j\\ -\sum_k(k-1)\nu_k+\sum_{jj'}(j-1)\{(3j+1)j'-j(j+1)\}s_{jj'}\\ +\sum_{jk}(j-1)(2j+1)(k-j)h_{jk},\end{aligned}$$

$$\begin{aligned}i=4n(n-1)(n-2)-2\sum_j j(j-1)\{2(3n-2j-2)m_j-(2j-1)r_j\}\\ -\sum_k(4k-5)\nu_k+4\sum_{jj'}j(j-1)(3j'-j-1)s_{jj'}\\ +8\sum_{jk}j(j-1)(k-j)h_{jk},\end{aligned}$$

where the $\sum\limits_{jk}$ extend to all the branches of the curves C_j which pass through P_k, and the $\sum\limits_{jj'}$ extend to all the intersections of two curves C_j outside the P_k, and to all the actual double points of a curve C_j.

For convenience of reference, we add here, also from Noether's paper, the following formulae, of which the meaning appears in the following chapter:

$$\begin{aligned} p_n = {} & \tfrac{1}{6}(n-1)(n-2)(n-3) \\ & -\tfrac{1}{6}\sum_j j(j-1)\{(3n-2j-5)m_j - \tfrac{1}{2}(2j-1)r_j\} - \tfrac{1}{6}\sum_k (k-2)\nu_k \\ & \qquad + \tfrac{1}{6}\sum_{jj'} j(j-1)(3j'-j-1)s_{jj'} + \tfrac{1}{3}\sum_{jk} j(j-1)(k-j)h_{jk}, \end{aligned}$$

$$\begin{aligned} p^{(1)} = {} & n(n-4)^2 + 1 - \sum_j (j-1)\{(3j-1)n - 2j(j+3)\}m_j + \sum_j j(j-1)^2 r_j \\ & -\sum_k \{k + (k-3)\nu_k\} + \sum_{jj'} (j-1)\{(3j-1)j' - j(j+1)\}s_{jj'} \\ & \qquad + \sum_{jk} (j-1)(2j-1)(k-j)h_{jk}. \end{aligned}$$

The reader may also consult Zeuthen, *Lehrbuch der abzählenden Methoden der Geometrie*, Leipzig, 1914, pp. 160–70.

CHAPTER V

INTRODUCTION TO THE THEORY OF THE INVARIANTS OF BIRATIONAL TRANSFORMATION OF A SURFACE, PARTICULARLY IN SPACE OF THREE DIMENSIONS

Preliminary; a general survey. The genus of a plane curve, it is well known, may be defined by drawing lines, in the plane of the curve, through an arbitrary fixed point, O, to meet the curve, say, in m variable points, and considering the coincidences of two (consecutive) points, belonging to the intersections of one of these lines, with the curve, which may arise as the line varies. If w be the whole number of such coincidences, for all the lines through O for which such coincidences occur, with suitable allowance for multiple coincidences, the genus of the curve is $\frac{1}{2}w-m+1$. In regard to this, it is proved, (i), that this number is the same for all positions of O in the plane, whether O be on the curve or not, and however intricate may be the multiplicities existing on the curve, if w be properly defined in each case; and, (ii), that this number is the same for any other plane curve which is in (1, 1) birational correspondence with the original curve. This definition of the genus can, it is known, be generalised, by considering, instead of the point series obtained on the curve by variable lines through a fixed point, any linear series of freedom 1 on the curve. The aggregate of the points of coincidence of two (consecutive) points of a set of the series, for all the sets in which coincidences occur, forms a set of points which is often called the Jacobian set of the series; if m be the number of points in any set of the series, and w the number of points in the Jacobian set, with suitable allowance for points at which more than two points of a set of the original series coincide, the genus of the curve is still $\frac{1}{2}w-m+1$. And more generally still, if g_r^m denote any linear series of sets of m points on the curve, with freedom r, we may choose from this a linear series of freedom 1, say g_1^m, and consider the Jacobian set of this series; all the Jacobian sets so obtainable, for all possible series g_1^m contained in the series g_r^m, are then known to belong to the same linear series; this we may speak of as the Jacobian series of g_r^m. If w be the number of points in any set of the Jacobian series, $\frac{1}{2}w-m+1$ is the genus of the curve.

When we seek to generalise these ideas to the case of surfaces, the primary definition of the genus of a curve, as here considered, which is effectively by the number of tangent lines which can be drawn to the curve from an arbitrary point, suggests the study of the enveloping cone to the surface, drawn from an arbitrary point, with a view to finding, if possible, numbers defined by the cone, or by the curve sections of the surface lying on planes through its vertex, which, (i), shall be independent of the position of the vertex of the cone, and, (ii), shall be equal to the numbers similarly defined for enveloping cones to another surface which is in (1, 1) birational correspondence with the original surface. This procedure, with the enveloping cone, was followed by Zeuthen, *Math. Ann.* IV, 1871, p. 21; and, with acknowledgments to Zeuthen and Clebsch, by Noether, *Math. Ann.* VIII, 1875, p. 495. A carefully revised introductory account is given by Zeuthen, *Lehrbuch der abzählenden Methoden der Geometrie*, 1914, pp. 160–7. But the sections of a surface, by planes through a fixed point, are a particular linear ∞^2 system, say a *net*, of curves on the surface; and it may be hoped to obtain invariants of the surface by consideration of any linear ∞^2 system, or net, of curves on the surface. For one invariant, I, it appears indeed that it is sufficient to consider a linear ∞^1 system, or *pencil*, of curves on the surface; this was shewn by Segre, *Atti... Torino*, XXXI, 1895–6, p. 485. For another invariant, ω, a definition was obtained with a net of curves by Severi, with special acknowledgments to Segre, in the *Atti...Torino*, XXXVII, 1901–2, p. 625. In many respects, the arguments followed by Segre and Severi are a direct generalisation of those of Zeuthen and Noether, and the results are the same. But they have a more geometrical form, and an elementary introduction will be given here from this point of view. The analytical work of Noether, however, provides a proof that the only possible invariants are those which are obtained here; and provides material for the consideration of intricate possibilities in the singularities of the surface considered; the account in Zeuthen's *Lehrbuch*, referred to above, provides the geometrical aspect of Noether's formulae, for the simpler cases. For greater complexity, Zeuthen's paper, *Math. Ann.* X, 1876, p. 446, and Noether's paper, *Ann. d. Mat.* V, 1871–3, p. 162, should be consulted. For the treatment of the canonical curves of a surface contained in this chapter, special acknowledgment is due to Enriques' paper, "Intorno ai fondamenti della Geometria sopra le superficie algebriche," *Atti...Torino*, XXXVII, 1901–2; as Enriques points out, the use of the Jacobian determinant is essentially involved in the detailed formulae of transformation of Clebsch's everywhere finite double integral, as developed by Noether. For the treatment of the invariant ω similar acknowledgment is due to Severi's paper, in the same volume of the *Atti...Torino*, referred to above, "Il genere aritmetico ed il genere lineare in relazione alle reti di curve tracciate sopra una superficie algebrica." The approach we make to the theory here is designedly of a tentative character, by a method of successive approximation, beginning with the simplest cases. Since this account was prepared (and given in lectures, substantially as here), there has appeared the volume of Enriques-Campedelli, *Lezioni sulla teoria delle superficie algebriche*, Parte prima, 1932, which contains a brilliant summary of the results obtained by Enriques, since his first " Ricerche" in the *Mem....Torino* of 1893, as well as by Castelnuovo and Severi; but, in the writer's view, an elementary introduction still remains desirable.

It may seem that it should be possible to evade the consideration of complex singularities on a surface, in a manner analogous to that followed for a plane curve, by the consideration of algebraic integrals associated with the surface. It was in fact the suggestion, by Clebsch (*Compt. rend.* LXVII, Dec. 1868, p. 1238), that we may consider everywhere finite *double* integrals for a surface, which gave the first assured hope of invariants of the birational transformation of surfaces. But more detailed consideration has shewn that the invariant so suggested (the number, p_g, of everywhere finite double integrals) does not always coincide with, and cannot be used to replace, another invariant p_n, which arises by such considerations as we have sketched; curiously enough, it is the difference of these two invariants, $p_g - p_n$, which finally emerges as the most important invariant of birational transformation. The existence of two invariantive genera, p_g and p_n, for a surface is one important difference of the theory from that of a curve. Another vital difference arises from the fact that, when one surface is transformed into another, a point of the original surface may fail to give a definite corresponding point of the other; the various methods of approach to the point on the first surface may all give rise separately to different points on the new surface, whereon there thus arises a *curve* to correspond to the *point* of the original. This possibility is familiar in the transformation of one plane into another plane. Let e' be the number of curves of the new surface which thus arise from points of the original surface; and e the number of curves of the original surface which similarly transform into points of the new surface. Curves thus arising by transformation from points are called *exceptional* curves; the name is often applied only to the case when the points in question are *simple* points of the surface on which they lie. We shall define invariants I, ω for the original surface, and, correspondingly, invariants I', ω' for the new surface, such that, subject to appropriate modification when either surface possesses multiple points, we have $I - e = I' - e'$ and $\omega + e = \omega' + e'$. Because of the corrective terms e, e' here, which depend on the particular transformation, I and ω are spoken of as relative invariants; from these an absolute invariant p_n can be defined by means of the equation $I+\omega=12p_n+9$. A form of this equation which is geometrically significant is $(I+4)+(\omega-1)=12(p_n+1)$. In regard to I and ω, we consider, apart, the two questions: first, of shewing that either can be defined in terms of systems of curves on a particular surface, and is the same whatever system be used, so that it is an invariantive property of this surface; and second, of finding the modification which arises by a birational transformation to another surface. The surfaces considered first of all are supposed to lie in ordinary space of three dimensions, and to have no multiple points other than a double curve with triple points, as explained in the preceding chapter; but the aim is to present the theory so that the definitions may be capable of generalisation to any surfaces whatever.

Definition and illustration of the invariant I. Consider, upon the surface considered, any linear ∞^1 system, or pencil, of curves; denote by p the genus of the general curve of the system, which is supposed to be irreducible; such a pencil of curves is given by the intersection of the surface with a system of surfaces having an equation of the form $U+\lambda V=0$, where λ is variable, but U, V are definite (of the same order). The curves of the system may have assigned base points, common to all; these are supposed, unless the contrary is said, to be at simple points of the surface, and the base point has an assigned multiplicity for the curves of the pencil, all the tangents of the curve thereat being free. The number of points of the surface at which such base points occur is denoted by σ. Among the curves of the pencil there will generally be a finite number possessing a double point, other than at the base points of the pencil, and other than on multiple curves of the surface; in such case let the whole number of double points so arising be denoted by δ. There is difficulty in case there are curves in the pencil which are multiple, or contain a multiple part; and the question arises whether in such case a consistent rule can be given for estimating the corresponding contribution to δ. In the simplest cases, however, it can be proved that the number $\delta-\sigma-4p$ is the same for any two linear pencils of curves on the surface; it is this number which is the invariant I spoken of. According to the plan we adopt, we consider various simple cases before passing to the proof of the invariance of I in general. The most elementary example is that of a pencil of plane sections of the surface, made by planes through a fixed line of general position. In this case, as the base points of the curves of the system are the intersections of the line with the surface, σ is the order (μ_0) of the surface; likewise δ is the number of planes through the line which touch the surface, the class (μ_2) of the surface; if p is the genus of a general plane section of the surface, the rank (μ_1) of the surface, equal to the class of an arbitrary plane section of the surface, is such that $\mu_1=2\mu_0+2p-2$, there being no cusps on an arbitrary plane section when the surface has no cuspidal curve. Hence we have $I, =\mu_2-\mu_0-4p, =\mu_2-2\mu_1+3\mu_0-4$. In particular, for a general surface of order n, with no multiple curve (and no multiple points), so that $\mu_2=n(n-1)^2$, and $p=\frac{1}{2}(n-1)(n-2)$, this gives $I=(n-2)(n^2-2n+2)$. Thus, for a general cubic surface $I=5$; for a quadric surface $I=0$, and, for a plane $I=-1$.

It will be interesting to verify at once, as an example of the general theorem of the invariance of I for a given surface, that the value $I=-1$, just obtained, essentially by a pencil of lines in the plane, is obtained also, from the original definition of I, by considering a general pencil of curves in the plane; we take the case in which the assigned base points are the intersections of any two curves of the pencil, supposed, first, all to be simple intersections. The curves being then given by an equation $\phi+\lambda\psi=0$, where ϕ and ψ are of order n, the double points arise for values of the parameter λ for which the three curves $\phi_1+\lambda\psi_1=0$, $\phi_2+\lambda\psi_2=0$, $\phi_3+\lambda\psi_3=0$ meet in a point, where $\phi_1=\partial\phi/\partial x$, etc. The order in λ, of the eliminant* of these three equations, is $3(n-1)^2$, so that there are $3(n-1)^2$ curves of the pencil which have double points; taking then $\sigma=n^2$, $p=\frac{1}{2}(n-1)(n-2)$, we obtain

$$I=3(n-1)^2-n^2-2(n^2-3n+2),$$

which is -1 whatever n may be. If, however, second, we suppose that all the curves of the pencil have a base point which is multiple, of order s, the remaining base points being simple, then $\sigma=n^2-s^2+1$ and $p=\frac{1}{2}(n-1)(n-2)-\frac{1}{2}s(s-1)$; the value $I=-1$ is then only obtained if we take $\delta=3(n-1)^2-(s-1)(3s+1)$. Thus, when we have proved that the value of I is independent of the pencil of curves with which it is defined, it will follow that the number of curves having double points, in a pencil which has a single s-fold common point, is less than the general number $3(n-1)^2$ by $(s-1)(3s+1)$; or, as we easily see, in the more general case of several multiple points, by the sum of the separate terms $(s-1)(3s+1)$ arising for these singly. For example, in the simple case of a pencil of plane cubic curves with one given double point, and five simple points common, a cubic with a double point is obtained, as a composite curve, by the conic through the double point and four of the simple points, taken with the line joining the simple point to the double point; and all the $12-7$, or 5, cubics of the pencil which have double points, not at the base points, are accounted for in this way.

---

* This is obtainable as an example of a rule which is often useful in general cases: Let $f_i(x,a)$ be a rational polynomial homogeneously of order n_i in the coordinates $x_0,\dots,x_r$, and homogeneously of degree e_i in the parameters $a_0,\dots,a_s$. Then the eliminant, in regard to $x_0,\dots,x_r$, of the $r+1$ equations $f_i(x,a)=0$, for $i=0,\dots,r$, is homogeneously of degree $n_0n_1\dots n_r\Sigma e_i/n_i$ in the parameters $a_0,\dots,a_r$. The eliminant may in fact be formed, for example, by taking the product of the $n_1\dots n_r$ factors $f_0(x^{(k)},a)$, in which $(x^{(k)})$ denotes one of the $n_1\dots n_r$ points common to the r loci $f_1(x,a)=0=\dots=f_r(x,a)$; this procedure shews the degree in which the parameters $a_1,\dots,a_r$ enter from $f_0(x,a)$; the other similar methods of forming the eliminant shew the influence of the other functions $f_1(x,a),\dots,f_r(x,a)$, in the aggregate entry of the parameters in this eliminant.

Ex. If $u=0$, $v=0$, $w=0$ be three general quadric surfaces, a linear pencil of elliptic curves, with 8 simple base points, is determined on the quadric surface $w=0$, by means of the system of quadric surfaces $u+\lambda v=0$. Find the number of curves of this pencil which have double points.

As another illustration, consider the case of a surface, of order n, which has a j-fold line, but is in other respects general. We can find directly the number of tangent planes to the surface through a general line, or the number through the j-fold line, and compute the value of I in both ways. But the assumption of the identity in the values of I in the two cases would enable us to infer the number of tangent planes to the surface from either line from the number from the other line. We shall see that the value of I is less than the value found above, in the general case, namely $(n-2)(n^2-2n+2)$, by $(j-1)\{(3j+1)n-2j(j+2)\}$. Take first the case of the tangent planes which can be drawn to the surface from the j-fold line. This line being taken to be $x=0=y$, the equation of the surface is $f=0$ where f is of the form $x^j u+x^{j-1}yv+\ldots+y^j w$, where $u, v, \ldots, w$ are of order $n-j$ in the coordinates x, y, z, t. The tangent plane at a point (x) of the surface, having an equation of the form $(x'\partial/\partial x)f=0$, in which (x', y', z', t') are current coordinates, will contain the multiple line when the point of contact (x) is such that $\partial f/\partial z=0$ and $\partial f/\partial t=0$. The points of contact of tangent planes, from the multiple line, thus lie on two surfaces of order $n-1$, with equations of the forms $x^j u_z+x^{j-1}yv_z+\ldots=0$, $x^j u_t+x^{j-1}yv_t+\ldots=0$, where $u_z=\partial u/\partial z$, etc., these representing the first polars of the two points $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ of the multiple line. These surfaces meet, outside the multiple line, in a curve of order $(n-1)^2-j^2$. At a point where this curve of intersection meets the multiple line, one of the j tangent planes of the former of the two polars coincides with one of the j tangent planes of the latter. The values of z/t at these points of meeting are therefore obtainable by putting $x=0$ and $y=0$ in the coefficients $u_z, \ldots, u_t, \ldots$, and then forming the eliminant, in regard to x/y, of the reduced equations. As $u_z, \ldots, u_t, \ldots$ are of order $n-j-1$, we thus see that the curve of intersection of the polars meets the j-fold line in $2j(n-j-1)$ points. But, further, at such an intersection, these two polars have a common tangent plane with $f=0$; for, when $x=0$, $y=0$, we have identities such as

$$zu_z+tu_t=(n-j)u, \quad zv_z+tv_t=(n-j)v, \quad \text{etc.};$$

the common curve of the two polars thus touches one of the j sheets of $f=0$ at the intersection, and meets the other $j-1$ sheets, thus having $j+1$ intersections with $f=0$. The number of intersections of this curve with $f=0$, other than on the multiple line, is thus $A-B$, where $A=n[(n-1)^2-j^2]$ and $B=2(j+1)j(n-j-1)$; it is easy to see that $A-B$ is the same as $(n-j-1)^2(n+2j)$, or as $C-D$, where $C=n(n-1)^2$ and $D=j[(3j+2)n-2(j+1)^2]$; this gives the number of tangent planes to the surface $f=0$ from the multiple line. Assuming now that the general plane section of $f=0$, through the multiple line, has no fixed intersections therewith, putting therefore $\sigma=0$ in the defining formula for I, we obtain

$$I=(n-j-1)^2(n+2j)-2(n-j-1)(n-j-2);$$

this is the same as that stated above. For instance, for a quartic surface with a double line, $I=8$. The number of tangent planes to the surface from an arbitrary line may be found in a similar way (given below) and is $C-E$, where $C=n(n-1)^2$ and

$$E=(j-1)\{(3j+1)n-2j(j+1)\}.$$

Calculating then I from this as the value of δ, with $\sigma=n$, and $2p=(n-1)(n-2)-j(j-1)$, we easily find the same value of I as before.

The proof of this value of δ may be given: The first polars of two points of an arbitrary line have the multiple line of the surface as $(j-1)$-fold line, and meet, beside, in a curve of order $(n-1)^2-(j-1)^2$; an arbitrary plane section of these polars, by a plane through the multiple line, gives curves of order $n-j$, with $(n-j)^2$ intersections; thus the curve of intersection of the two polars meets the multiple line in a number of points given by $(n-1)^2-(j-1)^2-(n-j)^2$, which is $2(n-j)(j-1)$. There will be no loss of generality in assuming the two first polars to be those of $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$; they will then have equations of which one is $U+U_x=0$, in which U denotes $jx^{j-1}u+(j-1)x^{j-2}yv+\ldots=0$, and U_x denotes $x^ju_x+x^{j-1}yv_x+\ldots=0$; thus their curve of intersection meets the multiple line ultimately on planes, through this line, satisfying both the equations $(U)=0$, $(V)=0$, where $(U)=jx^{j-1}(u)+(j-1)x^{j-2}y(v)+\ldots$, $(V)=x^{j-1}(v)+\ldots+jy^{j-1}(w)$, where (u), (v), ..., (w) are u, v, ..., w, with x, y both replaced by zero; namely, this curve meets the multiple line at points where two of the j tangent planes of $f=0$ coincide; the number of such points is $2(j-1)(n-j)$. And, as before, the two polars have a common tangent plane at this point, which coincides with the pair of coincident tangent planes of $f=0$; so that the curve of intersection of the two polars meets $f=0$ in 3 coincident points on this plane, and in $j-2$ other points, or, in all, in $(j+1)$ points. The total number of intersections, of the common curve of the two polars, with $f=0$, on the multiple line, is thus $2(j+1)(j-1)(n-j)$; whence, the number of remaining intersections of this curve with $f=0$, each corresponding to a tangent plane to the surface from the arbitrary line, is $n\{(n-1)^2-(j-1)^2\}-2(j^2-1)(n-j)$; this is the same as the number $C-E$ stated above. For the number of tangent planes to the surface through the multiple line, Berzolari (in Pascal, *Repertorium*, II, 2, 1922, p. 665) refers to Fouret, *Rend. Palermo*, VIII, 1894, p. 202, and to Godeaux, *Nouv. Ann.* IX, 1909, p. 162.

We may illustrate the definition of I, also, by considering a pencil of curves on a perfectly general surface, given by an equation $\phi+\lambda\psi=0$, where λ is variable, in which the surfaces $\phi=0$, $\psi=0$ have no common base curve or common point on $f=0$. The double points of the curves of the pencil, being such that

$$(\phi_1+\lambda\psi_1)/f_1=\ldots=(\phi_4+\lambda\psi_4)/f_4,$$

where $\phi_1=\partial\phi/\partial x$, etc., are the intersections of $f=0$ with the curve common to the surfaces each represented by the vanishing of a determinant, of three rows and columns, in the matrix of three rows and four columns of which the general column consists of the elements ϕ_i, ψ_i, f_i. If m be the order of the surfaces $\phi=0$, $\psi=0$, this curve, in accordance with a known rule (p. 109), which we associate with a formula $[(c)/(-r)]_{t^{q+1-p}}$, has an order equal to the coefficient of t^2 in the ascending expansion of $(1-\mu t)^{-2}(1-\nu t)^{-1}$, where $\mu=m-1$ and $\nu=n-1$; this order is thus $3\mu^2+2\mu\nu+\nu^2$; the number, δ, of double points of curves of the pencil is n times this. The value of I is given by $\delta-m^2n-4p$, where

$$p=1+\tfrac{1}{2}mn(m+n-4);$$

this is easily verified to be independent of m, and the same as the value first given, $(n-2)(n^2-2n+2)$. It would be proper to verify the value of I in this way when the surfaces $\phi=0$, $\psi=0$ have a common base curve on $f=0$; or conversely, the assumption of the invariance of I would enable us to compute the number of double points of curves of such a pencil. For instance, if a pencil of surfaces of order m be drawn through a fixed base curve, of order N and genus P, on a cubic surface, the resulting pencil of curves on the cubic surface has a number of double points given by

$$\tfrac{3}{2}m(5m-3)-N(5m-4)+5P+3,$$

when m is large enough in regard to N.

Note. Definition of I in terms of an irrational pencil of curves. In order to obtain a formula which it is interesting to compare with the results of this chapter, we anticipate here ideas which are more fully explained later. The fundamental surface, which we suppose to be without multiple points, and possibly lying in space of higher than three dimensions, may be such that there exists upon it a system of curves which, though not given by the intersection of it with a linear pencil of primals $\phi+\lambda\psi=0$, still has the property that a single curve of the system passes through an arbitrary point of the surface. The generators of a ruled surface, when this is not a rational surface, form an obvious example of this possibility. More generally, if ξ, η be variables connected by a definite irreducible polynomial equation $\chi(\xi, \eta)=0$, which may then be regarded as representing a plane curve, there may be rational functions of the coordinates, say u and v, such that the primals given by $u=\xi$, $v=\eta$, have a common curve of intersection with the fundamental surface when ξ, η are connected by $\chi(\xi, \eta)=0$; this requires that the equation $\chi(u, v)=0$ should be identically satisfied in virtue of the equations of the fundamental surface. When this is so, an arbitrary point of the surface, by determining the values of ξ and η, determines a definite curve of the system passing through this point. Such a system of curves is called an *irrational pencil of curves*, under the hypothesis that the curve $\chi(\xi, \eta)=0$ is not a rational curve; and the genus of $\chi(\xi, \eta)=0$ is called the *genus of the pencil of curves*. It can be shewn that when $\chi(\xi, \eta)=0$ is a rational curve, the curves of the pencil are curves of a linear pencil as defined before. We may suppose that there is a definite (1, 1) correspondence between the curves of the irrational pencil, and the points of the representative curve $\chi(\xi, \eta)=0$. Denote the genus of this curve by ρ', and the genus of each curve of the pencil by ρ. Under the hypothesis $\rho'>0$, we can suppose that the curves of the irrational pencil have no common base point, since they would else be in (1, 1) correspondence with the rational pencil of lines formed by the tangents of the fundamental surface at this (necessarily simple) point. We prove that the invariant I of the surface can be defined by $I=\delta+(2\rho-2)(2\rho'-2)-4$, where δ is the number of double points of curves of the irrational pencil. If ρ' were zero, this would be $\delta-4\rho$, but would then be replaced by $\delta-\sigma-4\rho$, where σ is the number of base points of curves of the pencil. This general formula is to be deduced from the case when I is defined by a linear pencil of curves, by remarking that a linear series g_1^m, of sets of m points, on the representative curve $\chi(\xi, \eta)=0$, of freedom 1, corresponds to a series of *sets of m curves* of the irrational pencil on the fundamental surface; and the composite curve, formed of these m curves of the pencil, is one of a linear system of freedom 1, that is of a linear pencil of composite curves, on the surface. From the known result, for a linear series g_1^m on $\chi(\xi, \eta)=0$, it follows that, among the sets of m curves of the pencil, there are $2m+2\rho'-2$ cases in which the two components of a set coincide (or, more than two curves may coincide, a possibility here disregarded). In the definition of I by a linear pencil of curves, we have supposed the curves of the pencil to be irreducible, and have definitely excluded the case in which particular curves of the pencil contain multiple parts; for our present purpose, we must assume a value for the number of double points to which the occurrence of a repeated part, in a curve of the linear system, is to be counted equivalent. Suppose that the preceding definition of I by a linear system of curves remains valid when, as here, the curves of the system are formed by sets of m curves of an irrational pencil, each of genus ρ; suppose further that when two curves of the set coincide, this is counted as equivalent to the occurrence of h double points in the curve of the linear system. Then, as there are $2m+2\rho'-2$ such coincidences, we have thus a contribution of $h(2m+2\rho'-2)$ to the number of double points; if δ be the number of actual double points of curves of the irrational pencil, the total number of double points in the linear system of reducible curves is therefore $\delta+h(2m+2\rho'-2)$. The genus of the composite curve of the linear pencil is found from a principle, arising below (p. 223), that the genus of a curve which degenerates into two curves of genera p_1, p_2, intersecting in i points, is p_1+p_2+i-1; as two curves of the irrational pencil do not intersect, the composite curve, formed of m of these, has a genus $m\rho-(m-1)$. The value of I is therefore given by $\delta+h(2m+2\rho'-2)-4m(\rho-1)-4$, or $\delta+h(2\rho'-2)-4+2m(h-2\rho+2)$. In this argument, m is arbitrarily great ($>\rho'$); and it is natural to suppose that the number, h, of double points, to which two coincident curves of the irrational pencil are to be counted equivalent, is independent of m; if, therefore, I have a definite value, from whatever linear series it be computed, it must be independent of m, and thus the value to be taken for h is $2\rho-2$. This gives the value of I stated above. The direct justification of this value of h belongs below, where we consider the proof that I is an invariant. We compare here a result previously found (Chap. I, p. 19), that if, on a curve of genus ρ, there be an irrational series, of genus π, of sets of k points, of freedom 1, the number of coincidences, of pairs of points of a set of this series, is $2k+2\rho-2-2k\pi$. Two curves of the irrational pencil here considered have no intersection; but, if two curves had k intersections, then, upon any one curve of the pencil, the other curves would determine an irrational series, of sets of k points, of freedom 1 and genus ρ', equal to the genus of the irrational pencil; there would then be $2k+2\rho-2-2k\rho'$ points, on this one curve, at which two of the intersections of another curve of the pencil coincided; this number is $2\rho-2$ when, as is the case, $k=0$. The extension of the definition of I to the case here considered is given by Castelnuovo and Enriques, *Ann. d. Mat.* VI, 1901, pp. 21–6 (see below, p. 209). If we apply it to the case of a ruled surface of genus p, the irrational pencil consisting of its generators, taking $\delta=0$, $\rho=0$, $\rho'=p$, we thus find $I=-4p$. This is true also for $p=0$, and can be obtained independently, by considering a linear pencil of plane sections through an arbitrary line, from $\delta-\sigma-4p$; since then $\delta=\sigma$, the class of the surface being equal to its order.

Introductory definition of the invariant ω. Consider, on the fundamental surface, $F=0$, a *net* of curves, that is, a linear system of curves of freedom 2. Such a net will be determined by the intersection of $F=0$ with a linear system of surfaces having an equation of the form $\lambda f+\mu\phi+\nu\psi=0$, in which λ, μ, ν are variable, and $f=0$, $\phi=0$, $\psi=0$ are definite surfaces, of the same order, which may have common points and common curves, on $F=0$. Then consider the locus of the double points of curves of the net, other than at the base points common to all of $f=0$, $\phi=0$, $\psi=0$, and other than those which may arise on multiple curves, or at multiple points, of $F=0$, if such exist. We have already, in the preceding chapter, been concerned with such a net of curves. As has been said, through such a double point of a curve of the net, there will pass a pencil of curves of the net, of which one curve may be taken to be the curve which has a double point thereat; all curves of the net through this point will thus have the same tangent line at this point (the particularity that all curves of the net have a double point thereat being excluded). Conversely, if, at a simple point of $F=0$, all curves, of the pencil of curves of the net which pass through this point, have the same tangent line, there is a curve of the net which has a double point at this point. The locus of such double points is a curve which we call the *Jacobian* of the net. More generally, if we have any linear system of curves on $F=0$, given by a system of surfaces of the same order expressed by an equation $\lambda_0\vartheta_0+\lambda_1\vartheta_1+\ldots+\lambda_r\vartheta_r=0$, of freedom $r(\geqslant 3)$, common points or curves of the surfaces $\vartheta_0=0, \ldots, \vartheta_r=0$ lying on $F=0$ not being considered, then we may form a net of curves $\lambda f+\mu\phi+\nu\psi=0$ from this system, in which $f=0$, $\phi=0$, $\psi=0$ determine particular curves of the system, given say by such equations as

$$f=\sum_{i=0}^{r} a_i\vartheta_i,\quad \phi=\sum_{i=0}^{r} b_i\vartheta_i,\quad \psi=\sum_{i=0}^{r} c_i\vartheta_i,$$

in which a_i, b_i, c_i are definite constants; and may then consider the Jacobian of this net. It can be proved, as below, that all Jacobian curves so obtainable, by taking different systems of values of a_i, b_i, c_i, belong to one linear system of curves on $F=0$; this we call the *Jacobian system* of the original linear system of curves.

We can form an equation satisfied by the Jacobian curve of a net of curves on $F=0$, given by $\lambda f+\mu\phi+\nu\psi=0$, by expressing that the latter surface touches $F=0$, and eliminating λ, μ, ν. The result is the vanishing of a determinant, say Δ, which we call the Jacobian determinant, of four rows and columns, whose rows consist respectively of the elements f_1, f_2, f_3, f_4; $\phi_1, \phi_2, \phi_3, \phi_4$; $\psi_1, \psi_2, \psi_3, \psi_4$; F_1, F_2, F_3, F_4, where $f_1=\partial f/\partial x$, etc.; but from the complete intersection of $F=0$, $\Delta=0$, must be omitted points expressly excepted in the definition of the Jacobian curve. And it is easy to see that if, in Δ, we suppose f to mean $\Sigma a_i\vartheta_i$, with similar values for ϕ and ψ, as explained above, then Δ is a sum of Jacobian determinants, formed with triads from $\vartheta_0, \ldots, \vartheta_r$, with determinant coefficients depending on the constant coefficients a_i, b_i, c_i; this shews that the Jacobian curves of all nets from a general linear system of curves belong to a linear Jacobian system. It can be shewn directly that, at a simple point of $F=0$, where the curves $\vartheta_i=0$ have a common base point of multiplicity s, the Jacobian curves have a base point of multiplicity $3s-1$. By the *complete* Jacobian system, of a given linear system of curves on $F=0$, is to be understood the most general linear system of curves which includes the Jacobian curves formed from all nets of the given linear system, and has a $(3s-1)$-ple point at every s-fold point, simple on $F=0$, of this given system*. This complete system may well include curves not expressed by the vanishing of the determinant we have used. For example, when the given linear system consists of all plane sections of the surface, the equation $\Delta=0$ is that of the first polar of a point in regard to the surface $F=0$; if the surface have a double curve, this first polar passes through the double curve, and a Jacobian curve, according to the definition, consists of the residual intersection; that is, of the curve which, in the last chapter, we have called a *proper* curve of contact. We have seen, however, that all proper curves of contact pass through the pinch points of the double curve; these, not being base points of the general plane section of the surface $F=0$, are not to be regarded as conditions for the complete Jacobian system of the system of plane sections, unless this be explicitly stated. The system of all first polars is of freedom 3; the freedom of the complete Jacobian system may well be greater than this. Another fact, in regard to which clearness is desirable, presents itself when we consider the Jacobian curve of a net of curves given by an equation of the form $u(\lambda f_1+\mu\phi_1)+\nu\psi=0$; for such a net there exists a curve, given by $u=0$, which meets the curves of the net only at base points, common to all the curves of the net; and the curves of the net which pass through an arbitrary point which lies on $u=0$ are those for which the coefficient ν is zero; all these curves are composite, consisting of the curve given by $u=0$, together with those of the pencil given by $\lambda f_1+\mu\phi_1=0$; of this pencil, one curve passes through the point in question, so that the point is a double point of a curve of the net. Correspondingly, it may be seen directly (see below, p. 218) that u is a factor of the Jacobian determinant of the net. More generally, for a system of curves given by an equation of the form $u(\lambda_0\vartheta_0+\ldots+\lambda_{r-1}\vartheta_{r-1})+\lambda_r\vartheta_r=0$, the complete Jacobian system, as defined above, is the linear system defined by particular Jacobian curves which have u as a part. Such a curve, given by $u=0$, which meets the curves of the given linear system only at base points of the system, may be called a *fundamental* curve of the system. We postpone the general discussion of the matter; for the present, we make the assumption that the original linear system of curves is such that the general curve of its complete Jacobian system is irreducible.

---

* It is understood that the tangents at a base point of the net are all variable, and no two base points are consecutive, in Noether's sense. For example, if two base points of the first order be consecutive, that is, if all the curves of the net have a given tangent line, the Jacobian curve has in fact a triple point, of which one tangent coincides with that of the curves of the net. We may agree, in this case, that, for the complete Jacobian system, only this tangent line is prescribed.

We may give, however, some introductory consideration to the case of systems of *plane* curves for which a fundamental curve exists; in the Jacobian determinant above, F is then to be replaced by t, and the determinant reduces to one of three rows and columns, in the plane coordinates x, y, z. (i), For a system of conics in a plane which have two points in common, and are expressible by an equation $z(2gx+2fy+cz)+x^2+y^2=0$, the line $z=0$, joining the base points, is evidently fundamental. Now, for a system of plane curves of order n, with s-fold base points, the Jacobian determinant would give rise to curves of order $3(n-1)$, with $(3s-1)$-fold base point at an s-fold base point of the system; when $n=2$, and there are two simple base points, this leads to cubic curves with two double points; such a system consists of the line $z=0$ joining the base points, together with all conics through the base points (if we consider the complete Jacobian system). (ii), Consider, next, the system of quartic curves in a plane which have two given double points. The general curve of this system is expressed by an equation $z(\lambda_0\vartheta_0+\ldots+\lambda_7\vartheta_7)+\lambda_8\vartheta_8=0$, where $\vartheta_0=0, \ldots, \vartheta_7=0$ are cubic curves through the two base points, which are supposed to lie on $z=0$, and $\vartheta_8=0$ is a particular quartic curve with the two given double points. The Jacobian system will then consist of curves of order 9, with a 5-ple point at each base point. Such curves necessarily have the fundamental line $z=0$ as part; and the system of all curves of order 9 with two 5-ple points consists of this line together with all the curves of order 8 having these two as 4-ple points. (iii), Consider now the system of ∞^3 cubic curves which have 6 common base points so situate as to lie on a conic; the conic is then a fundamental curve, the cubic curves being given by an equation $u(ax+by+cz)+dU=0$, where $u=0$ represents the conic, and $U=0$ is a particular cubic curve of the system. The Jacobian system then consists of sextic curves with the 6 base points as double points, and the conic $u=0$ is fundamental also for the system of these sextic curves; for sextic curves with 6 double base points are of tale $\frac{1}{2}7.8-18$, or 10, while quartic curves with 6 simple points are of tale $\frac{1}{2}5.6-6$, or 9; the complete Jacobian system is thus expressed by an equation $u(\lambda_0\psi_0+\ldots+\lambda_8\psi_8)+\lambda_9\psi_9=0$, where $\psi_0=0, \ldots, \psi_8=0$ are quartic curves, and $\psi_9=0$ is a particular sextic. Evidently, in this case, the six base points, though lying on the conic, are not independent conditions for the conic; and this is the reason why the fundamental curve of the original system is a fundamental curve of the Jacobian system, instead of being part of every Jacobian curve, as in the two former examples. Of this statement it is easy to give a preliminary enumerative justification. For this, consider a system of plane curves of order n, with base points typically of multiplicity s. Let $u=0$ be a curve, of order k $(k\leqslant n)$, having a base point of the original system multiple of order s_1 $(s_1\leqslant s)$, but having no intersection with the curves of the original system other than at these base points, so that $\Sigma ss_1=nk$. Let λ be the redundance of the base points for the curves of the original system, so that the tale of this system is

$\frac{1}{2}(n+1)(n+2)-\frac{1}{2}\Sigma s(s+1)+\lambda$, say L. Similarly, let μ be the redundance of the base points, supposed now of typical multiplicity $s-s_1$, for curves of order $n-k$; so that the tale of such curves, with these multiple points, is

$$\tfrac{1}{2}(n-k+1)(n-k+2)-\tfrac{1}{2}\Sigma(s-s_1)(s-s_1+1)+\mu,$$

say M. It is then easy to compute that if the genus of the curve $u=0$, defined by $\frac{1}{2}(k-1)(k-2)-\frac{1}{2}\Sigma s_1(s_1-1)$, be equal to $\lambda-\mu$, then $L=M+1$. In this case, the system of curves of order n has an equation of the form $u\Theta+\lambda_\rho\vartheta_\rho=0$, where Θ is of the form $\lambda_0\vartheta_0+\ldots+\lambda_{\rho-1}\vartheta_{\rho-1}$, and $\Theta=0$ represents the complete system of curves of order $n-k$ having the base points with multiplicity $s-s_1$. Thus $u=0$ is a fundamental curve for the former system. The Jacobian system of this system consists of curves of order $3(n-1)$, with $(3s-1)$-ple base points; let N be the tale of the complete system of such curves. Similarly, let N_1 be the tale of the complete system of curves of order $3(n-1)-k$, with base points, typically of multiplicity $3s-1-s_1$ at the original base points. Then, when $N=N_1$, the Jacobian system of the former system breaks up into $u=0$ taken with all curves of the latter system. But, if $N=N_1+\xi$, with $\xi>0$, this will not be so; the complete Jacobian system will be more ample than the system directly defined as the locus of double points, or by the Jacobian determinants. We have, in fact,

$$N=\tfrac{1}{2}(3n-3+1)(3n-3+2)-\tfrac{1}{2}\Sigma 3s(3s-1)+\lambda',$$
$$N_1=\tfrac{1}{2}(3n-3-k+1)(3n-3-k+2)-\tfrac{1}{2}\Sigma(3s-s_1)(3s-s_1-1)+\mu',$$

where λ', μ' are the redundancies of the base points for the curves of order $3(n-1)$ and $3(n-k)$, respectively. Wherefore ξ, or $N-N_1$, is equal to $\frac{1}{2}\Sigma s_1(s_1+1)+\lambda'-\mu'-\frac{1}{2}k(k+3)$; thus, if both λ' and μ' be zero, ξ will be greater than zero when the conditions to be satisfied by the fundamental curve, $u=0$, at the base points, are not independent conditions for this curve, but exceed the number $\frac{1}{2}k(k+3)$ of conditions which a curve of order k can satisfy. The case of a conic through 6 points, cited above, is an example. For the theory of fundamental curves, here indicated, the reader should consult Castelnuovo's paper, *Memorie Torino*, XLII, 1892, p. 3.

We pass now to the definition of the invariant ω. We consider, on the fundamental surface $F=0$, a linear system of curves, of freedom at least 2, with base points only at simple points of the surface; and then the complete Jacobian system of this. We suppose this Jacobian system to consist of generally irreducible curves (in particular excluding the consideration of the case where the Jacobian curves have a common part arising from a fundamental curve of the original system). At an s-ple base point of the original system, the Jacobian curves are prescribed to have a $(3s-1)$-ple point. We denote by β the number of (simple) points of the surface at which base points of the original system of curves are found. We denote also by p the effective genus of a general curve of the original system, and by π the effective genus of a curve of the Jacobian system. Then we define ω in the first instance by

$$\omega-1=\pi-1-9(p-1)+\beta;$$

it can be shewn that this gives a value independent of the system of curves used for its computation.

As a simple illustration, suppose $F=0$ to be without multiple points, of order n, and the original system of curves to be given by $\lambda f+\mu\phi+\nu\psi=0$, the surfaces $f=0$, $\phi=0$, $\psi=0$ being of order m, and the curves to have multiple points typically s-fold. The Jacobian determinant is of order $3m+n-4$, and thus (recalling the formula for the genus of the complete intersection of two surfaces)

$$p-1=\tfrac{1}{2}nm(n+m-4)-\tfrac{1}{2}\Sigma s(s-1),$$

and $$\pi-1=\tfrac{1}{2}n(3m+n-4)(3m+2n-8)-\tfrac{1}{2}\Sigma(3s-1)(3s-2);$$

hence we find $\omega-1=n(n-4)^2$, which is thus independent of the system of curves employed. It is in fact equal to the *grade* of the system of curves in which $F=0$ is met by the general system of surfaces of order $n-4$, namely the number of intersections of any two such curves. And we notice that if, with the value of I found above for a non-singular surface, $I=(n-2)(n^2-2n+2)$, we define a number p_n by the equation $I+\omega-1=12p_n+8$, then

$$p_n=\tfrac{1}{6}(n-1)(n-2)(n-3),$$

which is the tale of all surfaces of order $n-4$. In particular the values of I and $\omega-1$, respectively for a plane ($n=1$), a quadric surface, and a general cubic surface, are $(-1, 9)$, $(0, 8)$ and $(5, 3)$, the value of p_n being zero in all these cases. It may indeed be proved directly, for a net of plane curves of genus p, without fundamental curves, having base points at β points of the plane, that the genus of the Jacobian curve of the net is $9p-\beta+1$.

More generally, when the surface $F=0$ has a double curve, with triple points triple for the surface, but no other multiple points, consider the system of plane sections of the surface. The Jacobian curves are then, as we have seen, the proper curves of contact of tangent planes to the surface drawn from points. With the notation employed in the last chapter we then have (p. 161), for $p-1$ and $\pi-1$ respectively, the values $\tfrac{1}{2}\mu_1-\mu_0$ and $\mu_2+\nu_2-\tfrac{2}{3}\mu_1$. Thus $\omega-1$ has the value $\mu_2+\nu_2-6\mu_1+9\mu_0$. In this chapter, also, we have found for $I+4$ the value $\mu_2-2\mu_1+3\mu_0$. Thus, for the number p_n defined by $\omega-1+I+4=12(p_n+1)$, we have $12(p_n+1)$ equal to $2\mu_2+\nu_2-8\mu_1+12\mu_0$. If p denote the genus of an arbitrary plane section of the surface, and i denote the number of stationary tangent planes of the surface which pass through an arbitrary point, that is the number of cusps in the net of curves formed by plane sections of the surface which pass through the point, this (see p. 162) is the same as $i=24(p_n+p)$. As a particular verification of this result it may be proved directly that among the curves of a net of plane curves (without fundamental curve), of genus p, there are $24p$ curves which have a cusp. (More general results are quoted below, p. 241; references in Pascal, *Repertorium*, II, 1, 1910, p. 340.)

The formula $\omega-1=\mu_2+\nu_2-6\mu_1+9\mu_0$, just obtained, has an interpretation which suggests a more general definition of $\omega-1$ than that we have given. We explain this, first, when, as in the derivation of the formula, the net of curves used for the definition of ω consists of the sections of the surface by planes through an arbitrary point. Use the notation (P, Q) for the number of intersections of two curves P, Q on the surface, other than at the base points prescribed for these curves; in particular use (P, P) for the number of intersections of two curves of a linear system, other than at the prescribed base points, this being the *grade* of the linear system. Then, if C, C' denote plane sections of the surface, this being of order μ_0, the term $9\mu_0$ in the expression for $\omega-1$ is the number $(3C, 3C')$; if J, J' denote the proper curves of contact of tangent planes to the surface from two different points, each of order μ_1, the term $6\mu_1$ is $(J, 3C')+(J', 3C)$. Further, μ_2, the number of tangent planes to the surface from an arbitrary line, is the number of intersections of two curves J, J' at ordinary points of the surface; we have seen, however, that all curves J, J' pass through the pinch points on the double curve of the surface, and touch the cotangent at this point (Chap. IV, p. 178); but we have also seen (*ibid.* p. 179) that of the two intersections of the curves J, J' thus arising, only one arises on the surface in space [4] by whose projection the double curve is generated. For this reason we may regard the complete value of (J, J') as being $\mu_2+\nu_2$. The formula for $\omega-1$ thus has the interpretation $(J, J')-(J, 3C')-(J', 3C)+(3C, 3C')$. This we may agree to write in the form $(J-3C, J'-3C')$, or the form $(3C-J, 3C'-J')$. As in the theory of linear series of sets of points on a curve, if $|A|$, $|B|$ denote two linear systems of curves on a surface, we denote by $|A+B|$ the complete linear system which contains all the composite curves, of which each is the aggregate of a curve of the system $|A|$ with a curve of the system $|B|$, the base points prescribed for $|A+B|$ being the aggregate of those prescribed for $|A|$ and $|B|$ singly. Similarly, if there exist a linear system $|C|$, such that $|C+B|=|A|$, then $|C|$ may be denoted by $|A-B|$. Now, let $|C_J|$ denote the complete linear system of curves containing the proper curves of contact J of tangent planes to the surface from all arbitrary points; the system $|C_J|$ will, if we maintain our definition, not have the pinch points of the double curve as base points, but the grade (C_J, C_J), of this system, will be the same as the number (J, J'), in which we have counted these points as simple intersections of curves J and J'; the number (C_J, C) will also be μ_1. Thus it appears that, if one of the two complete systems $|C_J-3C|$, $|3C-C_J|$ exist, then $\omega-1$ is the grade of this system. This important result is proved when $|C|$ is the system of plane sections of the surface; we shall see below that it can be extended to the case when $|C|$

is any linear system of curves, of which $|C_J|$ is the Jacobian system.

In terms of the characters of the double curve of the surface (see p. 162 of the preceding chapter) the value of $\mu_2+\nu_2-6\mu_1+9\mu_0$ can be expressed in the form

$$\omega-1=\mu_0(\mu_0-4)^2-5(\mu_0-4)\epsilon_0+2\epsilon_1+9t;$$

similarly the value of $\mu_2-2\mu_1+3\mu_0$ can be expressed by

$$I+4=\mu_0(\mu_0{}^2-4\mu_0+6)-(7\mu_0-16)\epsilon_0+4\epsilon_1+15t.$$

From these the value of p_n, given by $\omega-1+I+4=12(p_n+1)$, is such that $p_n+1=(\mu_0-1, 3)-(\mu_0-4)\epsilon_0+2t+P$, where $(\mu_0-1, 3)$ is the binomial coefficient, and P is the genus of the double curve. Thus p_n is the theoretical tale of surfaces of order μ_0-4 which pass through the double curve, valid when μ_0 is sufficiently great in respect to ϵ_0 (Vol. v, p. 236). We have remarked (p. 168, Ex. 11) that surfaces of order μ_0-4, put through the triple points of this curve, cut thereon a non-special series; when this series is incomplete, the number, p_g, of actual surfaces of order μ_0-4 through the double curve, is greater than p_n. A surface of order μ_0-4, through the double curve, has, as residual intersection with the given surface, a curve of order $\mu_0(\mu_0-4)-2\epsilon_0$, which is $\mu_1-3\mu_0$. This is the order of a curve of the system $|C_J-3C|$, when this system exists.

Effect of isolated nodal points of the fundamental surface upon the definition of I and ω. The values of the invariants I and ω have been given under the hypothesis that the only multiple points of the surface are a double curve with triple points. The singularities of a surface may be of great complexity, and, in general, as has been suggested, it must be supposed that the surface has been transformed into the form adopted above, before the theory can be applied. But, when the only singularities beside a double curve consist of isolated nodes of general kind, a modified definition of I and ω can be applied directly, which is interesting because of the frequent occurrence of such surfaces. We limit ourselves to supposing I defined by a pencil of sections by planes through an arbitrary line and ω defined by a net of sections by planes through an arbitrary point. The modified definitions must be such that they include the case when no such nodes occur; they must be invariant for birational transformation into other surfaces with isolated nodes, save for corrections indicative of these nodes; and we choose them such that the value p_n, arising by the equation $I+4+\omega-1=12(p_n+1)$, is the theoretical tale of surfaces of order μ_0-4, passing through the double curve, which have a $(k-2)$-ple node at each isolated k-ple node of the given surface. It has in fact been shewn by Noether and Zeuthen that the number p_n so defined is an absolute invariant for birational transformation of the surface; but we do not give the algebraical proof of this fact, since a more general proof emerges at a later stage of the theory of surfaces. As in what has been said, our present aim is introductory and provisional.

Consider first the invariant I. It may be shewn (for the literature, cf. Pascal, *Repertorium*, II, 2, 1922, pp. 663, 690; also Segre, *Annal. d. Mat.* XXV, 1897, p. 26) that, for a surface with isolated k-fold nodes, at which the osculating cone is of class k', the number of actual tangent planes which can be drawn from an arbitrary general line is less than $\mu_0(\mu_0-1)^2$ by the sum of the values of $(k-1)k'$ at these nodes; we consider only the general case, in which $k'=k(k-1)$. Then the class of the surface is given by $\mu_2=\mu_0(\mu_0-1)^2-\Sigma k(k-1)^2$. We define then a number I_0, to be subsequently modified to I, by the equation $I_0=\delta-\sigma-4p+\Sigma(k-1)$, where, as before, p is the genus of a general plane section of the surface, σ is the number of base points of a section of the surface by planes through an arbitrary line, δ is the number of double points, that is of actual contacts with the surface, on such plane sections, and the summation $\Sigma(k-1)$ extends to all the isolated nodes of the surface not lying on the arbitrary line. If this line be quite general we thus have

$$I_0=\mu_0(\mu_0-2)^2-\Sigma k(k-1)^2-\mu_0-4p+\Sigma(k-1),$$
$$=\mu_0^2(\mu_0-2)-4p-\Sigma(k-1)(k^2-k-1).$$

And we can prove that the same value is obtained if we take a line which contains a single k_0-fold node, say P_0. For the line joining P_0 to an arbitrary point, A, is $k_0(k_0-1)$-fold on the enveloping cone drawn from A to the surface, so that there are $2k_0(k_0-1)$ tangent planes of the osculating cone at P_0 which pass through the line AP_0. The number of proper tangent planes to the surface from this line is therefore $\mu_0(\mu_0-1)^2-\Sigma k(k-1)^2-2k_0(k_0-1)$; this is the value to be taken for δ when I_0 is computed by planes through the line AP_0; the values for σ and the genus are respectively μ_0-k_0+1 and $p-\frac{1}{2}k_0(k_0-1)$. Thus we have

$$I_0=\delta-(\mu_0-k_0+1)-4[p-\tfrac{1}{2}k_0(k_0-1)]+\{\Sigma(k-1)-(k_0-1)\},$$

where the $\Sigma(k-1)$ extends to *all* the isolated nodes; with the value of δ put down, this is the same as was obtained by sections through a general line. A similar argument shews that the same result is obtained if we take sections by planes through a line which contains two of the isolated nodes. In particular for a surface with isolated nodes but without a double curve,

$$I_0+4=\mu_0(\mu_0^2-4\mu_0+6)-\Sigma(k-1)(k^2-k-1).$$

In general, the value given by the definition is

$$I_0+4=\mu_2-2\mu_1+3\mu_0+\Sigma(k-1),$$

where μ_2 is the actual class of the surface, as estimated above.

Consider next the invariant ω. For a surface with a double curve (and triple points thereon), and k-ple isolated quite general nodes, we define a number ω_0-1 given by

$$\omega_0-1=\pi-1-9(p-1)+\beta+\Sigma(k-1)(2k-1)-\tfrac{1}{2}\Sigma k(k-1),$$

where p is the genus of the curves of the net used for the definition (which we take to be given by planes through an arbitrary point), β is the number of (simple) points of the surface where curves of this net have prescribed base points of any orders (in the case taken, $\beta=0$), and π is the genus of the Jacobian curve of this net (in this case the proper curve of contact of tangent planes from the arbitrary point). From the formulae given in the last chapter (p. 163), we have $2p-2=\mu_1-2\mu_0$, and $2\pi-2=2(\mu_2+\nu_2)-3\mu_1+\Sigma k(k-1)$; thus we deduce for ω_0-1 the value $\mu_2+\nu_2-6\mu_1+9\mu_0+\Sigma(k-1)(2k-1)$. The correction for the isolated nodes is introduced so that the value of p_n deduced from $I_0+4+\omega_0-1=12(p_n+1)$, taking account of $\mu_2=\mu_0(\mu_0-1)^2-\Sigma k(k-1)^2$, should be

$$(\mu_0-1,\ 3)-(\mu_0-4)\epsilon_0+2t+P-1-\Sigma(k,\ 3),$$

where $(\mu_0-1,\ 3)$ and $(k,\ 3)$ denote binomial coefficients. This is the theoretic tale of surfaces of order μ_0-4, when μ_0 is large enough, which pass through the double curve, and have a $(k-2)$-ple point at every isolated k-ple point of the surface; we have in fact $k(k-1)^2-k(k-1)=6(k,\ 3)$. The value of p_n is such that

$$12(p_n+1)=2\mu_2+\nu_2-8\mu_1+12\mu_0+2\Sigma k(k-1).$$

Another point of view is to define ω by its value on another surface, in birational correspondence with the given one, in which the k-ple point of the original is replaced by a curve of order k. See below, p. 234.

Ex. 1. Prove directly, in accordance with Noether's formula quoted in Note II to the preceding chapter, that, for a general surface of order n, the presence of a j-ple line involves a diminution in the value of p_n given by $\tfrac{1}{6}j(j-1)(3n-2j-5)$. Cf. also Cayley, *Papers*, VIII, p. 395, for a direct calculation of p_n which includes the case of a cone.

Ex. 2. For sufficiently large μ_0, that a surface of order μ_0 should have a given curve of order n, and genus p, with δ double points, as a j-ple line, involves a number of conditions given by

$$[\tfrac{1}{2}j(j+1)\mu_0-\tfrac{2}{3}j(j^2-1)]n-\tfrac{1}{6}j(j+1)(2j+1)(p+\delta-1).$$

Cf. Noether, *Ann. d. Mat.* v, 1871–3, § 7.

Ex. 3. For sufficiently large μ_0, that a surface of order μ_0 should touch a given surface of order m, along a given curve of order n and genus p, lying thereon, involves a number of conditions given by

$$(2\mu_0+m-4)n-4(p-1).$$

Cf. Hudson, *Cremona Transformations* (Cambridge, 1927), p. 253.

We now illustrate the definitions of I_0, ω_0 which we have given, by applying them to several simple examples. These definitions are $I_0+4=\mu_2-2\mu_1+3\mu_0+\Sigma(k-1)$, and

$$\omega_0-1-\Sigma(k-1)(2k-1)=\pi-1-9(p-1)-\tfrac{1}{2}\Sigma k(k-1)$$
$$=\mu_2+\nu_2-6\mu_1+9\mu_0.$$

(*a*) For a general cubic surface we have seen that $I_0=5$, $\omega_0-1=3$ (p. 196). Consider a cubic surface with a single double point; then $k=2$, $\mu_0=3$, $\mu_1=6$, $\nu_2=0$, $\mu_2=3\,.\,4-2$, $=10$, and $p=1$, the value of the class, μ_2, being found from the general formula

$$n(n-1)^2-\Sigma k(k-1)^2.$$

The value of π, the genus of an enveloping cone having one double generator, is 3. Thus we find $I_0=4$, $\omega_0-1=4$, and $p_n=0$ as for a general cubic surface. Consider the correspondence of this nodal cubic surface with a plane: In general, a cubic surface has its plane sections representable by plane cubic curves passing through 6 points of the plane; but, if these 6 points lie on a conic, given, say by $u'=0$, such cubic curves are given by an equation

$$u'(ax+by+cz)+U'=0,$$

where a, b, c are arbitrary constants, and $U'=0$ is a particular cubic curve through the 6 base points. The cubic surface, with x, y, z, t as coordinates, is then obtained by the transformation

$$x/x'u'=y/y'u'=z/z'u'=t/U',$$

and the conic $u'=0$ corresponds to the neighbourhood of the double point on the cubic surface; the transformation is obtainable by projection from the double point. There is thus a *curve* on the plane corresponding to a *point* on the cubic surface, namely the conic $u'=0$; this we may call an *exceptional* curve, and, if e' denote the number of exceptional curves in the plane, we have $e'=1$. As in the case of the general cubic surface, there are 6 lines on the cubic surface corresponding each to one of the 6 base points in the plane; in this case they lie on the osculating quadric cone at the double point of the cubic surface; if e denote the number of exceptional curves on the cubic surface, we thus have $e=6$. The values of I_0, ω_0-1 for the plane are, we have seen, given by $I'=-1$, and $\omega'-1=9$. Thus we have the equations $I_0-e=I'-e'$, $\omega_0+e=\omega'+e'$, for which $I_0+\omega_0=I'+\omega'$.

More generally, for any surface of order n, with a single node of multiplicity $n-1$, given by an equation of the form

$$t(x, y, z)_{n-1}+(x, y, z)_n=0,$$

we have

$$\mu_0=n,\quad \mu_1=n(n-1),\quad \nu_2=0,$$
$$\mu_2=n(n-1)^2-(n-1)(n-2)^2=(n-1)(3n-4),\quad k=n-1,$$

and hence $I_0 = n^2 - n - 2$, and $\omega_0 - 1 = -n(n-1) + 10$; while, for the transformation of this surface to a plane, which is obtainable by projection from the node, we have $e = n(n-1)$ and $e' = 1$, as the respective numbers of exceptional curves on the surface and on the plane. Thus, in this case also, $I_0 - e = I' - e'$ and $\omega_0 + e = \omega' + e'$. In view of a subsequent theorem, we may remark further that if two surfaces of order $n-4$ be taken, each having a node of multiplicity $n-3$ at the node of the given surface, then their curve of intersection meets the surface, other than at the node, in

$$n(n-4)^2 - (n-1)(n-3)^2$$

points, which is $\omega_0 - 2$. This is then the number of free intersections of any two curves of the system determined, on the surface, by all surfaces of order $n-4$ having a $(n-3)$-fold node at the given node.

(*b*) Next consider the cubic surface with four nodes (of multiplicity 2). For this surface we have $\mu_0 = 3$, $\mu_1 = 6$, $\mu_2 = 4$, $\nu_2 = 0$, $p = 1$, $\pi = 0$ (the enveloping cone from an arbitrary point being the dual of a plane quartic curve with three nodes, since the dual of the surface is the Steiner quartic surface, considered in the following example); also $\Sigma(k-1) = 4$, $\Sigma(k-1)(2k-1) = 12$, $\frac{1}{2}\Sigma k(k-1) = 4$. Hence we have $I_0 = 1$, $\omega_0 - 1 = 7$, $p_n = 0$. The plane representation of this surface is by cubic curves through the six intersections of four lines; if such lines be $x' = 0$, $y' = 0$, $z' = 0$, $u' = 0$, such a cubic curve has an equation of the form

$$u'(ay'z' + bz'x' + cx'y') + x'y'z' = 0,$$

where a, b, c are parameters, and the formulae of transformation are $x/u'y'z' = y/u'z'x' = z/u'x'y' = t/x'y'z'$. There are e, $=6$, exceptional lines on the surface, joining the nodes in pairs, and e', $=4$, exceptional lines in the plane, corresponding to the nodes of the surface. Thus, with $I' = -1$, $\omega' - 1 = 9$ for the plane, we have $I_0 - e = I' - e'$; $\omega_0 + e = \omega' + e'$.

(*c*) For the Steiner quartic surface, with three double lines meeting in a triple point of the surface, we have

$$\mu_0 = 4,\ \ \mu_1 = \mu_0(\mu_0 - 1) - 2\epsilon_0 = 6,\ \ \nu_2 = 6,$$

there being two pinch points on each double line; further $\mu_2 = 3$, this being the order of the dually corresponding surface considered in the preceding example. There are no isolated nodes, and the formulae give $I_0 = -1$, $\omega_0 - 1 = 9$, as for a plane. The formulae of transformation to a plane (x', y', z') may be taken to be

$$x'/yz = y'/zx = z'/xy$$

with $$x/y'z' = y/z'x' = z/x'y' = t/(x'^2 + y'^2 + z'^2),$$

there being no exceptional lines on the surface, or on the plane.

(*d*) The Kummer quartic surface, with 16 nodes and 16 tropes (tangent planes each having contact along a conic), has $\mu_0=4$, $\mu_1=12$, $\nu_2=0$, $\mu_2=4$, $p=3$, $\pi=3$, the values of μ_2 and π being the same as the order and genus of the plane section of the dually corresponding surface, which is also a Kummer surface. Thus we find $I_0=4$, $\omega_0-1=16$, $p_n=1$.

(*e*) For the Weddle surface, a quartic surface with six nodes, the locus of the vertices of quadric cones passing through six points, we have $\mu_0=4$, $\mu_1=12$, $\nu_2=0$, $\mu_2=24$, $p=3$, $\pi=13$; and hence $I_0=14$, $\omega_0-1=6$, $p_n=1$. It can in fact be shewn (see Note III at the end of this chapter) that the Kummer surface can be birationally transformed to the Weddle surface so that ten of the sixteen nodes of the Kummer surface correspond to ten lines lying on the Weddle surface (which has also fifteen other lines, the joins of the nodes in pairs); the other six nodes of the Kummer surface correspond to the six nodes of the Weddle surface. Thus, in this transformation, the Kummer surface has no exceptional curves, or say $e=0$; but the Weddle surface has 10 exceptional curves, say $e'=10$. From $I_0=4$, $\omega_0-1=16$ for the Kummer surface, and $I_0'=14$, $\omega_0'-1=6$ for the Weddle surface, we thus have $I_0-e=I_0'-e'$, $\omega_0+e=\omega_0'+e'$.

(*f*) Next consider the general surface, ψ^{2k}, of order $2k$, which has a k-ple node at each of the 8 intersections of three general quadric surfaces $u=0$, $v=0$, $w=0$. If f_k be a general homogeneous polynomial of order k in the three quantities u, v, w, the equation $f_k(u, v, w)=0$ represents such a surface; we prove conversely that every surface ψ^{2k} is expressible by an equation of this form. Such a surface was considered for $k=2$ by Cayley (*Papers*, VII, p. 147); and the general case by Castelnuovo (*Rend. Ist. Lombardo*, 1891, Nota I). Let $U=0$ be a general quadric surface containing the 8 nodes; consider the curve determined thereon by the general surface ψ^{2k}; by definition this surface has a k-ple node at each of the 8 points, and it meets the quadric $U=0$ in a curve of order $4k$. Take k arbitrary points, P_i, on the quadric $U=0$, and let c_i denote the elliptic quartic curve on $U=0$ through the 8 given base points and through P_i; this curve c_i is the intersection of $U=0$ with another quadric surface through the 8 base points and P_i. This curve c_i meets any surface ψ^{2k} in $8k$ points at the base points, and is thus contained in any surface ψ^{2k} which meets it in a point other than the base points. The general surfaces ψ^{2k} have at least the tale of the particular surfaces $f_k(u, v, w)=0$, namely $\frac{1}{2}(k+1)(k+2)$, which is certainly $>k$; such a surface can therefore be put through the k points P_i, and will then, by what we have said, contain the k curves c_i, whose aggregate order, $4k$, is equal to that of the curve in which a ψ^{2k} meets $U=0$. The complete curve of intersection of a surface ψ^{2k} with $U=0$ is thus determined by k points of $U=0$. Among surfaces

ψ^{2k}, however, will be ($k \geqslant 2$) surfaces consisting of surfaces ψ^{2k-2} taken with the quadric surface $U=0$; and the freedom of the curves cut by surfaces ψ^{2k} on $U=0$ will be less than that of all surfaces ψ^{2k} only in consequence of this possibility. Thus, if τ_{2k} denote the tale of all surfaces ψ^{2k} we have $\tau_{2k}-1-\tau_{2k-2}=k$. As $\tau_2=3$, we thus have τ_{2k} equal to $(k+1)+k+\ldots+3+3$, or $\frac{1}{2}(k+1)(k+2)$, which is the tale of surfaces $f_k(u, v, w)=0$. This last equation thus represents the general surface ψ^{2k}. From this also it follows that the redundance of the 8 k-ple nodes, for surfaces of order $2k$, is $\frac{1}{2}k(k+1)$; for, if we subtract, from the general tale of surfaces of order $2k$, eight times the conditions necessary for a k-ple node, whose number is $\frac{1}{6}k(k+1)(k+2)$, we find $k+1$, which is then the *virtual* or theoretical number of surfaces ψ^{2k}. If the actual number of surfaces ψ^{2k-4}, having then a $(k-2)$ point at each of the k nodes, namely $\frac{1}{2}k(k-1)$, be denoted by p_g, and the virtual number, $k-1$, of these surfaces, be denoted by p_n (cf. p. 198 above), we have $p_g-p_n=\frac{1}{2}(k-1)(k-2)$. This is equal to the genus of the plane curve $f_k(\xi, \eta, 1)=0$. The surface ψ^{2k} possesses an irrational pencil, of this genus $\frac{1}{2}(k-1)(k-2)$, of elliptic curves, the quartic curves through the 8 base points, given by the equations $u-\xi w=0$, $v-\eta w=0$ (cf. p. 190, preceding). For this surface we have $\mu_0=2k$, $\mu_1=2k(2k-1)$,

$$\mu_2=2k(2k-1)^2-8k(k-1)^2=2k(4k-3);$$

thus the numbers I_0, ω_0-1, defined as above, are $I_0=12(k-1)$, $\omega_0-1=8$, leading to $p_n=k-1$, as we have found. It can be computed that the genus, π, of the proper curve of contact for ψ^{2k}, is $(k-1)(6k-1)$.

We may remark at once that this result for I_0 is in accord with the definition of I given above (p. 190) for the case of a surface, with an irrational pencil of curves, which was supposed to be without multiple points. For we shall see (p. 235), that, for a surface with κ isolated general nodal points, the value of I is, under certain conditions, to be taken to be $I=I_0+\kappa$. This would give in this case, ($\kappa=8$), $I+4=12(k-1)+8+4=12k$; and this is the value of $\delta+(2\rho-2)(2\rho'-2)$, where ρ, $=1$, is the genus of each curve of the irrational pencil, and ρ', $=\frac{1}{2}(k-1)(k-2)$, is the genus of the pencil, provided $\delta=12k$. Now a curve on ψ^{2k}, given by $u-\xi w=0$, $v-\eta w=0$, has a double point if the quadric surfaces $u-\xi w=0$, $v-\eta w=0$ have contact. The condition of contact of two quadric surfaces is however of degree 12 in the coefficients of either. Thus the curves of the pencil which have a double point correspond to the intersections of $f_k(\xi, \eta, 1)=0$ with a curve of order 12 in $(\xi, \eta, 1)$ and are $12k$ in number. This result is also clear because the locus of vertices of quadric cones contained in the net of quadric surfaces $lu+mv+nw=0$ is a curve of order 6; this will have $12k$ intersections with ψ^{2k}. It is convenient also to make another remark at this place. The general surface ψ^{2k-4}, with a $(k-2)$-ple node at each of the 8 base points, meets ψ^{2k} in a curve of order $2k(2k-4)$ or $4k(k-2)$; this is evidently the aggregate of the $k(k-2)$ quartic curves $u-\xi_i w=0$, $v-\eta_i w=0$, where $(\xi_i, \eta_i, 1)$ are the points common to the curve $f_k(\xi, \eta, 1)=0$, which determines ψ^{2k}, whose equation is $f_k(u, v, w)=0$, and to the curve $f_{k-2}(\xi, \eta, 1)=0$ which determines ψ^{2k-4}. Thus two surfaces ψ^{2k-4} meet ψ^{2k} in curves which have no common intersection save at the 8 nodes. And (recalling p. 197 above), this is in accord with $\omega-1=\omega_0-1-\kappa=0$; we shall in fact be led, correspondingly to $I=I_0+\kappa$, to $\omega=\omega_0-\kappa$. The case when, on a surface of order n, the surfaces of order $n-4$, passing $(j-1)$ times through every j-ple curve of the given surface, and $(k-2)$ times through every k-ple isolated node, have as intersection with the given surface an aggregate of elliptic curves, is considered by Noether, *Math. Annal.* VIII, 1875, § 11. Further information in regard to such surfaces is found in Enriques, *Rend. Lincei*, Dec. 1906, Jan. 1912, Feb. 1914; also *Rend. Bologna*, Dec. 1906. The result $p_g-p_n=\frac{1}{2}(k-1)(k-2)$ is also an illustration of a general theorem; cf. Castelnuovo, "Alcuni risultati...," *Mem. Soc. Ital. d. Sci.* 1896, p. 102.

(g) Consider a septimic surface ψ^7, with a single triple point, and a triple conic. By the formulae given (on p. 163), in the preceding chapter, putting $\mu_0=7$, $j=3$, $\epsilon_0=2$, $\epsilon_1=2$, $k=3$, we find $\mu_1=30$, $\mu_2=92$, $\nu_2=20$; and hence $I_0=51$, $\omega_0-1=5$, $p_n=4$. The equation of the surface can be supposed of the form

$$t^4(x, y, z)_3+t^3(x, y, z)_4+t^2[x, y, z]_3u+t(x, y, z)_2u^2 \\ +(lx+my+nz)u^3=0,$$

where the triple conic, in $t=0$, is given by $u=0$; we can suppose $u=yz+zx+xy$. If we now put $x'/xt=y'/yt=z'/zt=t'/u$, which, with $u'=y'z'+z'x'+x'y'$, lead to $x/x't'=y/y't'=z/z't'=t/u'$, the given surface is transformed to the quintic surface expressed by

$$t'^4(lx'+my'+nz')+t'^3(x', y', z')_2+t'^2[x', y', z']_3+t'(x', y', z')_4 \\ +(x', y', z')_3u'=0,$$

having neither multiple point nor multiple curve. For this surface, $\mu_0'=5$, $\mu_1'=20$, $\mu_2'=80$, $\nu_2'=0$ and therefore $I_0'=51$, $\omega_0'-1=5$. There is, however, an exceptional line on the septimic surface, given by $t=0$, $lx+my+nz=0$, which transforms to the simple point $x'=y'=z'=0$, $t'=1$ on the quintic surface; and there is an exceptional curve on the quintic surface, namely the conic $u'=0$, $t'=0$, which arises from the cubic node of the septimic surface. Thus $e=e'=1$, and $I_0=I_0'$, $\omega_0=\omega_0'$. The septimic surface is considered by Noether (see below, p. 231).

(h) For a sextic surface ψ^6 with a double elliptic quartic curve and a double line not meeting this, by the same formulae (p. 163 of preceding chapter), we find $\mu_0=6$, $\mu_1=20$, $\mu_2=32$, $\nu_2=24$, and hence $I_0=6$, and $\omega_0-1=-10$, $p_n=-1$. The surface can in fact be birationally transformed into an elliptic ruled surface (cone). Cf. Noether, "Ueber eine Fläche 6-ter Ordnung," *Math. Ann.* XXI, 1883.

(i) For a quintic surface ψ^5 with a double conic, and two triple nodes (Cayley, *Papers*, VIII, p. 397), we find by the same formulae (p. 163 of preceding chapter) $\mu_0=5$, $\mu_1=16$, $\mu_2=18$, $\nu_2=8$, and hence $I_0=1$, and $\omega_0-1=-5$, $p_n=-1$. This surface can in fact also be birationally transformed to an elliptic cone.

(*j*) We may add the values of I_0 and ω_0-1 for (i), a ruled surface whose plane section is of genus p; (ii), a quartic surface with a double line; (iii), a quartic surface with a double conic. These are, respectively, for (i), $-4p$ and $-8(p-1)$; for (ii), 8 and 0; for (iii), 4 and 4.

Sketch of the proof of the invariance of I. The preceding examples may suffice to render the invariants I and ω familiar. We now sketch the proof of the invariance of I in the geometrical form given by Segre. The surface considered is supposed to have no multiple points beyond a double curve with triple points, which are triple for the surface, and, possibly, isolated nodal points of general character. On this surface we consider a linear pencil of curves, of which the general curve is supposed irreducible unless the contrary be stated, the curves of the pencil having base points only at simple points of the surface. These base points may be multiple, say s-fold, for curves of the pencil, but it is supposed in general that at such a point, the s tangents are variable from curve to curve of the pencil; we denote by σ the number of such base points, without regard to their multiplicity. The condition for a curve of the pencil to have a double point is a single condition; we suppose then that there is a finite number of curves of the pencil which have a double point. We expect to find a proper correction for the cases when a curve of the pencil has several double points, or a multiple point of higher order, or when a curve of the pencil becomes reducible. The whole number of double points arising, *at simple points of the surface which are not base points of the pencil*, we denote by δ. Denoting by p the effective genus of a general curve of the pencil, we seek to establish the invariance of the number $\delta-\sigma-4p$, subject to a correction for the isolated nodes of the surface, when such exist, whatever be the pencil of curves by which this number is found; and the invariance of this number in the passage to another surface, in birational correspondence with the original, subject then to a correction for curves of the surface which are thereby transformed to single points.

On the original surface, consider two such linear pencils of curves as we have described, say γ and γ_1, which are independent of one another; and denote by m the number of intersections of a general curve of the pencil γ with a general curve of the pencil γ_1. That a curve of one pencil should, at an intersection with a curve of the other pencil, have three coincident intersections, requires two conditions; and there are two disposable parameters, one for each pencil. We assume therefore that there exists a finite number of such three-pointic intersections. This number we denote by τ; it is reckoned with the exclusion of possible intersections at the base points of the pencils γ or γ_1, and of points which are multiple points for curves of either pencil, or are multiple points of the surface. Now, on each curve of the pencil γ, there is a finite number of points at which there is coincidence of two points of a set, of the linear series $g_1{}^m$ determined by all the curves of γ_1 upon this curve γ, these points constituting the Jacobian set of this series $g_1{}^m$. There is therefore a curve C which is the locus of points at which two of the m intersections, of a curve of γ with a curve of γ_1, coincide with one another. This curve C contains every ordinary contact of a curve of γ with a curve of γ_1, not generally touching either of these curves at this point. The curve C also passes through every base point of the pencil γ_1 (supposed not also a base point of the pencil γ); for there is a definite curve of γ passing (simply) through this point, and a definite curve of γ_1 touching this curve of γ at this point. It can in fact be shewn that, if the point is s_1-fold for the curves of γ_1, then C has there a $(2s_1-1)$-ple point, its tangents consisting of the tangent of the curve of γ which passes through this point, together with the $(2s_1-2)$ double rays of the pencil of sets of s_1 tangent lines of curves γ_1 at this point; similarly for base points of the pencil γ. Apart from the base points, the curve C is met by any curve of the pencil γ in $2m+2p-2$ points, where p is the genus of a curve of γ; these $2m+2p-2$ points are in fact, on this curve γ, the Jacobian set of the linear series of sets of m points cut thereon by all curves γ_1. Thus, if $N=2m+2p-2$, we have, on C, a linear series $g_1{}^N$ cut by all curves γ. The curves γ_1 likewise cut on C a linear series $g_1{}^{N_1}$, where $N_1=2m+2p_1-2$, if p_1 be the genus of a curve of γ_1. Consider now the Jacobian set, on C, of the series $g_1{}^N$; this consists of $2N+2\pi-2$ points, if π be the genus of C. A point of this set arises at an ordinary contact of a curve γ with the curve C, and thus at a contact of a curve γ with a curve γ_1 if at this point the tangent of C coincides with the common tangent of the curves of γ and γ_1. Consider first a simple base point, O_1, of the pencil γ_1; there is one curve of γ through this point, and, of the m intersections of every curve of γ_1 with this curve γ, one falls at O_1; there are therefore only $2(m-1)+2p-2$, or $N-2$, other points of coincidence of pairs of intersections of curves γ_1 with this particular γ; thus, of the series $g_1{}^N$ upon C, by curves of γ, the set by this particular γ has two points at O_1; the Jacobian set of $g_1{}^N$ has thus one point at O_1; there is a definite curve γ_1 through O_1 which touches this particular γ at O_1, and the curve C touches both these at O_1. If, next, we consider a point O_1, which is an s_1-ple base point for curves γ_1, there will still be a particular curve γ through O_1, and a definite curve γ_1 of which one tangent coincides with that of this particular γ; hence also a single point of the Jacobian set of $g_1{}^N$ on C will fall at O_1. But, further, among the pencils, each of s_1 lines, formed by the tangents at O_1 of a curve γ_1, there are $2(s_1-1)$ pencils for each of which two of the s_1 tangents coincide; these give $2(s_1-1)$ tangents of C, as we have already indicated. And, reasoning as in the case when O_1 is a simple base point of curves γ_1, we can prove that there are no other than the $1+2(s_1-1)$, or $2s_1-1$, tangents of C at O_1 thus obtained; for, the particular curve γ through O_1 has $2(m-s_1)+2p-2$, or $N-2s_1$, contacts with curves γ_1, other than at O_1; and this is the number of points of the set of the series $g_1{}^N$ on C determined by this particular curve γ; thus C meets this γ in $2s_1$ points at O_1; these are those accounted for by the $2s_1-1$ tangents of C we have spoken of. Thus whatever be the multiplicity of O_1 for curves γ_1, there is just one point of the Jacobian set of the series $g_1{}^N$ on C which falls at O_1.

Take as illustration the case when $s_1=2$, the curves γ_1 having double points at O_1. There will then be two curves γ_1 which have a cusp at O_1; on the particular curve γ which passes through this point, the curve C has four points at O_1, two of them arising by the contact, with this particular curve γ, of a certain γ_1 through O_1; each cusp gives another, and hence a tangent for C at O_1; but there is no coincidence of pairs of otherwise distinct intersections of a curve γ with C arising therewith, and no point of the Jacobian set of $g_1{}^N$ on C arising corresponding to the cusp.

If then σ_1 be the number of points at which the pencil γ_1 has base points, there are on C just σ_1 points of the Jacobian set of the $g_1{}^N$ on C which fall at these base points. We prove further that, beside these, there are $\delta+\tau$ points in this Jacobian set, where, as stated, δ is the number of double points of curves γ, and τ is the number of points where a curve γ has three coincident intersections with a curve γ_1. Consider a curve γ with a double point (not at base points of the pencil, or at double points of the surface); this curve will then be of genus $p-1$. Curves γ_1 thus cut, on this γ, a series of sets of m points, among which there are $2m+2(p-1)-2$ coincidences of pairs of a set, other than at the double point, which number is $N-2$. The curve C contains the double point, since, on the curve γ_1 which passes through this point, there is here a coincidence of two points of a set of the series cut on this γ_1 by all curves γ. Thus, in the series $g_1{}^N$, cut on C by curves γ, the set cut by the curve γ with the double point has two intersections coinciding at this double point. This double point is thus a point of the Jacobian set, upon C, of the series $g_1{}^N$; and thus the Jacobian set contains the δ double points of the pencil γ. That the Jacobian set also contains the τ points of osculation of a curve γ with a curve γ_1 may be regarded as obvious. We assume now that all the points of this Jacobian set on C have been enumerated. Thus we obtain the equation $2N+2\pi-2=\sigma_1+\delta+\tau$. By a similar argument for the linear series $g_1{}^{N_1}$ cut on C by curves γ_1, we obtain

$$2N_1+2\pi-2=\sigma+\delta_1+\tau.$$

By subtracting these equations, replacing N, N_1 respectively by

$2m+2p-2$ and $2m+2p_1-2$, we arrive at the equation we desired, namely $\delta-\sigma-4p=\delta_1-\sigma_1-4p_1$.

It may be worth while to illustrate the preceding proof of the number $\sigma_1+\delta+\tau$, for the Jacobian points of the series g_1^N on C, by taking the case when both the pencils γ, γ_1 are formed by sections of the surface made by planes through fixed lines. Let the former pencil be by planes through the axis a, and the latter be by planes through the axis a_1, this not meeting a. A contact, or two-pointic intersection of a curve γ with a curve γ_1, will then arise at a point of the surface where there is a tangent line which meets both a and a_1; the curve C is the locus of such points. The number N, of intersections of a curve γ with the curve C, will be the order of C diminished by the number of base points of the pencil γ, which are the intersections of the line a with the surface (here supposed all simple). The number $2N+2\pi-2$ will be the number of contacts, or two-pointic intersections, of plane sections γ with the curve C. Such a point evidently arises at a point of contact, with the surface, of a tangent plane through the axis a, at which point the curve γ has a double point. This corresponds to the term δ in the number of Jacobian points. The term σ_1 arises from the fact that a Jacobian point arises at a point where the axis a_1 meets the surface. There is clearly a curve γ through such a point, and also a curve γ_1 which touches this curve at this point; it is necessary to see that the curve C, which evidently passes through this point, touches both the curves γ and γ_1; of this an algebraic verification may be given: Let the axis a be $z=0$, $t=0$, and the axis a_1 be $x=0$, $y=0$, the surface being $f=0$; the curve C is then the locus of a point (x, y, z, t) of $f=0$, at which the tangent plane passes through the line of intersection of the planes $x'y-y'x=0$, $z't-t'z=0$ (x', y', z', t' being current coordinates); the curve C is thus the intersection of $f=0$ with either of the surfaces $xf_1+yf_2=0$, $zf_3+tf_4=0$, where now x, y, z, t are current coordinates, and $f_1=\partial f/\partial x$, etc. Now, if $(0, 0, 0, 1)$ be a base point for the pencil γ_1, the equation $f=0$ is of the form $(ax+by+cz)t^{n-1}+\ldots=0$, and the tangent line of the curve C at this point is the intersection of $ax+by+cz=0$ with the tangent plane of $zf_3+tf_4=0$ at this point, that is, with

$$zc+(n-1)(ax+by+cz)=0;$$

or, the tangent line of the curve C at this point is $z=0$, $ax+by=0$. The curve γ through this point is on $z=0$, and thus touches the curve C; as was to be shewn.

As has been said, the preceding investigation of the invariance of $\delta-\sigma-4p$ assumes, not only that the multiple points of the curves γ are only double points, and that the tangents at a base point are all variable, but also that the curves γ are irreducible. Suppose, to indicate the necessary modifications when the last condition is not satisfied, that there is a curve γ which breaks up into a curve χ of genus ρ occurring doubly, and a curve ψ. Then the curve C, the locus of contacts of curves of the pencil γ with curves of the pencil γ_1, which we have supposed in general to be irreducible, contains the curve χ as part; let τ be its other part, which we suppose irreducible. We consider the series g_1^N cut upon the curve τ, by the curves γ, and the series $g_1^{N_1}$ cut upon τ by the curves γ_1. If μ be the number of intersections of the curve χ with any curve of the pencil

γ_1, it can then be proved that $N=2m+2p-2$, as before; but $N_1=2m+2p_1-2-\mu$. Consider the Jacobian sets of these series on the curve τ; we have as before

$$2N+2\pi-2=\delta'+\tau+\sigma,\quad 2N_1+2\pi-2=\delta_1+\tau+\sigma,$$

where, however, δ' consists of two parts $\Delta+\xi$, in which Δ is the number of double points of the curves of the pencil γ other than on the particular degenerate curve, and $\xi=2\mu+2\rho-2+2i$, where $2\mu+2\rho-2$ is the number of contacts with χ of curves of the pencil γ_1, and $2i$ is twice the number of intersections of the parts χ, ψ of the particular curve γ which degenerates. Thus the same formula holds as in the general case if for δ we put $\Delta+2i+2\rho-2$ (cf. above p. 189, the calculation of I from an irrational pencil).

To illustrate the wide application of the formula for I, we may consider in a plane the pencil of curves of order kn, given by an equation $u^k+\lambda v^k=0$, where $u=0$, $v=0$ are two general curves of order n. Each curve of the pencil is then an aggregate of k curves of the pencil $u+\mu v=0$, among which there are $3(n-1)^2$ curves having a double point. The genus p of the curve $u^k+\lambda v^k=0$, which may be computed from $\frac{1}{2}(kn-1)(kn-2)-\frac{1}{2}n^2k(k-1)$, is given by $p=\frac{1}{2}kn(n-3)+1$. Counting each of the curves $u^k=0$, $v^k=0$ as furnishing $(k-1)(2p_0-2)$ double points, where p_0 is $\frac{1}{2}(n-1)(n-2)$, we find for I the value

$$3(n-1)^2+2(k-1)n(n-3)-n^2-2kn(n-3)-4,$$

which is -1, the proper value for a plane.

Having thus obtained a number I, $=\delta-\sigma-4p$, for the given surface, which is independent of the pencil of curves by which it is defined, we consider the modification of this number in passing to another surface which is in (1, 1) birational correspondence with the original surface. If the transformation be such that no point of either surface is changed into a curve of the other surface, and no simple point of either into a multiple point of the other, the value of I is the same for both surfaces; for, any pencil of curves of either surface, with base points only at simple points of the surface, is transformed into a pencil of like kind, with an equal number of base points, and the same genus; and there will be the same number of curves in both pencils having double points at simple points of the surfaces respectively. But, taking first the commonest and simplest case of exception to this, suppose that the transformation is such that there is one point, O, which we suppose to be a simple point, of the original surface, which is represented by a curve on the derived surface. Consider in turn the two possibilities: (a), that the value of I on the original surface is computed by a pencil of curves of which O is not a base point; (b), that this pencil has O for a base point. Call the curve, on the derived surface, which arises from O, the *exceptional* curve; denote the pencil of curves of the original surface by (ϕ), and the corresponding pencil on the derived surface by (ϕ'). In case (a), as the curves (ϕ) do not pass through O, the curves (ϕ') will not be met by the exceptional curve, except, possibly, at the base points of (ϕ'); the exceptional curve will be what, for this reason, is called a *fundamental* curve of the pencil (ϕ'). There is, however, one curve, say ϕ_0, of the pencil (ϕ), which passes through O, with a definite tangent at O; while, the various directions of approach to O, on the surface, correspond to the various points of the exceptional curve on the derived surface. Thus, the curve on the latter surface which corresponds to ϕ_0 will break up into the exceptional curve, and another curve intersecting the exceptional curve in a definite point. This result will be considered, in a more detailed algebraic manner, below (pp. 212, 218). The pencil (ϕ') will thus contain a curve which must be considered to have a double point, on the curve corresponding to ϕ_0; and this is in addition to those derived by transformation of the double points arising on curves (ϕ). Thus, if δ' denote the number of double points of curves of the pencil (ϕ'), and δ of curves of (ϕ), we have $\delta' = \delta + 1$. Next, consider the case (b), when O is a base point of the pencil (ϕ). Then all the curves (ϕ'), instead of having as a base point a point arising from O, will have the exceptional curve as component, but have as base points the transformations of the other $\sigma - 1$ base points of the pencil (ϕ). Thus, if σ' be the number of base points of the pencil (ϕ'), we have $\sigma' = \sigma - 1$. In both cases, therefore, if I, I' be the invariants of the original and derived surfaces, computed from the pencils (ϕ) and (ϕ'), we have $I' - 1 = I$. This argument, though only for the simplest case, suggests the conclusion which we have verified in many examples, that if there be e curves of the first surface which, by the transformation, become simple points of the derived surface, and e' curves of the latter arising from simple points of the original, then $I' - e' = I - e$. In both cases (a), (b), a similar argument may be employed when the exceptional curve arises by transformation of an isolated *multiple* point of the original surface.

The algebraic formulation of the matter is on lines which we now indicate. Let (x, y, z, t) be coordinates for the original surface $F = 0$, and (x', y', z', t') for the derived surface $F' = 0$; and let the formulae of transformation be $x'/f = y'/\phi = z'/\psi = t'/\chi$, where f, ϕ, ψ, χ are polynomials of the same order in x, y, z, t. Suppose that these polynomials all vanish at the same point O of the original surface, which we take to be $(0, 0, 0, 1)$. Consider the effect of this transformation upon any point of the surface $F = 0$ lying in the neighbourhood of O, say the point $(\xi, \eta, \zeta, 1)$, where ξ, η, ζ are small. Retaining, in f, ϕ, ψ, χ, only terms of the lowest order in ξ, η, ζ, these will be replaced by polynomials, of the same order, say s, in ξ, η, ζ. To the first approximation, F is likewise a polynomial in ξ, η, ζ, say of order k. In particular, when O is a simple point of the surface, $k = 1$; and, if f, ϕ, ψ, χ vanish only to the first order at this point, also $s=1$. Thus, for variation of $(\xi, \eta, \zeta, 1)$ on the surface, in the neighbourhood of O, the point (x', y', z', t') describes a curve, of which the coordinates of a point are rational functions, of order s, of the coordinates of a point, (ξ, η, ζ), which varies upon a plane curve of order k. In particular when $(k=1)$ O is a simple point of $F=0$, the point (x', y', z', t') varies on a rational curve, of order s, whose points correspond to the directions on $F=0$ issuing from O; this is the case we have described above. The reverse transformation which by hypothesis exists and is expressible in a form

$$x/f' = y/\phi' = z/\psi' = t/\chi',$$

where f', ϕ', ψ', χ' are homogeneous polynomials of the same order in x', y', z', t', must be such as to lead to the point $(0, 0, 0, 1)$ when (x', y', z', t') is any point of the exceptional curve on the derived surface $F'=0$; thus the surfaces $f'=0$, $\phi'=0$, $\psi'=0$ all pass through the exceptional curve. We shall suppose that the *geometrical* relations are sufficiently represented by the hypothesis that, by means of $F'=0$, the equations are capable of the forms $x/u'f_1' = y/u'\phi_1' = z/u'\psi_1' = t/\chi'$, where $u'=0$ is a surface passing through the exceptional curve on $F'=0$ (cf., for further detail, Noether's fundamental paper, *Math. Ann.* VIII, § 9). Now consider a linear system of curves, of freedom r, on the surface $F=0$, given by surfaces of order m, whose equation, with x, y, z, 1 as current coordinates, is of the form

$$\lambda(A+Bx+Cy+Dz+Ex^2+\dots)+\dots$$
$$+\lambda_r(A_r+B_rx+C_ry+D_rz+E_rx^2+\dots)=0;$$

and denote the constant $\lambda A+\dots+\lambda_r A_r$ by Δ. The linear system of curves on $F'=0$, derived by direct transformation with the equations we consider, is then given by an equation

$$\Delta\chi'^m+u'(\beta f_1'+\gamma\phi_1'+\delta\psi_1')\chi'^{m-1}$$
$$+u'^2(\epsilon f_1'^2+\dots)\chi'^{m-2}+\dots+u'^m(\quad)=0,$$

where $\beta=\lambda B+\dots+\lambda_r B_r$, etc.; and this equation is of the form

$$u'(\lambda V+\dots+\lambda_r V_r)+\Delta\chi'^m=0,$$

where $V, \dots, V_r$ are certain polynomials in the coordinates x', y', z', t', independent of $\lambda, \dots, \lambda_r$. Thus, curves of the original linear system which do *not* pass through the point O, or $(0, 0, 0, 1)$, for which therefore $\lambda, \dots, \lambda_r$ are such that Δ is not zero, become curves on the derived surface for which the curve given by $u'=0$ is *fundamental*; namely, this exceptional curve does not meet the curves on the derived surface, except at points for which $\chi'=0$; these are then base points common to all these curves. These points are also base points of all the transforming curves of the reverse transformation, namely $u'f_1'=0$, $u'\phi_1'=0$, $u'\psi_1'=0$, $\chi'=0$; thus, as we see next, they correspond to exceptional curves on the original surface. For, to curves of the linear system on $F=0$ which are such that $\Delta=0$, that is, which pass through O, correspond on the derived surface, by the direct transformation, the curves given by $u'(\lambda V+\ldots+\lambda_r V_r)=0$; these break up into the curve given by $u'=0$, with a linear system of freedom $r-1$.

Ex. 1. The circumstances we have dwelt on are familiar for the transformation from a plane, with coordinates x, y, z, to a cubic surface, with coordinates x', y', z', t'. Let $\phi=0$, $\psi=0$ be two conics in the plane, containing respectively the points (1, 0, 0) and (0, 1, 0); for definiteness let $\phi=by^2+cz^2+fyz+gzx+hxy$ and $\psi=a_1x^2+c_1z^2+f_1yz+g_1zx+h_1xy$. Take the transformation expressed by $x'/x\phi=y'/z\phi=z'/y\psi=t'/z\psi$, wherein the four cubic curves, $x\phi=0$, $z\phi=0$, $y\psi=0$, $z\psi=0$, have six points in common, namely the four intersections of $\phi=0$, $\psi=0$, and the two points (1, 0, 0), (0, 1, 0); the reverse transformation is given by $x/x't'=y/y'z'=z/y't'$. By this transformation the points of the plane correspond to the cubic surface whose equation is

$$x'\{z'(ht'-h_1y')+t'(gt'-a_1x'-g_1y')\}$$
$$+y'\{z'(bz'+ft'-f_1y')+t'(ct'-c_1y')\}=0.$$

It is easy to see that the point (1, 0, 0) of the plane changes into the line of the cubic surface given by $y'=0$, $gt'+hz'-a_1x'=0$; and that this is fundamental for the curve of the surface corresponding to any curve of the plane which does not contain the point (1, 0, 0). So, the line $t'=0$, $f_1y'+h_1x'-bz'=0$ of the surface corresponds to the point (0, 1, 0) of the plane.

Ex. 2. Prove that a quartic surface having $t=0$, $yz+zx+xy=0$ as double conic, and also containing the line $y=0$, $z=0$, has an equation of the form

$$t^3(my+nz)+t^2(by^2+cz^2+2fyz+2gzx+2hxy)+(yz+zx+xy)^2=0.$$

We may then obtain a transformation between this surface and the plane $x=0$, by drawing the line through the point $(0, \eta^{-1}, \zeta^{-1}, \tau^{-1})$ to meet the line $y=z=0$, and also the double conic, and considering the fourth intersection of this line with the surface. If this be (x, y, z, t), prove that $x/D=-y/\zeta N=-z/\eta N=t/K$, where

$$D=\tau^3+\tau(b\zeta^2+c\eta^2+2f\eta\zeta)+\eta\zeta(m\zeta+n\eta),$$
$$N=\tau(2g\eta+2h\zeta)+(\eta+\zeta)(m\zeta+n\eta),$$
and
$$K=(\tau^2+b\zeta^2+c\eta^2+2f\eta\zeta)(\eta+\zeta)-(2g\eta+2h\zeta)\eta\zeta,$$

so that $\tau K-(\eta+\zeta)D+\eta\zeta N=0$. The reverse transformation is then $\eta/zt=\zeta/yt=\tau/(yz+zx+xy)$. The curves $D=0$, $N=0$, in the plane (η, ζ, τ), have 5 common points, beside the point $\tau=0$, $m\zeta+n\eta=0$, which does not lie on $K=0$; the four cubic curves $D=0$, $\zeta N=0$, $\eta N=0$, $K=0$ have 5 points in common. These transform into lines of the quartic surface, and are the only exceptional curves arising in the transformation.

Since for a plane $I=-1$, this gives for the quartic surface $I_0=4$. We can verify indeed that $\mu_0=4$, $\mu_1=8$, $\mu_2=12$, $\nu_2=4$, and so obtain, from the early formulae of this chapter, both $I_0=4$, and $\omega_0-1=4$. The quartic surface is obtainable by projecting the intersection of two quadrics in space [4] from a point; and this surface in [4] can be projected on to a plane from a line lying thereon. The quartic surface considered in this example is obtained by projecting from the point (0, 0, 0, 0, 1) in space

[4], where the coordinates are x, y, z, t, u, the surface which is the intersection of the quadric $yz+zx+xy+ut=0$ with the quadric

$$t(my+nz)+by^2+cz^2+2fyz+2gzx+2hxy+u^2=0.$$

Remark. Before passing from the consideration of the invariant I, it should be said that the equation $I=\delta-\sigma-4p$ is regarded by Segre (*Atti...Torino*, XXXI, 1896) as the first of a series of equations connecting the values of invariants I belonging to manifolds of successive dimensions, of the form $I_n=\delta-I_{n-2}-2I_{n-1}$, for $n=2, 3, \ldots$, with $I_2=I$. If we put $J_2=I+4$, $J_n=(-1)^n I_n+2n$, this general equation becomes $J_n-2J_{n-1}+J_{n-2}=(-1)^n\delta$. In a remarkable paper, which the reader should consult, J. W. Alexander (*Rend. Lincei*, XXIII, 1914) has identified J_n with Poincaré's topological invariant of a manifold (*J. d. l'Éc. polyt.* 1895) which is expressed by $J_n=\sum_{i=0}^{n}(-1)^i\alpha_i$ (see the chapter on Riemann Surfaces, Vol. V, preceding, p. 135).

The invariance of ω in a birational transformation. Consider first the case when the surface for which ω is defined has no multiple points other than a double curve, with triple points. We have given a general definition of ω, and reached (p. 197) the equation $\omega-1=(C_J-3C, C_J-3C)$, where the system of curves $|C|$ consists of the plane sections of the surface, and $|C_J|$ is the complete system defined by the proper curves of contact of tangent planes drawn to the surface from an arbitrary point, or, is the complete Jacobian system of $|C|$. We consider this result now in more detail, with the object of shewing, first, that ω can be defined by this equation also when $|C|$ denotes any linear system of curves on the surface for which the general curve of the Jacobian system is irreducible, and, second, to find (p. 226) the modification in ω arising in any birational transformation of the surface. For the first purpose we need a result which, for clearness, we state without proof and apply to the consideration of ω. The proof of this result, given below, p. 222, requires discussions not immediately concerned with the invariant ω, which have application however to the consideration of the modification of ω in a birational transformation. The result referred to, when, as on p. 197, (P, Q) denotes the number of intersections of two curves P, Q of the surface, other than at base points prescribed for these curves, is, that if $|C|$ be *any* linear system of curves without prescribed base points, whose complete Jacobian system $|C_J|$ consists of generally irreducible curves, the number $(C_J, A)-3(C, A)$, when A is any curve of the surface, is independent of the system $|C|$, and is thus a number characteristic of the curve A. Without any assumption as to the existence of the system $|C_J-3C|$, we may denote this number by (C_J-3C, A); and, more generally, a symbol $(P-Q, R-S)$ may be used to denote the number $(P, R)-(Q, R)-(P, S)+(Q, S)$. With the assumption of this result, if $|C|$, $|C'|$, $|B|$, $|D|$ be any systems of curves without prescribed base points, whose Jacobian systems $|C_J|$, $|C_J'|$, $|B_J|$, $|D_J|$ are irreducible, we have

$$\begin{aligned}(B_J-3B, D_J-3D)=(C_J-3C, D_J-3D)&=(D_J-3D, C_J-3C)\\ &=(C_J'-3C', C_J-3C);\end{aligned}$$

if we suppose that both $|C|$ and $|C'|$ denote the system of plane sections of the surface, this shews that $\omega-1$ can be computed from the first form. It is thus independent of the linear system, and in particular is independent of the net of curves, from which it is computed. We pass then to the discussion of the result assumed.

The canonical system of curves of a surface, and the definition of ω. Still supposing the surface to have no singularities except a double curve, with triple points, let $|A|$, $|B|$ denote any two linear systems of curves, whose base points, supposed to be independently assigned, are at simple points of the surface. Following a phraseology used by Enriques (*Memorie...Società Italiana d. Scienze*, x, 1896, pp. 1–81), we look upon an s-fold multiple point of a curve A, lying on the surface, as a break, or gap, in the curve; and regard a curve which has the same point as multiple point of multiplicity $s-1$, as derivable from A by the addition of this point of the surface counted once. Then, if this point be P, we may denote by $|A+P|$ the complete system of curves which differs only from the complete system $|A|$ in the fact that the point P is prescribed to be multiple of one less multiplicity for $|A+P|$ than for $|A|$. With this notation, denote by α the aggregate of the prescribed base points of the system $|A|$, which are not prescribed base points of the system $|B|$, each to be taken simply; likewise, denote by β the base points of $|B|$ which are not base points prescribed for $|A|$, each taken simply; and denote by μ the aggregate of points, each taken simply, which are prescribed as base points, generally of different orders, for both $|A|$ and $|B|$. Now consider the two complete systems of curves $|(A+B)_J|$, $|A_J+3B+\beta|$, of which the former is the Jacobian system of the system $|A+B|$, while $|A_J|$ is the complete Jacobian system of $|A|$. As already said, the only prescribed base points for $|A_J|$ are at the base points prescribed for $|A|$, and the prescription is that $|A_J|$ have a $(3s-1)$-ple point at an s-ple point of $|A|$. The two systems named will have the same base points. For a point, simple on the surface itself, which is an s-fold base point for $|A|$, and a t-fold base point for $|B|$, will be $(s+t)$-fold for $|A+B|$, and therefore $[3(s+t)-1]$-fold for $|(A+B)_J|$. On the other hand, a point which is s-fold for $|A|$, and not a base point for $|B|$, will be a base point of multiplicity $3s-1$ for $|A_J+3B+\beta|$; a point which is not a base point for $|A|$, but is

t-fold for $|B|$, will be $(3t-1)$-fold for $|A_J+3B+\beta|$, being $3t$-fold for $|3B|$; and a point which is s-fold for $|A|$, and t-fold for $|B|$, with $s>0$, $t\geqslant 0$, will be $(3s-1+3t)$-fold for $|A_J+3B+\beta|$. And we may notice that a point which is accidentally a base point of the same order for all Jacobian curves of a linear system, though not a base point of this system (and, therefore, we have agreed, is not a prescribed base point of the complete Jacobian system), enters equally in curves defining the two complete systems which we are comparing. To complete the proof that these two systems are the same it is only necessary to prove now that they consist of curves of the same order. But it may be more convincing to develop the algebraic basis of the theorem. For this, we enunciate an algebraic identity, which has other applications: let the given surface be $F=0$; let the Jacobian determinant, for a net of curves given by the variable part of the intersection of $F=0$ with surfaces $\lambda f+\mu\phi+\nu\psi=0$, in which λ, μ, ν are variable parameters, namely the determinant $\partial(f, \phi, \psi, F)/\partial(x, y, z, t)$, be denoted by $J(f, \phi, \psi, F)$, or simply by J. Also let f, ϕ, ψ be of order m; and let u, U be any two homogeneous polynomials of respective orders k and $k+m$; and let p, q, r be constants. The identity in question expresses the Jacobian determinant of the three forms $uf+pU$, $u\phi+qU$, $u\psi+rU$, which are all of order $k+m$, in a certain way; namely

$$J(uf+pU,\ u\phi+qU,\ u\psi+rU, F)=(1+km^{-1})(u^3J-uK),$$

where J denotes $J(f, \phi, \psi, F)$, and K is a determinant of 5 rows and columns formed by suitably "bordering" J; the five rows of K consist respectively of the elements f_1, f_2, f_3, f_4, p; $\phi_1, \phi_2, \phi_3, \phi_4, q$; $\psi_1, \psi_2, \psi_3, \psi_4, r$; $F_1, F_2, F_3, F_4, 0$; and $w_1, w_2, w_3, w_4, 0$, in which $f_1=\partial f/\partial x$, etc., and $w_i=uU_i-Uu_i$, for $i=1, 2, 3, 4$. Assuming this, let the net of curves on $F=0$, given by $\lambda f+\mu\phi+\nu\psi=0$, define the system $|A|$. The Jacobian curve of this net satisfies the equation $J=0$, which is satisfied, however, also at all the multiple points and curves of the surface $F=0$, as well as at base curves, and base points thereon, of the surfaces $f=0$, $\phi=0$, $\psi=0$. Also let a definite single curve of another system $|B|$ be given by the equation $b=0$, parts of the intersection of this with $F=0$ which belong to the prescribed base system of $|B|$ being disregarded. We may then form the Jacobian determinant of the net formed from the three composite curves which are given by $fb=0$, $\phi b=0$, $\psi b=0$. Referring to the algebraic identity we have remarked, taking therein $p=q=r=0$, and hence $K=0$, we see that the Jacobian curve of this composite net $(\lambda f+\mu\phi+\nu\psi)b=0$, defined in this way, consists of the Jacobian curve of the net (f, ϕ, ψ) compounded with the relevant part of the curve $b=0$ taken three times over. And it is proper, and easy, to consider the geometrical meaning of this when the Jacobian curve is defined, as was originally done, by a locus of double points. This

Jacobian curve evidently belongs to the Jacobian system $|(A+B)_J|$, and also, by the identity employed, also to the system $|A_J+3B|$; but, it is special, as having, at any t-ple base point of the system $|B|$ which is not a base point of $|A|$, a $3t$-ple base point. As the system $|(A+B)_J|$ has $3t-1$ for its prescribed multiplicity at such a point, and a complete system, of prescribed base, is determined by one of its curves, we infer that the systems $|(A+B)_J|$, $|A_J+3B+\beta|$ are the same; β here denoting the aggregate of all the base points of $|B|$, which are not base points of $|A|$, each taken once.

From this result, with a like definition for α, we infer that the systems $|A_J+3B+\beta|$, $|B_J+3A+\alpha|$ are the same. From this, the systems $|A_J-2A+B+\beta|$, $|B_J-2B+A+\alpha|$, both supposed to exist, are also the same, both being identical with $|(A+B)_J-2A-2B|$. But, further, if μ be the aggregate of the common base points of $|A|$ and $|B|$, each taken once, so that $\alpha+\mu$ denotes all the base points of $|A|$, each simply, and $\beta+\mu$ those of $|B|$, we can infer that the three complete systems

$$|(A+B)_J-3A-3B-\alpha-\beta-\mu|, \quad |A_J-3A-\alpha-\mu|, \quad |B_J-3B-\beta-\mu|,$$

are the same, if they exist. Conventionally, we say that two linear systems of curves, $|P-Q|$, $|R-S|$ are the same, even when the subtractions are impossible, provided there exists a system $|U|$ for which $|P-Q+U|$ and $|R-S+U|$ exist and are the same. It can be shewn that this definition is independent of the system employed (cf. Severi, *Rend. Ist. Lombardo*, XXXVIII, 1905). Hence finally we have the conclusion; let $|C|$ be any complete linear system, with base points only at simple points of the surface, $|C_J|$ the complete Jacobian system of this, and γ the aggregate of the base points prescribed for $|C|$, each taken simply; then the system $|C_J-3C-\gamma|$ is independent of the system $|C|$, and has no base points at base points of $|C|$. It is this system, supposed existing, which is called the *canonical system* of the surface.

The modification of the canonical system of curves in a birational transformation. If a birational transformation of the given surface be made to another surface, also having no multiple points other than a double curve, with triple points, and, in this transformation, no exceptional curves of either surface arise from points of the other surface, it is clear, from the geometrical meaning of the Jacobian system of a given linear system, that the canonical system of curves of one surface changes into the canonical system of the other. But suppose, to take the simplest hypothesis, that a simple point, O, of the original surface, $F=0$, is replaced, on the derived surface, $F'=0$, by an exceptional curve, given thereon by a surface $u'=0$. The point O may, (a), *not* be a base point of the system $|C|$ used to define the canonical system on $F=0$; or (b), it may be. In the first case (a), the system $|C'|$, of curves on $F'=0$, arising by direct transformation of the system $|C|$, will, by what was seen earlier (p. 212), be given by surfaces having an equation of the form $\lambda_0(u'f'+pU')+\lambda_1(u'\phi'+qU')+\lambda_2(u'\psi'+rU')+\ldots=0$, in which U' denotes what was before written as χ'^m, and p, q, r are constants. Thus, by the algebraic identity given above, the Jacobian system of the transformed system $|C'|$ will be *defined by* curves all containing, as part, the curve given by $u'=0$; this is *fundamental* for the system $|C'|$, intersecting the curves of this system only at its base points, if at all. These conclusions are sufficiently obvious without appeal to the algebraic formulation; for, since the curves $|C|$ do not have O as a base point, the condition for such a curve to contain O is one-fold; thus the condition for a curve of the corresponding system $|C'|$ to contain the curve $u'=0$ is likewise one-fold, which shews that this curve is fundamental for this system. The curves of any net contained in $|C'|$ which pass through a point of $u'=0$ consist therefore of this curve together with a pencil of curves, of which one can be supposed to pass through this point; this shews that the curve $u'=0$ is part of the Jacobian curve of this net. The Jacobian system of $|C'|$ is thus *defined by* curves containing $u'=0$ as part. The canonical curves of the derived surface are thus defined by curves $|C_J'-3C'-\gamma'|$ containing $u'=0$ as part (cf. however the discussion, p. 194, above). Consider next the case, (b), when the point O is a base point of the system $|C|$ by which the canonical system for the surface $F=0$ is defined. Then, if, on the derived surface, we consider the system $|C'|$ which is obtained *by direct algebraical transformation* from $|C|$, every such curve will contain $u'=0$ as part; thus, by the algebraic lemma, the Jacobian system of $|C'|$ will be defined by curves which contain the curve $u'=0$ three times over, the Jacobian determinant containing the factor u'^3 (the constants p, q, r in this lemma are zero, and K vanishes). In this case the factor u'^3 enters both in C_J' and in $3C'$, and the canonical curves defined by $|C_J'-3C'-\gamma'|$ are not proved in this way to contain the curve $u'=0$. But we have seen that the canonical curves may be determined from any system of curves, and, therefore, from a system $|D'|$, on the new surface, of which $u'=0$ is not a part. Thus the conclusion must be: *The canonical system of the derived surface, formed from curves $|C'|$ which are the direct transformation of a system $|C|$ of the original surface, is defined by curves having as part all the exceptional curves of the new surface arising from (simple) points of the original surface.* This is proved as here when these points are not base points of the system $|C|$ considered. When, however, they are such base points, the curves $|C'|$ obtained by direct transformation are, as we have seen, *composite*; they consist of the curve $u'=0$, and another system, say $|D'|$, which, in the simplest case, will not contain $u'=0$. Thus we infer from the former case that, if we compute the canonical system from the system $|D'|$, as $|D_J' - 3D' - \delta|$, the resulting curves will contain $u' = 0$. But this result is also indicated geometrically; for a curve D', and hence also a curve D_J', will be in (1, 1) correspondence respectively with a curve C and a curve C_J; if C have an s-ple point at O, C_J will have a $(3s-1)$-ple point, and thus D' and D_J' will meet $u' = 0$ in respectively s and $3s-1$ points; thus the curves $|D_J' - 3D' - \delta|$, if existing and meeting $u' = 0$ in points, would meet this curve in -1 points, which is impossible. The conclusion must therefore be that the curves, if existing, break up into the curve $u' = 0$ together with a part not intersecting $u' = 0$ (cf. Enriques-Campedelli, p. 83). We may compare a familiar result on a cubic surface: If $|C|$ denote the linear system of curves of order $3m$ thereon, obtained by its intersection with surfaces of order m, and $|D|$ the system of curves of order $3m-1$ obtained by its residual intersection with surfaces of order m which pass through a definite line lying on the cubic surface, the curves $|C|$ have m intersections with this line, and the curves $|D|$ have $m+1$ intersections with this line; the curves $|C-D|$ consist of the line only. Later (p. 222) we shall have occasion to consider the system $|C_J - 2C|$, and the corresponding system $|C_J' - 2C'|$ on the derived surface. When the point O is not a base point of $|C|$, the preceding argument shews that the curve $u' = 0$ is part of the curve defining the system $|C_J' - 2C'|$, as arising in the Jacobian curve of any net of curves contained in $|C'|$. When O is a base point of $|C|$, if $|C'|$, containing $u' = 0$ as part, be the direct transformation of $|C|$, the preceding argument would lead to $u' = 0$ as part of $|C_J' - 2C'|$, because u'^3 enters into C_J'; in this case, however, the curve D', which is the variable part of C', meets $u' = 0$ in s points (with the notation above), while D_J' meets $u' = 0$ in $3s-1$ points; thus a curve of $|D_J' - 2D'|$ meets $u' = 0$ in $s-1$ points, and does not usually contain $u' = 0$.

In the preceding sketch, the separation of the curves which we have called exceptional, from the canonical system of curves, has been reached in connexion with the theory of the transformation of the surface. But consider a surface, say of order n, possessing also, in order to make a general statement, multiple curves, say, typically, j-fold curves, and isolated nodal points, say, typically, k-fold points. Let $|C|$ denote the system of plane sections of this surface; the Jacobian curve of a net of such sections is formed by the first polar of a point in regard to the surface. Such a first polar will have a j-ple curve of the surface as a $(j-1)$-ple curve, and a k-ple node of the surface as a $(k-1)$-ple node. The system of curves $|C_J - 3C|$ will then lie on surfaces of order $n-4$, passing $(j-1)$ times through each j-ple curve of the surface. We have, so far, not dealt with the question of defining the canonical system in the case of a surface with isolated nodes; we consider the surfaces of order $n-4$, passing $j-1$ times through every j-ple curve, and $k-2$ times, (not $k-1$ times), through each k-ple node of the surface (the reason for this will be made clearer later; historically the behaviour first presented itself in the transformation of a double integral associated with the surface); the intersections of these surfaces with the original surface are the canonical curves of the surface as defined by Noether in his fundamental paper, "Zur Theorie des eindeutigen Entsprechens, u.s.w.," *Math. Ann.* VIII, 1875. Such curves may be composite, consisting of a variable, and a fixed part, common to all. It is this fixed part which constitutes, in Noether's definition, the aggregate of the exceptional curves of the surface. These curves are then determined, for any given surface, by the surface itself, without reference to any transformation of the surface into another. Noether states, however, and indicates a proof of the conclusion, that a transformation of the surface is possible, whereby these exceptional curves are changed to points (*loc. cit.* § 10). Without considering this question*, we shall distinguish the exceptional curves so arising by indicating their origin, and denote the number of such curves on a given surface by e_N. Two examples may be given. (1), Consider a sextic surface having a double conic θ, and a double line l, which does not meet θ. If $|C|$ denote the system of plane sections, the proper Jacobian system $|C_J|$ will be given by quintic surfaces containing both θ and l. Thus the system $|C_J-2C|$ will be given by cubic surfaces, also containing θ and l. The system $|C_J-3C|$ will be given by quadric surfaces, also through these; and these surfaces, as meeting the plane of θ outside this conic, namely on l, will contain this plane as part, and will thus consist, beside this plane, of planes through the line l. The plane of θ, meeting the quintic surface also in a conic with a double point on l, will contain two lines of the surface intersecting on l. These two lines, being a fixed component (beside the multiple curves) of the intersection of the surface with all the Noether canonical surfaces, are the exceptional curves of the surface in Noether's sense; the variable parts of the canonical curves are ∞^1 plane quartic curves, in planes through l, each having two double points, where its plane meets the conic θ. The quintic surfaces by which the curves $|C_J|$ are determined have here two intersections with each exceptional line, other than on the multiple curves of the surface; but the cubic surfaces determining the curves $|3C|$ have three such intersections; accordingly the curves $|C_J-3C|$ contain the line. As a second example, (2), consider a quintic surface with two triple nodes, of which the joining line, therefore, lies on the surface. For the system $|C|$, obtained by arbitrary plane sections, the Jacobian system $|C_J|$ is obtained by quartic surfaces having each of the triple points as a quadratic node; whence the system $|C_J-C|$ is obtained by cubic surfaces likewise having these quadratic nodes, and therefore containing their joining line. Thus the system $|C_J-3C|$ is obtained by planes through these nodes. These planes then contain the join of the nodes, which is an exceptional line in Noether's sense, and the reduced canonical system consists of quartic curves, in planes through the exceptional line, each having a node at both triple points of the surface. In both these examples, the exceptional curve may be said to arise from the presence of the term $3C$ in $|C_J-3C|$, and not from the term C_J, as in the preceding discussion. In the former example, the exceptional lines are not a necessary part of the curves $|C_J-2C|$, which are given by cubic surfaces containing the double line and double conic. These two examples are among those discussed in the note at the end of this chapter (p. 226).

---

* Enriques (*Mem. Soc. Ital. d. Scienze*, x, 1896, p. 72) states that it is not always possible to make a transformation whereby a Noether exceptional curve is changed to a simple point of the derived surface, quoting as an example, suggested by Castelnuovo, the case of a quintic surface having 3 tacnodes, whose plane meets the surface in a cubic curve and a conic touching this in three points. The case of a tacnode is examined in Picard-Simart, *Fonct. algéb.* I, 1897, pp. 76, 185, with reference to Noether, *Gött. Nachr.* 1871. Cf. Ex. 6, p. 241, below.

Return to the definition of ω. From this necessary discussion of the canonical curves of a surface, we return now to the question of the invariance of ω, limiting ourselves in the first place, as before, to the case when the surface has no multiple points other than a double curve with triple points thereon. From the theorem obtained, that the linear system expressed by $|C_J-3C-\gamma|$ is the same whatever the system $|C|$ may be, it follows, as was stated above for the case when $|C|$ has no prescribed base points, that the number of intersections of a curve U, of the surface, with a curve of $|C_J-3C-\gamma|$, depends on U only. We proceed to express this number in terms of the characters of U. Suppose, first, that U is not an isolated curve, but, with prescribed base points, at simple points of the surface, determines a linear system $|U|$, of freedom at least 2. Let n be the effective grade of this linear system, namely the number of intersections of two curves of the system other than at the base points, and ν the *completed* grade (cf. p. 139), the number of intersections including those at the base points, so that if t be the effective multiplicity of all curves of the system at one such base point $\nu=n+\Sigma t^2$; similarly let p be the effective genus of the curves $|U|$, in Riemann's sense (which depends on the multiple base points of the curves), and call $p+\frac{1}{2}\Sigma t(t-1)$ the *completed* genus, momentarily, denoting this by π. Then we prove that the number of intersections of a curve of the canonical system with the curve U is $2\pi-2-\nu$; by prescription the canonical system has no base points, and we can suppose that none of the intersections in question fall at base points of the system $|U|$. We call this number the ***canonical*** number* of the curve U, or of the system $|U|$; and we remark that, by the definition, the canonical number of a linear system $|U+V|$, formed by the sum of two systems $|U|$ and $|V|$, is the sum of the canonical numbers of these. To obtain the result stated, we suppose the canonical system to be computed from the system $|U|$; if

$$au+bv+cw=0$$

determine a net of curves of this system, this net determines on the particular curve $u=0$, outside the base points of the system, a linear series of freedom 1, given by the pencil $bv+cw=0$; and it is well known that the Jacobian set of this series, where two points of a set coincide, is equivalent, on the curve $u=0$, to two sets of the linear series, taken with a set of the canonical series on this curve; the linear system $|U_J-2U|$, where $|U_J|$ denotes the Jacobian system of $|U|$, thus determines on $u=0$, outside the base points of the system $|U|$, a set of $2p-2$ points, this being an immediate consequence of the definition of the Jacobian curve of the net

$$au+bv+cw=0.$$

If then, for two curves P, Q, the symbol (P, Q) denote the number of intersections other than at base points prescribed for either of these curves, and ω denote the aggregate of the base points of the system $|U|$, each taken simply, we have† $(U_J-2U, U)=2p-2$; and the number of intersections of U with the canonical system, not at the base points of $|U|$, or the number $(U_J-3U-\omega, U)$, is $2p-2-(U+\omega, U)$. Here $(U+\omega, U)$ will be the number of intersections, other than at the base points of $|U|$, of the curve U, with a curve of the same order having, at a t-ple base point of U, only a $(t-1)$-ple point, and will therefore be $n+\Sigma t^2-\Sigma t(t-1)$, or $n+\Sigma t$, the summation being extended to all the base points of $|U|$. Putting however $\nu=n+\Sigma t^2$, $\pi=p+\frac{1}{2}\Sigma t(t-1)$, we find at once that $2p-2-n-\Sigma t$ is the same as $2\pi-2-\nu$.

The system $|U_J-2U|$ is called the *adjoint* system of $|U|$ (cf. Vol. v, p. 213).

---

* Cf. Severi, "Il genere aritmetico ed il genere lineare", *Atti...Torino*, xxxvii, 1902, No. 1; also "Su alcune questioni", *Rend. Palermo*, xvii, 1903, No. 4; and "Sulle intersezioni", *Mem. Torino*, lii, 1903, No. 8. We define a "class of immersion" below (p. 225, Ex. 5, and Chap. vi, p. 251), and connect it with the canonical number.

† This remains true when the curves $|U|$ have variable double points on the double curve, p being the effective genus taking account of such. By a theorem, due to Bertini, a linear system of curves can have variable multiple points only on the multiple curves of the surface. But such points arise for instance when the system $|U|$ consists of plane sections of the surface, the system $|U_J|$ being then composed of proper curves of contact of enveloping cones to the surface.

Some particular remarks should be made in connexion with this result: (1), for the case when $|U|$ is the canonical system itself, supposed now to be of freedom 2 at least, if N denote the grade and Π denote the genus for this system, which has no base points, we infer $2\Pi-2-N=N$, or $N=\Pi-1$ (cf. the result for the canonical system, as defined by Noether, given in the preceding chapter, pp. 162, 168); (2), If ν_1, π_1 be respectively the completed grade and genus of one linear system of curves $|A_1|$, and ν_2, π_2 the corresponding numbers for a system $|A_2|$, while ν_{12}, π_{12} similarly belong to the sum system $|A_1+A_2|$, and i is the complete number of intersections of a curve of $|A_1|$ with a curve of $|A_2|$, since the complete number of intersections of two curves of $|A_1+A_2|$, say $[A_1+A_2, A_1+A_2]$, is clearly equal to $[A_1, A_1]+2[A_1, A_2]+[A_2, A_2]$, we have

$$\nu_{12}=\nu_1+\nu_2+2i;$$

(3), We also have $2\pi_{12}-2=2\pi_1-2+2\pi_2-2+2i$. For if, from an arbitrary point, a curve of each of $|A_1|$, $|A_2|$, $|A_1+A_2|$ be projected, on to a plane, into curves of respective orders m_1, m_2, m_1+m_2, and we denote by h_1, h_2 and k the numbers of lines from this arbitrary point which respectively meet A_1 in two points, A_2 in two points and meet A_1 and A_2 each once, we have the three equations

$$2\pi_1-2+2\delta_1=m_1(m_1-3)-2h_1,\quad 2\pi_2-2+2\delta_2=m_2(m_2-3)-2h_2$$

and $2\pi_{12}-2+2\delta_1+2\delta_2=(m_1+m_2)(m_1+m_2-3)-2h_1-2h_2-2k$,

where δ_1, δ_2 represent the number of double points of $|A_1|$, $|A_2|$ on the double line of the surface (and no correction of π_{12} is necessary in the last equation owing to the double points of a degenerate plane curve A_1+A_2 which arise from an intersection of A_1 and A_2 on the surface; for the sum system $|A_1+A_2|$ on the surface has no variable double points not included in $\delta_1+\delta_2$). Taking account then of $m_1m_2=k+i$, where i is the completed number of intersections of $|A_1|$ and $|A_2|$, we obtain the result in question. This proof, suggested in Picard-Simart, *Fonct. algéb.* (cf. Vol. v, p. 209), is valid when either $|A_1|$ or $|A_2|$ consists of an isolated curve, supposed irreducible. When both $|A_1|$ and $|A_2|$ have freedom 2 at least, another proof arises from an equation found above (p. 217), expressing that the system $|(A_1+A_2)_J-2A_1-2A_2|$ is equivalent either to $|A_{1J}-2A_1+A_2+\alpha_2|$, or to $|A_{2J}-2A_2+A_1+\alpha_1|$; this shews, putting down the number of intersections of a curve of the first system with a curve of $|A_1+A_2|$, other than at base points, as equivalent to the sum of

$$(A_{1J}-2A_1+A_2+\alpha_2, A_1) \text{ and } (A_{2J}-2A_2+A_1+\alpha_1, A_2),$$

that $2\pi_{12}-2-\Sigma(s_1+s_2)(s_1+s_2-1)$ is equal to the sum of

$$2\pi_1-2-\Sigma s_1(s_1-1)+i-\Sigma s_1s_2 \text{ and } 2\pi_2-2-\Sigma s_2(s_2-1)+i-\Sigma s_1s_2;$$

and this gives the formula in question. See also Noether, *Acta*

Math. VIII, 1886, p. 182; (4), The formula in (2) enables us to give a meaning to ν_2 when the system $|A_2|$ consists of an isolated curve; for, if this be given, and $|A_1|$ be a system of freedom at least 2, it is easy to see that the system $|A_1+A_2|$ is then of freedom at least as great as that of $|A_1|$; and then, further, that the value of ν_2 thence defined is independent of the system $|A_1|$ adopted. Further the formulae (2), (3) enable us to define ν_2 and π_2 when $|A_2|$ consists of reducible curves, and it can be shewn that the definition leads to definite values. It is then to be remarked that these formulae lead to $2\pi_{12}-2-\nu_{12}=2\pi_1-2-\nu_1+2\pi_2-2-\nu_2$; this is an immediate consequence of the result $(C_J-3C-\gamma,\ U)=2\pi-2-\nu$, proved above for the case when $|U|$ is a system of freedom at least 2. But, by what is now said, this result is now proved even when U is an isolated curve; (5), The formulae in (2) and (3), but not the formula $(C_J-3C-\gamma,\ U)=2\pi-2-\nu$, continue to hold when the completed grade, genus, and number of intersections, ν, π, i are respectively replaced by their effective values $\nu-\Sigma t^2$, $\pi-\frac{1}{2}\Sigma t(t-1)$, $i-\Sigma t_1t_2$, where t is an actual multiplicity; and also when these are replaced by their *prescribed*, or *virtual*, values $\nu-\Sigma s^2$, $\pi-\frac{1}{2}\Sigma s(s-1)$, $i-\Sigma s_1 s_2$, where s is a prescribed multiplicity, at a prescribed base point. (The actual multiplicity may exceed the prescribed multiplicity.) This is obvious because both the functions x^2, $x(x-1)$ satisfy the equation $\phi(x_1+x_2)=\phi(x_1)+\phi(x_2)+2x_1x_2$.

After this long discussion it is clear that *we may define the number* $\omega-1$ *as the grade of the canonical system* of the surface, itself defined from any linear system $|C|$ as the system $|C_J-3C-\gamma|$. And as we have considered the modification of the canonical system in the case of a birational transformation when a simple point of the original surface is replaced by a curve on the derived surface, we are in a position to find the modification of $\omega-1$ due to the transformation.

Ex. 1. The equation for the canonical number

$$(C_J-3C-\gamma,\ U)=2\pi-2-\nu,$$

if modified to the form $(3C+\gamma-C_J,\ U)=\nu-(2\pi-2)$, will be found to hold in cases when the canonical system consists of ideal curves, as for instance in the case when the surface is a plane or a rational surface. For example, if we consider a system of curves of order k in a plane with prescribed s-ple points, the system $|C_J|$ is that of curves of order $3k-3$ with prescribed $(3s-1)$-ple points, and the system $|3C+\gamma-C_J|$ is that of the cubic curves of a plane. These meet the curves of order n in the plane in $n^2-n(n-3)$, or $3n$ points.

Ex. 2. As stated above (p. 222), the equation $(U_J-2U,\ U)=2p-2$ is unaffected by the presence of a double curve on the surface. For example, if $|U|$ be the system of plane sections of a Steiner quartic surface $(p=0)$, the system $|U_J|$ consists of curves of order 6, and $|2U-U_J|$ of conics, meeting a plane in $2-2p$ points. Or, if $|U|$ be the system of plane sections of a quartic surface which has a double line, the system $|U_J|$ consists of curves of order 10, and the system $|U_J-2U|$ consists of conics, which meet a plane in $2p-2$ or 2 points. Or again the example (p. 229) of a sextic surface with a double conic and a double line may be quoted.

Ex. 3. If the canonical system $|C_J-3C-\gamma|$ break up into a system of irreducible curves $|K|$, together with other isolated curves $\epsilon_1, \epsilon_2, \ldots$, of grade $\nu_1, \nu_2, \ldots$ and genera $\pi_1, \pi_2, \ldots$, the equation $N=\Pi-1$ for the unreduced canonical system leads to the corresponding equation $N_0=\Pi_0-1$ for the system $|K|$, if, and only if,

$$\Sigma(\pi_i-1-\nu_i)=\Sigma(K, \epsilon_i)+\Sigma(\epsilon_i, \epsilon_j).$$

This equation is true if every π_i be zero and every ν_i be -1, and if the isolated curves do not intersect the curve K, nor one another (intersections on the double curve of the surface being neglected).

Ex. 4. If N, Π be the complete grade and genus of the system $|C_J-3C-\gamma|$, and ν, π of the system $|C|$, while ν', π' are the complete grade and genus of $|C_J-2C|$, we have from $C_J-2C=C_J-3C-\gamma+C+\gamma$, the results $\nu'=N+\nu+2(2\pi-2-\nu)$, and $\pi'=\Pi+\pi+2\pi-2-\nu-1$; hence as $N=\Pi-1$, we have $\nu'-\pi'=\pi-2$. And since

$$(t-1)^2-\tfrac{1}{2}(t-1)(t-2)=\tfrac{1}{2}t(t-1),$$

this remains true when the grades and genera involved are the actual, or are the prescribed (or virtual) values.

Ex. 5. We have considered, in the last chapter, the curve, of order μ_1, on a surface in space [4], which is the locus of points whereat the tangent plane of the surface meets an arbitrary general line, l. This is the Jacobian C_J, of the net of curves $|C|$, on the surface, determined by primes through the line l, all such prime sections through a point of this curve having a common tangent line which meets the line l. We shall have occasion, in the next chapter, to consider the number of points of a given curve U of the surface whereat the tangent plane of the surface meets an arbitrary line l; it will be called the *class of immersion* of the curve U on the surface, and usually denoted by z. From what we have said, z is equal to the number of points of intersection of U with the curve C_J arising from the line l. The canonical number of the curve U has, however, been defined as the number of its intersections with a curve of the canonical system of the surface, when this exists; which is given by $|C_J-3C|$. Thus if ϵ_0 be the order of U, its canonical number is equal to $z-3\epsilon_0$. This then is equal to $2\pi-2-\nu$, where π, ν are the completed genus and grade of U. As a simple example, consider a line on the Segre quartic surface of intersection of two quadrics in [4]. Here z is at once seen to be 2; thus $\nu=-1$. More generally, for an isolated curve, of order s, on the surface, for which $\pi=0$, $\nu=-1$, we infer $z=3s-1$. This is in agreement with the fact that for a linear system of curves on a surface, with a common s-fold base point at a simple point of the surface, the Jacobian system has a $(3s-1)$-fold point at this point (cf. p. 254, Ex. 2).

The computed characters of an exceptional curve. Suppose that in the birational transformation of a surface a simple point of the original surface becomes an exceptional curve on the derived surface. Let the point be O, and the curve into which it is transformed be ϵ'. Take two curves A, B on the original surface, without prescribed base points, intersecting at O and in n other points. On the derived surface these become, by *direct* transformation, as we have seen (p. 218, above), two composite curves $A'+\epsilon'$, $B'+\epsilon'$, whose total number of intersections is the same as of A and B on the original surface. Thus we have

$$n+1=(A, B)=(A'+\epsilon', B'+\epsilon')=(A', B')+(A', \epsilon')+(B', \epsilon')+(\epsilon', \epsilon');$$

here (A', B') is equal to the number of intersections of A and B which are not at O, namely n, and (A', ϵ'), (B', ϵ') are both equal to 1 because A and B pass simply through O. Thus we deduce $(\epsilon', \epsilon')=-1$. The canonical number of the curve ϵ' is also -1; for let K denote a canonical curve of the original surface, which we suppose to be irreducible, and not to pass through O; we have proved above (p. 218) that the direct transformation of this, on the derived surface, is a curve $K'+\epsilon'$, whereof K' will be in (1, 1) correspondence with K, so that $(K', \epsilon')=0$, as K does not contain O; thus $(K'+\epsilon', \epsilon')=-1$. Thus if π' be the complete genus of ϵ', we have $2\pi'-2-\nu'=-1$, wherein ν' has been shewn equal to -1. Whence also $\pi'=0$. As ϵ' is the representative of a simple point of the original surface, its effective genus is 0; thus it has no multiple points. A particular consequence (see Ex. 3 preceding) is that, if N', Π' denote the grade and genus of the curves of the reduced canonical system K', on the derived surface, we have $N'=\Pi'-1$. More generally this is true for the reduced canonical system on any surface if this is formed from the system $|C_J-3C-\gamma|$ by the omission of curves which can all be regarded as arising by transformation from simple points of another surface (see Ex. 3, preceding).

The modification of ω in a birational transformation. In the simplest case, just considered, where one simple point O of the original surface becomes an exceptional curve ϵ' of the derived surface, we have, with the same notation,

$$\omega'-1=(K'+\epsilon', K'+\epsilon')=(K', K')+2(K', \epsilon')+(\epsilon', \epsilon'),$$

wherein $(K', \epsilon')=0$, $(\epsilon', \epsilon')=-1$, and (K', K'), the number of intersections of two curves which are in (1, 1) correspondence with the canonical curves of the original surface, is equal to $\omega-1$. Thus we have $\omega=\omega'+1$. Thus, if, on the first surface, there be e curves which change into simple points of the second surface, and, on this surface, there be e' curves arising by the transformation from simple points of the original surface, we may infer that $\omega+e=\omega'+e'$.

Note I. Examination of examples cited by Noether. It will be interesting, before pursuing the general theory in more detail (see Note II, p. 232, below), to examine the application of what has been said to a set of examples quoted by Noether (*Math. Ann.* VIII, 1875, § 11), *and from his point of view*. Of his sixteen examples one involves intersecting multiple curves on the surface (a septimic surface with a triple line, and two double conics in planes through this line); we consider the other fifteen, numbering these as in his paper. He computes in each case two numbers: (i), The number p_g of actually existing surfaces of order $\mu_0 - 4$, where μ_0 is the order of the given surface ψ^{μ_0}, which pass $(j-1)$ times through every j-ple curve of the surface, and have a $(k-2)$-ple node at every k-ple node (supposed general) of the surface. Such so-called *adjoint surfaces* meet ψ^{μ_0} in the curves called by Noether the *canonical* curves; it may happen however that all these adjoint surfaces have common an undesigned curve, or curves, lying on ψ^{μ_0}; such curves we shall call Noether's *exceptional* curves, and the number of irreducible curves of this character will be denoted by e_N. The variable parts of the canonical curves constitute a linear system which we call the *reduced* canonical system. Noether also computes (ii), the effective grade of this reduced system, namely the number of variable intersections of two curves of the system other than at the multiple points of the surface, denoting this by $p^{(2)}$; and he computes the effective genus of a curve of this system (which takes account of multiple points of curves of this system, if any, which lie at the multiple points of ψ^{μ_0}). It appears in all cases that $p^{(2)} = p^{(1)} - 1$; this is a consequence of what has been given above (see p. 223) when there are no exceptional curves, and the surface has no isolated nodes, and is inferred to hold in general from an indicated proof that the exceptional curves can be transformed to simple points of another surface. Beside p_g and $p^{(1)}$ we shall, in the following examples, compute p_n and I, ω, or, in case the surface have isolated nodes, the numbers I_0, ω_0 which we have used above (p. 199) to mark the presence of such nodes. That surfaces exist for which $p_g > p_n$ is a discovery, not taken account of in Noether's paper, dating from Cayley's remark (1871, *Papers*, VIII, p. 394) that, for a non-rational ruled surface, p_n is negative, while $p_g = 0$ (see preceding chapter, Ex. 9, p. 167). One of the earliest examples of a surface with $p_g > p_n$ which is not ruled and not transformable to a ruled surface, appears to be the surface of order $2k$ with k-ple nodes at 8 associated points, considered by Castelnuovo (1891; see p. 203, above). But other cases are the hyperelliptic surfaces, representing the pairs of points of a curve of genus 2, for which $p_g = 1$, $p_n = -1$ (Picard, *Liouville's Jour.* I, 1885 and V, 1889; Humbert, *ibid.* IX, 1893); and, more generally, the surfaces (considered in some detail below, in Chap. VII) which represent the pairs of points of two curves, which may coincide (De Franchis, *Rend. Palermo*, XVII, 1903; Severi, *Atti...Torino*, XXXVIII, 1902-3). In all Noether's cases the so-called *irregularity* $p_g - p_n$ is zero. It is apposite to quote here the theorem (of Castelnuovo-Enriques, p. 140, above), that a surface on which there is a linear series of curves (of freedom $r \geqslant 0$), with complete genus $\pi > 0$, having a negative canonical number, $2\pi - 2 - \nu$, can be birationally transformed to a ruled surface.

In the first seven of the following examples (3), (6), (9), (10), (11), (13), (15), there is an ordinary double curve, and no isolated node, or exceptional curve; in the next four cases (1), (5), (7), (14) there is no isolated node, but exceptional curves arise; in the last four cases (2), (4), (8), (12) there are nodes, and, except in (8), also exceptional curves. When there are e_N exceptional curves, and κ nodes (of any orders), it will be verified in every case that $\omega_0-\kappa+e_N=p^{(1)}$ (see Note II, p. 232, below. Above, pp. 215–24, nodes were excluded).

(3) For a sextic surface, ψ^6, with a double elliptic quartic curve, using the formulae which have been previously given

$$\mu_1=\mu_0(\mu_0-1)-2\Sigma\epsilon_0,\quad \nu_2=\Sigma[(2\mu_0-4)\epsilon_0-2\epsilon_1-6t],$$

$$\mu_2=\mu_0(\mu_0-1)^2-\Sigma[(7\mu_0-12)\epsilon_0-4\epsilon_1-15t],$$

$$I+4=\mu_2-2\mu_1+3\mu_0,\quad \omega-1=\mu_2+\nu_2-6\mu_1+9\mu_0,$$

$$12(p_n+1)=I+4+\omega-1,$$

we find

$$\mu_0=6,\ \mu_1=22,\ \mu_2=62,\ \nu_2=16,\ I+4=36,\ \omega-1=0,\ p_n+1=3.$$

There are two quadric surfaces through the double curve, or $p_g=2$, $=p_n$; two such quadric surfaces have no further intersection, so that $p^{(2)}=0$; and $p^{(1)}=1$, $=\omega$.

(6) A ψ^6 with a double rational cubic curve. There are three quadric surfaces through the double curve, or $p_g=3$; any two of these quadric surfaces meet in a chord of the cubic curve, which has two further intersections with the surface, so that $p^{(2)}=2$. A quadric surface through the cubic curve meets ψ^6 further in a sextic curve, for which (Ex. 10, p. 168, Chap. IV) the genus is $p^{(1)}=3$, so that $p^{(2)}=p^{(1)}-1$. Moreover, the formulae quoted give $\mu_0=6$, $\mu_1=24$, $\mu_2=76$, $\nu_2=16$, $I+4=46$, $\omega-1=2$, $p_n+1=4$, so that $\omega=p^{(1)}$, and $p_g=p_n$.

(9) A ψ^6 with two skew double lines. There are four quadric surfaces through these lines, or $p_g=4$; moreover, two such quadric surfaces meet further in two skew lines, each meeting both the given lines and hence meeting ψ^6 residually in two points; thus $p^{(2)}=4$. A quadric surface through the double lines meets the surface further in an octavic curve for which (Ex. 10, p. 168, Chap. IV) the genus $p^{(1)}=5$, so that $p^{(2)}=p^{(1)}-1$. The formulae give $\mu_0=6$, $\mu_1=26$, $\mu_2=90$, $\nu_2=16$, $I+4=56$, $\omega-1=4$, $p_n+1=5$, so that $\omega=p^{(1)}$, $p_g=p_n$.

(10) A ψ^7 with a double elliptic quartic curve, and a double line not meeting this. Cubic surfaces through the quartic curve and the line are, in tale, $20-12-4$, or 4, so that $p_g=4$; two such cubic surfaces meet, beside, in a curve of order 4, which meets the quartic curve in 8 points, and the line in 4 points (as we may see, for example, by taking for such a cubic surface the aggregate of a quadric

surface through the quartic curve and a plane through the line); such cubic surfaces have therefore $28-24$, or 4, further intersections with the ψ^7; thus $p^{(2)}=4$. The formulae give $\mu_0=7$, $\mu_1=32$, $\mu_2=99$, $\nu_2=34$, $I+4=56$, $\omega-1=4$, $p_n+1=5$. Thus $p_n=p_g$ and $\omega-1=p^{(2)}$.

(11) A general ψ^5. The tale of all planes is 4; and two planes meet in a line having 5 intersections with the surface; thus $p_g=4$, $p^{(2)}=5=p^{(1)}-1$. The formulae give $\mu_0=5$, $\mu_1=20$, $\mu_2=80$, $\nu_2=0$, $I=51$, $\omega-1=5$, $p_n=4$. Whence $p_n=p_g$ and $\omega=p^{(1)}$. Cf. No. (12) below.

(13) A ψ^7 with a double sextic curve of genus 3. The tale of cubic surfaces through the sextic curve is $20-(18-3+1)$, $=4$; so that $p_g=4$. Two such cubic surfaces meet further in a rational cubic curve having 8 intersections with the sextic curve, and hence $21-16$, or 5, other intersections with the surface. Thus $p^{(2)}=5$. The formulae give $\mu_0=7$, $\mu_1=30$, $\mu_2=94$, $\nu_2=28$, $I=51$, $\omega-1=5$, $p_n=4$; so that $p_n=p_g$ and $\omega-1=p^{(2)}$. Evidently, if $U=0$, $V=0$, $W=0$, $K=0$ be independent cubic surfaces through the sextic curve, the equations $x'/U=y'/V=z'/W=t'/K$ transform the surface ψ^7 into a ψ^5; which explains the equality of the values of I and ω in this case with those in the preceding No. (11).

(15) A ψ^6 with a plane cubic double curve. The quadric surfaces through this curve break up into the plane of the curve and a further arbitrary plane. Thus $p_g=4$ and $p^{(2)}=6=p^{(1)}-1$. The formulae give $\mu_0=6$, $\mu_1=24$, $\mu_2=84$, $\nu_2=12$, $I=50$, $\omega-1=6$, $p_n=4$, so that $p_n=p_g$ and $\omega-1=p^{(2)}$.

(1) A ψ^6 with a double conic, and a double line not meeting this. The plane of the conic meets the surface again in a curve of order 2 with a double point on the double line, that is, in a pair of lines. These are exceptional lines according to Noether's definition, since a quadric surface through the double conic and the double line must break up into the plane of the conic and a plane through the line. The reduced canonical curves of the surface are then quartic curves in planes through the double line, each with two double points on the double conic. Thus $p_g=2$, $p^{(1)}=1$. The formulae give $\mu_0=6$, $\mu_1=24$, $\mu_2=68$, $\nu_2=20$, $I=34$, $\omega-1=-2$, $p_n=2$. Thus $p_n=p_g$ and $\omega+e_N=p^{(2)}$, where e_N, $=2$, is the number of exceptional lines. As was remarked earlier (p. 225), the exceptional lines do not lie on the adjoint cubic surfaces, $|C_J-2C|$, drawn through the double conic and the double line, which meet the surface residually in curves of order 12. Of such cubic surfaces the tale is 9, and they may be used to transform the given surface (as in No. (13) above) into a surface lying in space [8]; of these 9 surfaces there are 8 which contain a particular one of the two exceptional lines, so that this line becomes a point of the transformed surface—as, similarly, does the other exceptional line.

(5) A ψ^7 with a double sextic curve of genus 1. Such a curve has 3 quadrisecant chords; these chords therefore lie entirely on the surface ψ^7, and also on any cubic surface drawn through the curve. Thus, two cubic surfaces through the double curve have no further intersection with the surface than these three lines, and $p^{(2)}=0$. Through the double curve there are $20-18$, or 2, cubic surfaces, and $p_g=2$. The formulae give $\mu_0=7$, $\mu_1=30$, $\mu_2=78$, $\nu_2=36$, $I=35$, $\omega-1=-3$, $p_n=2$. Thus $p_n=p_g$ and $\omega+e_N=p^{(2)}+1$, where ϵ_N, $=3$, is the number of exceptional lines. As in the preceding case, No. (1), the adjoint quartic surfaces $|C_J-2C|$ through the sextic curve are such that each of the quadrisecants imposes a single condition for such a quartic surface to contain this line, which can then be transformed to a point.

(7) A ψ^7 with a double sextic curve of genus 2. In this case the curve has one quadrisecant which lies on the surface and on all cubic surfaces through the double curve. There are 3 such cubic surfaces, or $p_g=3$. Further, two such cubic surfaces meet again in a conic, beside the quadrisecant, and this conic meets the double curve in 6 points; whence $p^{(2)}=14-12$, $=2$. The formulae give $\mu_0=7$, $\mu_1=30$, $\mu_2=86$, $\nu_2=32$, $I=43$, $\omega-1=1$, $p_n=3$, so that $p_n=p_g$ and $\omega-1+e_N=p^{(2)}$, where e_N, $=1$, is the number of exceptional lines. And, as in the last example, the quadrisecant lies on all but one of the independent quartic surfaces containing the sextic curve.

(14) A ψ^7 with a rational quintic double curve. This curve has one quadrisecant line, which lies on the surface, and on all cubic surfaces through the quintic curve. Such surfaces are, in tale, $20-(15+1)=4$, so that $p_g=4$. Two such cubic surfaces meet further in the quadrisecant line and in a cubic curve having 8 intersections with the double curve; this cubic curve thus meets the surface, not on the double curve, in $21-16$ points, or $p^{(2)}=5$. The formulae give $\mu_0=7$, $\mu_1=32$, $\mu_2=99$, $\nu_2=34$, $I=52$, $\omega-1=4$, $p_n=4$. Whence $p_n=p_g$ and $\omega-1+e_N=p^{(2)}$, where $e_N=1$.

(2) A ψ^5 with two triple points. The join of these points lies on the surface; it also lies on any surface of order μ_0-4 (or plane) passing $k-2$ times (or once) through the triple points, and is an exceptional line. The canonical curves are the plane quartic curves with a double point at each triple point, so that

$$p_g=2, \quad p^{(2)}=p^{(1)}-1=0.$$

Assuming, as in the earlier part of this chapter, that an isolated k-ple point diminishes the class of the surface by $k(k-1)^2$, we have here, for $k=3$ twice over, $\mu_2=5.4^2-2.3.2^2=56$; while $\mu_1=20$. Whence, from the definitions (p. 200, preceding),

$$I_0+4=\mu_2-2\mu_1+3\mu_0+\Sigma(k-1),$$
$$\omega_0-1=\mu_2+\nu_2-6\mu_1+9\mu_0+\Sigma(k-1)(2k-1),$$

with $\nu_2 = 0$, using κ, $=2$, for the number of isolated nodes, we have $I_0 + \kappa = 33$, $\omega_0 - \kappa - 1 = -1$, $p_n = 2$. Whence $p_n = p_g$ and

$$\omega_0 - \kappa + e_N = p^{(1)},$$

where e_N is the number of exceptional lines.

(4) A ψ^6 with a rational non-singular cubic as double curve, and also a cubic node. Quadric surfaces through the cubic curve and the node are two in all, and these surfaces contain the chord of the cubic curve which can be drawn from the node, which lies on the surface and is thus an exceptional line. Thus $p_g = 2$, $p^{(2)} = 0$. With the formulae of p. 162 of the preceding chapter, and those in the preceding example, we find $\mu_0 = 6$, $\mu_1 = 24$, $\mu_2 = 64$, $\nu_2 = 16$, $I_0 = 32$, $\omega_0 - 1 = 0$, $p_n = 2$. Thus $p_n = p_g$, and $\omega_0 - 1 - \kappa + e_N = p^{(2)}$, where $\kappa = 1$, $e_N = 1$. In this case also, the exceptional line is one condition for cubic surfaces containing the cubic curve and the node.

(8) A ψ^6 with a double plane cubic curve, and a cubic node. Here, quadric surfaces through the plane cubic curve contain the plane, and the canonical curves are given by arbitrary planes through the node. Whence $p_g = 3$, $p^{(2)} = p^{(1)} - 1 = 3$. Here there is no exceptional line. The formulae referred to in the preceding example lead to $\mu_0 = 6$, $\mu_1 = 24$, $\mu_2 = 72$, $\nu_2 = 12$, $I_0 = 40$, $\omega_0 - 1 = 4$, $p_n = 3$. Whence $p_n = p_g$ and $\omega_0 - \kappa = p^{(1)}$, where κ, $=1$, is the number of nodes.

(12) A ψ^7 with a triple conic, and one cubic node. In this case, cubic surfaces having the conic as a double curve must contain the plane of the conic; thus the reduced canonical curves are given by quadric surfaces containing the conic and the node, and $p_g = 4$. Two such quadric surfaces meet further in a conic, through the node, having two intersections with the given conic; this conic then has $14 - 9$, or 5, intersections with the surface, other than at multiple points of the surface, so that $p^{(2)} = 5$. By the formulae referred to in the last example we find $\mu_0 = 7$, $\mu_1 = 30$, $\mu_2 = 92$, $\nu_2 = 20$, $I_0 = 51$, $\omega_0 - 1 = 5$, $p_n = 4$. The plane of the conic meets the surface further in an exceptional line. Thus, with $\kappa = 1$, $e_N = 1$, we have

$$\omega_0 - 1 - \kappa + e_N = p^{(2)}.$$

In this case also, the exceptional line is one condition for the adjoint quartic surfaces $|C_J - 2C|$ which pass through the triple point and have the conic as double curve. But the ψ^7 can be transformed to a quintic surface by considering the ∞^3 quadrics through the triple point and the double conic, these becoming planes in the new space; a particular plane section of the quintic surface breaks up into a conic, and a cubic curve, which is the transformation of the triple point of the ψ^7, but is not an exceptional curve of the quintic surface in Noether's sense. The values of the invariants for the quintic surface are found in No. (11) above, say $I' = 51$, $\omega' - 1 = 5$. Thus, in particular, $I_0 + \kappa - e_N = I'$, which can also be considered in the form $I_0 - e_N = I' - \kappa$, where κ would then refer to the cubic curve on the quintic surface, as on pp. 201 ff. See Note II following.

This completes the set of examples we quote from Noether. For these, see also D. W. Babbage, *Proc. Camb. Phil. Soc.* XXIX, 1933, p. 319. The following case by E. A. Maxwell may also be adduced: A surface ψ^9, of order 9, having an isolated 4-ple node and a rational quartic curve as triple curve. This quartic curve lies on only one quadric surface, and the residual intersection of this with the ψ^9 consists of six lines; the three chords of the quartic curve which can be drawn from the 4-ple node also lie on the ψ^9. The adjoint surfaces of order 5, passing doubly through the quartic curve, which have the node as a quadratic node, break up into the unique quadric surface through the quartic curve, together with cubic surfaces, passing through the quartic curve, which have a quadratic node at the 4-ple node of ψ^9. Thus we can prove that $p_g = 3$ and $p^{(2)} = 2$. The formulae give, in this case, $\mu_0 = 9$, $\mu_1 = 48$, $\mu_2 = 120$, $\nu_2 = 60$, $I_0 = 50$, $\omega_0 - 1 = -6$, $p_n = 3$. Thus again we have $p_n = p_g$ and

$$\omega_0 - 1 - \kappa + e_N = p^{(2)},$$

since there is one node ($\kappa = 1$), and 9 exceptional lines ($e_N = 9$).

Note II. The adjoint surfaces of a given surface; the exceptional curves. In the present state of the theory of the resolution of the multiple elements of a surface (cf. for example, Enriques-Chisini, *Teoria geometrica*, II, pp. 649–58; and the references there given) it seems desirable to approach the theory of the invariants of a surface from the historical side, following Noether, as we have done. The conclusion that the adjoint surfaces of the given surface should be taken to have a $(k-2)$-ple point at an isolated k-ple point, and not a $(k-1)$-ple point as have the first polars, was first suggested, for the case of the canonical surfaces, by the conditions of finiteness of Clebsch's double integral, by which the invariant p_g was defined. But a geometrical aspect can be given to this conclusion, at least in simple cases, in two ways, which we now consider in turn (cf. Enriques-Campedelli, *Superficie algebriche*, 1932, p. 129). Consider a general isolated k-ple node of the surface, O; denote by D any section of the surface by a plane through O. The first polar of the point O, in regard to the surface, is a surface with a k-ple node at O, whose asymptotic cone coincides with that of the given surface; the intersection of this first polar with the given surface is therefore effectively that of a surface with a $(k+1)$-ple node at O. Hence, if $|D_J|$ denote the Jacobian curve of the net of curves $|D|$, the system $|D_J - 3D|$ on the given surface will be given by surfaces having at O a multiple point of multiplicity $[k(k+1) - 3k]/k$, or $k-2$. This is for the case when the k-ple point is quite general, but the same suggestion avails in other cases.

Consider, for example, first, the case of a uniplanar double point; for this, the equation of the given surface, in non-homogeneous coordinates, for the neighbourhood of O, is of the form

$$z^2+\phi(x, y, z)+\ldots=0,$$

where ϕ is homogeneously of order 3 in x, y, z. The first polar of the point O intersects the surface, near O, as does a surface of equation $\phi(x, y, z)+\ldots=0$; the behaviour of D_J-3D near the origin is then that of a fraction $\phi(x, y, z)/\psi(x, y, z)$, where ψ is also homogeneously of order 3 in x, y, z, and hence, that of a fraction $u(x, y)/v(x, y)$, where u, v are homogeneously of order 3 in x and y only. Thus the uniplanar point imposes no condition for the adjoint surface. If, however, we consider a tacnode, for which the equation of the surface is $z^2+z\phi(x, y, z)+\ldots=0$, in which ϕ is homogeneously quadratic in x, y, z, we shall be similarly led to consider the fraction $z\phi(x, y, z)/\psi(x, y, z)$, where ψ is of order 3, and hence a fraction zu/v, where u, v are homogeneously of orders 2 and 3, respectively, in x and y only; in this case, however, z is of an order of smallness comparable with that of a *quadratic* polynomial in x and y, and the fraction therefore vanishes to the first order. Thus the canonical curves of the surface are given by adjoint surfaces subject to the condition of passing through the tacnode. That the adjoint surfaces should be given near O by considering the system $|D_J-3D|$ is of course suggested by the general theory given above for the case when the surface has no isolated multiple point. In Noether's theory ("Theorie des eindeutigen Entsprechens, u.s.w." § 3, *Math. Ann.* II, 1870) the adjoint surfaces determining the canonical curves, for the surface, of order n, expressed by $f(x, y, z, t)=0$, are those surfaces, $\phi=0$, of order $n-4$, for which the double integral $\iint\phi\Delta\,du\,dv$ remains finite, where u, v are two parameters in terms of which the coordinates (x, y, z, t), of a point of the surface, are supposed to be expressible in the neighbourhood of a point considered, and Δ is given by $(a, y, z_1, t_2)/(a\partial f/\partial x)$, wherein (a, y, z_1, t_2) means the determinant of four rows, namely a, b, c, d; x, y, z, t; x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2, with $x_1=\partial x/\partial u$, $x_2=\partial x/\partial v$, etc., and $(a\partial f/\partial x)$ means $a\partial f/\partial x+b\partial f/\partial y+c\partial f/\partial z+d\partial f/\partial t$; and it is shewn that Δ is independent of a, b, c, d. In particular, if $a=b=c=0$, Δ reduces to $(x, y_1, z_2)/\partial f/\partial t$, and, speaking in general terms, the determinant (x, y_1, z_2) here plays the part denoted by the term $3D$ in $|D_J-3D|$. A similar formulation is possible for a $(r-1)$-ple multiple integral attached to a primal $f(x, y, \ldots)=0$, of order n, in space of r dimensions, the ϕ then being a polynomial of order $n-r-1$; and the condition of finiteness of the integral, when there is on $f=0$ a k-fold multiple locus of dimension s, *in the most general case*, is that $\phi=0$ should have this locus to multiplicity $k-(r-s-1)$; for example, for a multiple k-fold point ($s=0$), to multiplicity $k-r+1$; for a multiple k-fold curve ($s=1$) to multiplicity $k-r+2$. When the multiple locus is of special character, these numbers may be altered; but it is possible to find them by considering the behaviour of Δ; for instance, for a uniplanar or tacnodal point of a surface in three dimensions, see Picard-Simart, *Fonctions algébriques*, I, 1897, pp. 77, 185. Another way of reaching the necessary behaviour of an adjoint surface at a multiple point (Enriques, *Soc. Ital. d. Sc.* x, 1896, p. 10; Enriques-Campedelli, *Superficie algebriche*, 1932, pp. 17, 66, 129) is based on the fact that it is possible to transform the surface birationally so that the neighbourhood of the multiple point of the surface is changed into a curve, ϵ, on the new surface; the sections $|C|$ of the surface by arbitrary planes are thereby changed into a system $|C'|$ on the new surface, for which the curve ϵ is fundamental, not intersecting the curves $|C'|$, except at base points of these. Such a curve ϵ will then be a fixed part of the Jacobian curve of any net of curves contained in the system $|C'|$, as explained above (p. 216). We may therefore regard the Jacobian system $|C_J|$, of the system $|C|$, as being of the form $|\Gamma+\Sigma O|$, where, under the hypothesis that the surface is transformed to one upon which all the isolated multiple points, O, of the original surface, are represented by curves, ϵ, the curve Γ is the curve of the original surface which transforms into the part of the Jacobian system $|C_J'|$ other than these curves ϵ. If D denote the intersection of the original surface with a plane through a general k-ple point O, it is natural to write the number of intersections (D, O) as k; and, therefore, as a general plane section, equivalent to $D+O$, does not pass through O, to write $(D+O, O)=\text{zero}$, and hence $(O, O)=-k$. We also suppose two curves ϵ, ϵ', arising by transformation from two multiple points O, O', not to intersect. Further, as the first polar of a point, in regard to the original surface, has a $(k-1)$-ple point at O, there are $k(k-1)$ points of its intersection which are in the immediate neighbourhood of O; thus we write $(\Gamma, O)=k(k-1)$. Hence we have

$$(C_J, O)=(\Gamma+\Sigma O, O)=(\Gamma+O, O)=k(k-1)-k;$$

or $(C_J, O)=k(k-2)$. Thus, identifying Γ with the proper curve of contact of tangent planes to the surface from an external point, with a recognised (or prescribed) $k(k-1)$-ple point at the k-ple point of the surface, and identifying $|C_J-3C|$ with the canonical system of the surface, we reach the conclusion that the adjoint surfaces determining these canonical curves have only a $(k-2)$-ple point at the multiple point. In this point of view we assign a grade $-k$ to the isolated multiple point O; if further, since this point transforms in general to a curve of order k, we assign to this point a genus $\frac{1}{2}(k-1)(k-2)$, we can reach the corrected value of the invariant $\omega-1$ which we have adopted above (p. 200) when the surface has isolated multiple points*. For let π be the prescribed genus of the proper curve of contact Γ; then, by the formula p_1+p_2+i-1 developed in the text (p. 223), the prescribed (or virtual) genus of C_J, that is of $\Gamma+\Sigma O$, is

$$\pi+\Sigma[\tfrac{1}{2}(k-1)(k-2)+k(k-1)-1],$$

which is the same as $\pi+\Sigma[(k-1)(2k-1)-\frac{1}{2}k(k-1)-1]$. Whence, with $(C, C)=\mu_0$, $(C_J, C)=(\Gamma, C)=\mu_1$, $(\Gamma, \Gamma)=\mu_2+\nu_2$, we have

$$\begin{aligned}(C_J, C_J)=(\Gamma+\Sigma O, \Gamma+\Sigma O)&=\mu_2+\nu_2+\Sigma[2k(k-1)-k]\\&=\mu_2+\nu_2+\Sigma[(2k-1)(k-1)-1],\end{aligned}$$

and, therefore,

$$(C_J-3C, C_J-3C)=\mu_2+\nu_2-6\mu_1+9\mu_0+\Sigma[(2k-1)(k-1)-1].$$

While, if ϖ be the prescribed genus of C_J,

$$\varpi-1-9(p-1)=\pi-1-9(p-1)+\Sigma[(2k-1)(k-1)-\tfrac{1}{2}k(k-1)-1].$$

These are the values found (p. 200) for $\omega_0-1-\kappa$, if κ denote the number of isolated multiple points. The use of ω, $=\omega_0-\kappa$, in place of ω_0, would involve the use of I, $=I_0+\kappa$, in place of I_0. It is this value of I, in the case of a surface with isolated quadratic nodes, which is used by Severi, *Ann. Sc. d. l'École norm. sup.* xxv, 1908, No. 8, and *Acta Math.* xxxii, 1907, p. 351 (cf. Zeuthen, *Lehrbuch d. abz. Meth.* 1914, p. 167). This I would, for instance, be 5, not only for a general cubic surface, but also for cubic surfaces with nodes. There seems convenience in making the distinction, but the essential facts are not modified whichever convention be employed.

In regard to the exceptional curves of a surface, which arise from or can be transformed solely to simple points of another surface, we may accept it as proved that these are part of the system $|C_J-3C|$, since such a curve would otherwise have $3s-1$ intersections with C_J and $3s$ intersections with $3C$ (see above, pp. 219, 226). But, pending a more complete examination of Noether's suggested proof that all curves breaking off as fixed parts from the system $|C_J-3C|$ are such as can arise by transformation from simple points of another surface, and of other connected questions, we have preferred to indicate, by the notation (e_N), the definition of Noether's curves. Cf. p. 220, above, footnote; p. 241, below, Ex. 6; and *Ann. d. Mat.* vi, 1901, p. 28.

* Although the fundamental ideas were subsequently found in the Enriques-Campedelli volume, referred to above, it should be stated that this was pointed out independently by D. W. Babbage; to him, and to E. A. Maxwell, the author is much indebted for criticism of the theory as first given, for the case of only double or triple points. The term $(k-1)(2k-1)$ was obtained, and the equation $\omega_0-\kappa+e_N=p^{(1)}$ was verified in many particular cases, by Mr Maxwell. See also Mr Babbage's paper, *Proc. Camb. Phil. Soc.* xxix, 1933, p. 212.

Ex. 1. We may formulate in general terms the proper genus and grade to be assigned to an isolated multiple point, which we have exemplified above for the case when this is of quite general character (cf. Enriques-Campedelli, as above, p. 65). Let n, π be the prescribed (or virtual) grade and genus of an arbitrary plane section C of the surface (n being the order of the surface). Denote the multiple point to be considered by O, and a section through this point by C_1, writing $C \equiv C_1 + O$. Let the grade and genus to be assigned to O be n_0 and π_0, the grade and genus of C_1 being n_1 and π_1. Let d be the loss of genus of C by passing through O, or $\pi_1 = \pi - d$; let the corresponding loss of n be k, so that k is the number of intersections of the surface with an arbitrary line through O which are absorbed at O, or k is the multiplicity of O, and $n_1 = n - k$. Let it be assumed that we can assign a definite meaning to the number, i, of intersections of the section of the surface by an arbitrary plane through O, with O itself regarded as a curve, that is, of the number of points of the surface in the neighbourhood of O, which lie on the arbitrary plane section. Then, from $C \equiv C_1 + O$, we have $n = n_1 + n_0 + 2i$, $= n - k + n_0 + 2i$, and $\pi = \pi_1 + \pi_0 + i - 1$, $= \pi - d + \pi_0 + i - 1$. Thus $n_0 = k - 2i$, $\pi_0 = d - i + 1$. Particular cases are: (1), A general k-ple point, for which we may assume $i = k$, $d = \frac{1}{2}k(k-1)$. Thence $n_0 = -k$, $\pi_0 = \frac{1}{2}(k-1)(k-2)$, as above, in the text; (2), A double point for which the asymptotic cone breaks up into two planes. For this, similarly, $n_0 = -2$, $\pi_0 = 0$; (3), A uniplanar double point, whereat the asymptotic cone consists of a repeated plane. Here $d = 1$, $k = 2$ and hence $n_0 = 2(1-i)$, $\pi_0 = 2 - i$. In this case, the terms of lowest order in the equation of the surface may be supposed to be of the form $z^2 + \phi + \ldots = 0$, where ϕ is of order 3 in x, y, z; of a surface passing through the point, which is one of a system which may be used to transform the uniplanar point to a curve, the lowest terms, though linear in x, y, z, are effectively linear in x and y only; thus the uniplanar point is transformed to a line, meeting the transformation of the plane section through the point in one point. Thus we see that $i = 1$, and hence $n_0 = 0$, $\pi_0 = 1$; (4), For a tacnodal double point, with $k = 2$, $d = 2$, the tacnode is likewise transformable to a line, which, however, is double on the new surface. Thus $i = 2$, and $n_0 = -2$, $\pi_0 = 1$ (cf. p. 270, Ex. 11); (5), For a triple point on a double curve of the surface, triple for the curve and the surface, we have $k = 3$, $d = 2$, $i = 3$, and $n_0 = -3$, $\pi_0 = 0$.

Ex. 2. In this connexion reference may be made to the relation of an exceptional curve on the surface to the adjoint system (p. 222) of a given linear system of curves thereon. If, in the birational transformation from one surface to another, a system $|C|$ be changed to a system $|C'|$, and a point O, which is not a base point for $|C|$, become a curve ϵ on the new surface, the curve ϵ is fundamental for $|C'|$; hence, taking a net of curves from $|C'|$, given, suppose, by an equation $u'(\lambda\phi + \lambda_1\phi_1) + \lambda_2\psi = 0$, where $u' = 0$ on the curve ϵ, there is a curve of $\lambda\phi + \lambda_1\phi_1 = 0$ passing through an arbitrary point of the curve ϵ; this point is then a double point of a (degenerate) curve of the net; hence ϵ forms part of the Jacobian of any net contained in $|C'|$; thus, if the complete Jacobian system $|C_J'|$, of $|C'|$, contain only curves of which ϵ forms a part, or even, this not being so, if the complete adjoint system $|C_J' - 2C'|$ contains only curves of which ϵ forms a part, we can infer that the exceptional curve ϵ, arising by transformation from a simple point which is not a base point of the original system $|C|$, is a fixed part of the adjoint system of the transformed system $|C'|$ (cf. Enriques, "Intorno ai fondamenti...", *Atti... Torino*, XXXVII, 1901–2, No. 21; Severi, in Pascal's *Repertorium*, II, 2, p. 749).

Ex. 3. A sextic surface having a double line, and a double conic which does not meet this, contains two exceptional lines, in the plane of the conic, meeting on the double line. These arise from simple points of a surface obtainable by transformation of the sextic surface by cubic surfaces containing the double conic and the double line; as a general plane section, C', of the sextic surface meets either of these lines, the simple point from which this line arises is a base point of the system $|C|$ which transforms into the plane sections $|C'|$ of the sextic surface; and the adjoint system, $|C_J' - 2C'|$, given by cubic surfaces through the double conic and the double line, does not contain the exceptional lines. These lines form part, however, of the system of adjoint curves defined by the plane of the conic and a variable quadric surface through the double line.

Ex. 4. For the surface $f=0$, given by $z^2t^{2n-2}-(x, y, t)_{2n}=0$, with a particular singularity at $x=y=t=0$, prove that for the integral $\iint\phi\, dx\, dy/\partial f/\partial z$ to be finite, it is necessary and sufficient that $\phi=(x,y,t)_{n-3}t^{n-1}$; so that the adjoint surfaces of order $2n-4$ contain the plane, $t=0$, $(n-1)$ times, and are otherwise cones with vertex at $x=y=t=0$. Hence also $p_g=\frac{1}{2}(n-1)(n-2)$. It is assumed that the curve $(x, y, t)_{2n}=0$ is without singular points. See Hodge, *Proc. Lond. Math. Soc.* xxx, 1930, pp. 133–40. A geometrical theory is given in the recent volume, Enriques-Campedelli, *Superficie algebriche*, 1932, p. 370.

Note III. Details of the algebra of a particular Cremona transformation of the Weddle surface and the Kummer surface. That there is a (1, 1) birational transformation between a Kummer surface and a Weddle surface is very well known (Cayley, *Papers*, VII, p. 133, Feb. 1870; Darboux, *Bull. Math. Astr.* I, 1870, p. 348; and other papers by Reye, R. de Paolis, Schottky); and this is immediately clear from Segre's generation of a Kummer surface (cf. *Proc. Lond. Math. Soc.* I, 1904, p. 249). Detailed algebra in close connexion with the fundamental binary sextic is given in the writer's *Mult.-periodic functions* (Cambridge, 1907), p. 39; an account is also given in Hudson's *Kummer's quartic surface* (Cambridge, 1905), pp. 160–8. The following is a self-contained simple account of a Cremona transformation between the surfaces.

Before proceeding to the formulae, it may be well to describe in general terms what the transformation is. In general, cubic surfaces through a sextic curve of genus 3 are ∞^3 in aggregate; and, as the residual curve intersection of two of these meets the sextic curve in 8 points, three of these cubic surfaces have a single free intersection; thus these cubic surfaces form a so-called homaloidal system, which, being taken to correspond to the planes of another space, determine a birational transformation; and it is easy to see that the reverse transformation is of the same character, determined by cubic surfaces through a sextic curve of genus 3. If we form a matrix, of three rows and four columns, of which each element is linear in the coordinates, the determinants of three rows and columns, in this matrix, determine four cubic surfaces having common such a cubic curve; and new coordinates, taken proportional to these four determinants, give such a transformation, expressible by three bilinear relations connecting the coordinates of the two spaces.

Thus, if a sextic of genus 3 can be taken on the Kummer surface, containing ten nodes of the surface, cubic surfaces through this sextic determine a transformation of the Kummer surface into another surface whose order is the number (12) of intersections of the Kummer surface with the variable cubic curve intersection of two of these cubic surfaces, less the number (8) of intersections of this cubic curve with the sextic curve; namely the new surface is of order 4. By considering the equations of the cubic surfaces for the neighbourhood of one of the ten nodes, supposed to be a simple point on these surfaces, we see that the node transforms into a line of the new surface; this new surface therefore has only six nodes.

In the actual transformation taken here, the base sextic curve on the Kummer surface consists of three conics, with one point (D) in common, a node of the surface, any two of these having a further common point, also a node of the surface (at A, or B, or C). The ten nodes of the surface which are transformed to lines of the Weddle surface are the nodes on these three conics other than A, B, C. The behaviour at A, B, C is exceptional, in that the cubic surfaces have a common tangent plane at each; and the behaviour at D is exceptional, in that the cubic surfaces have a quadric node, whose cones have three common generators. On the Weddle surface, the base sextic curve degenerates into a rational cubic curve and three coplanar lines, each meeting this curve once. That, in both cases, the composite sextic curve has genus 3, follows by applying the formula p_1+p_2+i-1 for the genus of a curve which consists of two curves of genera p_1 and p_2 with i common points.

In the space (ξ, η, ζ, τ), with $PQRO$ as tetrahedron of reference, O being (0, 0, 0, 1), we consider the *base-cubic-curve* which is given by taking ξ, η, ζ, τ respectively proportional to $p^2(qc-rb)$, $q^2(ra-pc)$, $r^2(pb-qa)$, pqr, wherein a, b, c are constants but p, q, r are any parameters subject to $p+q+r=0$. This curve contains the points (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0), lying respectively on the edges QR, RP, PQ. Through this curve, and these three edges, forming a composite sextic curve, pass ∞^3 cubic surfaces, whose system may be taken to have the equation

$$\lambda U\tau+\mu V\tau+\nu W\tau+\rho\Psi=0,$$

where λ, μ, ν, ρ are variable parameters, $U=0$ is the quadric surface, through the cubic curve, given by

$$U=a(a+b+c)\tau^2+[a(\eta-\zeta)+(b-c)\xi]\tau+\xi(-\xi+\eta+\zeta),$$

V, W being formed from this by cyclical changes, and $\Psi=0$ is the cubic surface given by $\Psi=(bc\xi+ca\eta+ab\zeta)\tau^2+\xi\eta\zeta$. This composite sextic base lies then also on the quartic surface $\Omega=0$, where

$$\Omega=(\eta c-\zeta b+6bc\tau)U\tau+(\zeta a-\xi c+6ca\tau)V\tau$$
$$+(\xi b-\eta a+6ab\tau)W\tau+[4e\tau-2(\xi+\eta+\zeta)]\Psi,$$

in which e is a further constant. The surface $\Omega=0$ is in fact a Weddle surface, being the locus of the vertices of the quadric cones containing the three points P, Q, R and three other points which we now define. Consider the *node-cubic-curve*, given by taking ξ, η, ζ, τ respectively proportional to $aq'r'(q'-r')$, $br'p'(r'-p')$, $cp'q'(p'-q')$, $p'q'r'$, in which p', q', r' are any parameters subject to $p'+q'+r'=0$. This curve contains P, Q, R, and can be shewn to meet the base cubic curve above in 4 points. Beside P, Q, R we consider the three points in which this node cubic curve is met by the plane $e\tau=\xi+\eta+\zeta$. It can be shewn that the equation of the general quadric surface through these six points is given by $xf_1+yf_2+zf_3+tf_4=0$, where x, y, z, t are variable parameters, $f_4=2\tau(e\tau-\xi-\eta-\zeta)$ and $f_1=3bc\tau^2+(\eta c-\zeta b)\tau+\eta\zeta$, while f_2, f_3 are obtained from this by cyclical changes. The conditions for this quadric surface to be a cone are four in number, and are linear both in ξ, η, ζ, τ and x, y, z, t. The equation $\Omega=0$ may easily be shewn to be the result of eliminating x, y, z, t from these four equations. Solving the first three of the four bilinear equations for x, y, z, t, however, we find $x/U\tau=y/V\tau=z/W\tau=t/\Psi$; and these equations are those of a Cremona transformation, leading, without reference to the equation $\Omega=0$, to $\xi/xL=\eta/yM=\zeta/zN=\tau/xyz$, where $L=\omega_2-\omega_3+xt$, etc., $\omega_1=(y-z)t+ayz$, etc. Thus, if x, y, z, t be coordinates in another space, with A, B, C, D as points of reference, D being $(0, 0, 0, 1)$, the Weddle surface is transformed by this Cremona transformation into another surface, whose equation is obtainable by eliminating ξ, η, ζ, τ from the four bilinear equations. The result is the symmetrical determinant whose rows consist of $0, z, y, u$; $z, 0, x, v$; $y, x, 0, w$; u, v, w, ϖ, in which $u=bz-cy-2t$, $v=cx-az-2t$, $w=ay-bx-2t$, $\varpi=6(bcx+cay+abz)+4et$. When expanded this is the same as $t^2X+2tY+Z^2=0$, in which

$$X=x^2+y^2+z^2-2yz-2zx-2xy,$$
$$Y=ayz(y-z)+bzx(z-x)+cxy(x-y)+exyz,$$
$$Z=ayz+bzx+cxy.$$

The order, 4, of this surface corresponds to the fact that the variable residual cubic intersection of two cubic surfaces of the family $\lambda U\tau+\mu V\tau+\nu W\tau+\rho\Psi=0$ meets the Weddle surface in 4 points not on the composite sextic base of the family. This surface is the Kummer surface, having 16 trope conics, of which 4 consist of $t=0$, $ayz+bzx+cxy=0$ together with 3 such as $x=0$, $\omega_1=0$.

Through the composite sextic base formed by the last 3 of these conics, there pass 4 cubic surfaces, belonging to the family given by $lxL + myM + nzN + kxyz = 0$, where l, m, n, k are variable, and $L = \omega_2 - \omega_3 + xt$, as above, etc. In the transformation, the three nodes of the Kummer surface at A, B, C, correspond to the nodes P, Q, R of the Weddle surface; the three other nodes of the Kummer surface which lie on the trope conic $t = 0$, $ayz + bzx + cxy = 0$, correspond to the nodes of the Weddle surface which are the intersections of the plane $e\tau = \xi + \eta + \zeta$ with the node cubic curve. The cubic surfaces $lxL + \ldots + kxyz = 0$ have a node at the point D, whose asymptotic cone contains the lines DA, DB, DC, and the residual variable sextic intersections of these surfaces with the Kummer surface all pass once through the 10 nodes of the Kummer surface other than those on $t = 0$. Each of these 10 nodes corresponds to a line on the Weddle surface (the intersection of a plane through 3 nodes of this surface with the plane through the other three nodes). The Weddle surface contains 15 other lines, each a join of two of its nodes; these correspond to the 15 trope conics of the Kummer surface other than that in $t = 0$; in particular, the lines QR, RP, PQ correspond to the trope conics in $x = 0$, $y = 0$, $z = 0$.

Ex. Verify that the 27 lines of the cubic surface $\Psi = 0$ are made up of three, say l, m, n, such as $\xi = 0 = \tau$, each counting four times; of three, say a_1, b_1, c_1, such as $\eta/b = -\zeta/c = \tau$, each counting twice; of three, say a_2, b_2, c_2, such as $\eta/b = -\zeta/c = -\tau$, each counting twice; and of three, say u, v, w, such as $\xi = 0$, $\eta/b = -\zeta/c$, each counting once. And specify the complete intersection of $U = 0$, $\Psi = 0$; and of $V = 0$, $W = 0$, $\Psi = 0$.

Note IV. Miscellaneous Examples.

1. For the incidental consequence of the theory of the invariant I, that, in a pencil of curves in a plane, of genus p, with base points at σ points of the plane, the number of curves of the pencil which have double points is $\sigma + 4p - 1$, compare Cremona, *Ann. d. Mat.* VI, 1864; Caporali, "Sopra i sistemi lineari triplamente infiniti di curve algebriche piane", *Collect. math. in mem. Chelini*, 1881, or *Mem. di Geom.*, 1888; also Guccia, *Rend. Palermo*, IX, 1895 (Mem. II).

2. For two pencils of plane curves, of orders n, n', the curve which is the locus of points of contact of a curve of one pencil with a curve of the other is of order $2n + 2n' - 3$, has a $(2s - 1)$-ple point at an s-ple base point of either pencil, and is of genus $4nn' + 6(p + p') - \sigma - \sigma' - 2$, where σ, σ' are the numbers of points of the plane at which the pencils have base points. The number of 3-pointic contacts of curves of the two pencils is $12(nn' + p + p') - 3(\sigma + \sigma') - 9$. See Steiner (for $\sigma = \sigma' = 0$), *Werke*, II, p. 500 (1848); Berzolari, *Atti... Torino*, XXXI, 1895–6. The number of double contacts of curves of the two pencils is $4N$, where

$$N = n^2 n'^2 + (nn' - 6)(p + p') + pp' - 7nn' + \sigma + \sigma' + 5.$$

3. For illustrations of the equation $I - e = I' - e'$ in the transformation of one plane into another, compare Cremona, *Mem. Accad. Bologna*, V, 1865, 2^a, § 5; Jung, *Ann. d. Mat.* XV, XVI, 1888, 1889 and *Rend. Ist. Lombardo*, XXI, 1888.

4. On an algebraic manifold of 3 dimensions, consider a linear pencil of surfaces, with base curves (simple or multiple) of genus p. Let δ be the number of these surfaces which have double points, and I the invariant for any one of these surfaces. Then the number $\delta-2p-2I$ is independent of the linear pencil (p. 214, above). For the case when the 3-fold manifold is ordinary space, this number is 2. Thus, in ordinary space, a pencil of surfaces of invariant I, with a base curve of genus p, has, in general, $2(I+p+1)$ surfaces with a double point outside the base curve. In particular, for general surfaces of order n, whose base is the complete intersection of two such surfaces, we have $I=(n-2)(n^2-2n+2)$, $p=n^2(n-2)+1$, and hence $\delta=4(n-1)^3$. For such a pencil of surfaces, with a single isolated k-ple point, Cremona gives (*Grundzüge einer allgem. Theorie der Oberflächen*, No. 119) $\delta=4(n-1)^3-2(k-1)^2(2k+1)$. Cf. Guccia, *Compt. rend.* CXX, 1895, p. 896; Pannelli, *Gior. d. Mat.* XLI, 1903, p. 97; XLII, 1904, p. 197; Pieri, *Gior. d. Mat.* XXIV, 1882.

5. For a net of irreducible curves, whose Jacobian curve is irreducible, on a surface of invariants I, p_n, the following are given by Severi, *Atti... Torino*, XXXVII, 1901–2. Let p be the genus of a curve of the net, n the grade, β the number of points of the surface at which the curves have base points. Then (a), in the net there are $24(p+p_n)$ curves which have cusps; (b), the number of curves of the net which have two double points is $\frac{1}{2}M$, where

$$M=(n+\beta+4p+I)^2-3n-\beta-78p-I-72p_n+2.$$

For example, for the number of planes from an arbitrary point which cut a general cubic surface in sections with 2 double points, we find 27 ($\beta=0=p_n$, $n=3$, $p=1$, $I=5$); (c), the number of pencils of curves of the net having two curves with 3-pointic contact is $3P$, where

$$P=n+6p-\beta-1+8p_n-2;$$

(d), the number of pencils having two curves with double contact is $2D$, where

$$D=(n+p)^2-17p-5n+2\beta+2I-18p_n+6.$$

Cf. Zeuthen, *Compt. rend.* LXXXIX, 1879 (2$^{\text{me}}$ sem.), pp. 899–901.

6. The canonical system of a surface has been taken here to have no *prescribed* simple base points. But, for example (Castelnuovo-Enriques, *Math. Ann.* XLVIII, 1897, p. 279; see also p. 281), on a quintic surface with two tacnodes, the canonical curves (by planes through the tacnodes) all have common the simple point of the surface on the join of the tacnodes; to preserve the equation $p^{(2)}=p^{(1)}-1$, this point must be counted in $p^{(2)}$. If the surface be transformed so that this point becomes a curve, the *reduced* canonical system of the new surface will have this as common part. Another case of this behaviour is a double plane of which the branch curve has the singularity described as two consecutive triple points (three ordinary branches with a common tangent); see Enriques-Campedelli, *Superf. alg.* pp. 389, 399. (Cf. also p. 220, above, footnote.)

CHAPTER VI

SURFACES AND PRIMALS IN FOUR DIMENSIONS. FORMULAE FOR INTERSECTIONS

The chord curve and trisecant curve for a surface in four dimensions. Rational surfaces. For surfaces ψ, in space of four dimensions, with no multiple points other than accidental double points, we have, in Chap. IV, found formulae in terms of four characteristics μ_0, μ_1, μ_2, ν_2, and, in Chap. V, we have added the formulae $I+4=\mu_2-2\mu_1+3\mu_0$, $12(p_n+1)=2\mu_2-8\mu_1+12\mu_0+\nu_2$. Conversely we can express the various numbers in terms of the four μ_0, p, I, p_n, where p is the genus of the prime section. Then we have

$$\mu_1=2\mu_0+2p-2,\quad \mu_2=\mu_0+4p+I,\quad \nu_2=2\mu_0+8p-4-2I+12p_n,$$

$$d=\tfrac{1}{2}(\zeta_0-\nu_2)=\tfrac{1}{2}(\mu_0-2)(\mu_0-3)-5p+I-6p_n,$$

$$\zeta_0=(\mu_0-1)(\mu_0-2)-2p,$$

$$t=\tfrac{1}{6}(\mu_0-2)(\mu_0-3)(\mu_0-4)-(\mu_0-8)p-I+8p_n,$$

$$2Q-2=2(\mu_0-2)(\mu_0-4)+4(\mu_0-10)p+6I-48p_n,$$

$$2P-2=(\mu_0-2)(\mu_0-5)+2(\mu_0-12)p+4I-30p_n;$$

here d is the number of double points, ζ_0 is the order of the curve in which the surface is met by the chords from an arbitrary point (*the chord curve*), t is the number of trisecant chords of the surface from an arbitrary point, Q is the genus of the chord curve, and P is the genus of the double curve, with t triple points, on the surface in space [3] into which the surface ψ projects. We have also denoted the order of this double curve by ϵ_0, and its rank by ϵ_1, using ζ_1 for the rank of the chord curve on the surface ψ, and have remarked that $\zeta_0=2\epsilon_0$, $\zeta_1=2\epsilon_1+\nu_2$, $2Q-2=2(2P-2)+\nu_2$. For the projected surface in [3] we also used three numbers, i, κ, ρ. Of these, i was the class of the developable formed by the stationary tangent planes of the surface, in general the same as the number of inflexional generators of the proper enveloping cone of the surface from an arbitrary point; it was given by $i=4(\mu_2-\mu_1)+2\nu_2$. The number κ was the number of inflexional tangents of the surface passing through an arbitrary point, in general the same as the number of cuspidal generators of the proper enveloping cone. The number ρ was the class of the developable formed by the tangent planes of the surface at points of the double curve. Expressed as above these are respectively

$$i=24(p_n+p),\quad \tfrac{1}{3}\kappa=\mu_0+6p-2-I+8p_n,$$

$$\rho=(\mu_0-2)(2\mu_0-5)+2(\mu_0-11)p+3I-24p_n.$$

For a ruled surface the formulae can be simplified by $p_n = -p$, $I = -4p$ (p. 206).

One incidental consequence of the equations, following from the facts that neither d nor t can be negative, is that

$$p \leqslant \tfrac{1}{6}(\mu_0 - 1)(\mu_0 - 2) + 2p_n/(\mu_0 - 3),$$

as we see by adding the forms for d and t. For a ruled surface this leads to $p \leqslant \frac{1}{6}(\mu_0 - 2)(\mu_0 - 3)$.

Another consequence of greater interest is that

$$2Q - 2 + 6t = \zeta_0(\mu_0 - 4).$$

This result we can now obtain in another way, by proving that canonical sets on the chord curve are obtained by primals of order $\mu_0 - 4$ passing through the $3t$ double points which this curve possesses (see Chap. IV, p. 172). We have in fact, in the preceding chapter (pp. 217, 222), by reasoning founded on the consideration of the Jacobian curve of a net of curves, which is equally applicable to a surface in space [4], defined the canonical system of curves, $|K|$, on a surface; and shewn that, with a linear system of curves $|C|$ without base, it gives a system $|K + C|$ defining canonical sets on any curve of $|C|$; the system $|K + C|$ is that called the adjoint system of $|C|$. Now, in the case in hand, if the surface ψ have a canonical system of curves, these will be of order $2p - 2 - \mu_0$; for, when added to the system of prime sections of the surface, they give canonical sets on a prime section. The adjoint system, of the system of curves on ψ defined by the chord curve, is obtained by adding the canonical system of the surface to curves of the system defined by the chord curve, and thus consists of curves of order $2p - 2 - \mu_0 + \zeta_0$. By the formula for ζ_0 this is the same as $\mu_0(\mu_0 - 4)$. Thus, sets of the canonical series on the chord curve are obtainable by its intersections with primals of order $\mu_0 - 4$, account being taken of the multiple points of the chord curve. This curve has $3t$ double points, intersections of branches of the curve which lie in the same sheet of the surface. It has also d double points, at the accidental double points of the surface, where branches of the curve, which lie in different sheets of the surface, cross one another; these latter points, on projection into space of three dimensions, give ordinary points of the double curve, not affecting the genus P of this, nor, therefore, the genus Q of the chord curve on ψ. Hence we have $2Q - 2 = (\mu_0 - 4)\zeta_0 - 6t$. The reader may compare the somewhat different interpretation of what is essentially the same relation, for the case of the double curve of the surface in [3], given in Chapter IV, p. 168, Ex. 11. The argument does not apply to surfaces without a canonical system of curves, in particular not to rational surfaces. But, with suitable modifications, numerical results such as this remain valid in many cases, with use of a fictitious (virtual) canonical system; and we have in fact obtained the present formula independently, from the preceding numerical results only.

We may also consider the trisecants of the surface which pass through a point P of the surface, each meeting the surface in two points beside P; the locus of these points is a curve on the surface which we may here call the *trisecant curve*. Denote the order of this curve by ζ_0', its genus by Q', and the number of quadrisecants of the surface which pass through P by t', these giving rise, as for the chord curve, to $3t'$ double points on the trisecant curve; also, let ν_2' be the number of lines from P which touch the surface elsewhere. By projecting from P, on to a space [3], we obtain from ψ a surface of order $\mu_0'=\mu_0-1$, with a double curve, the projection of the trisecant curve, of order ϵ_0', equal to the order of the conical sheet formed by the trisecants, and of genus P', with ν_2' pinch points. The trisecant curve has a multiple point at the point of ψ from which the trisecants are drawn, say of order k. To state the values of these numbers it is convenient to treat the case when the surface ψ is ruled apart from the general case. In the general case $k=\mu_0-4$ and

$$\zeta_0'=\zeta_0-\mu_0,\quad \epsilon_0'=\epsilon_0-(\mu_0-2),\quad t'=t-(\epsilon_0'-k),\quad \nu_2'=\nu_2-4,$$
$$2Q'-2+6t'=2(\mu_0-5)\epsilon_0',\quad 2(2P'-2)+\nu_2'=2Q'-2;$$

but when ψ is a ruled surface, $k=\mu_0-3$, and, using p for the genus of a prime section,

$$\zeta_0'=\zeta_0-\mu_0+1,\quad \epsilon_0'=\epsilon_0-(\mu_0-2),\quad t'=t-(\epsilon_0'-k),\quad \nu_2'=\nu_2-2,$$
$$2Q'-2=2(\mu_0-5)(\mu_0-3+2p),\quad 2P'-2=(\mu_0-6)(\mu_0-3+2p).$$

It seems unnecessary to give proofs of these formulae (a detailed consideration will be found in the *Proc. Camb. Phil. Soc.* XXVIII, 1932, p. 62); but some remarks may be made. In the case when ψ is not ruled, for ϵ_0', t', P', ν_2' we may consider the surface in space [3], obtained by projection from the point P, and apply thereto the Salmon formulae, with $p_n'=p_n$ and $I'-1=I$, because the point P becomes a line on the projected surface. The same is true when ψ is a ruled surface, except that $I'-1=I-1$, because the generator through P is an exceptional line of the surface ψ. Then Q' may be found by the Zeuthen correspondence formula. In the general case, sets of the canonical series on the trisecant curve are given by primals of order μ_0-5, cutting the surface so as to have a (μ_0-5)-ple point at the point P from which the trisecants are drawn, and also passing through the $3t'$ double points of the trisecant curve. The same is true for the case when ψ is a ruled surface, only then the generator through P is a fixed component of the complete adjoint system of the trisecant curve. In both cases, the proper adjoint system is given by primals of order μ_0-5, but there are absorbed at P $k(k-1)$ or $k(k-1-1)$ intersections, with k respectively μ_0-4 or μ_0-3. The value of Q' for the case when ψ is a ruled surface forms an interesting application of a formula for a curve on a ruled surface which was given by Segre: For a curve of order ν and genus π, which is a simple curve on a ruled surface of order n whose prime section is of genus p, this curve meeting every generator in k points $(k>1)$ and having the equivalent of d double points (neglecting those which may exist at accidental double points of the surface, arising from the nature of these points), we have

$$k(2p-2)=2\pi-2-\{(k-1)(2\nu-nk)-2d\}.$$

Now, it can be shewn that the trisecant curve of ψ, when this is a ruled surface, meets every generator in μ_0-3 points; and, as has been said, it has a point of multiplicity μ_0-3 at the point from which the trisecants are drawn. We can then put, in Segre's formula,

$$n=\mu_0,\quad k=\mu_0-3,\quad \pi=Q',\quad d=\tfrac{1}{2}(\mu_0-3)(\mu_0-4)+3t',\quad \nu=\zeta_0-\mu_0+1;$$

the result will be $2Q'-2+6t'=(\mu_0-5)(\zeta_0-2\mu_0+4)$, which leads to the value independently obtained for Q'.

Ex. 1. When the surface ψ is rational, its prime sections being represented on a plane by curves of given order with given multiple points, the formulae put down enable us to find the characters of the surface. For the value of p_n, equal to that of the plane, will be zero, and the value of I will be $-1+\sigma-e$, in which σ is the number of points of the plane which are base points of the plane system of curves, and e is the number of points of the surface (supposed simple) arising from exceptional curves of the plane. The formulae also enable us to obtain the curve in the plane which represents the chord curve of the surface, or that representing the trisecant curve. To obtain, for instance, the representation of the chord curve, when ψ is not ruled, supposing, for simplicity, that the system of representing plane curves has no fundamental curve, we may argue thus: Let the plane curves representing the prime sections be of order m, and their base points be of multiplicities denoted by k_i, so that the system of curves may be denoted by $c^m(k_i)$. Let the plane curve representing the chord curve be of order M, and have multiplicity K_i at the (k_i) base point of the system $c^m(k_i)$. An accidental double point of the surface ψ, which is a point intersection of two sheets, will correspond to two generally distinct points of the plane; these will lie on the curve representing the chord curve, and be such that every curve of the system $c^m(k_i)$ which passes through one of them also passes through the other. The three double points of the chord curve which lie on a trisecant of the surface through the vertex of the chord cone, will be represented by three double points of the plane curve which represents the chord curve, and will be such that every curve of $c^m(k_i)$ which passes through two of them will also contain the third. We may suppose that the plane curve representing the chord curve has no multiple points other than the $3t$ double points spoken of, and the multiple points (K_i) at the base points of the system $c^m(k_i)$. Canonical sets on the plane curve representing the chord curve correspond to canonical sets on the chord curve; and these, we have seen, are determined by primals of order μ_0-4 passing through the $3t$ double points of the chord curve. But, canonical sets on the plane curve representing the chord curve are obtained by curves of order $M-3$ passing K_i-1 times through a base point (k_i), and once through each of the $3t$ double points; a base point (k_i) corresponds, generally, to a curve of order k_i on the surface. Thus we infer, for each point (k_i), that $K_i-1=(\mu_0-4)k_i$. Further, as the curves of the surface which correspond to the lines of the plane are of order m, we also infer that $M-3=(\mu_0-4)m$. The curve in the plane which represents the chord curve of the surface ψ is thus one of a system which may be represented by $c^{3+(\mu_0-4)m}[1+(\mu_0-4)k_i, (2^3)_t]$, where the entry $(2^3)_t$ is to recall the three double points which correspond to each of the t trisecants of the chord curve. This result, for the representation of the double curve of a rational surface in ordinary space [3], was obtained by Clebsch, *Math. Ann.* I, 1869, p. 253. See also Zeuthen, *ibid.* IV, 1871, p. 26.

The representation of the trisecant curve in the plane is similarly given by a curve belonging to the system which may be denoted by

$$c^{3+(\mu_0-5)m}[1+(\mu_0-5)k_i, \mu_0-4, (2^3)_{t'}],$$

the trisecant curve having a multiple point of order μ_0-4.

These results assume that, for the plane curves representing the prime sections of the surface, the adjoint system of curves, which determine the canonical series on a curve of this representative system $c^m(k_i)$, are not reducible. But, if this system $c^m(k_i)$ have a fundamental curve, with base points furnishing independent conditions for the fundamental curve, then the adjoint system of $c^m(k_i)$ contains this fundamental curve as fixed part, and the canonical series on a curve of the system $c^m(k_i)$ is determined by the variable residual curves of this adjoint system. We have already illustrated this fact (Chap. V, p. 194), referring for the general proof to Castelnuovo, *Mem.... Torino*, XLII, 1892. If the fundamental curve be of character $c^\mu(\kappa_i)$, and its base points, supposed to be only at the points (k_i), be so independent, the preceding equations for the chord curve must evidently be modified to

$$M-3-\mu=(\mu_0-4)m, \quad K_i-1-\kappa_i=(\mu_0-4)k_i.$$

In particular this is true when ψ is a ruled surface. For the representation of the trisecant curve of the ruled surface, another correction arises: in this case, the generator of the ruled surface ψ, through the point from which the trisecants are drawn, is represented in the plane by a fundamental line of the curve representing the trisecant curve; this is additional to the fundamental lines of the representative system $c^m(k_i)$. For details we refer to the Note, *Camb. Phil. Soc. Proc.* XXVIII, 1932, p. 62. The result given in the following Ex. 2, is also proved in this Note.

Ex. 2. If the system of plane curves, representing the prime sections of a rational surface in space $[r]$, have an equation of the form

$$\phi(\lambda_0 Q_0+\ldots+\lambda_{r-2}Q_{r-2})+\lambda_{r-1}U+\lambda_r V=0,$$

and the curve $\phi=0$ be of genus q, then the surface contains a line corresponding to this curve, and this line is $(q+1)$-fold on the surface. A simple example is the quartic surface with a double line in ordinary space, whose prime sections are represented by plane quartic curves having one double base point and 8 simple base points. Another example is the quartic intersection of two quadrics in space of four dimensions, whose prime sections are represented by the conics in a plane which are outpolar to two flat pencils of lines. Examples of octavic surfaces with a double line, in space [5], are considered by D. W. Babbage, *Camb. Phil. Soc. Proc.* XXIX, 1933, pp. 95, 405; cf. Roth, *ibid.* p. 186. The five surfaces considered are those represented by the respective systems of plane curves $c^5(2^2, 1^9)$, $c^6(4, 1^{12})$, $c^6(2^6, 1^4)$, $c^7(3, 2^8)$, $c^9(3^8, 1)$, all the base points being taken on a plane cubic curve.

Ex. 3. We have found, in Chap. IV, p. 175, the values of $\mu_0, \mu_1, \mu_2, \nu_2$ for the surface which is the complete intersection of two primals of orders ρ, ρ', in space of four dimensions, namely $\mu_0 = \rho\rho'$, $\mu_1 = \mu_0(k+k')$, $\mu_2 = \mu_0(k^2+kk'+k'^2)$, $\nu_2 = \mu_0 kk'$, where $k = \rho - 1$, $k' = \rho' - 1$. We have given the values of I and p_n in terms of these characters at the beginning of this chapter, and we have $I + 4 + \omega - 1 = 12(p_n + 1)$. Hence prove that, for this complete intersection,

$$I + 4 = \mu_0[k^2 + kk' + k'^2 - 2k - 2k' + 3],$$

$$12(p_n + 1) = \mu_0(2k^2 + 3kk' + 2k'^2 - 8k - 8k' + 12),$$

and $\omega - 1 = \mu_0(k + k' - 3)^2$. This last is in accord with the fact that $\omega - 1$ is the grade of the canonical system, of which the curves are (as remarked above) of order $2p - 2 - \mu_0$, where p is the genus of the prime section, that is, of order $\rho\rho'(\rho + \rho' - 4) - \mu_0$, or $\mu_0(k + k' - 3)$. They are in fact given by primals of order $k + k' - 3$.

Intersections of loci in space of four dimensions. We now pass to some formulae* for the intersections of surfaces and primals, mainly, but not exclusively, in space [4]. These results may be provisionally classified as relating either to usual or to exceptional intersections. Thus, in space $[r]$, two algebraic manifolds of dimensions k and $r - k + s$ usually meet in a manifold of dimension s; and we may be required, given a manifold of dimension s, through which two manifolds of dimensions k and $r - k + s$ both pass, to find the characters of the remaining intersection of these, supposed to be of dimension s, and to find, when $2s > r$, the points of this lying on their given partial intersection. This is a problem of usual intersection. A simple case, already considered (Vol. V, Chap. VIII), is that of two surfaces in space [3] passing through a curve, when we are required to characterise the remaining curve common to these surfaces, and the points of this which lie on the former curve. But it may happen that two manifolds of dimensions k and $r - k + s$ have common a manifold of dimension greater than s, and we may be required to find the character of their remaining intersection, supposed to be of dimension s, and the points of this, if any, which lie on their given partial intersection. For instance, two surfaces in space [4] may have a common *curve*, and we may be required to find the number of common points of the surfaces which do not lie on this curve. Or, it may happen that several manifolds, in space $[r]$, of dimensions $k_1, k_2, \ldots$, have common a manifold of dimension greater than $k_1 + k_2 + \ldots - r$, and we may be required to find their remaining intersection of this dimension. We may provisionally classify such cases as relating to exceptional intersections.

---

* For the more difficult of the results special acknowledgment is due to Severi's paper, "Sulle intersezioni delle varietà algebriche e sopra loro caratteri e singolarità projettive", *Mem.... Torino*, LII, 1903—who refers to Caporali's previous results for space [4], *Mem. di Geom.*, Napoli (Pellerano), 1888.

In the first place, we deal with the cases which arise in space [4], though not entirely in logical order. In two respects the problems are more difficult than for higher space; first because the surfaces arising may have (accidental) double points, which we do not regard as usual in higher space; second because a primal, a manifold of dimension 3, passing through a surface, has generally isolated multiple points at ordinary points of the surface. This latter may be illustrated by the case when the given surface lies on two primals $U=0$, $V=0$, and the primal, through the surface, which is considered, has an equation of the form $uU+vV=0$; the first polars of this primal vanish, on the surface, where the four equations $uU_i+vV_i=0$ are satisfied ($U_i=\partial U/\partial x_i$ etc.); and the four equations $U=0$, $V=0$, $u=0$, $v=0$ have common solutions. We may also refer at once to a character of a curve which lies on a surface in space [4], of which generalisation is possible to higher cases, namely *the class of immersion* of the curve on the surface; this is the number of points of the curve at which the tangent plane of the surface meets an arbitrary line; it is also the order of the manifold, of three dimensions, described by the tangent planes of the surface at the points of the curve. By use of this character we may distinguish between curves of the same order lying on the surface. (Cf. p. 225, Ex. 5.)

Ex. On the ruled cubic surface of space [4], prove that a generator is of class of immersion 1, but the common transversal of the generators is of class of immersion 2.

I. Residual intersection of three primals with a common curve. Let three primals, of orders ρ, ρ', ρ'', have in common a curve of order ϵ_0 and class ϵ_1. Let their remaining intersection be a curve of order ϵ_0' and class ϵ_1'; it will be proved that this meets the former curve, say in i points, and that

$$\epsilon_0+\epsilon_0'=\rho\rho'\rho'', \quad \epsilon_1+i=\epsilon_0\sigma, \quad \epsilon_1'+i=\epsilon_0'\sigma,$$

where $\sigma=\rho+\rho'+\rho''-3$. From these ϵ_0', ϵ_1' and i can be found when, beside ρ, ρ', ρ'', we are given ϵ_0 and ϵ_1. The first of these equations is obvious. For the other two, consider the Jacobian primal of the three given primals and two arbitrary primes. This is expressed by the vanishing of a determinant of five rows and columns, of which the elements of any one of the three first rows are the derivatives, in regard to the coordinates, formed from the equation of one of the three primals, and the last two rows are similarly formed from the equations of the two arbitrary primes; these primes meet in a plane, say ϖ. This Jacobian is of order σ; evidently it vanishes at a point, common to the three given primals, at which the tangent line of their curve of intersection is such as to meet the plane ϖ; also, evidently, this Jacobian vanishes at a point which is common to the two curves into which the intersection of the three primals breaks up—for, at such a point, the tangent primes of the three primals, having common the plane of the tangent lines of the two curves, belong to a pencil of primes, and the minor determinants formed from the first three rows of the Jacobian all vanish. We assume that the intersections of the two curves are of general character, and that the Jacobian does not vanish on either at other points than those described. By considering the intersections of the Jacobian primal with each of the two curves in turn, we then obtain the second and third of the equations put down.

Cor. It follows that if, exceptionally, four primals, of orders $\rho, \rho', \rho'', \rho'''$, have common a curve (ϵ_0, ϵ_1), then they intersect further, in general, in a number of points given by $\epsilon_0'\rho''' - i$, that is, in $\rho\rho'\rho''\rho''' - \epsilon_0(\rho+\rho'+\rho''+\rho'''-3)+\epsilon_1$ points.

More generally, if three primals (ρ), (ρ'), (ρ''') have common an aggregate of m curves $(\epsilon_0^{(r)}, \epsilon_1^{(r)})$, of which the two curves $(\epsilon_0^{(r)}, \epsilon_1^{(r)})$, $(\epsilon_0^{(s)}, \epsilon_1^{(s)})$ have $i_{r,s}$ intersections, then the three primals have further common a curve (ζ_0, ζ_1), having j_r intersections with the curve $(\epsilon_0^{(r)}, \epsilon_1^{(r)})$, where, writing ϵ_0 for $\sum_1^m \epsilon_0^{(r)}$ and ϵ_1 for $\sum_1^m \epsilon_1^{(r)}$, with

$$\sigma = \rho+\rho'+\rho''-3, \quad \zeta_0+\epsilon_0 = \rho\rho'\rho'', \quad \zeta_1 = \sigma(\rho\rho'\rho''-2\epsilon_0)+\epsilon_1+2\sum_{r,s} i_{r,s},$$

$$j_r+\epsilon_1^{(r)}+(\sum_s i_{r.s} - i_{r,r}) = \sigma\epsilon_0^{(r)}.$$

And, if four primals, (ρ), (ρ'), (ρ''), (ρ''') have in common an aggregate of m curves $(\epsilon_0^{(r)}, \epsilon_1^{(r)})$, they meet further, in general, in a number of points specified by saying that the *point-equivalence* of their common curve is $(\rho+\rho'+\rho''+\rho'''-3)\epsilon_0 - \epsilon_1 - 2\sum_{r,s} i_{r,s}$.

Ex. 1. Three quadrics drawn through a rational quartic curve in space [4] meet further in a rational quartic curve, having 6 intersections with the former curve. Four quadrics through the original curve have two further common points. Conversely, through two rational quartic curves which have 6 common points, there can be drawn three linearly independent quadrics.

Through a rational quartic curve, in space [4], there can be drawn in all six linearly independent quadrics; if the quadratic functions giving the equations of these quadrics be supposed proportional to the coordinates in a space of five dimensions, the original space [4] is thereby represented by a primal, which, since two quadrics through the quartic curve have two further intersections, is a quadric, Ω, in this space [5]; a plane conic section of this quadric Ω, given by three primes of the space [5], represents then a quartic curve in the original space [4], having 6 points common with the fundamental quartic curve therein. It can be shewn that, in this representation, the neighbourhood of any point of this quartic curve, consisting of points lying in the ∞^2 planes through the tangent line of the curve at this point, is represented, in the space [5], by a plane, which then will lie on the quadric Ω; the aggregate of such planes, corresponding to all the points of the quartic curve, defines a manifold on Ω, of dimension 3, whose order, equal to the number of the component planes which meet three arbitrary primes of the space [5], is equal to the number of intersections of the fundamental quartic of the original space with the residual curve of intersection of three arbitrary quadrics through this fundamental curve; namely this order is 6. But this manifold V_3^6, of order 6 and dimension 3, on Ω, is in fact the intersection of Ω with a primal M_4^3, in the space [5], of order 3 and dimension 4. To see this we consider the representation in the space [5] of the chords of the quartic curve in [4]; any one of these chords lies on five of the quadrics containing the curve, and is represented by a single point of Ω; the surface formed by the ∞^2 points so arising is of order 4 because, as we may prove, two quadrics containing the quartic curve in [4] contain four chords of this curve; the ∞^1 chords of this curve which pass through one of its points are represented in the space [5] by a conic, there being two chords of the quartic curve through any one of its points which lie on a particular quadric containing the curve. The surface of order 4 representing the chords of the quartic curve is in fact a Veronese surface lying on Ω; and the M_4^3 spoken of contains the planes of conics thereon (of which we have spoken only of ∞^1). In the reverse transformation, any prime of the original space [4] is represented by an ∞^3 locus lying on Ω, whose order, equal to the number of points of this prime which lie on the residual curve of intersection of three quadrics through the original quartic curve in [4], is four. This locus is again the intersection of Ω with a primal, a quadric. And as the prime contains a point of any chord of the original quartic curve, we see that the primes of the original space [4] are represented by the intersections of Ω with the five quadrics, independent of Ω, which contain the Veronese surface. It should be added, however, that the transformation we have referred to, is derivable from the symmetric transformation between two spaces [5], in which the primes of either space represent the quadrics of the other space which contain a Veronese surface lying therein. It can be shewn that three quadrics through a Veronese surface meet in another Veronese surface, having an elliptic sextic curve common with the former Veronese surface. The neighbourhood of a point of the fundamental Veronese surface of one of the two spaces, in this transformation, corresponds to a plane of the other space; and this is the plane of a conic lying on the fundamental Veronese surface of this other space; etc. Cf. Severi, "Intersezioni...", *Mem. Torino*, LII, 1903, No. 24; also Semple, *Trans. Roy. Soc.* A, CCXXVIII, 1929, p. 351.*

---

* The theorem here twice exemplified, that on a quadric in space [r], an algebraic manifold of $r-2$ dimensions is the complete intersection with a single primal, was recognised and proved by Klein (1872), *Ges. Abh.* I, 153, with reference to Cayley, *Papers*, IV, 1860, p. 455; see also Noether, *Berl. Abh.* 1882, §§ 11, 12, for the curves on a general surface, of ordinary space, of order >3. In Klein's theorem it is necessary that $r>3$, and the quadric, if a cone, must be general enough to require at least 5 homogeneous coordinates to express its equation. For more general cases of primals in space [r] whereon any manifold of dimension $r-2$ is a complete intersection, see III, p. 295, below. For a surface on a general quadric of space [4], Klein's argument may be summarised as follows: The surface must meet the tangent prime T at any point O of the quadric Ω in a curve; and every point of this curve, lying on Ω and on T, is on the quadric conical sheet Σ in which T meets Ω. Let m be the number of points

Ex. 2. If three quadrics be drawn through three skew lines in space [4] —and therefore also through the common transversal line of these—they will meet further in a rational quartic curve, having 6 intersections in all with the four lines common to the quadrics. If four quadrics be drawn through the 3 skew lines (and their common transversal) they have 2 intersections common, not on any of the 4 given lines. In accordance with the formulae p_1+p_2+i-1 for the genus of a composite curve, the 3 skew lines and their common transversal make a degenerate quartic curve of genus zero, and the results may be regarded as cases of Ex. 1.

Ex. 3. Four quadrics through a conic in space [4], and through four lines each meeting the conic once, the lines consisting of two pairs each of intersecting lines, meet further in $16-5.6+2.6+2$ points, that is, not at all.

Ex. 4. Consider 10 lines in [4], of which each meets 3 others, there being 15 intersections in all. Then 4 cubic primals through the 10 lines have $81-9.10+30$, or 21 external intersections. Cf. Salmon, *Higher Algebra*, 1885, p. 298. Find the number of quadric surfaces in ordinary space which can be drawn through 5 given points to touch 4 given planes. Cf. Schubert, *Abzähl. Geom.* 1879, p. 105; and Ursell, *Proc. Lond. Math. Soc.* xxx, 1930, p. 322.

Ex. 5. In space $[r]$, if $r-1$ primals, (ρ_i), meet in two curves (ϵ_0, ϵ_1) and $(\epsilon_0', \epsilon_1')$, we have, as for the case $r=4$, for the number i of the common points of these two curves $\epsilon_1+i=\epsilon_0\sigma$, $\epsilon_1'+i=\epsilon_0'\sigma$, where $\sigma=\Sigma(\rho_i-1)$. Cf. Veronese, "Behandlung, u.s.w.", *Math. Ann.* xix, 1882, No. 37.

II. Residual intersection of two surfaces with a common curve. To find the point equivalence of the curve in which two surfaces in space [4] exceptionally intersect, namely the number of external points common to these surfaces.

Let the surfaces be of respective orders μ_0, μ_0'; and the curve in which they intersect be of order ϵ_0 and class ϵ_1. Suppose that the tangent plane of either surface at a point of their common curve does not touch this surface at any other point of this curve; and that the tangent planes of the two surfaces at a point of this curve define a definite prime. Let the order of the three-fold constituted by the tangent planes of one surface at the points of the common curve be z, the corresponding number for the other surface being z'; also denote by X the number of the primes, each defined by the tangent planes of the two surfaces at a point of the common curve, which pass through an arbitrary point; we may call this the class of the manifold constituted by these primes. We suppose z, z' to be given, as well as μ_0, μ_0', ϵ_0, ϵ_1; and prove that the point equivalence of the

in which this curve meets any generator of Σ. The order of the surface, equal to the number of its intersections with a plane of the space [4], is equal to the number of its intersections with a plane through O lying in T; and such plane contains two generating lines of Σ; thus the order of the surface is $2m$. Projected from O we thus obtain, in an arbitrary three-fold space Π, a surface of order $2m$ having the conic ω, in which Σ meets Π, as an m-fold curve; such a surface is entirely defined by a single equation. Expressing that this surface has ω as an m-fold curve, this equation is at once seen to be that of the intersection of Ω with a primal.

common curve is $X+\epsilon_0$, and that $X=z+z'-\epsilon_1$. Thus the number N, of external points common to the two surfaces, is given by $\mu_0\mu_0'=N+X+\epsilon_0$.

We first prove, in two ways not essentially independent, that $X+\epsilon_1=z+z'$. For the former proof, consider the dual figure in which the tangent planes of the first surface, ψ, at the points of the common curve, c, are replaced by the generating lines of a ruled surface, ρ; and, likewise, the tangent planes of the second surface, ψ', at the points of this common curve, c, are replaced by the generating lines of a second ruled surface, ρ'; then any generating line of ρ will have a point of intersection with a proper corresponding line of ρ'. Of the first surface ρ, there will be z generating lines meeting an arbitrary plane, namely ρ is of order z; and, similarly, ρ' is of order z'. Of the curve locus of the intersection of a generator of ρ with the corresponding generator of ρ', there will be X points in an arbitrary prime, namely this curve is of order X. Further, as there are ϵ_1 tangent lines of the curve c, or (ψ, ψ'), which meet an arbitrary plane, there will be ϵ_1 of the planes, each defined as containing a generator of ρ and the corresponding intersecting generator of ρ', which meet an arbitrary line in the dual figure; namely, the planes of these pairs of intersecting generators, of ρ and ρ', constitute a three-fold of order ϵ_1. Now take a section of this dual figure by an arbitrary prime; in this space we shall then have two curves, of orders z and z', the sections of ρ and ρ', these curves being in $(1, 1)$ correspondence, with X coincidences of corresponding points; and the joins of corresponding points of these curves will give a ruled surface of order ϵ_1. Whence, by a well-known theorem of space [3], we have $\epsilon_1=z+z'-X$, as was stated. For the alternative proof spoken of, effectively the dual of the former, take an arbitrary plane, ϖ, and therein an arbitrary point, O; any line through O, in the plane ϖ, is met by z tangent planes of the surface ψ, whose points of contact are on the curve c; and the tangent planes of ψ' at these points meet the plane ϖ in z points, which can be joined to O; in this way to every line through O in the plane ϖ correspond z other lines; reversely, any one of these lines is met by z' tangent planes of ψ' whose points of contact are on c, and the tangent planes of ψ at these points give z' points of ϖ which can be joined to O. Thereby we define a (z', z) correspondence between rays of the pencil of lines through O in the plane ϖ, in which therefore there are $z+z'$ coincidences of corresponding rays. Such a coincidence can arise in two ways; either because the tangent planes of ψ, ψ' at a point of the curve c meet ϖ in the same point, namely for the ϵ_1 cases in which the tangent line of the curve c meets the plane ϖ; or because the prime determined by the tangent planes of ψ and ψ' at a point of the curve c contains the point O, in which case the line in which this prime meets ϖ intersects two corresponding tangent planes of ψ and ψ'; and this occurs in X cases. Thus as before we have the result $z+z'=\epsilon_1+X$. We next prove the formula

$$\mu_0\mu_0'=N+X+\epsilon_0.$$

For this we recur to the consideration, given in a preceding chapter (II, p. 92), of the correspondence between two surfaces, in which to any point of one surface, say ψ, of order μ_0, corresponds every point of another surface, say ψ', of order μ_0', and conversely. For this we have proved the formula $\mu_0\mu_0'=\epsilon(0, 4)(4)+\epsilon(1, 4)(3)$, where, it will be recalled, the symbol ϵ denotes a coincidence of corresponding points, namely, here, refers to a common point of the surfaces ψ, ψ'; and, in $\epsilon(0, 4)(4)$, the (4) indicates that we consider such a common point lying anywhere in the space $[4]$, while the $(0, 4)$ indicates that the ultimate join of two corresponding points, which are to coincide at the common point of the two surfaces, passes through an arbitrary point of the space $[4]$. Such a coincidence of corresponding points arises at an isolated common point of the two surfaces, any line through this point being a possible position for the ultimate join of the two points which coincide there; it also arises at a point of the common curve c of the two surfaces, if the prime, determined by their tangent planes at this point, passes through the arbitrary point of the space $[4]$. Thus the term $\epsilon(0, 4)(4)$ has the value $N+X$. In $\epsilon(1, 4)(3)$, the (3) indicates coincidences lying on an arbitrary prime of the space $[4]$, and the $(1, 4)$ indicates that the ultimate join of the two coinciding points meets an arbitrary line of the space $[4]$; now an arbitrary prime will contain a common point of ψ and ψ' only at the ϵ_0 intersections of this prime with the curve c; ultimate joins of coinciding points, for such a point, are all the lines through this point which lie in the prime determined by the tangent planes of ψ and ψ' at this point; and there is one such line passing to the point in which this prime is met by the arbitrary line of the space $[4]$, and no other such meets this last line. Thus $\epsilon(1, 4)(3)$ is ϵ_0. On the whole then we have, as stated, $\mu_0\mu_0'=N+X+\epsilon_0$ (cf. p. 102, Ex. 7).

Ex. 1. As a simple example, suppose ψ is the surface of order 4 obtained by projecting the Veronese surface into space $[4]$, and ψ' is a plane meeting this surface in a conic. Here z', the number of tangent planes of ψ' which meet an arbitrary line, is zero; to obtain z, we may assume that the tangent planes of the projected Veronese surface are duals of the chords of a rational quartic curve in $[4]$; then, the tangent planes of this surface ψ at points, of a conic section, are duals of chords of the quartic curve which meet a (properly taken) line; such chords are known to determine an involution on the curve, and to generate a ruled surface of order 3. Whence $z=3$. These are in accord with the remark made in Ex. 5 (p. 225) of Chap. V, the grade of the conic being respectively 4 and 1 on the plane and on the surface. This leads to $X=3+0-2$ or $X=1$. This is equivalent to saying that the primes, through the plane of the conic section ψ, each containing the tangent plane of the surface at one point of the conic, consist of all the primes through this plane. From $X=1$, we have $N=\mu_0\mu_0'-X-\epsilon_0=4-1-2=1$. Thus the plane of a conic section of the projected Veronese surface meets the surface in one point not on the conic. This follows also from the fact that a solid, or space [3], drawn through the plane of a conic section of the Veronese surface in space [5], has one further intersection with the surface; by such solids the surface is projected on to a plane.

Ex. 2. Another simple example is given by taking ψ to be the Segre quartic surface which is the intersection of two quadrics in [4], and ψ' to be a plane containing a line of the surface ψ. Here, as before, $z'=0$; but z, the number of tangent planes of ψ, at points of the line, which meet an arbitrary line, is 2; for the tangent primes, of the two quadrics which meet in ψ, at points of the line on ψ, meet an arbitrary line in related ranges. In fact the line is of grade 1 on the plane, but of grade -1 on the surface. Wherefore $X, =z+z'-\epsilon_1=2$, and $N=\mu_0\mu_0'-X-\epsilon_0=1$. Thus the plane meets the surface in one point not on the line. By such planes the surface is projected on to a plane.

Ex. 3. A plane through a generator of a ruled surface in [4], of order μ_0, meets the surface again in μ_0-2 points.

Ex. 4. The formula $\mu_0\mu_0'=N+X+\epsilon_0$ may be employed to obtain the formula (III), p. 159, of Chap. IV. For this, we take the surface ψ' to be the cone of chords, drawn from an arbitrary point O, to the surface ψ, of order μ_0, so that μ_0' is what is called ϵ_0 in Chap. IV, and the ϵ_0 of the formula here is what is called ζ_0 in Chap. IV, the order of the chord curve. Thus the formula is $\mu_0\epsilon_0=N+X+\zeta_0$ in the notation of Chap. IV. There is an isolated intersection of the surface ψ with the cone of chords at each of the intersections with ψ of the trisecants from O, as we have seen, Chap. IV, p. 172; or $N=3t$. To find X, the number, through an arbitrary point, P, of primes which are each defined by the tangent plane of ψ at a point of the chord curve, and the tangent plane of the chord cone at this point, we notice that, as O, on all the tangent planes of the chord cone, lies in all such primes, when P is in such a prime, the line OP will meet the tangent plane of ψ by which the prime is defined; and conversely. Thus, on projection from O into space [3], X is the number of tangent planes, of the surface obtained by projection, at a point of the double curve of this, which pass through an arbitrary point of the space [3], namely the point where OP meets this space. Thus X is the number denoted in Chap. IV by ρ. Thus we have $\mu_0\epsilon_0=3t+\rho+\zeta_0$, or, as $\zeta_0=2\epsilon_0$, we have $\mu_0(\epsilon_0-2)=\rho+3t$, which is the formula (III) in question. We may note, however, that, when we consider the two surfaces consisting of the surface ψ and the chord cone, the formula $X+\epsilon_1=z+z'$ is to be replaced by $X+\zeta_1=z+2z'$, since a tangent plane of the chord cone has *two* contacts with the chord curve. Recurring to the argument given above, the correspondence in the pencil of rays in the plane ϖ is of indices z and $2z'$. Here $z=\rho+\nu_2$, and z' is $2\epsilon_0+2P-2$, or ϵ_1, where ϵ_0, ϵ_1, P refer to the double curve of the projected surface. Thus we have $X+\zeta_1=\rho+\nu_2+2\epsilon_1$, in agreement with $X=\rho$, $\zeta_1=2\epsilon_1+\nu_2$ (IV, p. 174).

Ex. 5. If a curve (ϵ_0) lie on a surface (μ_0) in space [4] which has d accidental double points, at each of which the curve has a four-fold point, then the conical sheet of lines, through an arbitrary point O of the space, to points of the curve, has a number, N, of isolated intersections with the surface, other than on the curve, given by $N=\epsilon_0(\mu_0-1)-z-4d$, where z is the class of immersion of the curve on the surface, namely the number of tangent planes of the surface, at points of the curve, which meet an arbitrary line. This follows as in the last example, from $N=\mu_0\mu_0'-\epsilon_0-X$, replacing μ_0' by ϵ_0 (the order of the cone) and X by z. Or, if we project from O on to a space [3], we are required to find the number of intersections, N, of the double curve of the surface in [3] with the curve, thereon, obtained by projection of the given curve (ϵ_0). This latter curve is such that z of the tangent planes of the projected surface at points of this curve, pass through an arbitrary point, say P', of the space [3]; namely, z is the number of intersections of this curve with the first polar of P', of order μ_0-1, other than those (N) on the double curve, and those $(4d)$ on the double curve, where the projected curve has a four-fold point.

III. Residual intersection of a primal and surface having a common curve. Now consider the residual curve intersection of a surface ψ and a primal Π, which intersect in a given curve, supposing that this curve has no multiple point. For ψ the usual characters are μ_0, μ_1; for Π we have the characters ρ_0, ρ_1, of which ρ_0 is the order, and ρ_1 is the order of the surface formed by points of contact of tangent primes of Π which pass through an arbitrary point, so that $\rho_1=\rho_0(\rho_0-1)$. The given curve of intersection, c, of ψ and Π, is of order ϵ_0, and rank ϵ_1, and has a class of immersion z in ψ, and a class of immersion Z in Π, the latter being the number of tangent primes of Π at points of c which pass through an arbitrary point. Thus, when the surface ψ, and the curve therein, are given, ϵ_0, ϵ_1 and z are definite; and Z is the number of intersections of the curve with the first polar of an arbitrary point taken in regard to Π. For the residual curve of intersection c', of ψ and Π, we denote the characters by ϵ_0', ϵ_1', z', Z'; and the number of its intersections with c by i. These unknown characters are to be found from the formulae

$$\epsilon_0+\epsilon_0'=\mu_0\rho_0,\quad \epsilon_1+i=z+Z,\quad \epsilon_1'+i=z'+Z',$$
$$z+z'=\mu_1\rho_0,\quad Z+Z'=\mu_0\rho_1,$$

of which one can be replaced, if desired, by

$$\epsilon_1+\epsilon_1'+2i=\mu_0\rho_1+\mu_1\rho_0.$$

The formula $\epsilon_0+\epsilon_0'=\mu_0\rho_0$ is obvious. The formula $z+z'=\mu_1\rho_0$ follows because the locus of points of contact of tangent planes of ψ which meet an arbitrary line is, by definition, a curve on ψ of order μ_1; and this meets Π in $\mu_1\rho_0$ points; while all points of the complete intersection (ψ, Π), at which the tangent plane of ψ meets the arbitrary line, are so obtainable. Similarly for $Z+Z'=\mu_0\rho_1$, the points of Π, whereat the tangent prime passes through an arbitrary point form a surface of order ρ_1, meeting ψ in $\mu_0\rho_1$ points. To obtain the formula $\epsilon_1+i=z+Z$, we use formulae of correspondence developed in an earlier chapter (Chap. I). Let ϖ be an arbitrary plane. This is met, by a tangent plane of ψ at a point of the curve c, in a point, say F, and, by the tangent prime of Π at the same point of c, in a line, say l; let L denote any point of l. We consider, in ϖ, the ∞^2 pairs of corresponding points in which to a point F corresponds any point L of the line l; every point F lies on a definite curve, γ, of the plane ϖ, that in which this plane ϖ is met by the tangent planes of ψ at the points of the curve c on ψ; and we assume, ϖ being an arbitrary plane, that, to every point of γ corresponds just one point of c, and hence a definite line l of points L; conversely, every point L in ϖ lies on the tangent prime at one of Z points of c, and each of these gives a definite point F. Coincidences in this correspondence arise when F lies on the associated line l. This may happen in two ways: Either, because the tangent line of the curve c, which is generally the intersection of the tangent plane of ψ with the tangent prime of Π at a point of c, meets the plane ϖ; and the number of such cases is ϵ_1; or, because the tangent prime of Π, at a point of c, contains the tangent prime of ψ at this point; this happens at one of the i intersections of the curves c, c', and does not happen otherwise if c have no double point. Thus the total number of coincidences in the correspondence is, in general, ϵ_1+i. On the other hand, it was shewn, in the chapter referred to, that the number of coincidences is $(0)(2)'+(1)(1)'+(2)(0)'$, where $(m)(n)'$ means the number of existing pairs of corresponding points in which F lies in an arbitrary space $[m]$, of the plane ϖ, and L lies at the same time in another arbitrary space $[n]$. Thus, here, the term $(0)(2)'$ vanishes, there being positions of F only on the curve γ of the plane ϖ. The term $(1)(1)'$ has a contribution when F is on an arbitrary line, and thus at one of the z intersections of γ with this line, while L lies on an arbitrary line; this latter will meet the line l, associated with F, in one point; whence $(1)(1)'$ is z. The term $(2)(0)'$ has a contribution when L is at an arbitrary point, the corresponding point F being anywhere in the plane ϖ; wherefore $(2)(0)'=Z$. Hence we have the result $\epsilon_1+i=z+Z$; and the proof of $\epsilon_1'+i=z'+Z'$ is similar.

Ex. 1. For a curve which is the complete intersection of a surface (μ_0, μ_1) and a primal (ρ_0, ρ_1), the characters are given by $\epsilon_0=\mu_0\rho_0$, $\epsilon_1=\mu_1\rho_0+\mu_0\rho_1$, $z=\mu_1\rho_0$, $Z=\mu_0\rho_1$, where $\rho_1=\rho_0(\rho_0-1)$.

Ex. 2. If a primal, of order $\rho+1$, be put through the curve which is a prime section of a surface ψ, or (μ_0, μ_1), we may find the characters of the residual curve of intersection of the primal with the surface. The order and rank of the prime section, and the class of its immersion in ψ are given by $\epsilon_0=\mu_0$, $\epsilon_1=\mu_1$, $z=\mu_1$; the class of immersion, of the prime section of ψ, in the primal of order $\rho+1$, is the number of intersections of this curve with the first polar of a point in regard to this primal. Hence, from $\epsilon_0'+\epsilon_0=\mu_0(\rho+1)$, $z+z'=\mu_1(\rho+1)$, $Z+Z'=\mu_0(\rho+1)\rho$, we have $\epsilon_0'=\mu_0\rho$, $z'=\mu_1\rho$, $Z'=\mu_0\rho^2$; while, from $\epsilon_1+i=z+Z$, $\epsilon_1'+i=z'+Z'$, we have $i=\mu_0\rho$, $\epsilon_1'=\mu_1\rho+\mu_0\rho(\rho-1)$. Thus, as was to be anticipated, ϵ_0', ϵ_1', z' are the same as for a complete intersection of the surface with a primal of order ρ. Test the formulae similarly, for the residual intersection with ψ, of a primal of order $\rho+\rho'$, put through the complete intersection of ψ with a primal of order ρ.

Ex. 3. Consider the modifications arising in the formulae when the given curve common to ψ and Π has multiple points.

IV. Residual intersection of two primals having a surface common. We consider now the residual intersection of two primals passing through the same surface. This is a more laborious task than the preceding, and the results obtained will be stated at the conclusion, when the notation has become familiar.

We consider a general surface ψ, in space [4], with characters μ_0, μ_1, μ_2, ν_2 and d accidental double points. Through this surface we put two primals Π, Π', of orders ρ, ρ', which intersect further in a surface ψ'. We shew that ψ' has double points, in general, only at the d double points of ψ, these being equally accidental for ψ'; also that the surfaces ψ, ψ' have no isolated intersections, their common points lying on a curve, c; of this curve the characters are denoted by ϵ_0, ϵ_1. We find these, as also the characters μ_0', μ_1', μ_2', ν_2' of ψ'; and the numbers δ, δ' of double points of Π and Π' which occur at simple points of ψ (and therefore of double points of Π, Π' at simple points of ψ').

That an accidental double point of ψ is an accidental double point of ψ' is clear enough when the tangent planes of the sheets of ψ at this point have only a point intersection and do not meet in a line. A general line through this point meets ψ in two points coincident thereat (is the limit of a chord of the surface), and so meets Π in two coincident points; or Π has a double point at this point; but a line through the point in one of the two tangent planes of ψ has a higher intersection with Π, and so lies on the osculating cone of Π at this point. Thus the tangent planes of ψ are planes, of the same system, of this osculating cone of Π; and the same may be said for Π'. The two osculating cones have therefore in common the two tangent planes of ψ at the double point; thus, being quadric cones, as is supposed, they also have common two other planes, both of the opposite system, likewise meeting only in this double point. These are then tangent planes of the residual intersection, ψ', of Π and Π', which therefore has also a double point, of accidental character. But, to meet the possibility of the tangent planes of ψ at the double point having a line in common, we may also proceed differently, and shew that the rank of the curve section of ψ', by a prime through the double point of ψ, is equal to the rank of the section of ψ' by any other general prime. From this, as, by the preceding argument, ψ' certainly has a double point at the point, it will follow that this is an accidental double point for ψ'. Denote the double point of ψ by D; denote by k the curve in which

ψ is met by a general prime through D; let the surfaces in which this prime meets the primals Π, Π' be called ϕ and ϕ'. Form the Jacobian, in this prime, of these two surfaces ϕ, ϕ', and of two arbitrary planes lying in this prime. The curve k has D for a double point, and (because ψ lies on both Π and Π') lies on both ϕ and ϕ'; we can shew that the Jacobian, say J, has D for a double point. Thus J has four intersections with k at D. Its other intersections with k are at the points of this where the tangent line meets the line of intersection of the two arbitrary planes taken in the prime, and at the points of this where the surfaces ϕ, ϕ' touch one another. The number of intersections of the former kind, by the property of an accidental double point, is the same as would be the case for an arbitrary prime section of ψ, and is μ_1. As the Jacobian is of order $\rho+\rho'-2$, we infer that the surfaces ϕ, ϕ', and hence also the primals Π, Π', touch, in this prime, other than at D, in

$$\mu_0(\rho+\rho'-2)-\mu_1-4$$

points. Now, when two surfaces of orders m, m', in space [3], intersect in two curves of orders n_1, n_2, and ranks r_1, r_2, with i common points, we have

$$n_1+n_2=mm', \quad i+r_1=n_1(m+m'-2), \quad r_1r_2=(n_1-n_2)(m+m'-2);$$

hence, considering the curves in which ψ and ψ' are met by a quite general prime, we have $\mu_1'-\mu_1=(\mu_0'-\mu_0)(\rho+\rho'-2)$, and these curves meet in $\mu_0(\rho+\rho'-2)-\mu_1$ points. This number, say ϵ_0, is the order of a curve, say c, of intersection of the surfaces ψ, ψ'; at points of this curve the primals Π, Π' have a common tangent prime, determined by the tangent planes of ψ, ψ'. Thus, the result obtained above may be expressed by saying that, in any prime through D, the primals Π, Π' touch, outside D, in ϵ_0-4 points. Such points are the intersections, not at D, of the curve c, common to ψ, ψ', with this prime. Thus, this curve, of order ϵ_0, passes through D, and has thereat a four-fold point. Now consider, in the same prime through D, the curve k' in which this prime meets the surface ψ', and the intersections of k' with the same Jacobian of ϕ, ϕ' and two planes. We have seen that the common curve of ψ and ψ' is met in four points, coinciding at D, by any prime though this point; thence, as k has there a double point, it follows that k' has also a double point at D; thus the Jacobian, which has a double point at D, meets k' in four points at D. It follows, then, as before, that the primals Π, Π' have contacts, not at D, in this prime, whose number is $\mu_0'(\rho+\rho'-2)-(\mu_1')-4$, where (μ_1') denotes the rank of the curve in which this prime meets ψ'. This number is then equal to the former, $\mu_0(\rho+\rho'-2)-\mu_1-4$. Whence, from the equation $\mu_1'-\mu_1=(\mu_0'-\mu_0)(\rho+\rho'-2)$, we infer that the rank (μ_1'), of the section of ψ' by a prime through D, is the same as the rank μ_1 of a section by an arbitrary prime. Hence D is an *accidental* double point of ψ'. A similar argument to the foregoing shews that the surfaces ψ, ψ' have no point in common which is not on the curve c. For, if D be a point common to ψ and ψ', which is an ordinary point of both, and is either simple for both Π and Π', or simple for one of these and double for the other, we prove in the same way that an arbitrary prime through this point D contains ϵ_0-1 other points where Π and Π' touch; thus the curve c, common to ψ and ψ', passes once through this point.

As the surfaces ψ, ψ' are thus proved to have the same accidental double points, it follows, from the formulae for the number of these, such as $\nu_2+2d=\mu_0(\mu_0-1)-\mu_1$, and the formula for $\mu_1'-\mu_1$ given above, that $\nu_2'-\nu_2=(\mu_0'-\mu_0)(\rho-1)(\rho'-1)$. Thus μ_0', μ_1', ν_2' are found. Next, if z, z' be the classes of immersion of the curve c, common to ψ, ψ', in these, respectively, and X the class of immersion of c in Π and Π', which have the same tangent prime at a point of c, determined by the tangent planes of ψ and ψ' at this point, it follows, as in a preceding case (pp. 252, 253, above) that* $\mu_0\mu_0'=X+\epsilon_0$, $z+z'=X+\epsilon_1$. Further, we have

$$\mu_2+\nu_2+z=\mu_1(\rho+\rho'-2).$$

For consider, on the surface Ψ, the curve of order μ_1, say θ, which is the locus of the points of contact of tangent planes of ψ which meet an arbitrary line, say l. This defines two series of primes, each ∞^1 in aggregate, in (1, 1) correspondence, namely the tangent primes of Π and Π' at points of this curve θ. Of these series of primes, respectively $\mu_1(\rho-1)$ and $\mu_1(\rho'-1)$ pass through an arbitrary point, as we see by taking the first polars of this point in regard to Π and Π'. Also, there are $\mu_2+\nu_2$ tangent planes of ψ, at points of the curve θ, which meet a further arbitrary line, say m; this we may see by taking the line m so as to intersect l; then there are μ_2 tangent planes of ψ which meet the plane (l, m) in a line, and so meet l, so that their points of contact are on θ; and there are ν_2

---

* The former of these gives an independent proof of Severi's formula for the number d of double points of the surface ψ. For, consider a surface which is the intersection of the first polars of an arbitrary point O in regard to the primals Π and Π'. This surface, of order $(\rho-1)(\rho'-1)$, meets ψ in $\mu_0(\rho-1)(\rho'-1)$ points, consisting of (i), the X points of the curve c whereat the common tangent prime of Π and Π' is such as to pass through O; (ii), the ν_2 points of ψ whereat the tangent plane of ψ, common to the tangent primes of Π and Π', passes through O; (iii), the accidental double points of ψ, which, we have seen, are double points for Π and Π', and lie in the first polars. Thus we have

$$X+\nu_2+2d=\mu_0(\rho-1)(\rho'-1);$$

this gives $\nu_2+2d=\mu_0(\rho-1)(\rho'-1)-\mu_0\mu_0'+\epsilon_0$; but we have $\mu_0'=\rho\rho'-\mu_0$, and, by an arbitrary prime section, we have $\epsilon_0=\mu_0(\rho+\rho'-2)-\mu_1$. Whence

$$\nu_2+2d=\mu_0(\mu_0-1)-\mu_1.$$

tangent planes of ψ which pass through the point (l, m); thus there are $\mu_2+\nu_2$ tangent planes of ψ which meet two arbitrary lines. Further, among the pairs of tangent primes of Π and Π' at points of the curve θ, there are z coincidences; for, we assume, coincidences of tangent primes of Π and Π' arise only at points of the curve c, or (ψ, ψ'); and the number of points of this curve at which the tangent plane of ψ meets the line l is z; these points are then on the curve θ. Now consider the dual figure; dual to the two series of primes, we shall have two curves, in (1, 1) correspondence, of respective orders $\mu_1(\rho-1)$ and $\mu_1(\rho'-1)$, with z common corresponding points; and the ruled surface formed by joins of corresponding points is of order $\mu_2+\nu_2$. Hence the formula $\mu_2+\nu_2+z=\mu_1(\rho+\rho'-2)$ follows, as an application of a well-known result. From this z can be found. The corresponding formula for ψ' is $\mu_2'+\nu_2'+z'=\mu_1'(\rho+\rho'-2)$, from which μ_2' could be found if z' were known; we have as yet, however, only obtained $z+z'=X+\epsilon_1, =\mu_0\mu_0'-\epsilon_0+\epsilon_1$. We proceed then further, and first prove that $X+4d+\delta=\epsilon_0(\rho-1)$, where δ is the number of double points of the primal Π which arise at simple points of ψ. Such a point, as lying on ψ, lies on Π', and counts doubly among the points common to Π and Π', which are the points of ψ and ψ'; such a point, simple on ψ, must then be also on ψ', or it would count only singly among the intersections of Π and Π'; being on ψ and ψ', which have no isolated common point, it is thus on the curve c. Thus δ is also the number of double points of Π which occur at a common simple point of ψ and ψ' lying on the curve c. We can then prove the result $X+4d+\delta=\epsilon_0(\rho-1)$ by considering the intersections of c with the first polar of Π in regard to an arbitrary point; these intersections consist of the X points of c for which the common tangent prime of Π and Π' passes through the arbitrary point, then of the double points of ψ and ψ', each a four-fold point for c, but simple on the first polar, and lastly of the δ double points of Π spoken of, each likewise simple on the first polar. Lastly we prove the simple result,

$$\epsilon_1+4d=\epsilon_0(\rho+\rho'-3);$$

but the proof is rather intricate. Consider the cone Γ, with vertex at an arbitrary point O, which passes through the curve c; let N, N' be the numbers of isolated intersections of this cone respectively with ψ and ψ'; also let γ be the curve, of order $\epsilon_0\rho-\epsilon_0$, which is the remaining intersection of this cone with the primal Π. We first prove that $N+N'+2X+8d+\delta=\epsilon_0(\rho-1)\rho'$. For this we consider the intersections of the curve γ with the primal Π'. The $N+N'$ isolated intersections of the cone Γ with ψ and ψ' are on Π, and hence on the curve γ, and on Π'; at one of the X points of c where the common tangent prime of Π and Π' passes through O, the curve γ touches Π'; at a common double point of ψ and ψ', where, by what has preceded, both c and γ have a four-fold point and Π' has a double point, there are 8 intersections of γ with Π'; lastly, at a simple common point of ψ and ψ' where Π has a double and Π' has a simple point, both c and γ have simple points, and there is one intersection of γ with Π'; there are δ such points. By considering all these intersections of γ with Π', the formula is proved. From this then, by the result last proved, namely $X+4d+\delta=\epsilon_0(\rho-1)$, we have $N+N'+X+4d=\epsilon_0(\rho-1)(\rho'-1)$. We have, however (see Ex. 5, p. 254, above), $N+z+4d=\epsilon_0(\mu_0'-1)$; and

$$N'+z'+4d=\epsilon_0(\mu_0'-1);$$

and, as $\mu_0+\mu_0'=\rho\rho'$, these give $N+N'=-z-z'-8d+\epsilon_0(\rho\rho'-2)$. Thus, eliminating $N+N'$ by means of

$$N+N'+X+4d=\epsilon_0(\rho-1)(\rho'-1),$$

we obtain $z+z'+4d=\epsilon_0(\rho+\rho'-3)+X$. It was seen above that $z+z'=X+\epsilon_1$. Thus finally we reach the simple result

$$\epsilon_1+4d=\epsilon_0(\rho+\rho'-3).$$

The equations we have found are just sufficient to give the characters of the residual surface ψ', and of the curve c in which this meets ψ. Beside $\mu_0+\mu_0'=\rho\rho'$, they give

$$\mu_1'-\mu_1=(\mu_0'-\mu_0)(\rho+\rho'-2), \quad \nu_2'-\nu_2=(\mu_0'-\mu_0)(\rho-1)(\rho'-1),$$

of which a particular consequence is that

$$\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2=\mu_0'(\rho-1)^2-\mu_1'(\rho-1)+\nu_2'.$$

The two equations such as $\mu_2+\nu_2+z=\mu_1(\rho+\rho'-2)$, using

$$z+z'=X+\epsilon_1, \quad \mu_0\mu_0'=X+\epsilon_0, \quad 2d=2d'=\mu_0(\mu_0-1)-\mu_1-\nu_2$$

and $\epsilon_1+4d=\epsilon_0(\rho+\rho'-3)$, lead to

$$\mu_2'+\mu_2=\mu_0'\mu_0+8d+\rho\rho'(k^2+kk'+k'^2)-3\epsilon_0(k+k'),$$

where $k=\rho-1$, $k'=\rho'-1$. Herein d, or d', is given by the formula just quoted, and $\epsilon_0=\mu_0(\rho+\rho'-2)-\mu_1$; also, we may notice, the equation $X+\epsilon_0=\mu_0\mu_0'$ may be replaced by the formula

$$X+\nu_2+2d=\mu_0(\rho-1)(\rho'-1),$$

which is proved in the footnote to p. 259, above. If we denote by p the genus of the prime section of the surface ψ, so that $\mu_1=2\mu_0+2p-2$, the equation $\epsilon_0=\mu_0(\rho+\rho'-2)-\mu_1$ is the same as $\epsilon_0+2p-2=\mu_0(\rho+\rho'-4)$; this expresses that a primal of order $\rho+\rho'-4$, passing through the curve c, would meet a prime section of ψ in a number of points equal to that of a canonical set on this prime section. Put into the form $2p-2-\mu_0=\mu_0(\rho+\rho'-5)-\epsilon_0$ it expresses that a canonical curve on ψ, including exceptional curves, and also including the curve c, has the order of the intersection of ψ with a primal of order $\rho+\rho'-5$ containing the surface ψ' (cf. p. 269 below; Severi, *Rend....Palermo*, XVII, 1903, No. 18). The equation $\epsilon_1+4d=\epsilon_0(\rho+\rho'-3)$, which we have found more difficult to establish, being equivalent, in terms of the genus, q, of the curve c, to $2q-2+4d=\epsilon_0(\rho+\rho'-5)$, expresses that a primal of order $\rho+\rho'-5$, put through the common accidental double points of ψ and ψ', meets the curve c in a residual set of points whose number is that of a canonical set* of c. Lastly, from the equations $\mu_0\mu_0'=X+\epsilon_0$, $X+4d+\delta=\epsilon_0(\rho-1)$, we see that the number of double points, δ, of Π, at simple points of ψ, on c, and the corresponding number δ' for Π', are given by $\mu_0'\mu_0+4d=\epsilon_0\rho-\delta=\epsilon_0\rho'-\delta'$ or by $\delta+2d=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2$; and we have had the auxiliary formulae $X=\mu_0'\mu_0-\epsilon_0=z+z'-\epsilon_1$, and $\mu_2+\nu_2+z=\mu_1(\rho+\rho'-2)$, and also $N+z+4d=\epsilon_0(\mu_0-1)$. We notice that, from the formulae for the number of double points of a primal of order ρ passing through a surface (μ_0, μ_1, ν_2), with d double points, and through a second surface (μ_0', μ_1', ν_2'), with d' double points, namely $\delta=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2-2d$ and

$$\delta'=\mu_0'(\rho-1)^2-\mu_1'(\rho-1)+\nu_2'-2d',$$

we have

$$\delta+\delta'-2i=(\mu_0+\mu_0')(\rho-1)^2-(\mu_1+\mu_1')(\rho-1)+\nu_2+\nu_2'-2(d+d'+i),$$

where i is the number of intersections of the two surfaces, assuming that the common points are not among the double points of either component surface. This gives the number of double points for a primal containing a composite surface.

Ex. 1. Denoting $\rho-1$, $\rho'-1$ by k and k', and denoting by p_n, ω the invariants of ψ and by p_n', ω', those of ψ', prove from the formulae found that (cf. Ex. 3, p. 247, preceding)

$$p_n+1+p_n'+1+\tfrac{1}{2}\epsilon_0(k+k'-3)-d$$
$$=\tfrac{1}{12}(\mu_0+\mu_0')(2k^2+2k'^2+3kk'-8k-8k'+12),$$

and $\omega-1+\omega'-1+2\epsilon_0(k+k'-3)+z+z'-6\epsilon_0=(\mu_0+\mu_0')(k+k'-3)^2$,

and in fact that $\omega-1+z-3\epsilon_0=(k+k'-3)(2p-2-\mu_0)$.

Ex. 2. If it be possible to put a primal of order $k+k'-3$ through the surface ψ' which does not contain ψ, its curve of intersection with ψ will meet the curve c in $z-3\epsilon_0$ points. For (by p. 255 preceding, under III), this number of intersections is given by $z+Z-\epsilon_1$, where Z is the number of variable intersections of the first polar of this primal with the curve c;

---

* If ϵ be the deficiency of the series, and $d-\eta$ be the number of independent conditions which the d accidental double points impose for a primal of order $\rho+\rho'-5$ put through them, it is proved by Severi (*loc. cit.*) that $\epsilon+\eta$ is equal to the sum of the irregularities (p. 227) of the surfaces ψ and ψ'.

this first polar, of a primal passing through the surface ψ' with d accidental double points, passes simply through each of these, which are four-fold points for the curve c. Thus $Z=(k+k'-4)\epsilon_0-4d$; we have, however, $\epsilon_1=\epsilon_0(k+k'-1)-4d$. It has already been seen (Chap. v, p. 225) that $z-3\epsilon_0$ is the number of intersections of the curve c with the canonical curves of the surface ψ, when these exist.

Ex. 3. If we now further assume that the primals Π of order $k+k'-3$ through ψ' cut canonical curves on ψ, and that two such canonical curves K_0, K cut in $\omega-1$ points (see Chap. v, p. 197), and also that canonical curves are of order $2p-2-\mu_0$ (since such a curve, added to a prime section, gives a curve belonging to the adjoint system of the prime sections); then, taking intersections on ψ, we have $(\Pi, K)=(K_0+c, K)=\omega-1+i$; which is therefore equal to $(k+k'-3)(2p-2-\mu_0)$. We thus have an alternative proof of the last formula of Ex. 1.

Several applications of the formulae for intersections here given will be found in the following chapter.

The postulation of a surface for primals containing it. It is convenient to add here a proof, of summary and preliminary character, of the formula expressing the number of conditions which a surface, without multiple points, imposes upon a primal that is to contain the surface, when the order, ρ, of the primal is sufficiently high. Let the surface be of order μ_0, of numerical genus p_n, and its prime sections be of genus p. Then the number of conditions is

$$\tfrac{1}{2}\mu_0\rho(\rho+1)-\rho(p-1)+p_n+1-k,$$

where k is some number $\geqslant 0$. It appears that the formula is exact when the surface has accidental double points, the number of these being k. In general k is zero if the primals of order ρ determine a complete system of curves on the surface, supposed without multiple points, and, when this is not so, is the deficiency of this system. In terms of the characters of the surface this postulation is $\frac{1}{2}\mu_0(\rho+1)(\rho+2)-\frac{1}{6}\mu_1(3\rho+4)+\frac{1}{6}\mu_2+\frac{1}{12}\nu_2-k$. To prove this, we assume a result due to Castelnuovo, *Ann. d. Mat.* xxv, 1897, p. 83: *The characteristic series* (p. 124), *for the complete system of curves determined on a surface without singular points, lying in space of any dimensions, by the primals of any order sufficiently high, is not complete, but of deficiency p_g-p_n, where p_g-1 is the freedom of the canonical system of curves on the surface.* We also assume that the canonical series, on a curve of a linear system of curves on the surface, which has no base points, is the sum of the characteristic series on this curve, and the series cut on this curve by the canonical system of curves of the surface (above, p. 222). From this it follows that any set of the characteristic series on the curve is contained in a number equal to p_g of canonical sets on the curve, diminished by the number of canonical curves of the surface containing the curve in question. Thus, when this curve is of sufficiently high order, the index of specialness of its characteristic series is p_g.

(Cf. *Proc. Lond. Math. Soc.* XII, 1913, p. 19.) Now consider a linear system of curves on a surface, cut by all the primals of sufficiently high order; let n be the grade of this system, and r the freedom of the complete system to which this system belongs; thence the characteristic series is of sets of n points, and is of freedom $r-1$. This series belongs then, by the Castelnuovo result above quoted, to a complete series on the curve of freedom $r-1+p_g-p_n$. Whence, by the Riemann-Roch theorem on the curve, we have

$$r-1+p_g-p_n=n-\pi+p_g,$$

where π is the genus of the curve. Wherefore $r=n-\pi+1+p_n$. Now let the primals be of order ρ, so that $n=\mu_0\rho^2$, if μ_0 be the order of the surface. We can obtain the genus π by a theorem for a composite curve, $\pi_1+\pi_2+i-1$, previously obtained (Chap. V, p. 223); for let the genus of a prime section of the surface be p; then the section by a primal of order ρ is equivalent to the section by a primal of order $\rho-1$, together with the section by a prime; thus, if π_ρ, $\pi_{\rho-1}$ be the genera of these sections, we have

$$\pi_\rho=\pi_{\rho-1}+p+\mu_0(\rho-1),$$

leading to π, or π_ρ, $=\frac{1}{2}\mu_0\rho(\rho-1)+\rho(p-1)+1$. Applying then the preceding result, for the freedom of the complete system of curves on the surface determined by its sections with primals of order ρ, we have $r=\mu_0\rho^2-\pi_\rho+1+p_n$, or $r=\frac{1}{2}\mu_0\rho(\rho+1)-\rho(p-1)+p_n$. The freedom of the system of curves, possibly not complete, cut by the primals, is this number less a correction, say k. In order that such a primal should contain the surface it is necessary and sufficient that the primal should contain one more point of the surface than is necessary to determine a curve of the system. Thus the postulation of the surface, for primals of order ρ, sufficiently great, is $\frac{1}{2}\mu_0(\rho+1)\rho-\rho(p-1)+1+p_n-k$.

The formula $\frac{1}{2}\mu_0\rho(\rho+1)-\rho(p-1)+p_n$, for the freedom of the complete system of curves on the surface determined by primals of order ρ, is the same as that for the freedom of the system determined by primals of order $\rho+1$ put through a prime section of the surface, if we assume the usual value, $\mu_0(\rho+1)-(p-1)$, of the postulation of a curve of order μ_0 and genus p for primals of order $\rho+1$, sufficiently great; such primals of order $\rho+1$ give a system of variable curves of the same characters as do the unrestricted primals of order ρ (cf. Ex. 2, p. 256). But suppose the surface has a single accidental double point; then the postulation, for primals of order $\rho+1$, of a prime section of the surface through the double point, which has a double point thereat, is one less than before, two intersections with the prime section at the double point being obtained by one condition for the primal. Hence we infer that, for a surface with such a double point, the system of sections by unrestricted primals of order ρ, has a freedom less by one than the system of sections by primals of order $\rho+1$, put through a fixed prime section containing the double point. This suggests that the formula above obtained for the postulation of a surface, when ρ is sufficiently great, which is obtained under the hypothesis that the surface has no double points, may be exact for a surface with d improper double points, if k be replaced by d; that is, that the only correction on account of the possible incompleteness of the system cut by primals of order ρ arises in virtue of the double points, when ρ is sufficiently great, and the surface has no other singularities. Examples in verification of this result arise below; for a complete result see Severi, *Rend....Palermo*, XVII, 1903, No. 32. We may remark that, for a composite surface, consisting of two surfaces with respectively d and d' double points, having i intersections, we have, denoting

$$\tfrac{1}{2}\rho(\rho+3)\mu_0-\tfrac{1}{2}\rho\mu_1+p_n+1-d$$

by $F(\mu_0, \mu_1, p_n, d)$, the equation

$$F(\mu_0, \mu_1, p_n, d)+F(\mu_0', \mu_1', p_n', d'+i)$$
$$=F(\mu_0+\mu_0', \mu_1+\mu_1', P_n, d+d'+i),$$

where $P_n+1=p_n+1+p_n'+1$. Cf. Severi, *Rend....Palermo*, XXVIII, 1909, No. 5.

Ex. 1. If two primals (ρ), (ρ') have a plane in common, their remaining surface of intersection has the characters

$$\mu_0=\rho\rho'-1,\quad \mu_1=(\rho\rho'-2)(\rho+\rho'-2),\quad \nu_2=(\rho\rho'-2)(\rho-1)(\rho'-1),$$

while, with $k=\rho-1$, $k'=\rho'-1$, $\mu_2=\rho\rho'(k^2+kk'+k'^2+1)-3(k+k')^2-1$. This surface meets the plane in a curve, c, for which $\epsilon_0=\rho+\rho'-2$, $\epsilon_1=(\rho+\rho'-2)(\rho+\rho'-3)$, that is, in a curve of order $\rho+\rho'-2$ without double points, upon which the primals have double points of respective numbers $\delta=(\rho-1)^2$, $\delta'=(\rho'-1)^2$.

Ex. 2. If two primals (ρ), (ρ') have two planes in common (together forming a composite surface of characters $\mu_0=2$, $\mu_1=0$, $\mu_2=0$, $\nu_2=0$, $d=1$), they meet further in a surface of characters

$$\mu_0=\rho\rho'-2,\quad \mu_1=(\rho\rho'-4)(\rho+\rho'-2),\quad \nu_2=(\rho\rho'-4)(\rho-1)(\rho'-1),$$

for which also, with $k=\rho-1$, $k'=\rho'-1$,

$$\mu_2=\rho\rho'(k^2+kk'+k'^2+2)-6(k+k')^2+4;$$

this surface meets the planes in a composite curve, c, for which $\epsilon_0=2(\rho+\rho'-2)$, $\epsilon_1=2(\rho+\rho'-2)(\rho+\rho'-3)-4$, that is, it meets each plane in a curve of order $\rho+\rho'-2$, with a single double point at the point of intersection of the two planes, upon which the primals have double points, other than at the intersection of the planes, of respective numbers $\delta=2\rho(\rho-2)$, $\delta'=2\rho'(\rho'-2)$; or, with the double point at the intersection of the planes, they have, respectively, $(\rho-1)^2$ and $(\rho'-1)^2$ double points on each plane.

Ex. 3. If, in space $[r]$, with $r>4$, there be put $r-k$ primals through a manifold of dimension k, M_k, where $k<\tfrac{1}{2}r$, these primals meeting again in a M_k', which has with M_k a manifold common of dimension $(k-1)$, say m_{k-1}, then the tangent $[k-1]$ of this m_{k-1} at any point of this lies in the two tangent spaces $[k]$ of M_k and M_k' at this point, and these two tangent spaces define a space $[k+1]$. Denote by X the number of such spaces $[k+1]$ which meet an arbitrary space $[r-2k]$; likewise denote by z the number of tangent $[k]$, of M_k, at points of m_{k-1}, which meet an arbitrary space $[r-2k+1]$. In particular, when $k=2$, and there is a surface M_2, through which there pass $r-2$ primals of orders $\rho_1, \ldots, \rho_{r-2}$, let s_1 denote $\Sigma(\rho_i-1)$, and s_2 denote $\Sigma(\rho_i-1)(\rho_j-1)$, for $i \neq j$; then for general $r \geqslant 4$, we also have

$$\mu_0+\mu_0'=\rho_1\ldots\rho_{r-2}, \quad \mu_1'-\mu_1=(\mu_0'-\mu_0)s_1, \quad \nu_2'-\nu_2=(\mu_0'-\mu_0)s_2,$$
$$\mu_2+\nu_2+z=\mu_1 s_1, \quad \mu_2'+\nu_2'+z'=\mu_1' s_1, \quad X=z+z'-\epsilon_1=\mu_0 s_2-\nu_2-2d,$$
$$\epsilon_0=\mu_0 s_1-\mu_1, \quad \epsilon_1=\epsilon_0(s_1-1)-4d,$$

where for $r>4$, we have $d=0$, and the primals are not supposed to have double points on the surface through which they are drawn.

Ex. 4. As an example of the formulae just given, prove that if, in [5], three quadrics be drawn through the Del Pezzo surface—that namely whose prime sections are represented by plane cubic curves with four common base points—for which $\mu_0=5$, $\mu_1=10$, $\mu_2=12$, $\nu_2=8$, then these quadrics meet again in a rational ruled cubic surface (lying actually in a space [4]) for which $\mu_0'=3$, $\mu_1'=4$, $\mu_2'=3$, $\nu_2'=2$, having common with the Del Pezzo surface an elliptic quartic curve, a prime section of this.

Ex. 5. In fact an irreducible algebraic manifold M_k^n, of order n and dimension k, is necessarily contained in a space $[r]$ for which $r<n+k$. Or, if the manifold be contained in a space $[r]$, and not in a space of less dimension, its order n is necessarily greater than the dimension $(r-k)$ complementary to that of the manifold. This fact was suggested in Clifford, "Classification of Loci", *Papers*, 1882, p. 307; a complete proof is given in Bertini, *Geom. d. Iperspazi* (Pisa, 1907), p. 193. But, summarily, we may assume that there is at least one space of complementary dimension, $r-k$, whose intersections with the manifold, which are n in number, do not lie in a space of less than $r-k$ dimensions, while the dimension of the linear space of least dimension which contains n points is certainly less than n; thus $r-k<n$.

V. Residual intersection of three primals having a surface in common. By a combination of III and IV above, we can now prove that if, in space [4], three primals Π, Π', Π'', of respective orders ρ, ρ', ρ'', have a surface, ψ, in common, of characters μ_0, μ_1, μ_2, but without double points, and these primals meet further in a curve of order ϵ_0' and rank ϵ_1', having i intersections with ψ, then

$$\epsilon_0'=\rho\rho'\rho''-\mu_0(s_1+1)+\mu_1, \quad \epsilon_1'=s_1\epsilon_0'-2i, \quad i=\mu_0 s_2-\mu_1 s_1+\mu_2,$$

where s_1, s_2 are respectively the sum, and the sum of the three products of pairs, of the three numbers $\rho-1, \rho'-1, \rho''-1$, which we may denote by k, k', k''.

The first formula is obvious by taking an arbitrary prime section, whereby we obtain three surfaces of orders ρ, ρ', ρ'', in space [3], having in common a curve of order μ_0 and rank μ_1; these curves intersect further in ϵ_0' points (Vol. V, Chap. VIII). For the second formula, we may consider the Jacobian of the three primals and of two arbitrary primes, meeting in a plane ϖ. This Jacobian is of order s_1, and meets the curve $(\epsilon_0', \epsilon_1')$ in the points whereat the tangent line meets the plane ϖ. It may also be proved to have two intersections with the common curve of the three primals at each of the points where this curve meets the surface ψ common to the primals. This second formula may also be found by the method we now employ to obtain the third formula: let the primals Π, Π' meet, beside in ψ, in the surface ψ', of characters $\mu_0', \mu_1', \ldots$, and let ψ' meet ψ in the curve c, of characters ϵ_0, ϵ_1, with a class of immersion in ψ' which we denote by z'. The primal Π'' passes through c, and meets ψ' also in another curve, say γ, which is the residual curve of intersection of Π, Π', Π''. As c has a class of immersion in the primal Π'' which is equal to the number of intersections of the first polar of this, taken in regard to an arbitrary point, with this curve, namely $\epsilon_0 k''$, we have, by the formulae of III above, $i = z' - \epsilon_1 + \epsilon_0 k''$. By the formulae in IV, however, we have

$$\epsilon_0 = \mu_0(k+k') - \mu_1, \quad z' - \epsilon_1 = \mu_0\mu_0' - \epsilon_0 - z, \quad z = \mu_1(k+k') - \mu_2 - \nu_2.$$

Thus

$$\begin{aligned} i &= \epsilon_0 k'' + \mu_0\mu_0' - \epsilon_0 + \mu_2 + \nu_2 - \mu_1(k+k') \\ &= \mu_2 + \nu_2 + (k''-1)[\mu_0(k+k') - \mu_1] - \mu_1(k+k') \\ &\qquad\qquad + \mu_0[(k+1)(k'+1) - \mu_0] \\ &= \nu_2 - \mu_0^2 + \mu_0 + \mu_1 + \mu_2 + \mu_0 s_2 - \mu_1 s_1, \end{aligned}$$

which has the value given above, since the number

$$d, = \tfrac{1}{2}\mu_0(\mu_0 - 1) - \tfrac{1}{2}\mu_1 - \tfrac{1}{2}\nu_2,$$

of double points of the surface ψ is, by hypothesis, zero. The value of ϵ_1' is given by the formula of III above

$$\epsilon_1' + \epsilon_1 + 2i = \mu_0' k''(k''+1) + \mu_1'(k''+1),$$

and will be found to lead to the value stated above in terms of ϵ_0'.

Corollary. Four primals (ρ), (ρ'), (ρ''), (ρ''') passing through a given surface (μ_0, μ_1, μ_2) have further in common a number of points given by $\rho\rho'\rho''\rho''' - [\mu_0\sigma_2 + (\mu_0 - \mu_1)(\sigma_1 + 1) + \mu_2]$, where σ_1, σ_2 are the sum, and the sum of the six products of pairs, of k, k', k'', k''', where $k = \rho - 1$, etc.

More generally if, in space $[r]$, there be $r-1$ primals through a surface, meeting further in a curve $(\epsilon_0', \epsilon_1')$, having i intersections with this surface, we have

$$\epsilon_0' = \rho_1 \cdots \rho_{r-1} - \mu_0(s_1 + 1) + \mu_1, \quad \epsilon_1' = s_1\epsilon_0' - 2i, \quad i = \mu_0 s_2 - \mu_1 s_1 + \mu_2,$$

where s_1, s_2 are formed as before from the numbers $\rho_1 - 1, \ldots, \rho_{r-1} - 1$. And the point equivalence of the surface is

$$\mu_0\sigma_2 + (\mu_0 - \mu_1)(\sigma_1 + 1) + \mu_2,$$

where σ_1, σ_2 are formed from r primals. The result is given by Severi, *Intersezioni...*, No. 29; it is assumed that the surface has no multiple points or multiple curves. Of the other results given here, I–V, wide generalisations will be found in this paper.

Ex. 1. More generally, in space $[r]$, the point equivalence of a manifold $M_k(k<\frac{1}{2}r)$, without multiple points, for primals of orders $k_1+1, \ldots, k_r+1$, is the coefficient of t^k in the ascending expansion of the function $(1+k_1t)\ldots(1+k_rt)/(1-t)(1+\mu t)$, if it be understood that $(1+\mu t)^{-1}$ means the series $\mu_0-\mu_1t+\mu_2t^2-\ldots$.

Ex. 2. Find the point equivalence of a surface with d accidental double points in space [4], for primals.

Ex. 3. If a manifold of order $\rho_1\ldots\rho_h$, and dimension $r-h$, in space $[r]$, without multiple parts of dimension $r-h$, but possibly with multiple parts (or points) of less dimension, be the complete intersection of h primals of orders $\rho_1, \ldots, \rho_h$, the postulation of this manifold for a primal of any order R is the coefficient of x^{-1} in the ascending expansion of $(1-x^{\rho_1})(1-x^{\rho_2})\ldots(1-x^{\rho_h})/x^{R+1}(1-x)^{r+1}$. The proof of this has already been indicated (Vol. v, p. 222); it will be found at length in Bertini, *Iperspazi*, 1907, p. 263. Applying to this rational function of x the theorem already referred to (Vol. v, p. 47), written for abbreviation as $[R(x)\,dx/dt]_{t^{-1}}=0$, the values of x beside $x=0$, to be considered are $x=1$ and $x=\infty$. Thus the postulation in question, say $\chi(R)$, is found to be equal to the sum

$$(-1)^{r-h}\left\{\frac{[(1+\xi)^{\rho_1}-1]\ldots[(1+\xi)^{\rho_h}-1]}{\xi^{r+1}(1+\xi)^{R+1}}\right\}_{\xi^{-1}}$$
$$+(-1)^{r-1-h}\left\{\zeta^{R-\Sigma\rho+r}\frac{(1-\zeta^{\rho_1})\ldots(1-\zeta^{\rho_h})}{(1-\zeta)^{r+1}}\right\}_{\zeta^{-1}};$$

of these the latter gives no contribution when $R\geqslant\Sigma\rho-r$, but gives $(-1)^{r-1-h}$ when $R=\Sigma\rho-r-1$; putting $k_i=\rho_i-1$, the former is equal to

$$(-1)^{r-h}\mu_0\{(1+\xi)^{-(R+1)}\Pi[1+\tfrac{1}{2}k_i\xi+\tfrac{1}{6}k_i(k_i-1)\xi^2+\ldots]\}_{\xi^{r-h}},$$

where $\mu_0=\rho_1\ldots\rho_h$. We consider the cases $h=r-1$, $h=r-2$.

For $h=r-1$, we obtain $\chi(\Sigma k-2)=\mu_0(\frac{1}{2}\Sigma k-1)+1$, in fact equal to the genus, p, of the curve which is the complete intersection in question; for $R>\Sigma k-2$, however, $\chi(R)=\mu_0R-p+1$. If the curve break up into two curves of orders m and m', with i intersections, of genera p and p', and we take account of the equations

$$2m+2p-2+i=m\Sigma k,\quad 2m'+2p'-2+i=m'\Sigma k,$$

the postulation $\chi(\Sigma k-2)$, which is $(m+m')(\frac{1}{2}\Sigma k-1)+1$, is equal to $p+p'+i-1$. For $R>\Sigma k-2$, we have

$$\chi(R)=(m+m')R-(p+p'+i-1)+1,$$

which is $\qquad (mR-i-p+1)+(m'R-i-p'+1)+i.$

For $h=r-2$, we obtain (cf. p. 247, Ex. 3, and p. 262, Ex. 1)

$$\chi(\Sigma k-3)=\tfrac{1}{12}\mu_0[2\Sigma k_i^2+3\Sigma k_ik_j-8\Sigma k_i+12]-1$$

and, denoting this by p_n, we have, for $R>\Sigma k-3$,

$$\chi(R)=\mu_0[\tfrac{1}{2}(R+1)(R+2)-\tfrac{1}{2}(R+1)\Sigma k_i+\tfrac{1}{6}\Sigma k_i(k_i-1)+\tfrac{1}{4}\Sigma k_ik_j],$$

which is the same as $\frac{1}{2}\mu_0R(R-1)-R(p-1)+p_n+1$, where p is the genus of a prime section of the surface, equal to $\mu_0(\frac{1}{2}\Sigma k-1)+1$. When the surface breaks up into surfaces ψ, ψ' of orders μ_0 and μ_0', this last formula remains true if we replace μ_0, p, p_n respectively by $\mu_0+\mu_0'$, $p+p'+\epsilon_0-1$

and $\chi(\Sigma k-3)$, where ϵ_0 is the order of the curve of intersection of the two surfaces.

When the surface ψ possesses a canonical system of curves, of at least one curve ($p_g>0$), these (together with the exceptional curves if there are such) are all given by the primals of order $\Sigma k-3$ through the surface ψ'.

Ex. 4. Assuming Severi's theorem quoted in the footnote, p. 262, above, prove in space [4] that if two cubic primals can be put through a regular surface of order μ_0, which has no double point, the irregularity of the residual surface of intersection is μ_0-p-3 if this is not negative, p being the genus of the prime section of the original surface. One example is when the surface is that obtained by projection of the Veronese surface, in which the residual intersection is a ruled elliptic quintic surface.

Ex. 5. The following results may also be proved (Severi, *Rend. Palermo*, XVII, 1903; also *Rend. Palermo*, XXVIII, 1909, p. 5): (*a*), An irreducible curve, without multiple points, which is the complete intersection of $r-1$ primals in space $[r]$, is normal; (*b*), A manifold M_h, which is irreducible, and without multiple M_{h-1}, and is a complete intersection of $r-h$ primals, is normal; (*c*), A surface, without multiple points, which is the complete intersection of $r-2$ primals, is regular ($p_g=p_n$), and without exceptional curves (except when $r=4$, $\rho_1=\rho_2=2$). On this surface the canonical system of curves is given by the primals of order $\Sigma k-3$, while primals of order $R>\Sigma k-3$ determine thereon a linear system of curves which is regular and complete.

Ex. 6. The following theorems, proved by Severi, *Atti...Torino*, XLI, 1906, have connexion with the results of Ex. 3: (*a*), The general one of a linear system of manifolds M_{k-1}, of dimension $k-1$, existing on a given manifold M_k, has no multiple points other than at the base points common to all the M_{k-1} or at multiple points of the M_k; (*b*), If $F_0, \ldots, F_r$ be $r+1$ homogeneous polynomials in the $r+1$ variables $x_0, \ldots, x_r$, of orders $\rho_0, \ldots, \rho_r$, any other such *form*, of order $>\Sigma(\rho_i-1)$, is expressible in the form $A_0F_0+\ldots+A_rF_r$, where $A_0, \ldots, A_r$ are suitably chosen forms; (*c*), If, for $h<r+1$, the $h+1$ forms Φ, $F_1, \ldots, F_h$ have only ∞^{r-h-1} common zeros, and there holds an equation $\Phi F=A_1F_1+\ldots+A_hF_h$, where F, $A_1, \ldots, A_h$ are suitable forms, then F is likewise expressible as $B_1F_1+\ldots+B_hF_h$, for suitable forms $B_1, \ldots, B_h$. Cf. also Lasker, "Zur Theorie der Moduln und Ideale", *Math. Ann.* LX, 1905.

Ex. 7. Recalling that to a linear system of curves $|C|$, without base points, of grade n and genus π, there exists an adjoint system $|C'|$, defined by $|C_{,}-2C|$, whose curves cut canonical sets on curves of $|C|$, and further that the system $|C'-C|$ is independent of $|C|$ (see Chap. V, pp. 217, 222), it follows, if $|C''|$ be the adjoint system of $|C'|$, that $|C''-C'|=|C'-C|$, and hence $|C''+C|=|2C'|$. Wherefore, if n', π' be the grade and genus for $|C'|$, we have

$$2n'=(2C', C')=(C''+C, C')=(C'', C')+(C', C), =2\pi'-2+2\pi-2;$$

so that $n'=\pi'-1+\pi-1$ (p. 225, Ex. 4).

Ex. 8. We have assumed above, in obtaining the equation

$$r=n-\pi+1+p_n,$$

for a linear system $|C|$, of curves of sufficiently high order, that, for this system the characteristic series is of deficiency p_g-p_n. It is convenient to enunciate also a theorem stated by Picard (*Crelle*, CXXIX, 1905; Picard-Simart, *Fonctions algéb.* II, 1906, p. 438. Cf. Severi, *Rend. Lincei*, XVII, 1908, p. 465), that for *any* general linear system $|C|$, the series of canonical sets cut on a curve of this system, by the curves $|C'|$ of the adjoint system, is of deficiency $p_g - p_n$. Hence, if r' be the freedom of the system $|C'|$, since p_g curves of this system contain any curve of $|C|$, we have $r' - p_g = \pi - 1 - (p_g - p_n)$, or $r' = \pi - 1 + p_n$. Hence, by the result of the preceding example, we have $r' = n' - \pi' + 1 + p_n$ (cf. p. 169).

Ex. 9. If, in space [4], two primals F, F', of orders ρ, ρ', which have in common no surface which is multiple, have as their intersection two surfaces ψ, ψ', and if ψ be regular $(p_g = p_n)$, then the formula

$$\tfrac{1}{2}\mu_0' R(R+1) - R(p'-1) + p_n' + 1 - d'$$

is valid to compute the postulation of the surface ψ' for primals of order $R = \rho + \rho' - 5$ put through it; and such primals, if $p_g > 0$, after subtraction of primals of this order containing both ψ and ψ' (which are given by an equation $UF + U'F' = 0$) give the complete canonical system on ψ (Severi, *Rend. Palermo*, XVII, 1903). Shew that this interprets the first identity of Ex. 1, p. 262.

Ex. 10. For a quintic surface with two tacnodes (cf. p. 241, Ex. 6) the adjoint system $|C'|$, of the plane sections $|C|$, is given by quadric surfaces through the tacnodes (verifying the relation $r' = \pi - 1 + p_n$ of Ex. 8 with $\pi = 6$, $p_n = p_g = 2$); and the adjoint system of the canonical system is given by quadric surfaces touching the given surface at the tacnodes (verifying $r'' = \pi' - 1 + p_n$, with $\pi' = 2$).

Ex. 11. It is easily seen that the acquisition of a tacnode lessens the class of a surface by 12 (as does an ordinary triple point). Hence, accepting the result of Ex. 1, p. 236, (4), the tacnode lessens the invariant I by 10, and the invariant ω by 2. Thus, for the quintic surface of the previous example we find $I = 31$, $\omega = 2$. More simply, for a quartic surface with one tacnode, we have $I = 10$, $\omega = -1$; this last can be tested in the manner of pp. 201–6 above (which are for *ordinary* multiple points only); for the quartic surface can be transformed to a cubic surface (Vol. V, p. 110); thus we arrive at $I_0 = 9$, $\omega_0 = 0$.

Ex. 12. By applying Clifford's theorem (Vol. V, p. 80) to the characteristic series on a canonical curve of a surface, prove that $p^{(2)} \geqslant 2(p_g - 2)$. Noether's proof of this (*Math. Ann.* VIII, 1875, § 11) is effectively a deduction of Clifford's theorem (1877) from the theory of special series.

CHAPTER VII

ILLUSTRATIVE EXAMPLES AND PARTICULAR THEOREMS

The present chapter contains various examples and results which may serve to illustrate foregoing theories, the earlier cases being very simple applications of the account given in the preceding chapter of the residual intersection of two primals, in space [4], passed through a given surface. In general a surface, in space $[r]$, of order n, with prime sections of genus p, will be denoted by ${}^{p}\psi^{n}[r]$, or, when $r=4$, by ${}^{p}\psi^{n}$.

I. Particular examples of intersections.

1. The formulae for the characters of a surface ${}^{4}\psi^{6}$ which is the complete intersection of two primals in space [4], Chap. VI, p. 247, when applied to the intersection of a quadric and a cubic primal, in general positions, lead to

$\mu_0=6,\ \mu_1=18,\ \mu_2=42,\ \nu_2=12,\ p_n=1,\ I=20,\ \omega=1,\ t=0,\ d=0.$ The adjoint system of the prime sections is given by the prime sections themselves (in accordance with the formulae $r'=\pi-1+p_n$, $n'=\pi'-1+\pi-1$ of Ex. 8, p. 269, Chap. VI), so that the canonical system of the surface is a single curve of zero order, or $p_g=1$. The postulation of the surface for primals of order ρ (Chap. VI, p. 263) is $3\rho^2+2$; thus for $\rho=2$ it is 14 and there are $15-14$, or only one quadric containing the surface, as is obvious, since the surface is of order 6; and, for $\rho=3$ it is 29, so that there are $35-29$, or 6, cubic primals containing the surface, evidently given by an equation $(a_0x_0+\ldots+a_4x_4)Q+U=0$, where $Q=0$ is the unique quadric and $U=0$ a cubic primal.

2. If we consider the surface ${}^{2}\psi^{5}$ which is the residual intersection of a cubic primal with a quadric primal when these have a plane in common, so that the quadric is a cone, the formulae (Chap. VI, p. 265) give

$$\mu_0=5,\ \mu_1=12,\ \mu_2=20,\ \nu_2=8,\ p_n=0,\ I=7,$$
$$\omega=2,\ d=0,\ \epsilon_0=3,\ \epsilon_1=6,\ \delta=1,\ \delta'=4,$$

and the canonical series on any prime section is given by primes through the fixed plane, and hence by planes through the trisecant chord of this section which lies in this plane. Thus $p_g=0$. The surface is in fact rational, its prime sections being representable in a plane by a system of quartic curves with one double and seven simple base points, say by a system $c^4(2,1^7)$, the equations

$I-8=-1$, $\omega+8=10$ being in accordance with the equations $I_0-e=I_0'-e'$, etc. of Chap. v (p. 198). Starting from such a system of plane curves we may verify the definition of the surface as the partial intersection of a quadric and cubic. More generally, it has been proved by Castelnuovo (*Rend. Palermo*, IV, 1890, pp. 73–88, No. 5; cf. Enriques, *Math. Ann.* XLVI, 1895, p. 179; Scorza, for manifolds of 3 dimensions, *Ann. d. Mat.* XV, 1908, p. 217; Castelnuovo-Enriques, *Ann. d. Mat.* VI, 1901, p. 38), that any surface of which the prime sections are hyperelliptic curves is necessarily rational. In this case, if the quadric and cubic primals have equations $xt-yz=0$, $xU+yV=0$, where U, V are quadric polynomials in the coordinates x, y, z, t, u, their common plane being $x=0$, $y=0$, the cubic curve in which the surface meets this plane is given by $zU_0+tV_0=0$, where U_0 is U with $x=0$, $y=0$, etc. The cubic primal has double points (as we may verify by forming the partial derivatives) at the four points of the cubic given by $U_0=0$, $V_0=0$. The adjoint system of the prime sections of the surface are the conics given by such equations as $y=mx$, $t=mz$, $U_1+mV_1=0$, where U_1 is U for $y=mx$, $t=mz$, etc.; each of these meets the cubic curve in the plane $x=0$, $y=0$ in two points collinear with the vertex of the quadric primal. These conics of the surface correspond to the lines of the plane passing through the common double point of the plane quartic curves, each of these lines being a rational curve having two variable intersections with the quartic curves of the system. The surface also contains seven families of rational cubic curves; each corresponding to the pencil of lines through one of the seven simple base points of the quartic curves. But the curves of the surface seem best studied by regarding the surface as the projection from a space [6] of a rational surface of order 12 in space [11], whose prime sections are represented by plane quartic curves with one double base point. For this study we refer to Castelnuovo's paper. The number of primals of order ρ containing the surface is (Chap. VI, p. 263) at least $(\rho+4, 4)-\frac{5}{2}\rho(\rho+1)+\rho-1$. Thus, for $\rho=2$, the surface lies on the one quadric cone; and, for $\rho=3$, the surface lies on 7 cubic primals, of which 5 will be given by the aggregate of the quadric cone and an arbitrary prime. It will be found (cf. Chap. VI, p. 242) that the chords drawn to the surface ${}^2\psi^5$, from an arbitrary point, meet the surface in an octavic curve of genus 5, forming a quartic conical sheet of genus 1. Thus the ${}^2\psi^5$ projects, on to space [3], into a quintic surface with an elliptic quartic curve as double curve. It will appear below, under (4), that if two cubic primals be put through the surface their residual intersection is the Segre quartic surface which is the intersection of two quadric primals.

3. A simple and familiar surface is the cubic ruled surface, ${}^0\psi^3$ (4), representable by equations $x_0/x_1=x_1/x_2=x_3/x_4$. The general

formulae for a ruled surface (Chap. IV, p. 165) lead in this case to $\mu_0=3$, $\mu_1=4$, $\mu_2=3$, $\nu_2=0$, $p=0$, $d=0$, $I=0$, $p_n=0$, $\omega=9$, in accordance with the plane representation of the prime sections of the surface, which is by conics having a common simple base point. The postulation of the surface for quadrics containing it is 12, so that the surface lies on ∞^2 quadrics; it is in fact projected by a quadric cone from any point of itself. The postulation of the surface for cubic primals containing it is 22, so that there are 13 such cubics. And in fact, with the coordinates indicated above, three quadrics containing the surface are given by

$$x_0x_2-x_1^2=0,\quad x_1x_3-x_0x_4=0,\quad x_1x_4-x_2x_3=0,$$

and, denoting these by $u=0$, $v=0$, $w=0$, thirteen cubics containing the surface are in fact given by the equation $uL+vM+wN=0$, where $L=0$, $M=0$, $N=0$ represent arbitrary primes, of which the last may be taken through the line $x_2=0$, $x_3=0$, $x_4=0$. It will be found on examination that this equation contains 13 independent primals. Thus two cubic primals through the ${}^0\psi^3$ intersect further in the sextic surface given by the vanishing of the determinants of the third order in a matrix of three rows and four columns,

$$\left|\begin{matrix} x_0, & x_1, & N, & N' \\ x_1, & x_2, & M, & M' \\ x_3, & x_4, & L, & L' \end{matrix}\right|$$

wherein L, M, N, L', M', N' are linear in the coordinates. Thus this residual surface is the locus of intersection of four corresponding primes $\nu x_0+\mu x_1+\lambda x_3=0$, $\nu x_1+\mu x_2+\lambda x_4=0$, $\lambda L+\mu M+\nu N=0$, $\lambda L'+\mu M'+\nu N'=0$, in four projective systems of primes, of which each system consists of the primes through a line. This surface is evidently rational, any point having its coordinates rational in λ, μ, ν; and it is easy to see that its prime sections are represented in a plane by quartic curves $c^4(1^{10})$ with 10 simple base points. The characters of this surface ${}^3\psi^6$ can then be found at once (Chap. VI, p. 242); or they can be found from the formulae for the residual intersection of two primals through a given surface (Chap. VI, p. 252). They are in fact $\mu_0'=6$, $\mu_1'=16$, $\mu_2'=27$, $\nu_2'=14$, $I'=9$, $p_n'=0$, and the ${}^3\psi^6$ meets the ${}^0\psi^3$ in a curve of order 8 and genus 5, say ${}^5c^8$; upon this curve the primes of the space [4] cut the complete canonical series (Chap. VI, p. 262, footnote). This curve, it may be seen, meets each generator of the ${}^0\psi^3$ in three points, and thus contains a linear series g_1^3; it is not the most general curve of genus 5, of which a plane representative is a sextic curve with 5 double points; but is that which can be represented by a plane quintic curve with only one double point. It can be determined on the

${}^0\psi^3$ by a cubic primal, not containing this surface but containing one generator of it. The surface ${}^3\psi^6$ was considered by Bordiga, *Mem....Lincei*, IV, 1887, p. 182 (see also Veronese, *Math. Ann.* XIX, 1882; White, *Proc. Camb. Phil. Soc.* XXI, 1923, p. 221; Room, *Proc. Roy. Soc.* A, CXI, 1926, p. 386). The sextic curve in [3] which is obtained by a prime section has been considered by Schur, *Math. Ann.* XVIII, 1881. By the plane representation of the surface ${}^3\psi^6$ we infer that the surface contains 10 lines, corresponding to the base points of the quartic curves; also 10 elliptic plane cubics, corresponding to the plane cubics through 9 of the base points, each of these curves meeting 9 of the lines; also 10 pencils of rational cubic curves, etc.

4. For the Segre quartic surface ${}^1\psi^4[4]$, the complete intersection of two quadrics, we have

$$\mu_0=4,\ \ \mu_1=8,\ \ \mu_2=12,\ \ \nu_2=4,\ \ d=0,\ \ p=1,\ \ I=4,\ \ \omega=5,\ \ p_n=0,$$

in accordance with the fact that the surface is rational, being represented by plane cubic curves having 5 simple base points, say by a system $c^3(1^5)$. The characters for a general Del Pezzo surface will be given below. The postulation of the surface for primals of order ρ (Chap. VI, p. 263) becomes here $\rho^2+(\rho+1)^2$; there are therefore at least 2 quadrics and 10 cubic primals containing the surface. Thus if $U=0$, $V=0$ be the defining quadrics, ten cubics are given by $uU+vV=0$, where $u=0$, $v=0$ are arbitrary primes. The formula (Chap. VI, p. 262) $\delta+2d=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2$ shews that a primal of order ρ through the ${}^1\psi^4$ has $4(\rho-2)^2$ double points at ordinary points of the surface; such a primal is given by an equation $uU+vV=0$, where u, v are of order $\rho-2$, and the double points in question are the intersections of $U=0$, $V=0$, $u=0$, $v=0$. Two cubic primals through the ${}^1\psi^4$ meet again in a quintic surface; if we use the formulae of Chap. VI, p. 261, we find for the characters of this residual surface $\mu_0'=5$, $\mu_1'=12$, $\mu_2'=20$, $\nu_2'=8$, which are the same as for the ${}^2\psi^5$ considered under (2), the residual intersection of a cubic primal and a quadric cone having common a plane. In fact two cubic primals $uU+vV=0$, $u'U+v'V=0$, meet, beside where $U=0$, $V=0$, also where $uv'-u'v=0$; this represents a quadric cone having the plane $u=0$, $v=0$ which lies on $uU+vV=0$. Further (Chap. VI, p. 261), the ${}^2\psi^5$ meets the ${}^1\psi^4$ in a curve ${}^5c^8$, of order 8 and genus 5, which is thus the intersection of the three quadrics $U=0$, $V=0$, $uv'-u'v=0$, and is a general curve of genus 5.

4*a*. There is a particular Segre quartic surface which has a double line, obtainable as the intersection of two quadric *line* cones, say $x_0x_4=x_2^2$, $x_1x_4=x_3^2$, the double line being $x_2=x_3=x_4=0$. This surface is representable, in terms of ξ, η, ζ, in the form

$x_0/\xi^2 = x_1/\eta^2 = x_2/\xi\zeta = x_3/\eta\zeta = x_4/\zeta^2$, and its prime sections project from the double line into conics meeting a given line in pairs of an involution. By equating these five ratios to a ratio $x_5/\xi\eta$ we obtain the parametric representation of a Veronese surface in space [5], and the particular Segre surface is obtainable by projecting this Veronese surface from the point common to the chords joining any two points $(\xi, \eta, 0)$, $(\xi, -\eta, 0)$ of the Veronese surface.

4*b*. We have already defined the characters μ_0, μ_1, μ_2, ν_2 for a surface in space [3] so as to correspond to characters denoted by these symbols for a surface in space [4]; we may similarly adopt definitions for higher space which, on projection into space [4], may give the characters as there defined. Thus, for space [5], μ_0 will be the order of a surface; μ_1 will be the order of the curve on the surface at which the tangent plane meets an arbitrary given plane (in a point), which is in fact the Jacobian curve of the net of curves on the surface determined by primes through the given plane; μ_2 will be the number of tangent planes of the surface which meet a given arbitrary solid, or space [3], in a line; ν_2 will be the number of tangent planes which meet an arbitrary line; d will be the number of chords of the surface which pass through an arbitrary point; t will be the number of trisecant chords of the surface which meet an arbitrary line, and so on. And similarly for space of higher dimension.

The general Del Pezzo surface in space $[\mu_0]$ may be understood to mean that surface of order μ_0 whose prime sections are represented by the cubic curves in a plane which pass through $9-\mu_0$ simple base points. For such a surface we then have $p_n = 0$, $p = 1$, $I = 8-\mu_0$, and hence, by the formulae of Chap. VI, p. 242, we have

$$\mu_1 = 2\mu_0, \quad \mu_2 = 12, \quad \nu_2 = 4(\mu_0 - 3), \quad d = \tfrac{1}{2}(\mu_0-3)(\mu_0-4),$$
$$t = \tfrac{1}{6}(\mu_0-2)(\mu_0-3)(\mu_0-4).$$

Thus, for the ${}^1\psi^5[4]$, obtained by projecting the Del Pezzo surface in space [5], we have $\mu_1 = 10$, $\mu_2 = 12$, $\nu_2 = 8$, $d = 1$, $t = 1$; and, by the formula (Chap. VI, p. 263) $\frac{1}{2}\mu_0\rho(\rho+1) - \rho(p-1) + p_n + 1 - d$, this surface lies on no quadrics, but on 5 cubics. The surface is thus obtainable as the partial intersection of two cubic primals. The remaining intersection of these will then be a quartic surface ${}^0\psi^4$, for which (by Chap. VI, p. 257) we have

$$\mu_0' = 4, \quad \mu_1' = 6, \quad \mu_2' = 4, \quad \nu_2' = 4, \quad p' = 0, \quad \epsilon_0 = 10, \quad \epsilon_1 = 26, \quad d' = 1,$$

while, by $\delta = \mu_0(\rho-1)^2 - \mu_1(\rho-1) + \nu_2 - 2d$, we have 6 for the number of double points, at ordinary points of ${}^1\psi^5$, which a cubic primal through this must possess. Comparing these formulae for μ_0', μ_1', ... with the results $\mu_1 = 2\mu_0 + 2p - 2$, $\mu_2 = \mu_0$, $\nu_2 = 2\mu_0 + 4p - 4$ holding for a ruled surface (Chap. IV, p. 165) we infer that two cubic primals through the projected Veronese surface meet again in a ruled ${}^0\psi^4$ (projection of the normal rational ruled quartic surface in [5]), which meets the projected Del Pezzo surface in a curve of order 10 and genus 4. In accordance with the footnote, Chap. VI, p. 262, the complete canonical series on this curve is determined by primes through its four-fold point. We shall see below (under (6)), that this ${}^0\psi^4$ is also the residual intersection of a quadric cone with a cubic primal containing two planes of the cone of the same system.

For the Del Pezzo surfaces in general, reference may be made to G. Timms, *Proc. Roy. Soc.* A, CXIX, 1928, p. 213. Those of these surfaces which are not representable on a plane by non-singular cubic curves are there referred to on p. 218 and in the diagram on p. 238.

5. For the quartic surface in [4] which is the projection of the Veronese surface in [5], we easily find

$$\mu_0=4,\ \ \mu_1=6,\ \ \mu_2=3,\ \ \nu_2=6,\ \ d=0,\ \ t=1,\ \ p_n=0,$$

and the postulation, $\frac{1}{2}\mu_0\rho(\rho+1)-\rho(p-1)+p_n+1$, for primals of order ρ is $(\rho+1)(2\rho+1)$; in accordance with this, the surface lies on no quadrics, and on 7 cubic primals. It can in fact be shewn that the surface can be represented on a plane (ξ, η, ζ) by taking the coordinates $x_0, \ldots, x_4$ respectively equal to ξ^2, $\xi\eta$, $\frac{1}{3}(\xi\zeta+2\eta^2)$, $\eta\zeta$, ζ^2; and that the coefficients in the sextic covariant of the quartic form $x_0t^4+4x_1t^3+6x_2t^2+4x_3t+x_4$, separately equated to zero, give 7 cubic primals containing the surface. Any two of these cubic primals meet further in a surface ${}^1\psi^5$, for which $\mu_0'=5$, $\mu_1'=10$, $\mu_2'=5$, $\nu_2'=10$, $\epsilon_0=10$, $\epsilon_1=30$, that is in a ruled quintic surface of genus 1, which meets the projected Veronese surface in a curve ${}^6c^{10}$, of order 10 and genus 6. And in fact, the definite trisecant chord which can be drawn to the Veronese surface from any point, outside this surface, which lies on a cubic primal containing the surface, will meet this primal in 4 points, and will therefore be entirely upon it; thus the residual surface of intersection of two cubic primals through the Veronese surface is composed of lines (cf. Bertini, *Iperspazi*, 1907, p. 232 ff.). In accordance with this (see Chap. VI, p. 266), it can be shewn that 3 of the cubic primals have as residual common intersection a set of five lines, and that 4 of them have no residual intersection; and further (as will be remarked below in more detail) the equation of the ∞^6 system of cubic primals is expressible in terms of four polynomial functions of the five coordinates. The surface ${}^1\psi^5[4]$ is obtainable, as is known (Segre, *Rend. Palermo*, II, 1888, No. 7), as the locus of lines meeting 5 arbitrary planes in the space [4] (cf. Vol. IV, p. 143); it is normal in the space [4], in accordance with the rule (Segre, *Rend. Palermo*, II, 1888, p. 45) that an elliptic ruled surface of order n is normal in space $[n-1]$. We have already (under (4)b) referred to a *rational* ${}^1\psi^5$, obtainable by projection of a Del Pezzo ${}^1\psi^5[5]$; in general, the normal space of a rational surface ${}^p\psi^n$ is $[n-p+1]$, as we see by considering the complete characteristic series of the representing plane curves. The cubic primals through the projected Veronese surface have a number of double points given by

$$\delta+2d=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2,$$

namely, for $\rho=3$, we have $\delta=10$. These primals are then Segre rational cubic primals with 10 nodes (cf. Semple, *Trans. Roy. Soc.* A, CCXXVIII, 1929, p. 253), and the trisecants of the Veronese surface involved are one of the 6 systems of lines on such a primal. The ∞^6 system of primals meets an arbitrary prime space [3] in cubic surfaces of this space containing the rational quartic curve in which the prime cuts the Veronese surface, and hence in the complete system of such cubic surfaces through this curve, which is also ∞^6; and hence, as a unique trisecant of the Veronese surface passes through an external point, the primals are in (1, 1) correspondence with these cubic surfaces (or with the cubic cones which project these, and the quartic curve, from an arbitrary point).

Ex. 1. If we use a, b, c, d, e for the coordinates in the space [4], the system of cubic primals through the Veronese surface may be represented by

$$\begin{aligned}\lambda_1(a^2d-3abc+2b^3)&+\lambda_2(a^2e+2abd-9ac^2+6b^2c)\\&+5\lambda_3(abe-3acd+2b^2d)+10\lambda_4(b^2e-d^2a)\\&-\lambda_7(e^2b-3cde+2d^3)-\lambda_6(ae^2+2bde-9ec^2+6d^2c)\\&-5\lambda_5(ade-3bce+2bd^2)=0,\end{aligned}$$

where the cubic expressions are those of the sextic covariant of the binary form $at^4+4bt^3+\ldots+e$; if h be the Hessian of this form f, and we recall that the sextic covariant $J_{\kappa\lambda}$ of the quartic form $\kappa f+\lambda h$ is a multiple of the covariant of f (in fact, with proper multiples of f and h, if I, J be the invariants of f, we have $J_{\kappa\lambda}/J_{10}=\kappa^3-\frac{1}{2}I\kappa\lambda^2-\frac{1}{3}J\lambda^3$ (Clebsch-Lindemann-Benoist, *Géométrie*, I, 1879, p. 302)), it follows that, save for a common factor, the seven cubic forms above are unaffected by writing for a, b, c, d, e respectively $\kappa a+ac-b^2$, $\kappa b+\frac{1}{2}(ad-bc)$, $\kappa c+\frac{1}{6}(ae+2bd-3c^2)$, $\kappa d+\frac{1}{2}(be-cd)$, $\kappa e+ce-d^2$ (cf. for instance, Salmon, *Higher Algebra*, 1885, p. 192), where κ is arbitrary. We may in particular take κ so that the middle term of these vanishes; the cubic forms are then in the ratios of the forms obtained by writing A, B, 0, D, E respectively for a, b, c, d, e, where $A=a(ae+2bd-3c^2)-6c(ac-b^2)$, etc.; namely

$$A=a^2e+2abd-9ac^2+6b^2c,\quad B=abe-3acd+2b^2d,$$
$$E=ae^2+2bde-9ec^2+6cd^2,\quad D=ade-3bce+2bd^2.$$

Thereby the ∞^6 cubic primals are represented by

$$\begin{aligned}\lambda_1(A^2D+2B^3)+10\lambda_4(B^2E-D^2A)-\lambda_7(E^2B+2D^3)&\\+(AE+2BD)(\lambda_2A+5\lambda_3B-5\lambda_5D-\lambda_6E)&=0.\end{aligned}$$

In the space [4], the coordinates A, B, D, E are in fact those of the intersection of the trisecant line of the Veronese surface from the point (a, b, c, d, e) with the prime $x_2=0$; and the last equation may then be regarded as that of the cubic surfaces in this prime which pass through the quartic curve given by $x_0/4\lambda^4=x_1/2\lambda^3=-x_3/\lambda=x_4$ (which has two points of inflexion). In order that a binary sextic form $\xi_0 t^6+6\xi_1 t^5+\ldots+\xi_6$ should be of the form of the sextic covariant of a binary quartic, that is, should have for roots three pairs each harmonic in regard to both the other pairs, the point $(\xi_0, \ldots, \xi_6)$, of a space [6], must lie on 5 quadrics (expressing the identical vanishing of the covariant $(ab)^4 a_x^2 b_x^2$ of a binary sextic a_x^6)

$$\xi_0\xi_4-4\xi_1\xi_3+3\xi_2^2=0,\quad \xi_2\xi_6-4\xi_5\xi_3+3\xi_4^2=0,\quad \xi_0\xi_5-3\xi_1\xi_4+2\xi_2\xi_3=0,$$
$$\xi_1\xi_6-3\xi_2\xi_5+2\xi_3\xi_4=0,\quad \xi_0\xi_6-9\xi_2\xi_4+8\xi_3^2=0$$

(Clebsch, ***Binäre Formen***, pp. 440, 447), and the point $(\xi_0, \ldots, \xi_6)$ then lies on a certain manifold of order 5 and dimension 3, M_3^5, in the space [6]. By what we have seen, the prime sections of this manifold are represented by the cubic surfaces through a rational quartic curve in space [3]. Precisely, with $\Omega=AE+2BD$, the coordinates $(\xi_0, \ldots, \xi_6)$ in [6] are respectively

$$\xi_0=A^2D+2B^3,\quad \xi_3=\tfrac{1}{2}(B^2E-D^2A),\quad \xi_6=-(E^2B+2D^3),\quad \xi_1=\tfrac{1}{6}\Omega A,$$
$$\xi_2=\tfrac{1}{3}\Omega B,\quad \xi_4=-\tfrac{1}{3}\Omega D,\quad \xi_5=-\tfrac{1}{6}\Omega E;$$

and the cubic surfaces in the space [3] are obtained by projection of the prime sections of the M_3^5 from the plane $\xi_1=\xi_2=\xi_4=\xi_5=0$. As the so-called octahedral relation of the six roots of the binary sextic form is unaffected by a linear transformation $t'=(mt+n)/(pt+q)$ of the variable t of the form, it follows that the M_3^5, and the original space [3], are capable of a continuous group of birational transformations involving 3 arbitrary parameters, which is linear in $\xi_0, \ldots, \xi_6$, and interchanges the cubic surfaces of the space [3] passing through a rational quartic curve among themselves. See Enriques-Fano ("Sui gruppi continui di transformazioni cremoniane dello spazio"), *Ann. d. Mat.* XXVI, 1897, p. 92, and Scorza, *Ann. d. Math.* XV, 1908, where M_3^5 is studied as a manifold M_k^n whose curve sections are elliptic. If we represent the lines of space [4] by points in a space where the 10 coordinates are the line coordinates, we obtain a M_6^5[9]. The M_3^6 is the section of this by a general space [6].

Ex. 2. Fano has proved (*Atti...Torino*, XXXIX, 1904; see below, III, p. 295) that on a general cubic primal in space [4] there exists no surface (meeting all the lines of the primal in the same number of points) which is not the complete intersection of the cubic primal with another primal. The projected Veronese surface considered here (of order 4) is not a complete intersection of the 10-nodal Segre cubic primal on which it lies. It is, however, when taken with a degenerate quintic surface, the complete intersection of the cubic primal with a cubic primal (a cone). It can be shewn that through a node of the Segre primal there pass six planes lying thereon, consisting of three planes, say ξ, η, ζ, which meet only at the node, and three planes, say α, β, γ, also meeting only at the node, but all meeting each of ξ, η, ζ in lines through the node (these being the lines joining the node to the other 9 nodes of the primal). Further that ∞^3 cubic cones, each a primal, can be described with vertex at the given node, to have the plane ξ as a double plane and to contain the planes α, β, γ. The residual intersection of such a cubic cone with the Segre primal is a projected Veronese surface. We may consider, for the Veronese surface, the plane ξ traced by the trisecants of the surface

7. Consider a surface Ψ_m for which

$$\mu_0=\tfrac{1}{2}m(m+1),\ \mu_1=\tfrac{2}{3}m(m^2-1),\ \mu_2=\tfrac{1}{8}m(m^2-1)(5m-6),$$
$$\nu_2=\tfrac{1}{12}m(m^2-1)(3m-2).$$

For this surface it can then be deduced that the number of accidental double points, the genus of a prime section, p, and the invariants I, p_n, are given by

$$d=0,\ p=\tfrac{1}{6}(m-1)(m-2)(2m+3),$$
$$I+4=\tfrac{1}{24}m(m+1)(15m^2-65m+86),$$
$$12(p_n+1)=\tfrac{1}{2}m(m+1)(3m^2-17m+26),$$

an alternative form for p_n being $\frac{1}{24}(m-1)(m-2)(m-3)(3m+4)$, and further that there are $4m+5$ primals F^{m+1} of order $m+1$ passing through this surface, each having $\frac{1}{12}m(m+1)^2(m+2)$ nodes at ordinary points of the surface.

Shew that two such F^{m+1} through Ψ_m intersect further in a Ψ_{m+1}, which meets Ψ_m in a curve of order ϵ_0 given by

$$\epsilon_0=\tfrac{1}{3}m(m+1)(m+2).$$

Shew that Ψ_m is the surface represented by the vanishing of all minors of order m in a matrix of m rows and $m+1$ columns in which each element is linear in the coordinates.

The surface Ψ_1 is a plane, the surface Ψ_2 is the ruled cubic, the surface Ψ_3 is the Bordiga surface ${}^3\psi^6$, and so on. The *quartic* primals through Ψ_4 are ∞^4 and give a Cremona transformation of the space [4], expressible by four lineo-linear relations connecting the coordinates of the corresponding points. In fact these quartic primals determine, on the Bordiga surface which is the residual intersection of two of them, the net of rational quartic curves corresponding to the lines of the plane on which the Bordiga surface may be represented. In general 4 primals of order m through Ψ_m meet further in $\frac{1}{24}m(m-1)(m-2)(m-3)$ points.

8. The surface which is the residual intersection of two cubic primals having a plane in common, has, by the results of Chapter VI, p. 265, for characters

$$\mu_0=8,\ \mu_1=28,\ \mu_2=68,\ \nu_2=28,\ \omega-1=0,\ I+4=36,\ p_n+1=3,\ d=0;$$

it is a surface ${}^7\psi^8$, meeting the plane in a general quartic curve, on which curves of the canonical system are given by primes through the plane.

9. If, in space [4], two cubic primals be taken through a quadric surface (in ordinary space, lying in the [4]), for which

$$\mu_0=2,\ \mu_1=2,\ \mu_2=2,\ \nu_2=0,$$

the formulae of the last chapter give, for the residual ${}^5\psi^7$,

$$\mu_0'=7,\ \mu_1'=22,\ \mu_2'=48,\ \nu_2'=20,\ p_n'=1,\ \omega'=0,\ I'=21,$$

and this meets the quadric surface in a curve ${}^4c^6$. Each of the two cubic primals contains, in the space of the quadric surface, also a plane; the line of intersection of these planes lies on the ${}^5\psi^7$. Canonical curves of this surface are given by the solid, or space [3], containing the quadric surface, which meets the ${}^5\psi^7$ (beside in the ${}^4c^6$) only on the line of intersection of these two planes. If the equations of the two cubic primals be written $u^3+u^2P_1+uP_2+Q_1\Omega=0$ and $u^3+u^2P_1'+uP_2'+Q_1'\Omega=0$, the surface is the projection of the surface ${}^5\psi^8$ in space [5], which is the intersection of the three quadrics $u^2+uP_1+P_2+Q_1v=0$, $u^2+uP_1'+P_2'+Q_1'v=0$, $\Omega-uv=0$ (the coordinates in the [5] being x, y, z, t, u, v). For this surface $p_g=p_n=1$, $\omega=1$, $I=20$; the exceptional line of ${}^5\psi^7$ lies in the tangent plane of the new surface at the point of projection $(0, 0, 0, 0, 0, 1)$, and is the transformation of this point.

10. The surface which is the residual intersection of two cubic primals which have two planes in common is, by the formulae of Chap. VI, p. 265, of characters

$$\mu_0=7,\ \mu_1=20,\ \mu_2=34,\ \nu_2=20,\ d=1,\ p_n=0,\ \omega=-2,\ I=11,$$

so that the surface is a ${}^4\psi^7$; it meets each plane in a quartic curve with a double point (at the intersection of the two planes); each of the cubic primals has also 3 double points at ordinary points of each plane quartic curve. The characters are those of the rational surface whose prime sections are represented by plane quintic curves with 2 double and 10 simple base points, $c^5(2^2, 1^{10})$. The formula for the postulation gives two cubic primals passing through the ${}^4\psi^7$.

The ${}^4\psi^7[4]$ is evidently the projection of a rational ${}^4\psi^8[5]$ represented by quintic curves $c^6(2^2, 1^9)$. When the 11 base points of this system of plane curves lie on a cubic curve, the surface ${}^4\psi^8[5]$ has a double line (Babbage, referred to, p. 246, Ex. 2, above). The reader may prove that the surface represented by the system of plane septimics $c^7(3, 2^8, 1)$, with one triple point, eight double points, and one simple base point, is a ${}^4\psi^7[4]$ which has a double line (cf. Ex. 2, p. 246).

11. Consider the residual intersection of two cubic primals passing through the aggregate of a quadric surface (in space [3]) and a plane not lying in the space of the quadric surface. For the plane we have $\mu_0=1$, $\mu_1=0$, $\mu_2=0$, $\nu_2=0$, and for the quadric surface we have $\mu_0=2$, $\mu_1=2$, $\mu_2=2$, $\nu_2=0$; thus as neither of these surfaces has a double point (though they intersect in two points, each therefore an accidental double point of the composite surface) we may take for the composite surface (cf. Ex. 1, p. 262), $\mu_0=3$, $\mu_1=2$, $\mu_2=2$, $\nu_2=0$ and $p_n=1$. Whence we compute for the residual intersection of two cubic primals through the composite surface

$\mu_0'=6$, $\mu_1'=14$, $\mu_2'=20$, $\nu_2'=12$, $\epsilon_0=10$, $\epsilon_1=22$, $d'=2$, $p'=2$, $p_n'=0$, $I'=6$, $\omega'=3$, which are the characters of the rational surface ${}^2\psi^6$ whose prime sections are represented by the system of plane quartic curves $c^4(2, 1^6)$, with one double point and six simple base points. Using the formula $\delta+2d=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2$, we find that a cubic primal through the composite surface has 4 double points not at the intersections of the quadric surface with the plane; and detailed algebra shews that two of these are on the quadric surface and two on the plane. Similarly the formula

$$\tfrac{1}{2}\rho(\rho+3)\mu_0-\tfrac{1}{2}\rho\mu_1+p_n+1-2$$

applied to the composite surface leads to 11 as the number of cubic primals containing this composite surface, which likewise may be directly verified. (Cf. Segre, *Mem. Torino*, XXXIX, 1888.)

Many of the simple cases considered here for intersections of cubic primals occur in the papers of J. G. Semple, *Trans. Roy. Soc.* A, CCXXVIII, 1929, p. 331; *Proc. Camb. Phil. Soc.* XXV, 1929, p. 145; *Proc. Lond. Math. Soc.* XXXII, 1931, p. 369. For the characters of composite surfaces, a paper by L. Roth, *Proc. Camb. Phil. Soc.* XXIX, 1933, p. 88, and a paper by J. G. Semple, *Proc. Royal Irish Academy*, 1933, may be consulted. (Also, Segre, *Atti Torino*, XXII, 1887.)

II. The surface representing pairs of points of one or two curves. As an example of an irregular surface, for which the geometric genus p_g exceeds the arithmetic genus p_n, we have already considered (Chap. V, p. 203), after Castelnuovo, the surface of order $2n$ having a n-ple node at each of the 8 associated intersections of three quadric surfaces. We consider now another wide class of irregular surfaces, those namely of which the general point represents a pair of points one on each of two given curves. A particular case is that in which the curves coincide in one curve, and we consider the *ordered* pairs of points thereon. But we may also consider the surface representing the pairs of points of a single curve, without regard to the order of the points of a pair; each point of this surface corresponds then to two points of the surface representing the *ordered* pairs of points. We give an account first of a method followed by De Franchis (*Rend. Palermo*, XVII, 1903; see also Maroni, *Atti...Torino*, XXXVIII, 1903); and then, for the surface representing the pairs of points of a single curve without regard to order, of a descriptive method followed by Severi (*Atti...Torino*, XXXVIII, 1903).

Consider two curves f, ϕ, of respective genera p, π. We suppose that, to every pair of points, one on each curve, corresponds a definite point of a surface (with coordinates rational in the coordinates of the two points of the curves), and, conversely, to any point of this surface corresponds a unique point of each curve (with coordinates rational in the coordinates of the point of the surface). Such a surface is obtainable by supposing the curves f, ϕ, to be in perfectly general positions, in space of sufficiently high order; then that every point of either curve is joined by a line to every point of the other curve; and then, that a section is taken by a prime of the space, which will meet every joining line in one point. The locus of these points is the surface in question. More analytically, if we suppose the two curves to be plane curves, in general positions in different planes of a space of 4 dimensions, with equations respectively $f(x, y)=0$ and $\phi(\xi, \eta)=0$, the surface in question may be that of which a point has (non-homogeneous) coordinates x, y, ξ, η. If A be any definite point of the curve f, and Q a point of the curve ϕ, the point (A, Q) of the surface considered, as Q describes the curve ϕ, will describe a curve on the surface, of genus π; and, for different positions of A on the curve f, we thus obtain a system of curves on the surface, with the property that one such curve passes through an arbitrary point (A, B) of the surface. These form then what has been previously defined as an irrational pencil of curves on the surface (Chap. v, p. 190), the genus of this pencil being that of the curve f, of which every point A determines one such curve; namely p. In particular when $p=0$, the pencil is rational. In the same way we have another irrational pencil, of curves each of genus p, the genus of this pencil being π, of which every curve is the locus of points (P, B) of the surface, wherein B is a definite point of the curve ϕ, as P describes the curve f. Through any point (A, B) of the surface there passes one curve of each pencil, and conversely any curve of either pencil has just one intersection with any curve of the other pencil. If we consider any (algebraical) curve on the surface, the curve of the first pencil spoken of which passes through an arbitrary point U of this curve, will cut thereon a set of points determined by U, or equally by any other point of the same set, that is an involution; and the genus of this involution will be that of the first pencil, namely p. On the same curve there will likewise be another involution of genus π, determined by curves of the second pencil. Unless both p and π be zero, one of these involutions is irrational; and no irrational involution exists on a rational curve (Vol. II, p. 136). Thus, unless both p and π be zero, the surface under consideration contains no rational curves. In particular, therefore, the surface contains no exceptional curves of the kind which can arise by transformation of simple points of another surface. In what follows we do not consider cases in which either p or π is zero, for a surface which contains a pencil of rational curves is either rational ($p=\pi=0$), or reducible to a ruled surface of genus equal to that of the pencil, as we have remarked (Chap. III, p. 147).

We can at once make statements, based on the representation of the surface as the locus of points (x, y, ξ, η) in space of four dimensions, where $f(x, y)=0$ and $\phi(\xi, \eta)=0$, in regard to the everywhere finite simple integrals belonging to the surface (Picard et Simart, *Fonctions algébriques*, I, p. 130), and in regard to the everywhere finite double integrals (*loc. cit.* p. 197). The number of the simple integrals is in fact $p+\pi$, and the number of the double integrals is $p\pi$. In regard to the former, if $\int l_i(x, y)\, dx/f'(y)$ and $\int \lambda_j(\xi, \eta)\, d\xi/\phi'(\eta)$ be finite integrals for the curves f, ϕ respectively ($i=1, \ldots, p$; $j=1, \ldots, \pi$), it is clear that the sum of these is a finite simple integral for the surface. Conversely, every such integral is necessarily of the form $\int P(x, y, \xi, \eta)dx+\Pi(x, y, \xi, \eta)d\xi$, where P and Π are rational functions; for this to be everywhere finite, the integral $\int P(x, y, \xi, \eta)dx$, for every definite (ξ, η) satisfying the equation $\phi(\xi, \eta)=0$, must be everywhere finite on $f(x, y)=0$; this integral must then be expressible as a sum of the integrals $\int l_i(x, y)dx/f'(y)$, each multiplied by a number independent of (x, y), and hence, here, depending on (ξ, η), namely by a rational function of (ξ, η). Any infinity of such a multiplier, regarded as a function of (ξ, η), cannot be compensated by infinities arising in the part $\int \Pi(x, y, \xi, \eta)d\xi$, since this must be finite everywhere in regard to (ξ, η). All the multipliers in question must then be absolute constants. In the same way the part $\int \Pi(x, y, \xi, \eta)d\xi$ must be a sum of constant multipliers of the everywhere finite integrals for the curve $\phi(\xi, \eta)$. We thus obtain $p+\pi$ everywhere finite simple integrals for the surface, and no more. A very similar argument shews that the general everywhere finite double integral for the surface is a sum of constant multipliers of the $p\pi$ integrals

$$\iint l_i(x, y)\, \lambda_j(\xi, \eta)\, dx\, d\xi / f'(y)\, \phi'(\eta).$$

From this it follows, defining the geometric genus p_g of the surface, in the manner originally proposed by Clebsch, as the number of linearly independent everywhere finite double integrals, that we have $p_g=p\pi$; and, defining the canonical curves of the surface as those for which the integrand of the double integral vanishes (that is, by the number of adjoint surfaces of appropriate order), we see that such a canonical curve is here equivalent on the surface to the composite curve consisting of $2p-2$ curves of the irrational pencil of which each curve arises from a particular point of the curve f, taken with $2\pi-2$ curves of the irrational pencil of which each curve arises from a particular point of the curve ϕ. From this last fact we can obtain the grade of the canonical system, and the genus of a canonical curve. For, if C, Γ denote respectively two such aggregates of respectively $2p-2$ and $2\pi-2$ curves from these two pencils, and C', Γ' two other such aggregates, since no two curves of the same pencil have an intersection, and any curve of either pencil has one intersection with any curve of the other pencil, this grade $p^{(2)}$ of the canonical system, $(C+\Gamma, C'+\Gamma')$, is equal to $(C, \Gamma')+(C', \Gamma)$, namely to $2(2p-2)(2\pi-2)$. For the genus, $p^{(1)}$, of a curve of the canonical system, if we use the formula p_1+p_2+i-1 for the genus of a curve consisting of two curves of genera p_1 and p_2 with i intersections, we have, first, for a composite curve consisting of $2p-2$ curves of the first pencil, the genus $(2p-2)\pi-(2p-3)$, and, similarly, for the aggregate of $2\pi-2$ curves of the second pencil, the genus $(2\pi-2)p-(2\pi-3)$; also, the total number of intersections of the former composite curve with the latter is $(2p-2)(2\pi-2)$. Whence, the genus of a canonical curve is given by $p^{(1)}=(2p-2)\pi-(2p-3)+(2\pi-2)p-(2\pi-3)+(2p-2)(2\pi-2)-1$, which is $2(2p-2)(2\pi-2)+1$. This verifies the relation previously (Chap. v, p. 223) remarked, $p^{(2)}=p^{(1)}-1$. From the same point of view we can obtain the invariant I of the surface, if we recall the formula previously obtained (Chap. v, p. 190), $I+4=\Delta+(2p-2)(2p'-2)$, for a surface on which there exists an irrational pencil, of genus p', of curves of genus p, among which there are Δ curves with actual double points. Applying this to either of the pencils on the surface under consideration, with the assumption that any point of this surface corresponds to only one point of both curves, so that $\Delta=0$, we have $I+4=(2p-2)(2\pi-2)$. The invariant ω of the surface, as there are no exceptional curves obtainable by transformation from simple points of another surface, is (see Chap. v, pp. 184, 228) given by $\omega=p^{(1)}$. Whence, Noether's relation $I+4+\omega-1=12(p_n+1)$ (*ibid.* p. 185), gives here

$$p_n+1=(p-1)(\pi-1),$$

for the arithmetic genus.

And we thus prove that the number of everywhere finite simple integrals belonging to the surface is equal to p_g-p_n. This is a very noticeable result, the proof that this is always the case, for any surface, having been completed only in 1905, and with the help of results not employed in this earlier theory. Summarising the results obtained we may say that

$$\tfrac{1}{8}(\omega-1)=\tfrac{1}{4}(I+4)=p_n+1=(p-1)(\pi-1),\quad p^{(1)}=\omega,\quad p_g=p\pi.$$

It has been assumed, as stated, that $p>0$ and $\pi>0$; if either p or π be 1, the results remain, as may be proved directly.

We pass now to consider the surface of which any point represents a pair of points of a given curve, without regard to the order of these points. We study this surface, ψ, in association with the surface, ψ_0, which represents the pairs of points of the same curve when the order of the points of a pair is regarded. Thus ψ_0 is the surface which we have just discussed in the case when the two curves f, ϕ are replicas of one another (or identical). Any general point of the surface ψ thus corresponds to two points of the surface ψ_0, or these surfaces (ψ, ψ_0) are in (1, 2) birational correspondence. But there is a curve c on the surface ψ which represents the pairs of coincident points of the fundamental curve, f, and there is a curve c_0 on the surface ψ_0 likewise representing the pairs of coincident points of the curve f; and the curves c, c_0 are birationally equivalent with one another, and with the curve f. Upon the surface ψ there will be a curve representing the couples of f which consist of a general point of f taken with all other points of f; this curve, γ, is birationally equivalent with f; and such curves γ will constitute a system, of which two pass through any general point of ψ (or a system of *index* 2), while any two of the curves have one point of intersection. In another point of view, thinking of f as a curve which does not lie in a plane, any curve γ, of this system on ψ, corresponds to the cone of chords projecting f from any point of itself; any chord of this cone belongs to two such cones, and any two such cones have one generator in common. The common generator of two such cones, with vertices at O and O', becomes the tangent line of the curve f at O when O' moves to coincidence with O. Thus two consecutive curves of the system (γ) intersect on the coincidence curve c. In other words all the curves γ touch c. To every curve γ on ψ corresponds two curves on the surface ψ_0, intersecting one another on the curve c_0, at the point which corresponds to the point of contact of γ with c.

To any linear system $|C|$ on the surface ψ, whereon the coordinates of a point are rational functions of the coordinates of either of the two corresponding points of the surface ψ_0, there corresponds a linear system $|C_0|$ on the surface Ψ_0, there being two points of a curve C_0 arising from a single point of C. And, if π, π_0 be the genera respectively of C and C_0, and so the number of coincidences on C_0 of a pair of points arising from the same point of C, it follows, from the Zeuthen correspondence formula (Chap. I, p. 19) that $2(2\pi-2)=2\pi_0-2-s_0$; the number s_0 is the number of intersections of the curve C_0 with the coincidence curve c_0 on ψ_0, and therefore also the number of intersections of C, on ψ, with the coincidence curve c on ψ; thus we write s for s_0. If ν, ν_0 be the respective grades of the systems $|C|$, $|C_0|$, we have $\nu_0=2\nu$; and if d, d_0 be the respective numbers of double points of linear pencils of curves contained in $|C|$ and $|C_0|$, the invariants I, I_0 of ψ and ψ_0 are such that $I=d-\nu-4\pi$, $I_0=d_0-\nu_0-4\pi_0$. To a double point of a curve of $|C|$ which does not lie on the coincidence curve c, correspond two double points of the corresponding curve of $|C_0|$, not lying on c_0; but to a double point of a curve C_0 which lies on c_0, corresponds a contact of the corresponding curve C with the curve c; and conversely. Thus we can express d_0 in terms of d; for a linear pencil of curves in the system $|C|$ determines, on the coincidence curve c of ψ, a linear series g_1^s, of sets of s points, of freedom 1; and there are $2(s+p-1)$ coincidences on c of pairs of a set of this linear series, where p is the genus of c, and therefore of the fundamental curve f; each of these coincidences determines, in the simplest case, a double point of a curve of the corresponding linear pencil of curves contained in $|C_0|$. Wherefore we have $d_0=2(d+s+p-1)$. From this then we deduce

$$\begin{aligned} 2(I+4)-(I_0+4) &= 2d-d_0-2\nu+\nu_0-2[2(2\pi-2)-(2\pi_0-2)] \\ &= -2(s+p-1)+2s=-2(p-1); \end{aligned}$$

we have proved, however, above, that $I_0+4=4(p-1)^2$. Wherefore,

$$2(I+4)=4(p-1)^2-2(p-1), \quad \text{or} \quad I+4=(p-1)(2p-3).$$

We consider now the adjoint systems of curves, on ψ and ψ_0, of the systems $|C|$ and $|C_0|$, denoting these adjoint systems respectively by $|C'|$ and $|C_0'|$. We have seen previously, in connexion with the proof of the Zeuthen correspondence formula quoted above, that a canonical set on any curve C_0 contains, beside the set directly corresponding to a canonical set on C, also the set of coincidences on C_0, where two points of C_0 corresponding to a point of C coincide with one another. Whence, if $t|C'|$ denote the linear series on ψ_0 arising by direct transformation of the system $|C'|$ on ψ, we have $|C_0'|=t|C'|+c_0$, it being assumed, to save explanations, that there are no exceptional curves on ψ or ψ_0. Whence, the canonical systems on ψ, and ψ_0, which we may define by $K=|C'-C|$ and $K_0=|C_0'-C_0|$, are also such that $|K_0|=t|K|+c_0$, where $t|K|$ denotes the direct transformation of $|K|$. From this equation we can infer an equivalence for the curves of $|K|$ which gives us much information in regard thereto. For, first, there are curves γ on ψ, each representing couples of f of which one point is fixed, each of which corresponds to two of the curves of ψ_0 which have a similar definition; and we have seen that a degenerate curve of K_0 consists of $2(2p-2)$ curves of this latter kind; this is then the direct transformation of $2p-2$ curves γ on ψ. And second, the direct transformation of the curve c on ψ is twice the curve c_0 on ψ_0. We may then regard a curve of $|K_0|-c_0$ as equivalent to the direct transformation of $(2p-2)$ curves γ on ψ less half the curve c on ψ. Hence we may regard any curve of $|K|$ as equivalent to $(2p-2)$ curves γ, less half the curve c, and write $K\equiv(2p-2)\gamma-\frac{1}{2}c$.

Hence, if ξ denote the virtual grade of the curve c, and s' the number of intersections of c with a curve of the adjoint system $|C'|$, recalling that the curves γ all touch c, we infer that

$$s'-s=2(2p-2)-\tfrac{1}{2}\xi.$$

This, however, being the number of intersections of c with a curve of the canonical system of ψ, is equal to $2p-2-\xi$ (Chap. v, p. 222), provided it be assumed that the curve c has no multiple points, so that its complete genus is the same as its actual genus. Whence we infer that $\xi=-2(2p-2)$, and $s'-s=3(2p-2)$.

In passing we remark that the virtual grade ξ_0, of c_0, on ψ_0, is given by the fact that the virtual grade of $t(c)=2\xi$, that is the virtual grade of $2c_0=2\xi$, so that $4\xi_0=2\xi$, leading to $\xi_0=-(2p-2)$. We can also find this directly on ψ_0, since $2(2p-2)=$ number of intersections of a canonical curve on ψ_0 with c_0, $=2p-2-\xi_0$.

From the equivalence for the canonical curve on ψ, we can find the invariant ω for the surface, assuming that $\omega-1$ is the grade of the canonical curves. For such curve being $\equiv(2p-2)\gamma-\frac{1}{2}c$, and because $(\gamma,\gamma)=1$, $(\gamma, c)=2$, $(c, c)=\xi$, we have

$$\omega-1=(2p-2)^2-2(2p-2)-\tfrac{1}{4}\xi=(p-1)(4p-9).$$

We have found however that $I+4=(p-1)(2p-3)$. Hence $p_n+1=\frac{1}{12}[I+4+\omega-1]=\frac{1}{2}(p-1)(p-2)$. A somewhat different method of obtaining $\omega-1$ is to remark that this is the difference, $\zeta'-\zeta$, of the canonical numbers, ζ' and ζ, of the adjoint system $|C'|$ and the system $|C|$, as is clear because the canonical number of a curve is the number of its intersections with a canonical curve. If, as before, C_0 be the curve on ψ_0 obtained by direct transformation of the curve C on ψ, and ν, ν_0 be the grades of $|C|$ and $|C_0|$, and ζ, ζ_0 their canonical numbers, we have

$$\zeta=2\pi-2-\nu,\quad \zeta_0=2\pi_0-2-\nu_0,\quad 2(2\pi-2)=2\pi_0-2-s,\quad \nu_0=2\nu;$$

these lead to $2\zeta=\zeta_0-s$. If this result be applied to the adjoint system $|C'|$, we have $2\zeta'=\zeta_1-s'$, where ζ_1 is the canonical number of the curve obtained by direct transformation of C'; this direct transformation, we have remarked, is $C_0'-c_0$, and we have found that the canonical number of c_0 is ($2p-2-\xi_0$ or) $2(2p-2)$. Whence $2\zeta'=\zeta_0'-s'-2(2p-2)$. But also $\zeta_0'-\zeta_0$, or ω_0-1, was proved to be $8(p-1)^2$, and also $s'-s=3(2p-2)$. Wherefore

$$\begin{aligned}2(\omega-1)=2(\zeta'-\zeta)&=\zeta_0'-\zeta_0-(s'-s)-2(2p-2)\\&=8(p-1)^2-5(2p-2),\end{aligned}$$

giving the equation $\omega-1=(p-1)(4p-9)$ found above.

This argument may be regarded as an introduction to the general investigation of the relation between the invariants of two surfaces in multiple correspondence (see Note, p. 294, below).

To obtain the invariant p_g of the surface ψ, we investigate the everywhere finite double integrals belonging to the surface. If the fundamental curve be regarded as a plane curve, given by $f(x,y)=0$, of which the general everywhere finite integral is u_i^x, of the form

$\int \phi_i(x)\,dx/f'(y)$, the general everywhere finite double integral of the surface, as in the case of the surface ψ_0, is $\iint \Sigma A_{ij}\,du_i^x\,du_j^\xi$, for $i, j=1, \ldots, p$, namely is $\Sigma A_{ij} \iint F_{ij}\,dx\,d\xi/f'(y)f'(\eta)$, where, subject to $f(\xi, \eta)=0$, we have $F_{ij}=\phi_i(x)\,\phi_j(\xi)-\phi_i(\xi)\,\phi_j(x)$. But as, in this case, the points (x, y), (ξ, η) may be taken in either order, we must have $A_{ji}=-A_{ij}$, and in particular $A_{ii}=0$. Hence $p_g=\frac{1}{2}p(p-1)$. And, with the value found above for p_n, this leads to

$$p_g-p_n=\tfrac{1}{2}p(p-1)-\tfrac{1}{2}(p-1)(p-2)+1,$$

which is p. This again is the same as the number of everywhere finite simple integrals belonging to the surface, for it may be proved as before that these integrals are

$$\int \phi_i(x)\,dx/f'(y)+\int \phi_i(\xi)\,d\xi/f'(\eta).$$

We may interpret the function F_{ij} as follows: In every set of the pencil of canonical sets given by the equation $\phi_i(x)+\lambda\phi_j(x)=0$, there are $(p-1)(2p-3)$ couples. The locus of the points of the surface ψ representing these couples is a canonical curve on ψ. For from $\phi_i(x)+\lambda\phi_j(x)=0$ and $\phi_i(\xi)+\lambda\phi_j(\xi)=0$, we have

$$\phi_i(x)\,\phi_j(\xi)-\phi_j(x)\,\phi_i(\xi)=0.$$

For instance, if the curve f be the intersection of a quadric and a cubic surface, a pencil of canonical sets is given on the curve by planes through an arbitrary line; and the points of the surface ψ representing the 15 couples of such a set describe such a curve. Thus, more generally, the canonical system of curves on ψ, in this case, represents the chords of the sextic curve which belong to a linear complex.

The values found above for a general value of p become, for $p=1$, respectively $I=-4$, $\omega=1$, $p_n=-1$, $p_g=0$, which (since, for a ruled surface of genus p, we have $I=-4p$, $\omega-1=-8(p-1)$) are the values appropriate to an elliptic ruled surface. For $p=1$ the surface ψ must contain an elliptic pencil of rational curves (there is an involution on a plane cubic curve determined by lines through an arbitrary point of the curve), and is hence (cf. Chap. III, p. 147) reducible to an elliptic ruled surface. The general formulae have been found on the assumption $p>0$; but, in fact, for $p=0$ they lead to the values $I=-1$, $\omega=10$, $p_n=0$, $p_g=0$, appropriate to a rational surface, in accordance with the fact that a rational symmetric function of the parameters θ, θ' of any two points of a rational curve, is a rational function of the sum and product, $\theta+\theta'$ and $\theta\theta'$, of these. In general, however, the preceding theory requires further examination when the fundamental curve is hyperelliptic with $p>1$, since there exists, then, a rational series g_1^2 of pairs of points on the curve, giving a rational curve on the surface ψ; while we have assumed that the surface has no (rational) exceptional curves.

Ex. Examine the canonical curves for the surface, in space (x, y, z), expressed by $x=t_1+t_2$, $y=t_1t_2$, $z=s_1+s_2$, where $s_1^2=F(t_1)$, $s_2^2=F(t_2)$, in which F is a polynomial of order 6. Shew that for this surface $p_g=1$, $p_n=-1$. Also shew that for the surface expressed by $x=t_1+t_2$, $y=t_1t_2$, $z=s_1s_2$, we have $p_g=1$, $p_n=1$.

We pass now to a more geometrical investigation (due to Severi, as has been said). Consider the canonical curve ${}^{p}c^{2p-2}[p-1]$, of order $2p-2$ and genus p, in space $[p-1]$. The ∞^2 chords of this curve are representable by a surface, lying in a space of dimension $\frac{1}{2}p(p-1)-1$, wherein any point has for coordinates the line coordinates of a line of the space $[p-1]$. The general problem is to find the order of this surface, which we denote by $p^{(2)}$ (with a subsequent justification of the notation), and the genus of its prime section, which we denote by $p^{(1)}$.

The character of the work will be sufficiently represented by taking the case when $p=4$, the canonical curve being the complete sextic intersection of a quadric and a cubic surface in ordinary space. The lines of this space are then represented by the points of a quadric Ω in space $[5]$; the surface to be examined is the surface lying on Ω which represents the chords of this sextic curve. The order of this surface is the number of its points lying on two primes of $[5]$; hence it is the number of chords of the curve ${}^{4}c^{6}[3]$ which belong to two arbitrary linear complexes; hence also it is the number of these chords which meet two arbitrary lines; and thus finally is the sum of the number of chords of the sextic curve which lie in an arbitrary plane and the number of chords which pass through an arbitrary point. For a curve of order n and genus p this would give $\frac{1}{2}n(n-1)+\frac{1}{2}(n-1)(n-2)-p$, or $(n-1)^2-p$; for $n=2p-2$ this is $(p-1)(4p-9)$, which for $p=4$ is 21. A prime section of the surface consists of points representing the chords of the sextic curve which belong to a linear complex; in particular chords meeting an arbitrary line; these form a ruled surface. These chords can be arranged as sets of a "linear series", g_1^{15}, of which one set consists of the chords in a plane through the arbitrary line. For a curve of order n and genus p one of these sets would consist of $\frac{1}{2}n(n-1)$ chords; a coincidence of two chords of such a set can arise only in a plane through the given line which contains a tangent line of the curve, and then there are $n-2$ such coincidences, the lines which join the point of contact, of the plane with the curve, to its remaining $n-2$ intersections. The total number of coincidences of pairs of lines in the same set of the linear series $g_1^{\frac{1}{2}n(n-1)}$ is thus $(n-2)(2n+2p-2)$. Thus, if q be the genus of the ruled surface formed by the chords of the curve which meet the given line, we have $2[\frac{1}{2}n(n-1)]+2q-2=(n-2)(2n+2p-2)$, leading to $q=\frac{1}{2}(n-2)(n-1+2p-2)$, which is $(p-2)(4p-5)$ when $n=2p-2$; this exceeds by 1 the number $(p-1)(4p-9)$ found for the order of the surface; and for $p=4$ it is equal to 22.

Ex. The chords of the sextic curve that meet a line may also be arranged as sets of 6, all passing through a point of the line, forming thus a linear series g_1^6. Determine the number of coincidences of lines of the same set, and hence find the genus q.

Let the curve under consideration, of which we have taken the sextic intersection of a quadric and a cubic surface as typical, be denoted by σ; and the ruled surface formed by the chords of the curve σ which meet a given arbitrary line, l, be denoted by ρ. The lines of ρ may be represented, as we have said, by points, in another space, forming a curve on the surface which represents all the chords of the given curve σ. It is therefore clear what is meant if we speak of a canonical set, K, of lines of the ruled surface ρ. We may now consider a multiple correspondence $(2, n-1)$ between the points of the curve σ, and the lines of the ruled surface ρ, that namely in which, to each point P of σ correspond the $n-1$ (or 5) lines of ρ which pass through P (that is, join P to the $n-1$ other points in which the plane Pl meets σ), and to each generator, G, of ρ, correspond the two points in which G meets σ. There is, on σ, a linear series g_1^n (or g_1^6), of sets of n points all lying in a plane through l; of this series the Jacobian set is formed by the points of σ whereat the tangent line meets l. This Jacobian set is then equivalent, on σ, to two sets of g_1^n together with a canonical set of σ, or, as σ is a canonical curve, to three times a canonical set of σ. Let a set of generators of ρ which lie in a plane (through l) be denoted by A; to a canonical set on σ, namely a set of coplanar points, corresponds in virtue of the correspondence spoken of, a set of $n(n-1)$ lines, forming a set $2A$ on the ruled surface ρ. Thus the points of σ whereat the tangent lines meet l give rise, by this correspondence, to a set on ρ equivalent to $6A$. The lines on ρ to which they give rise consist however of the $(2n+2p-2$, or 18) tangents (say B) of σ which meet l, together with the $18(n-2)$ chords of σ through a point of contact of such a tangent which lie in a plane through l; these latter chords, however, are the double elements of the various sets A, and are thus equivalent, on ρ, to a set $2A+K$. Thus we infer that on ρ we have the equivalence $6A \equiv B+2A+K$, or $K \equiv 4A-B$, where B denotes the lines of ρ which touch σ.

If, however, C denote a set of (6) chords of σ through any point of l, we can shew that, on ρ, $B \equiv 2A-2C$. For the lines of ρ which meet an arbitrary line form a set equivalent to the aggregate $A+C$ (of a set A lying in a plane through l, and a set C passing through a point of l); thus the lines of ρ which touch the quadric surface containing the curve σ (all such tangents forming a quadratic complex), being equivalent on ρ to twice a set belonging to a linear complex, are equivalent to a set $2A+2C$. On the other hand, the lines of ρ which touch this quadric surface consist of the set B touching the curve σ, and the generators of this quadric surface through the two points where it is met by l; the two generators through one of these points constitute the whole of a set C of chords of σ through this point; but, beyond touching the quadric surface, they lie upon it, and are, therefore, we assume, equivalent to a set $2C$. On the whole, therefore, we reach the conclusion

$$2A+2C\equiv B+4C,$$

as was stated.

Hence a canonical set of lines on ρ satisfies the equivalence $K\equiv 2A+2C$. Thus, on the surface in space [5] which represents the chords of the curve σ, whereof a prime section represents the generators of a ruled surface such as ρ (or more generally the chords of σ which belong to a linear complex), the canonical series on a prime section consists of sets, $2A+2C$, each the double of a set, $A+C$, of the characteristic series of the prime section. The canonical system of curves of the surface thus cuts a prime section in sets equivalent with a characteristic set. Wherefore the canonical system of the surface contains the prime sections, there being no exceptional curves on the surface.

That the canonical system is not more ample than, and is therefore equivalent with, the system of prime sections, follows from the analytical theory given above; for it was shewn, from consideration of the everywhere finite double integrals of the surface, that the canonical system of the surface is linearly dependent on curves of which one is the locus of points representing the pairs of points, on the original curve, which can be chosen from a set of a series g_1^{2p-2} of freedom 1, chosen from the canonical series (in the notation employed, curves formed from functions $\phi_i(x)\phi_j(\xi)-\phi_i(\xi)\phi_j(x)$). In our case, such curve of the canonical system represents the pairs of points of the curve σ whose joining line meets an arbitrary general line; thus the canonical system consists of curves representing pairs of points, on chords of σ, which belong to a general linear complex (whose equation is linearly dependent from that of six special linear complexes); and such linear complex gives a prime section of the representative surface.

The geometric proof is however instructive. Denote by γ a curve on the surface which represents the generators of a cone projecting the curve σ from a point of itself, and by c the curve which, similarly, represents the tangents of σ. This curve is touched by all the curves γ, in accordance with the fact that the osculating plane at any point of the curve σ is tangent to the cone projecting σ from this point. It can be shewn that the curves γ are plane quintic curves, of genus 4, each with two double points, the joining line of which contains the point where the curve γ touches c; and further that the surface ψ contains two conics (corresponding to the trisecants of the curve σ), while through any point of either conic pass three curves γ, each having a double point thereat. Two curves γ have a point in common, and thus lie in a prime. We have seen that any prime section of the surface ψ is a canonical curve; we can estimate the freedom of canonical curves which contain two of the curves γ, by their intersections with a third curve γ, and hence infer that the only such canonical curve is the prime section through the two curves γ. From this it can be proved that the canonical curves are all prime sections.

We can also prove that a canonical curve on the surface ψ is equivalent to ($2p-2$ or) 6 curves γ, less a curve equivalent to half the curve c (with resulting order $6.5-8$ or 22). For consider a set of 6 cones projecting the curve σ from the 6 points of a special coplanar set of points thereon. The vertices of these cones project from any other point of σ into a canonical set of generators of the cone whose vertex is this other point. Hence, on the surface ψ, 6 curves γ which form a special set cut, on an arbitrary curve γ, a set of 6 points equivalent with (the 6 intersections of this γ with a conic in its plane through its two double points, and hence with) the set constituted by 5 collinear points of this γ, taken with the point where this γ touches the curve c; that is equivalent to the intersection of this γ with the aggregate of a prime section and a curve equivalent to half the curve c. We may assume however that two curves (of the same order), which cut equivalent sets on every curve of an algebraic system of curves, are equivalent, or are equivalent save for curves which are fundamental for the algebraic system, that is have no variable intersections with these*.

Particular consequences are, (a), for the virtual grade of the curve c, we have $2\pi-2-\xi$ equal to the number of its intersections with a prime, equal to the rank of the curve σ, which is $3(2p-2)$, so that $\xi=-2(2p-2)$, as was previously found; and (b), the order of the surface ψ is $p^{(2)}=(2p-2)^2-2(2p-2)-p-1$ or $(p-1)(4p-9)$, equally found before on the supposition that ψ has no exceptional curves. For $p=4$ this gives $p^{(2)}=p^{(1)}-1=21$.

For the invariant I we may proceed directly on the surface ψ, considering a pencil of curves obtained by primes through a fixed solid, or space [3], which meets the quadric Ω in two planes. It can be shewn that there are 96 prime sections with a double point, beside 6 prime sections which degenerate into a curve γ and a residual curve which meets this in 4 points, so leading to 24 further double points. Hence we infer

$$I=96+24-p^{(2)}-4p^{(1)}=120-21-88,\ =11;$$

and, from Noether's equation, $I+p^{(1)}=12p_n+9$, we can infer $p_n=2$. For, the prime sections, of the surface ψ, spoken of, are those representing the ruled surfaces ρ formed by chords of the curve σ which meet a variable line of a flat pencil of lines, say (O, ϖ), formed of lines in the plane ϖ which pass through a point O of this plane. Now, through O there pass 96 bitangent planes of the curve σ (the number of bitangents of the projection of σ on to a plane, which is $d+\frac{1}{2}(n'-n)(n'+n-9)$, where $d=6$, $n=6$, $n'=30-2d=18$, Vol. v, Chap. VIII), each giving a double generator in the particular ruled surface ρ associated with the line, of the pencil, in which the bitangent plane meets ϖ. Again, the ruled surface associated with one of the six lines OP, where P is a point of the curve σ lying on ϖ, breaks up into the cone γ projecting σ from P, and other chords of σ meeting OP, of which 4 belonging to this cone γ are obtained by the plane joining O to the tangent line of σ at P.

---

* Severi, "Alcune relazioni di equivalenza", *Atti...Veneto*, LXX, 1911, p. 380, enunciates the general result: If on a manifold of k dimensions, M_k, two manifolds M_{k-1}, of $k-1$ dimensions, cut equivalent manifolds on the M_h of an algebraic system Σ, which is ∞^{k-h}, where $h \leqslant k-1$, and is of index $\nu \geqslant 1$, then they are equivalent or differ by manifolds fundamental for Σ.

The results obtained by the geometrical reasoning thus agree with those which were found more analytically above.

Remark. Another irregular surface extremely worth study, is the surface ${}^{46}\psi^{45}[9]$, of order 45, with prime sections of genus 46, in space [9], whose points are those representing the lines lying on a general cubic primal in space [4]. On this also, the canonical curves are those given by the primes of its space. For it $p_g=10$, $p_n=5$, $I=23$, $\omega=46$. Of these numbers, p_g and p_n are the same as for the surface representing the chords of a curve of genus 5, for which, however, $I=24$, $\omega=45$. For the system of lines of a cubic primal in space [4] which has $5-p$ independent nodes, with $0 \leqslant p<5$, all the numbers $p_g=\frac{1}{2}p(p-1)$, $p_n=\frac{1}{2}p(p-3)$, $I=(p-1)(2p-3)-4$, $\omega=(p-1)(4p-9)+1$, are the same as for the system of chords of a curve of genus p. See Fano, *Atti...Torino*, XXXIX, 1904, p. 778; *Ann. d. Mat.* x, 1904, p. 251; *Rend....Lombardo*, XXXVII, 1904, p. 554.

Note on the multiple correspondence of two surfaces. We have had occasion to consider the (1, 2) correspondence between the surfaces ψ, ψ_0, in the preceding. We quote now the general results, first for an $(1, \alpha')$ correspondence between two surfaces ψ, ψ', and then for an (α, α') correspondence (Severi, "Sulle relazioni...di due superficie in corrispondenza algebrica", *Rend.... Lombardo*, XXXVI, 1903). In the case of an (α, α') correspondence, we denote *the branch curve* on ψ, locus of points for which two of the α' corresponding points of ψ' coincide with one another, by b; and the coincidence curve on ψ', which corresponds to this, by c', with a similar notation, b' for the branch curve on ψ', c for the corresponding curve of coincidence on ψ. Unless the contrary is stated we suppose that there is no point of either surface transforming, by the relations which express the correspondence, into a curve of the other surface, which is stated by saying that, unless remarked, there is no *fundamental point of the correspondence* on either surface. We denote the *unreduced* canonical system on ψ by K; this is of the form $K_0+\epsilon$, if ϵ denote the exceptional curves on ψ and K_0 the reduced canonical system; the n-canonical system will then be $K_{n,0}+n\epsilon$, where $K_{n,0}$ denotes the reduced n-canonical system; this we denote by K_n. Similarly for canonical systems on ψ'.

We recall the theorems previously (Chap. I, p. 19) considered for two curves C, C' in $(1, \alpha')$ correspondence: (*a*), on C', a canonical set is equivalent with the set obtained by direct transformation of a canonical set on C, augmented by the set of coincidence points on C'; (*b*), on C, the aggregate of α' canonical sets, augmented by the set of branch points on C, corresponds to a canonical set on C'. We have similar results in the $(1, \alpha')$ correspondence of the surfaces ψ, ψ', for the canonical systems of curves; namely, using the symbol t for transformation, (*a*), *on* ψ', $K'\equiv t(K)+c'$; (*b*), *on* ψ, $K_{\alpha'}+b\equiv t(K')$; hence in case of an (α, α') correspondence, we have, *on* ψ', $K_{\alpha}'+b'\equiv t(K)+c'$. This latter is true if there be, on ψ', fundamental curves of the correspondence, say ϕ', when we put $c'+\phi'$ in place of c'. Neglect this possibility, as before, and denote by ϖ, θ, κ respectively the genus, canonical number, and number of cusps of the branch curve b, putting $\varpi-1$, θ and κ zero when there is no branch curve, with similar notation for the branch curve b' on ψ'. Then we have, for the invariants ω, I and the arithmetic genus p_n,

$$\alpha(\omega'-1)+\varpi'-1+\tfrac{3}{2}\theta'=\alpha'(\omega-1)+\varpi-1+\tfrac{3}{2}\theta,$$

$$\alpha(I'+4)+2(\varpi'-1)-\kappa'=\alpha'(I+4)+2(\varpi-1)-\kappa,$$

$$12\alpha(p_n'+1)+3(\varpi'-1)+\tfrac{3}{2}\theta'-\kappa'=12\alpha'(p_n+1)+3(\varpi-1)+\tfrac{3}{2}\theta-\kappa,$$

and, in particular, for a $(1, \alpha')$ correspondence, putting $\varpi'-1$, θ' and κ' zero, we have

$$\omega'-1=\alpha'(\omega-1)+\varpi-1+\tfrac{3}{2}\theta, \quad I'+4=\alpha'(I+4)+2(\varpi-1)-\kappa,$$

which can be modified, by addition of terms, to include the case when there are fundamental points for the correspondence.

III. On complete sections of a non-singular primal. It was suggested as probable by Cayley in 1859 (*Papers*, IV, p. 455) that on a surface in space of three dimensions, represented by the vanishing of a perfectly general polynomial in the coordinates, of order >3, there exist no algebraical curves other than complete intersections of this surface each with one other surface. For consideration of this result, see Noether, "Zur Grundlegung, u.s.w.", *Berlin. Abh.* 1882, §§ 11, 12). It was then proved by Klein, in 1872 (*Ges. Abh.* I, p. 153) that upon a quadric primal of sufficient generality, in space $[r]$, $(r>3)$, any algebraic manifold of one less dimension, that is, of dimension $r-2$, is the complete intersection of the quadric with another primal; the necessary generality of the quadric, which may be a cone, may be briefly stated by saying that its equation must involve 5 (homogeneous) coordinates at least. It was next proved by Fano, in 1904 (*Atti...Torino*, XXXIX, 1904, p. 613), that a general cubic primal in space [4] contains only surfaces each a complete intersection with another primal, the surfaces considered in the first instance being supposed to cut every one of the ∞^2 lines lying on the cubic primal in the same number of points. (The theorem proved however is more general than here stated; for instance, the ruled surface of order 15, formed by the lines of the primal which meet any one of the lines, is a complete intersection. And the primal may have nodes. Cf. Ex. 2, p. 278, above.) It was then proved by Severi, in 1906 (*Rend. Lincei*, XV, p. 691), that *on any primal, which is without multiple points, in space* [r], *if* $r \geqslant 4$, *every algebraic locus of dimension* $r-2$ *is a complete intersection of the primal with one other primal.* This was extended in 1909 by Fano (*Atti...Torino*, XLIV, p. 633) to apply, *in general*, to a (non-singular) manifold M_k, of k dimensions, which is the complete intersection of $r-k$ primals in space [r], with $k \geqslant 2$ (excluding *rational* surfaces if $k=2$), any M_{k-1} thereon being the complete intersection with one further primal. And, in 1915, it was proved by Severi (*Ann. d. Mat.* XXIV) that on a Grassmannian (the locus of points, in space of appropriate dimension, each representing a linear space [k] of space [r]), of dimension t, where $t=(k+1)(r-k)$, any algebraic locus of dimension $t-1$ is the complete intersection with one primal. In particular, when $r=3$, $k=1$, this last gives Klein's theorem for a general quadric in [5].

In the present place we limit ourselves to giving a summary account of Severi's proof of the theorem for surfaces, without double points, which can exist upon a non-singular primal in space [4]; it is to be shewn that such a surface is the complete intersection of the primal with another primal.

Denote the primal by V, and the surface in which this is met by a prime ϖ by (ϖ, V). On V there is a surface, denoted by ψ. We limit ourselves to the case when ψ has no double points. We shew, first, as a consequence of the fact that V is without double points, that the curve (ϖ, ψ), in which ψ is met by a prime ϖ, is the complete intersection of the surface (ϖ, V), on which it necessarily lies, with a surface, lying in the prime ϖ. The proof depends on two results due to Noether (*Berlin. Abh.* 1882; see Vol. V, Chap. VIII): (a), that the genus, p, of a curve, lying on a surface of order ρ, in ordinary space [3], when the order of the curve is $\rho m - \nu$ with $\nu < \rho$, is such that $p \leqslant \frac{1}{2}(\nu-1)(\nu-2)+\frac{1}{2}(\rho m-2\nu)(\rho+m-4)$ (*loc. cit.* p. 25); (b), that if the order and genus of the curve are respectively ρm and $1+\frac{1}{2}\rho m(\rho+m-4)$, then the curve is the complete intersection of the surface on which it lies with a surface of order m (*loc. cit.* p. 42). Let the surface ψ, given on V, have the characters μ_0, μ_1; it can have no accidental double points, since these would be nodes of V, and V cannot have nodes at ordinary points of ψ. Whence, by the formula $\delta+4d=\rho(\rho-2)\mu_0+\mu_0^2-\rho\mu_1$, of the preceding chapter, where ρ is the order of V, we have

$$\rho\mu_1=\mu_0[\mu_0+\rho(\rho-2)].$$

From μ_0 and ρ we can obtain uniquely two integers m, ν by putting $\mu_0=m\rho-\nu$, with $0\leqslant\nu<\rho$ and $m\geqslant 1$; whence, considering the curve (ϖ, ψ) as lying on the surface (ϖ, V), which is of order ρ, we have, for its genus p, the inequality above. As the curve is (ϖ, ψ) we also have $\mu_1=2\mu_0+2p-2$, which is $\mu_1=2(m\rho-\nu)+2p-2$. Utilising $\mu_0=m\rho-\nu$, $\mu_1=\mu_0[\mu_0+\rho(\rho-2)]/\rho$, and the inequality for p, we find on reduction the simple result $\nu(\rho-1)(\rho-\nu)\leqslant 0$. But $\rho-\nu>0$, and, if $\nu>0$, this gives also $\rho-1>0$. Thus we infer $\nu=0$, and $\mu_0=m\rho$; thence $2p-2$, equal to $\mu_1-2\mu_0$, is found equal to $m\rho(m+\rho-4)$. Wherefore, by (*b*) above, the curve (ϖ, ψ) is the complete intersection of the surface (ϖ, V) with another surface, in the prime ϖ, of order m.

Now, the surface ψ, regarded as having no prescribed base loci, determines on V a complete system of surfaces, which we denote by $|\psi|$; just as a curve on a surface determines a complete series of curves thereon, or a set of points on a curve determines a complete series. This we assume; we also assume that primals of order m determine on V a complete system of surfaces (each the complete intersection with such a surface); this latter is in consequence of the fact that V is a primal and without multiple points; the same is true of a manifold of dimension k, in space $[r]$, which is the complete intersection of $r-k$ primals and without multiple points (Severi, ***Rend. Palermo***, XVII, 1903, and XXVIII, 1909, already referred to in the last chapter). This assumption is equivalent to the statement that if, through the intersection of V with a primal of order m, there be drawn a primal of order m', $>m$, having a residual intersection, of order $(m'-m)\rho$, with V, then any other primal of order m', through this residual intersection, meets V again in a surface which is the complete intersection of V with a primal of order m. From these two assumptions it follows that, to prove ψ to be the complete intersection of V with a primal of order m, it is sufficient to shew that the linear system $|\psi|$ is identical on V with that determined by primals of order m. This last system we denote by $|m\varpi|$. The identity of these linear systems of surfaces on V follows then if we prove the identity of two systems $|H+\psi|$, $|H+m\varpi|$, where $|H|$ is any other system of surfaces on V. In virtue of another theorem (p. 293, above; Severi, *Atti... Veneto*, LXV, 1906, LXX, 1911) these systems of surfaces will be identical if they cut equivalent (not necessarily complete) systems of curves on any prime section, ϖ, of V; this assumes, what is evidently true, that there is no surface on V, common to all its prime sections, which is part of all the surfaces of either of the two systems. For the system $|H|$ we employ the system of surfaces cut on V by primals of order $\rho-5$ (thus assuming $\rho \geqslant 5$); thus we are to prove that the systems of surfaces on V expressed by $|(\rho-5)\varpi+\psi|$ and $|(\rho+m-5)\varpi|$ cut equivalent systems of curves on the surface of intersection of V with an arbitrary prime. We shall not consider the removal of the limitation $\rho \geqslant 5$ which is convenient for the statement of the proof.

We have shewn that the curve (ϖ, ψ) is the complete intersection of the surface (ϖ, V), of order ρ, with a surface of order m, in the prime ϖ. Thus (Vol. v, Chap. VIII), the canonical series on this curve is determined by its intersections with surfaces of order $\rho+m-4$ lying in the prime ϖ. From this we see, recalling the definition of the canonical system of curves on a surface (pp. 217, 222, above), that primals of order $\rho+m-5$ cut canonical curves on *the surface* ψ, though not necessarily, in default of proof, the complete canonical system of *curves*. On the other hand, the complete system of canonical *surfaces*, on the primal V, is known (after Noether, *Math. Annal.* II, 1870, § 3, reasoning with a multiple integral) to be cut thereon by primals of order $\rho-5$; and we assume from this (arguing in a manner analogous to that employed for surfaces) that canonical *curves*, on the surface ψ, though not necessarily the complete system, are cut thereon by curves on ψ, given by $|\psi+(\rho-5)\varpi|$. Whence, on the *surface* ψ, which is without multiple points, the two complete systems of curves expressed by the last, and by $|(\rho+m-5)\varpi|$, are the same. Hence the two series, of sets of points, on the *curve* (ϖ', ψ), section of ψ by any prime ϖ', cut by the surfaces

$$|\psi+(\rho-5)\varpi|, \quad |(\rho+m-5)\varpi|$$

are equivalent. This curve (ϖ', ψ) however, we have seen, is the complete intersection of the surface (ϖ', V) with a surface of order m of the prime ϖ', and is thus, regarded as belonging to the surface (ϖ', V), one of the complete system cut thereon by surfaces of order m. The fact that the surfaces $|\psi+(\rho-5)\varpi|$, $|(\rho+m-5)\varpi|$ cut equivalent series on this curve, whatever ϖ' may be, suffices (by the theorem of equivalence to which reference has been made) to shew that these surfaces cut equivalent systems of curves on the surface (ϖ', V). And, by what has been said, this is sufficient for our purpose.

IV. Miscellaneous theorems and examples.

Ex. 1. (Chap. I, pp. 13, 14.) In regard to the double tangents of a plane curve, it was remarked by Dersch, *Math. Ann.* VII, 1873, p. 497, that if the tangent at a point (x) of a plane curve $a_x^n=0$ meet the curve again in $(n-2)$ points (y), then there exists the relation

$$(abc)^2 \sum_{\lambda+\mu+\nu=n-2} (a_y b_x c_x)^\lambda (b_y c_x a_x)^\mu (c_y a_x b_x)^\nu = 0;$$

this generalises Salmon's theorem for a plane cubic curve, Vol. v, p. 22. The equation is of order $n-2$ in (y), and of order $2n-4$ in (x). Thus (x) will be one of the points of contact of a bitangent if this curve, in (y), be touched by the tangent line $a_x^{n-1}a_y=0$.

The condition that a curve of order m be touched by a line $u_y=0$ is of degree $2(m-1)$ in the coefficients of the curve, and of order $m(m-1)$ in the line coefficients (u) (Vol. v, p. 86). Hence, in the case in question, the condition resulting is, in (x), of order

$$2(n-3).2(n-2)+(n-2)(n-3)(n-1), \text{ or } (n-2)(n^2-9);$$

the curve has therefore $\frac{1}{2}n(n-2)(n^2-9)$ bitangents, whose points of contact (x) lie on the curve resulting from the condition of contact. Cf. Cayley, *Papers*, IV, pp. 186, 347; Salmon, *Higher Plane Curves*, 1879, pp. 342–57. Klein, *Math. Ann.* XXXVI, 1889, p. 20, remarks that the left side of Dersch's relation is, in virtue of $a_x^n=0$, the same as

$$6a_x^{n-1}a_y.N.(hxy)^{-2},$$

where

$$N=\sum_{\kappa=1}^{n}(a_yb_x)^{\kappa-1}(b_ya_x)^{n-\kappa}a_hb_h-\sum_{\kappa=1}^{n-1}a_y^{\kappa-1}a_x^{n-\kappa-1}a_h^2b_y^{n-\kappa}b_x^{\kappa},$$

and that N is characterised by being covariant, of degree 2 in the coefficients of a_x^n, and of order $n-1$ in both x and y, and such that $N.(hxy)^{-2}$ is independent of (h). See Pick, *Math. Ann.* XXIX, 1887, p. 261.

Ex. 2. (See p. 85, above.) The Grassmann manifold whose points have for coordinates the coordinates of a general linear space $[k]$ containing $k+1$ points of a rational normal curve of order n, in space $[n]$, may be regarded as representing the primals of order $n-k$ of space $[k+1]$ (Brown, *Journ. Lond. Math. Soc.* v, 1930, p. 168). On a curve of genus p, of general moduli, the number of special series g_r^n, when $p=(r+1)(r'+1)$, in which $r=n-p+r'+1$, is finite (Vol. v, p. 91); this number is equal to the order of the Grassmann manifold which represents all the linear spaces $[r]$ existing in $[r+r'+1]$, namely, equal to the number of spaces $[r]$, in $[r+r'+1]$, which meet $(r+1)(r'+1)$ spaces $[r']$ (Castelnuovo, *Rend. Lincei*, v, 1889, p. 130). See p. 85 above.

Ex. 3. (See p. 121, above.) According to Wiman (*Math. Ann.* XLVIII, 1897, p. 195) every finite group of birational transformations of a plane is one of nine types: (1), a group of collineations; (2), groups with one, or two, flat pencils invariant; (3), a group in which a system of cubic curves with 3, ..., 7 fixed points, is invariant; (4), a group in which a sextic curve with 8 fixed double points, and the adjoint pencil of cubic curves, is invariant.

The collineation groups contain: (*a*), groups which have three points, or a point and a line invariant; (*b*), the Hessian group of order 216, in which the nine inflexions of a plane cubic curve are invariant; (*c*), the group of order 168, changing the curve $y^3z+z^3x+x^3y=0$ into itself; (*d*), the Valentiner group of order 360 leaving the sextic curve

$$10x^3y^3+9(x^5+y^5)z-45x^2y^2z^2-135xyz^4+27z^6=0$$

invariant. This last is of the same structure as the even permutation group of six elements.

Ex. 4. (See p. 272, above.) In connexion with Castelnuovo's paper, prove, if u_i denote a homogeneous polynomial of order i, in x and y, with variable coefficients, and ω_k be such a polynomial of order k with fixed coefficients, that the equation

$$\omega_k^2u_{p-k}z^2+\omega_ku_{p+1}z+u_{p+k+2}=0, \qquad (k=0, 1, \ldots, p),$$

represents, for any k, a system of hyperelliptic curves of genus p, of grade $4p+4$, and freedom $3p+5$. The multiplicity at $x=0, y=0$ is described in Noether's phraseology as a $(p+k)$-ple point, with k double points infinitely near, in its first neighbourhood, in the directions given by $\omega_k=0$. A simplifying birational transformation is given by

$$\xi/xy^k = \eta/y^{k+1} = \zeta/z\omega.$$

Also, another system of hyperelliptic curves, of the same genus, grade and freedom, is given by

$$u_{p+1}z^2 + v_{p+1}zy + w_{p+1}y^2 = 0,$$

where v_{p+1}, w_{p+1} are homogeneous in x, y, of order $p+1$, with variable coefficients. (See also Castelnuovo, *Atti...Torino*, XXV, 1890, and *Ann. d. Mat.* XVIII, 1890, p. 119; Scorza, *Ann. d. Mat.* XVI, 1909; and dal Re, *Rend. Napoli*, XXX, 1924, p. 80.)

Ex. 5. (See pp. 140, 222.) Prove the following theorems:

(*a*), A surface containing a pencil of irreducible rational curves can be transformed to a ruled surface (Noether, *Math. Ann.* III, 1870; Enriques, *Math. Ann.* LII, 1899. Cf. Picard-Simart, *Fonct. algéb.* I, p. 194).

(*b*), A surface containing a linear pencil of irreducible elliptic curves, with a simple or double base point, is rational, or transformable to an elliptic ruled surface. A surface containing a linear pencil of irreducible curves of genus 2, with one or more base points of multiplicities $s_1, s_2, \ldots$, where $\Sigma s > 2$, is rational, or transformable to a ruled surface, of genus 1 or 2 (Castelnuovo-Enriques, *Rend. Palermo*, XIV, 1900; Castelnuovo, *Atti...Torino*, 1901).

(*c*), A surface containing a linear pencil of irreducible curves of genus p, >2, of freedom $r \geqslant 3p-5$, is rational, or reducible to a ruled surface (Castelnuovo-Enriques, *Ann. d. Mat.* VI, 1901, p. 38).

Ex. 6. (See p. 282.) Elaborate the theory of the surface representing the pairs of points on a curve of genus 3. (Cf. Severi's paper, referred to, No. 7; Humbert, *Compt. rend.* 1895, pp. 365, 425.)

INDEX

The references are to the pages of the volume

Accidental double point of a surface in four-fold space, 157, 170

Adjoint system of a linear system of curves, 219, 222, 246; relation connecting grade and genus with genus of original system, 225

Albanese, elimination of multiple points of a surface, 158

Alexander, on Castelnuovo's reduction of Cremona transformations, 117; identification of the Zeuthen-Segre and the Poincaré topological invariants, 214

Babbage, transformation of surfaces considered by Noether to non-singular surfaces, 232; deduction of values for invariants when a surface has nodes, 235; normal surfaces in five-fold space with a double line, 246, 281

Benoist, *see* Clebsch

Bertini, representation of the pairs of a de Jonquières involution, 119; theorem for certain hyperelliptic plane curves, 120; reduction of plane involutions of two points to four types, 121; Bertini involution in a plane, 124; on reduction of plane involutions and rationality of double planes, 131; theorem for the variable multiple points of the curves of a linear system, 222; Clifford's theorem for least order of a manifold, 266; on the postulation of a complete intersection, 268; primals whose intersection consists of spaces, 276; *Geometria proiettiva degli iperspazi*, 1907 (second edition, 1923), referred to, 120, 268, 276

Berzolari, in Pascal's *Repertorium der höheren Mathematik*, referred to, 31, 34, 38, 188; on contacts of curves of two plane pencils, 240

Bitangents of a plane curve, 13, 298

Bordiga's surface, 273, 274

Brill, on quadrisecants of curves, 34; on multiple contacts, 42; Brill-Noether, Report to German Mathematical Society on algebraic functions, 1894, referred to, 117, 130

Brown, the Grassmann representation of the secant spaces of a rational curve, 299

Burnside, a particular group of Cremona transformations, 121

Calculus of conditions, 69

Campedelli, *see* Enriques-Campedelli

Canonical series on a curve, transformation in a correspondence of two curves, 18; canonical number of curve, 140, 224; canonical system of curves on a surface, 215, 217, 262; canonical curves as defined by Noether, 220, 227; canonical system of a general double plane, 237; undesigned base points of canonical system, 241

Caporali, a theorem for plane curves, 120; class of an involution, 140; number of double points of a plane pencil, 240; on intersections in four-fold space, 247

Castelnuovo, condition for an involution on a curve to belong to a linear series, 37; enumerative formulae for curve, 45; reduction of Cremona transformation to quadratic transformations, 117; surface whose prime sections are hyperelliptic, 120, 272; rationality of plane involutions, 139; theorem used by Picard, 169; on fundamental and exceptional curves, 195, 246; and Enriques, the Zeuthen-Segre invariant deduced from an irrational pencil, 191; surface with multiple points at eight associated points, 203, 227; surface having an irrational pencil of curves is irregular, 205; example in regard to transformation of exceptional curve to a simple point, 220; and Enriques, undesigned base points of the canonical system, 241; theorem for deficiency of characteristic series, 263; number of special series on a curve as the order of a Grassmannian, 299; hyperelliptic plane curves representing sections of a rational surface, 299; and Enriques, section conditions for a surface to be rational or ruled, 140, 300

Cayley, on correspondence, 12; on quadrisecants of a curve, 34; on ruled surfaces, 38; on multiple contacts of curves, 42; reduction of plane Cremona transformations to quadratic transformations, 117; pinch points of a double curve of a surface, 158; use of name *cotangent*, 177; example of pinch point, 179; direct calculation of numerical genus of a surface, 200; quartic surface with nodes at eight associated points, 203; on a particular quintic surface, 205; the numerical genus of a ruled surface, 227; birational relation of Kummer and Weddle surfaces, 237; curves on a general surface are complete intersections, 250; on double tangents of a plane curve, 299

Characteristic series of a linear system of curves, 124; deficiency of, 263

Chisini, resolution of multiple points of a surface, 158; *see also* Enriques-Chisini

Chord, common to two curves, 29; chord-curve and trisecant-curve of a surface in four-fold space, 242

Class of an involution (Caporali), 140; of immersion and canonical number, 222, 225, 251

Clebsch, on multiple tangents of a surface, 90; in lectures, on Cremona transformation, 117; everywhere finite double integral of a surface, 183, 184; plane representation of double curve of a rational surface, 246; Hessian of a pencil of binary quartics, 277; conditions for roots of binary sextic to be three pairs mutually harmonic, 278

Clifford, reduction of plane Cremona transformation to quadratic transformations, 117; least order of a manifold belonging to a given space, 266

Coble, Cremona transformations, 117

Coincidences in a correspondence with valency, 8; of $r+1$ points in a linear series of freedom r, 10; in a general correspondence, 54, 142

Completed grade and genus of a curve of a linear system, 139, 140, 221; complete system of sections of a non-singular primal, 295

Composite curve, genus and grade in terms of components, 223

Correspondence, of points on a curve, elementary methods, 1; transcendental methods, 46; connexion with theory of degenerate integrals, 59; correspondence of points of two manifolds, 92; of linear spaces, 105; enumerative formulae for, 108; multiple, between two curves, analytic treatment of, 20; of two surfaces, relations of invariants of, 294

Cotangent of a surface at a pinch point (Cayley), 177

Coxeter, on lines of a cubic surface, 15

Cremona, transformations in a plane, 112; on a certain curve on a cubic surface, 130; the Zeuthen-Segre invariant in plane transformations, 240; the Zeuthen-Segre invariant for pencils of surfaces, 241; transformations, groups of, 121, 278, 299

Curve, fundamental, 194, 246

Cyclical sets in a correspondence, 143

Darboux, birational relation of Kummer and Weddle surfaces, 237

Dedekind, on an algebraical reversion formula, 144

Defective integrals, in the correspondence of two curves, 22, 59; regular system, 62

Deficiency, of the characteristic series on a surface, 263; of series on a curve by curve of adjoint system (Picard's theorem), 269

Dersch, on bitangent curve, 13, 298

Double, curve of a surface, general property for, 169; points of a surface, in four-fold space, accidental, 171, 259; double plane, canonical system of general, 237

Edge, on a certain curve on a cubic surface, 130; treatise on ruled surfaces, 166

Eliminant, rule for degree of, in parameters which enter, 186

Enriques, and Chisini, *Teoria geometrica delle equazioni*, referred to, 34, 117, 232; on cubic primal in four dimensions, 137; on rationality of surface representing an involution, 139; on surface with an irrational pencil of rational curves, 147; on ruled surfaces, 168; use of Jacobian of net of curves to obtain canonical system of surface, 183; and Campedelli, *Superficie algebriche*, 1932, referred to, 184, 232, 234, 236, 237, 241; researches on theory of surfaces of 1893, referred to, 184; exceptional curves in a plane, 113; exceptional curves on a surface, 184, 195, 246; and Castelnuovo, the Zeuthen-Segre invariant from an irrational pencil, 191; surface whose canonical system consists of elliptic curves, 205; phraseology for multiple base points of a linear system of curves, 215; on transformation of exceptional curve to simple point, 220; relation of exceptional curve to transformed adjoint system, 236; on surfaces whose prime sections are hyperelliptic, 272; and Fano, groups of Cremona transformations in three-fold space, 278; and Castelnuovo, section conditions for a surface to be rational or ruled, 300

Enumerative formulae for manifolds defined by matrices, 109

Equivalence of two manifolds, Severi's theorem for, 293, 297

Exceptional curves, in plane transformations, 113; on a surface, 184, 195, 246; arising in birational transformation, 201, 202, 203, 210; as considered by Noether, 220, 227

Fano, on the rationality of a three-fold manifold, 137; and Enriques, groups of Cremona transformations in ordinary space, 278; theorem for surfaces lying on a cubic primal in four-fold space, a particular case, 278; lines lying on cubic primal in four-fold space, 294; manifold which is complete intersection contains only complete intersections, 296

Fouret, common normals of two primals, 104; tangent planes to a surface from a multiple line, 188

Franchis, De, on surface representing points of two curves, 227, 282

Fundamental curve of a linear system of curves, 194, 210, 246

Geiser involution in a plane, 122

Genus of a curve, completed, 139, 221

Giambelli, number of spaces meeting a curve in higher space, 44; manifolds defined by matrices, 111

Godeaux, tangent planes to a surface from a multiple line, 188

Grade of a curve of a linear system, completed, 139, 221

Grassmannian, of linear spaces, order of, 85; giving number of special series on a curve, 299

Groups of Cremona transformations in a plane, 299; in space, 278

Guccia, number of double points of plane pencil, 240; Zeuthen-Segre invariant for pencil of surfaces, 241

Halphen, common lines of two congruences, 28; results anticipatory of Schubert's ideas, 69

Hessian of a ruled surface, Salmon on, 27

Hilton, on a certain curve on a cubic surface, 130

Hodge, canonical system of a double plane, 237

Hudson, H. P., treatise on Cremona transformations, referred to, 117, 124, 128, 200

Hudson, R. W. H. T., treatise on Kummer's surface, referred to, 237

Humbert, on a particular curve, 24; on hyperelliptic surfaces, 227

Hurwitz, transcendental methods for theory of correspondence, 46, 54, 56

Hyperelliptic curves representing sections of a rational surface, 299

Immersion, class of, and canonical number, 222, 225 (and 251)

Improper double points of a surface in four-fold space, 157, 170
Inflexional lines of a surface, 148
Intersections, residual, of three primals through a curve, 248; of two surfaces, 251; of a primal and a surface, 255; of two primals, 257; of three primals through a surface, 266
Invariants, the Zeuthen-Segre, 185, 186, 189; the Segre-Severi, 191, 214, 221; modified for isolated nodes, 199, 200; proof of invariance, 206, 210, 221; of a surface which is determined by a matrix, 280; of two surfaces in multiple correspondence, relations of, 294; of an involution, 135
Involution, not rational, on a curve, 17; number of sets belonging to a linear series, 35; in space of three dimensions, leading to the Geiser involution, 124; rationality of, 132; surface representing, 133; in ordinary space, presumably not rational, 137
Irrational pencil of curves, not existing on a surface representing an involution, 138; used for defining Zeuthen-Segre invariant, 190; existence involves irregularity of surface, 205
Irregularity of surface, 227; of surface with nodes at eight associated points, 204; of surface representing points of two curves, 285

Jacobian, of a net of curves on a surface, 152, 192
Join of two spaces, 70
Jonquières, de, on multiple contacts of curves, 39, 42; manifolds defined by matrices, 111; transformation, 115; involution, 118
Jung, the Zeuthen-Segre invariant in plane transformations, 240

Kantor, on periodic Cremona transformations, 131, 145
Klein, manifold on quadric in higher space, 250, 295; on double tangents of a plane curve, 299
Kohn, manifolds defined by matrices, 111
Kummer surface, transformation to a Weddle surface, 237

Lasker, on the theory of Moduls and Ideals, 269
Lateral, direct, correspondence, 11
Lefschetz, on correspondence between two curves, 59
Levi, Beppo, resolution of multiple points of a surface, 158
Lindemann, *see* Clebsch
Line, formula for product of two line conditions, 81
Lüroth, rationality of involutions and surfaces, 131, 132

Maroni, surface representing points of two curves, 282
Matrices, manifolds defined by, enumerative formulae for, 109; invariants of, 280
Maxwell, surface with nine exceptional lines, 232; a general formula for invariants, 235
Meet of two spaces, 70
Milinowsky, on the Geiser involution, 122
Moebius, an algebraic reversion formula, 144
Moore, and Slaught, a particular group of Cremona transformations, 121

Nanson, manifolds defined by matrices, 111
Net of curves on a surface, 152
Netto, on a theorem for symmetrical functions, 136
Noether, reduction of Cremona transformations to quadratic transformations, 117; on a certain curve of genus three, 130; on the double planes which are rational, 130; on cubic primal in four-fold space, 137; on surfaces with a rational pencil of rational curves, 146; determination of canonical series on a curve in space, 161; on multiple curves of a surface, 163; formulae for the class, stationary points, genus and curve-genus of a surface, 180; procedure by enveloping cone for invariants of a surface, 183; number of conditions for a surface imposed by a given multiple line, 200; surface whose canonical curves are aggregates of elliptic curves, 205; on a particular sextic surface, 205; on

conditions for an adjoint surface at a tacnode, 220; fifteen examples of the invariants of surfaces, 226; on condition of adjointness at an isolated multiple point, 233; incidental proof of Clifford's theorem for special series on a curve, 270; curves on a general surface are complete intersections, 295; particular theorems for a curve in space, 296; canonical surfaces on a primal in four-fold space, 298

Painlevé, surface with a finite group of birational transformations, 147
Palatini, manifolds determined by matrices, 111
Pannelli, Zeuthen-Segre invariant for pencil of surfaces, 241
Pascal, *Repertorium of higher mathematics, geometry*, 1922, referred to, 199, etc.
Pencil, irrational, of curves on a surface, 138, 189
Pezzo, Del, surfaces whose order is one less than the dimension of the space to which they belong, 120; Del Pezzo surfaces, characters, 275
Picard, on deficiency of series by adjoint system, 169, 269; influence of tacnode on adjunction, 220, 234; on hyperelliptic surfaces, 227
Pick, on a covariant arising for bitangents of a plane curve, 299
Pieri, formula for multiple correspondence, 99; applications, 102, 104; coincidences of corresponding lines, 106; manifolds defined by matrices, 111; Zeuthen-Segre invariant for pencils of surfaces, 241
Pinch points of a double curve of a surface, 158, 176
Poincaré, theorem of complementary systems of degenerate integrals, 68; topological invariant generalising Euler's, 214
Poisson, on a theorem for symmetrical functions, 136
Postulation, of a surface for primals, 263; of a manifold which is complete intersection, 268
Primals, intersection of three, with a common curve, 248; point equivalence of common curve of four, 249; and surface, residual intersection of, 255; intersection of two with a common curve, 257; intersection of three with a common surface, 266
Product, and sum, of two correspondences, 6; product of two curves, 57

Quadratic transformation, reduction of Cremona transformations to, 117
Quadrisecants of a curve, number of, 33

Rank and grade for a complete intersection, 75
Rationality, of an involution, 132, 137; of surface representing a plane involution, 139; rational surfaces, their plane representation, 245
Re, dal, theorem for surfaces, 300
Reduced canonical system on a surface, 227
Richmond, on cubic primal in four-fold space, 137
Roberts, manifolds defined by matrices, 111
Rohn, on a plane sextic curve with six nodes, 156
Room, on generalisation of the Bordiga figure, 274
Rosanes, on reduction of Cremona transformations to quadratic transformations, 117
Rosati, correspondence and degenerate integrals, 68
Roth, multiple tangents, 91; a numerical property of the characters of a surface, 164; inflexional lines for a surface in four-fold space, 175; a surface with a double line, in five-fold space, 246; on composite surfaces, 282
Ruled surface, by joins of corresponding points of two curves, 17; genus of a curve thereon, 25; number of torsal generators, 26; condition for a surface to be transformable to, 140; characters of 164, 167

Salmon, bitangent curve, 13, 299; Hessian of a ruled surface, 27, 156; quadrisecants of a curve, 34; multiple tangents of a surface, 90, 91; multiple correspondence, 92;

manifolds defined by matrices, 111; general formulae for a surface, 159, 169; generalisation to four-fold space, 174; use of four-fold space to solve a problem of ordinary space, 251; treatise on Higher Algebra, 277

Schottky, on a certain curve, 130

Schubert, torsal generators of a ruled surface, 26, 102, 166; refers to Halphen as precursor, 28; enumerative formulae for contacts of spaces with curves, 46; reference to his treatise, 28, 69, 96, 102, 103, 104; composite condition for a linear space, 71; on multiple tangents of a primal, 90, 91; notation for calculus of conditions, 98; on umbilici, 102; coincidences of corresponding spaces, 106; manifolds defined by matrices, 111; on quadric surfaces satisfying nine conditions, 251

Schur, on sextic curve, 274

Scorza, correspondence and degenerate integrals, 68; on three-folds with curve sections of given genus, 272; on a manifold of order five and dimension three in six-fold space, 278

Segre, genus of a curve on a ruled surface, 25; sets common to an involution and a linear series, 37; Report on higher space in *Encyklopädie der Mathematischen Wissenschaften*, III, C 7, 1912, referred to (under *Enzyk. Math. Wiss.*), 44, 110, 111, 174; enumerative formulae for manifolds defined by matrices, 110, 111; reduction of Cremona transformations to quadratic transformations, 117; on resolution of multiple points of a surface, 158; inflexional lines of a ruled surface in four dimensions, 174; use of pencil of curves on a surface to obtain invariant, 183; the Zeuthen-Segre invariant, 185; from irrational pencil, 189; Segre-Severi invariant, 191; as grade of canonical system, 197; class of a surface with isolated nodes, 199; influence of nodes on invariants, 199, 200; proof of the invariance, 206, 210; generalisation of the Zeuthen-Segre invariant, 214; Segre-Severi invariant defined by grade of canonical system, 224; modification in birational transformation, 226; Segre quartic surface with double line by projection of Veronese surface, 274; surface formed by lines meeting five planes in four-fold space, 276; elliptic ruled surface normal in space of dimension one less than its order, 277; duality of an elliptic ruled surface and a manifold of order five, 279

Semple, on a birational transformation in four and five dimensions, 250; cubic primals through a projected Veronese surface are rational cubic primals, 277; on a birational transformation in four-fold space, 279; papers on transformation, and composite surfaces, 282

Series, characteristic for a linear system of curves, 124, 263; on double curve of a surface by surfaces through the triple points, 169

Severi, *Trattato di geometria algebrica*, 1926, referred to, 37; enumerative formulae for contacts of spaces with curves, 45; correspondence between two curves, by considering their product, 59; formula for coincidences of corresponding spaces, 105; number of cyclical sets in a correspondence, 145; on improper double points of a surface in four-fold space, 157, 171; elimination of multiple points of a surface, 158; on residual intersection of a surface with a surface through its double curve, 168; use of net of curves on a surface to obtain invariant, 183; on virtual curves, 217; class of immersion and canonical number, 222; on surface representing pairs of points of a curve, 227, 282; modification of Zeuthen-Segre invariant in case of isolated nodes, 235; relation of exceptional curves to transformed canonical system, 236; enumerative formulae for a net of curves on a surface, 241; memoir on intersection of manifolds, 247; application of theory of intersections, 250; canonical curves on part intersection of two primals, 262;

postulation of surface residual intersection of two primals, 265; postulation of composite surface, 265; point equivalence of a surface, 267; theorems for a complete intersection, 269, 270; extension of a theorem of Bertini, 269; on Picard's theorem for deficiency of series cut by adjoint system, 269; on modular expression of a manifold, 269; fundamental theorem of equivalence of two manifolds, 293, 297; relations of invariants of two surfaces in multiple correspondence, 294; theorem that non-singular primal contains only complete intersections of appropriate dimension, 296; for Grassmannian, 296; theorem of completeness of system of surfaces on a primal, 297

Simart, *see* Picard

Slaught, *see* Moore and Slaught

Spaces, linear, numbers satisfying given conditions, fundamental theorems, 70

Stationary contact, of a tangent plane with a surface, 149

Steiner, locus of points of contact of curves of two plane pencils, 240

Sturm, normals of a surface, 104

Sum, and product, of two correspondences, 6; sum, and difference, of linear systems of curves, 197

Surface, representing the sets of an involution, 133; containing a pencil of rational curves, 145; in ordinary space, preliminary properties of, 148; Salmon's formulae for, 159, 164; with only double curve and triple points, 157; in space of four dimensions, 169; in four dimensions which is complete intersection, characters of, 175, 247; residual intersection of two with a common curve, 251; a characteristic equation proved from theory of intersections, 254; and primal, residual intersection of, 255; in four-fold space, number of accidental double points, 259; two, forming complete intersection of two primals, sum of invariants, 262; whose prime sections are hyperelliptic is rational 272; representing points of two curves, 282; representing pairs of points of one curve, 285, 289; representing pairs of points of a curve of genus two, 289; representing lines of a cubic primal in four-fold space, 294

Tacnode, effect on invariants of surface, 220, 234, 270

Tangent, multiple, of a manifold, 86; tangent planes to a surface from a multiple line, 187

Timms, on the general Del Pezzo surface, 276

Torelli, on de Jonquières formula for multiple contacts of curves, 42

Torsal chords of a curve, 17; torsal generators of a ruled surface, 26

Transformations, Cremona, in a plane, 112; groups of, 278, 299

Trisecants, ruled surface by, 32; to a surface in four-fold space, 172; trisecant curve of a surface in four-fold space, 242

Ursell, on quadric surfaces satisfying nine conditions, 251

Vahlen, manifolds defined by matrices, 111

Val, Du, on bitangent curve for a plane quartic curve, 14

Valency, of a correspondence, 3, 54

Veronese, relations of two curves forming a complete intersection, 251; on a rational surface, 274

Voss, umbilici of a general surface, 102

Weddle surface, transformation to a Kummer surface, 237

Welchman, on contact primes of the canonical curve, 43

White, example of a (2, 1) correspondence between two curves, 24; on the Bordiga surface, 274

Wiman, groups of Cremona transformations in a plane, 121, 299

Zeuthen, formula for correspondence between two curves, 19; on torsal generators of a ruled surface, 27; on ruled surface by trisecants of a curve, 32; on quadrisecants, 34; *Lehrbuch der abzählenden Methoden der Geometrie*, 1914, referred to, 19, 27, 34, 42, 69, 91, 181, 183, 235;

number of triple points of the nodal curve of a given curve, 38; on de Jonquières' formula for multiple contacts of curves, 42; on multiple tangents, 91; on multiple correspondence, fundamental results, 92; on pinch points and close points, memoir, 178; procedure by enveloping cone for invariants of a surface, 183; Zeuthen-Segre invariant, 185, 189, 199, 206, 210, 233, 247; for pencil of surfaces, 214, 241; enumerative formulae for a net of curves on a surface, 241

CAMBRIDGE: PRINTED BY WALTER LEWIS, M.A., AT THE UNIVERSITY PRESS